On the spectrum of gauge periodic point perturbations on the Lobachevsky plane

J. Brüning and V. A. Geyler

Introduction

Let H be the periodic Schrödinger operator with a uniform magnetic field on the Euclidean plane $\mathbb{R}^2$; the spectral structure of H depends drastically on the flux η of the field through an elementary cell of the period lattice: If η is a rational number, then the spectrum of H has a band structure, whereas for irrational η regions with Cantor spectrum may appear [1]. The situation is different in the case of the Lobachevsky plane. Indeed, if the group of periods of H is the modular group $SL(2, \mathbb{Z})$, then the spectrum of H has band structure for any value of the flux η [2],[3]. This result is obtained under the condition that the periodic perturbation of the free magnetic Hamiltonian is the operator of multiplication by a periodic function. On the other hand, an interesting class of periodic Schrödinger operators is obtained by so-called point perturbations since these perturbations give a broad collection of explicitly solvable models [4], [5]. In particular, the point perturbations of the two-dimensional magnetic Schrödinger operator are widely used in theoretical physics to investigate the transport properties of two-dimensional systems [6], [7].

In the present paper, the results of the articles [2], [3] are extended to periodic point perturbations of magnetic Schrödinger operators on the Lobachevsky plane. It is proved that these operators have band spectrum, too, if the associated C^*-algebra has the Kadison property. This result seems to be relevant in studying how the geometry of a two-dimensional electron system influences its spectral and transport properties [8], [9].

1. The free Hamiltonian

We consider a two-dimensional complete Riemannian manifold X of negative curvature (the Lobachevsky plane). We suppose that X is realized as the Poincaré upper half-plane, $\mathbb{H}$,

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}, \tag{1.1}$$

endowed with the metric

$$ds^2 = \frac{a^2}{y^2}\,(dx^2 + dy^2); \tag{1.2}$$

thus, the curvature of X is equal to $R = -1/a^2$. The geodesic distance between points $z = x + iy, z' = x' + iy' \in X$ has the form

$$d(z, z') = a \cosh^{-1}\left[1 + \frac{|z - z'|^2}{2yy'}\right], \tag{1.3}$$

and the volume form is given by

$$d\sigma = \frac{a^2}{y^2}\, dx \wedge dy. \tag{1.4}$$

By definition, a constant uniform magnetic field $\mathbf{B}$ perpendicular to X is a 2-form

$$\mathbf{B} = B\frac{a^2}{y^2}\, dx \wedge dy, \quad B \in \mathbb{R}, \tag{1.5}$$

where B is the strength of the field. The form $\mathbf{B}$ is exact, i.e. $\mathbf{B} = d\mathbf{A}$, where the 1-form $\mathbf{A}$ is called a vector potential of $\mathbf{B}$. The vector potential $\mathbf{A}$ is definded up to a gauge term; we shall use the so-called Landau gauge,

$$\mathbf{A} = \frac{Ba^2}{y}\, dx. \tag{1.6}$$

The Hamiltonian of a free quantum-mechanical particle (of mass m and charge e) moving in the plain X subjected to the external field $\mathbf{B}$ has the form [10]

$$H^0 = \frac{y^2}{2ma^2}\left\{\left(-i\hbar\frac{\partial}{\partial x} - \frac{e}{c}\frac{b}{y}\right)^2 - \hbar^2\frac{\partial^2}{\partial y^2}\right\}, \tag{1.7}$$

where we write

$$b := Ba^2 \tag{1.8}$$

(as usual, c denotes the velocity of light and $\hbar$ is the Planck constant). In what follows we use a system of units in which $e = c = \hbar = 1$ and $m = 1/2$. In this case, H^0 is a self-adjoint operator in $L^2(X)$ namely the closure of the symmetric operator τ^0,

$$\tau^0 = \frac{1}{a^2}\left\{-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + 2iby\frac{\partial}{\partial x} + b^2\right\}, \tag{1.9}$$

with domain $\mathcal{D}(\tau^0) = C_0^\infty(X)$. It is well known (see e.g. [11]) that $\mathcal{D}(H^0) \subset C(X)$. It is useful to note that with

$$Df := df + i\mathbf{A} \wedge f, \quad f \in C_0^\infty(X), \tag{1.10a}$$

we obtain

$$\tau^0 = D^* D. \tag{1.10b}$$

The spectrum of H^0, $\operatorname{spec} H^0$, consists of two parts. The first one is the pure point spectrum formed by the finitely many eigenvalues (the Landau levels)

$$\lambda_l = \frac{1}{a^2}\left[\frac{1}{4} + b^2 - \left(l + \frac{1}{2} - |b|\right)^2\right], 0 \leq l < |b| - \frac{1}{2}\,. \tag{1.11}$$

The second part of the spectrum is the absolutely continuous spectrum which fills out the whole semi-axis $\left[\frac{1}{a^2}\left(\frac{1}{4} + b^2\right), \infty\right)$[12].

The main role in this paper is played by the resolvent, $R^0(\zeta) = (H^0 - \zeta)^{-1}$, of H^0. The integral kernel of $R^0(\zeta)$ (i.e., the Green's function $G^0(z; z'; \zeta)$ of H^0) is determined in [10]; it has the form

$$G^0(z, z'; \zeta) = \frac{\sigma^{-t}}{4\pi}\exp(ib\varphi)\frac{\Gamma(t+b)\Gamma(t-b)}{\Gamma(2t)}F\left(t+b, t-b; 2t; \frac{1}{\sigma}\right). \tag{1.12}$$

Here $F(\alpha, \beta; \gamma; z)$ is the Gauss hypergeometric function,

$$\sigma := \cosh^2\frac{d(z, z')}{a}, \tag{1.13}$$

$$\varphi := 2\operatorname{arctg}\frac{x - x'}{y + y'}, \tag{1.14}$$

and $t = t(\zeