Simple model of self-organized biological evolution as a completely integrable dissipative system

Yu. M. Pis'mak [1] [2]
Institut für Theoretische Physik
Freie Universität Berlin, Arnimallee 14, 14195 Berlin, Germany

June 26, 1998

Abstract

The Bak-Sneppen model of self-organized biological evolution of an infinite ecosystem of randomly interacting species is represented in terms of an infinite set of variables which can be considered as an analog to the set of integrals of motion of completely integrable system. Each of this variables remains to be constant but its influence on the evolution process is restricted in time and after definite point in time its value is excluded from description of the system dynamics.

[1] Permanent address: Department of Theoretical Physics State University of Sankt-Petersburg Ul'yanovskaya 1, Stary Petergof, 198904 Sankt-Petersburg, Russia
[2] e-mail: pismak@snoopy.phys.spbu.ru, pismak@physik.fu-berlin.de

1 Introduction

Investigations of complex dynamical systems play an important role in the modern theoretical and mathematical physics. In the recent years, the essential achievements have been arrived in studies of two dimensional field theoretical models. A large class of the exactly solvable and completely integrable models has been found [1]. It enabled one to understand the lows of the soliton dynamics creating many non-ordinary phenomena in several physical systems.

The non-trivial completely integrable models like sine-Gordon ore non-linear Schödinger equations are characterized by the infinite set of the conservations lows [1]. If the dynamics of such a system is considered in the framework of Hamilton's formalism and the integral of motions are chosen as momenta , then evolution of the system in such a description appears to be similar to one of the mechanical system of non-interacting material points: the momenta ate constant and the coordinates are linear functions of the time.

Many theoretical physicist hope that there are universal types of dynamics of natural processes, which can be classified. A possible theoretical approach for constructing such a classification could be the following. For the model under consideration one needs to find its "principal" representation, i.e. to formulate the dynamics of the system in terms of "principal" variables for which the dynamics is as simple as possible. If the principal representations of two models are the same, it is naturally to consider these models as representative of the same type of dynamics. From this point of view the system of free particles is the representative of the class of all the completely integrable systems.

If the construction of such a typology could be possible it would be very useful for understanding the most essential features in dynamics of complex natural systems. However, to find that what we call the principal representation is not an easily problem for complex dynamical system. Probably, the set of ones for which this problem can be solved is restricted but it seems to be important to reveal the non-trivial systems of such a kind.

In the present paper the proposed by Bak and Sneppen model (BSM) of biological evolution [2] is considered. It is a dynamical system describing the ecosystem evolution as mutation and natural selection of interacting species. The most essential property of dynamics in this model is the self-organized criticality (SOC). The main characteristic of the SOC dynamics is that the system evolves to a critical state without fine tuning of its parameters. The SOC type in the BSM has the main specific features of real biological evolution considered in the framework of the Gould-Eldridge "punctuated equilibrium" conception [3]. One can hope that this model represent an important universal type of critical dynamics of avalanche-like processes, realizing in many systems [4].

The simplest version of the BSM with random interaction structure [5] appeared to be exact solvable both in the thermodynamical limit and for finite system [6], [7], [8]. For infinite system it is possible to construct its principal representation. It is done in this paper. The dynamics in terms of principal variables looks as follows: each variable conserves its initial value the definite (different for several variables) length of time and after that becomes to be equal to zero. Thus the principal variables in this model are not exact integral of motion but in the process of the system evolution they transform (to zero) only ones. For an alternative set of principal variables, one of them is loosed at each time step and the others are renumbered.

2 Formulation of the model

In the framework of the BSM the biological evolution of ecosystem is described as follows [1]. The state of the ecosystem of N species is characterized by a set $\{x_1, ..., x_N\}$ of N number, $0 \leq x_i \leq 1$. The number x_i represents the barrier of the i-th species toward further evolution. Initially, each x_i is set to a randomly chosen value. At each time step the barrier x_i with minimal value and $K-1$ other barriers are replaced by K new random numbers. In the random neighbor model (RNM) [5] which will be considered in this paper the $K-1$ replaced non-minimal barriers are chosen at random. In the local or nearest neighbor model (LM) these are the barriers of the nearest neighbors to the species with minimal barrier. For each species in the LM the nearest neighbors are assumed to be defined.

For the LM t he most of results are obtained by numerical experiments or in the framework of mean field approximation. The RNM is more convenient for analytical studies. The master equations obtained in [6] for RNM are very useful for this aim. These equations appeared to be exact solvable. The stationary solution was found in [6]. The time dependent solutions were obtained in [7] (infinite system), [8] (finite system).

In this paper we construct the principal representation of the RNM master equation for infinite system. We restrict ourselves with the simplest case of two interacting species, i.e $K = 2$.

The obtained in [6] master equations for the RNM are of the form:

$$\begin{aligned} P_n(t+1) = A_n P_n(t) + B_{n+1} P_{n+1}(t) + C_{n-1} P_{n-1}(t) + \\ D_{n+2} P_{n+2}(t) + (B_1 \delta_{n,0} + A_1 \delta_{n,1} + C_1 \delta_{n,2}) P_0(t) \end{aligned} \tag{1}$$

Here, $P_n(t)$ is the probability that n is the number of barriers having values less than a fixed parameter λ at the time t; $0 \leq n \leq N$, $0 \leq \lambda \leq 1$, $0 \leq t$. The initial probability distribution $P_n(0)$ is supposed to be given. For $0 < n \leq N$

$$\begin{aligned} A_n &= 2\lambda(1-\lambda) + \frac{n-1}{N-1}\lambda(3\lambda - 2), \\ B_n &= (1-\lambda)^2 + \frac{n-1}{N-1}(1-\lambda)(3\lambda - 1), \\ C_n &= \lambda^2 - \frac{n-1}{N-1}\lambda^2, \quad D_n = (1-\lambda)^2 \frac{n-1}{N-1}, \end{aligned} \tag{2}$$

and $A_n = B_n = C_n = D_n = 0$ for $n = 0, n > N$.

In virtue of the definition of $P_n(t)$,

$$P_n(t) \geq 0, \tag{3}$$

$$\sum_{n=0}^{N} P_n(t) = 1. \tag{4}$$

Making summation in (1) over n and taking into account (2) it is easy to prove that

$$\sum_{n=0}^{N} P_n(t+1) = \sum_{n=0}^{N} P_n(t). \tag{5}$$

Therefore, if $P_n(0)$ are chosen in such a way that (4) is fulfilled for $t = 0$, then in virtue of (5) it is the case for the solution of (1) for $t > 0$ too.

For analysis of (1) it is convenient to introduce the generating function $q(z,u)$:

$$q(z,u) \equiv \sum_{t=0}^{\infty} \sum_{n=0}^{N} P_n(t) z^n u^t. \tag{6}$$

In virtue of (3),(4) for finite system $q(z,t)$ is polynomial in $z$and analytical in u for $|u| < 1$. For infinite system $q(z,t)$ is analytical in z, u for $|u| < 1$, $|z| < 1$. The master equations (1) can be rewritten for the generating function $q(z,u)$ as follows:

$$\frac{1}{u}(q(z,u) - q(z,0)) = (1 - \lambda + \lambda z)^2 \left(\frac{1}{z} \left(1 - \frac{1-z}{N-1} \left(\frac{1}{z} - \frac{\partial}{\partial z} \right) \right) \times \right.$$
$$\left. \times (q(z,u) - q(0,u)) + q(0,u) \right). \tag{7}$$

In (7) the function

$$q(z,0) = \sum_{n=0}^{N} P_n(0) z^n$$

is assumed to be given.

3 Principal representation

For infinite system the master equation (7) has the following form

$$(z - u\rho^2) q(z,u) = (z-1) u \rho^2 q(0,u) + z q(z,0). \tag{8}$$

Here, $\rho \equiv 1 - \lambda + \lambda z$. Let us denote

$$Q(z,u) \equiv \frac{(z - u\rho^2) q(z,u)}{(z-1)\rho^2}, \quad T(z,u) \equiv \frac{q(z,u)}{(z-1)\rho^2} = \frac{Q(z,u)}{z - u\rho^2}. \tag{9}$$

Then rewriting the equation (8) as

$$\frac{(z - u\rho^2) q(z,u)}{(z-1)\rho^2} = u q(0,u) + \frac{z q(z,0)}{(z-1)\rho^2}$$

we see that the function $Q(z,u)$ fulfill the relation

$$Q(z,u) = Q(0,u) + Q(z,0) \tag{10}$$

In virtue of (10)

$$\frac{\partial^2}{\partial u \partial z} Q(z,u) = 0. \tag{11}$$

Substituting $Q(z,u) = (z - u\rho^2)T(z,u)$ in (11) we obtain the equation for $T(z,u)$

$$\frac{\partial}{\partial z}\left[\rho^2 \left(1 + u\frac{\partial}{\partial u}\right) T(z,u)\right] = \frac{\partial}{\partial z}\left[z\frac{\partial}{\partial u} T(z,u)\right] \tag{12}$$

In virtue of analytical properties of the generating function $q(z,u)$ and definitions (9) the functions $Q(z,u)$, $T(z,u)$ are analytical in the variables u, z for $|u| < 1$, $|z| < \min\{1, (1-\lambda)/\lambda\}$. Hence the function $T(z,u)$ can be presented as

$$T(z,u) = \sum_{n=0}^{\infty} T_n(z) u^n \tag{13}$$

where the series is convergent for $|u| < 1$ and $|z| < 1$, $|z| < (1-\lambda)/\lambda$. Substituting in (12) the function $T(z,u)$ in the form of (13) we obtain the equation for the functions $T_n(z)$

$$\frac{\partial}{\partial z}\rho^2 T_n(z) = \frac{\partial}{\partial z} z T_{n+1}(z). \tag{14}$$

From (14) it follows that

$$\rho^2 T_n(z) = z T_{n+1}(z) + c_n, \quad n \geq 0. \tag{15}$$

where c_n are constants. In terms of the function

$$\tau(z,n) = \frac{z^n}{\rho^{2(n-1)}} T_n(z) \tag{16}$$

the equation (15) is written in the form

$$\tau(z, n+1) - \tau(z,n) = -\frac{z^n}{\rho^{2n}} c_n, \quad n \geq 0. \tag{17}$$

Let us denote

$$y = \frac{z}{\rho^2}, \quad \zeta(y,t) = \tau(z(y), t). \tag{18}$$

where the function $z(y)$ is the analytical in the neighborhood of $y = 0$ solution in respect to y of the first of the relations (18)

$$z(y) = \frac{1 - 2y\lambda(1-\lambda) - \sqrt{1 - 4y\lambda(1-\lambda)}}{2\lambda^2 y}. \tag{19}$$

Substituting in (17) $z = z(y)$ and using the notations (18) we obtain the equation

$$\zeta(z, n+1) - \zeta(z, n) = -c_n y^n, \quad n \geq 0. \tag{20}$$

Summing the first $t-1$ equations (20) we have

$$\zeta(y, t) = \zeta(y, 0) - \sum_{n=0}^{t-1} c_n y^n. \tag{21}$$

In virtue of convergence of the series (13) for small z, u, it follows that $T_n(z) \to 0$, $\zeta(y, n) \to 0$ if $n \to \infty$ and $|z|$, $|y|$ are sufficiently small. Taking $t \to \infty$ in (21) we obtain

$$\zeta(y, 0) = \sum_{n=0}^{\infty} c_n y^n. \tag{22}$$

Substituting (22) in (21) we see that the function $\zeta(y, t)$ can be expressed in terms of the constants c_n as a series

$$\zeta(y, t) = \sum_{n=t}^{\infty} c_n y^n. \tag{23}$$

Thus, if we define the functions $Z_k(t)$ as the coefficients of the power series expansion for $\zeta(y, t)$

$$\zeta(y, t) = \sum_{n=0}^{\infty} Z_n(t) y^n, \tag{24}$$

then comparing (23) with (24) we see that the $Z_n(t)$ are very simple functions on t:

$$Z_n(t) = c_n \ \text{ for } \ t \leq n, \quad Z_n(t) = 0 \ \text{ for } \ t > n. \tag{25}$$

Using (9), (13), (16), (18), (24) one can find the explicit expressions of variables $P_n(t)$ in terms of $Z_n(t)$. In compact form they look as follows:

$$\sum_{n=0}^{\infty} P_n(t)z^n = (z-1)\sum_{n=0}^{\infty} Z_n(t)\frac{z^{n-t}}{\rho^{2(n-t)}}. \tag{26}$$

The expressions $Z_n(t)$ in terms of $P_n(t)$ can be obtained with help of relation

$$\sum_{n=0}^{\infty} Z_n(t)y^n = y^t \sum_{n=0}^{\infty} P_n(t)\frac{z(y)^n}{z(y)-1}, \tag{27}$$

where $z(y)$ is defined by (19)

The variable transformation defined by (26), (27) is explicitely dependent on the time t. For variables

$$C_n(t) = Z_{n+t}(t)$$

the dynamics is defined by homogeneous equation

$$C_n(t+1) = C_{n+1}(t). \tag{28}$$

with initial conditions

$$C_n(0) = c_n. \tag{29}$$

It has the simple solution:

$$C_n(t) = c_{n+t}. \tag{30}$$

The analogs of (26), (27) for $C_n(t)$ are of the form

$$\sum_{n=0}^{\infty} P_n(t)z^n = (z-1)\sum_{n=0}^{\infty} C_n(t)\frac{z^n}{\rho^{2(n)}}, \tag{31}$$

$$\sum_{n=0}^{\infty} C_n(t)y^n = \sum_{n=0}^{\infty} P_n(t)\frac{z(y)^n}{z(y)-1}. \tag{32}$$

The variable transformations defined by (31), (32) are not independent in an evident way on the time.

The variables $Z_n(t)$, $C_n(t)$ can be considered as the principal ones for the RNM. Principal representation of its dynamics is defined by equations (25), (28), (29).

4 Conclusion

We obtained the following results. In terms of in a special way chosen variables the SOC dynamic in BSM can be described by very simple equations (25), (28). These variables are expressed straightforwardly in terms of infinite set of the constants defined explicitly by the initial conditions. For each time step one of this constant is forgotten, i.e its value do not influence on the further stages of the system evolution. The consequent loose of the information about initial state is all what happens in the self-organization process described by RNM. The system of such a kind could be called completely integrable dissipative system. It would be interesting to understand how robust this dynamics is. Is it the inherent property of the SOC processes or an artifact of the considered model.

Acknowledgments

The author is grateful to R.Schrader and J.Plän for kind hospitality at Institut für Theoretische Physik of Freie Universität Berlin. This work was supported in part by Russian Foundation for Basic Research (Grant No 97-01-01152) and Russian State Committee for High Education (Grant No 97-14.3-58).

References

[1] L.D. Faddeev and L.A. Takhtadjan, *Hamiltonian Methods in the Theory of Solitons* (Springer, Berlin, 1987)

[2] P. Bak and K. Sneppen, *Phys. Rev. Lett.* **71** (1993) 4083

[3] S.J.Gould and N.Edredge, *Paleobiology* **3** (1977) 115; *Nature* (London) **336** (1993) 223;

[4] K.Sneppen, P.Bak. H.Flyvbjerg and M.H.Jensen, *Proc.Natl. Acad.Sci.USA* **92** (1995) 5209

[5] H.Flyvbjerg, P.Bak and K.Sneppen, *Phys. Rev. Lett.* **71** (1993) 4087

[6] J. de Boer, B.Derrida, H.Flyvbjerg, A.D.Jackson, and T.Wettig, *Phys.Rev. Lett.* **73** (1994) 906.

[7] Yu.M.Pis'mak, *J.Phys.A* **28** (1995) 3109

[8] Yu.M.Pis'mak, *Phys. Rev. E* **56** (1997) R1325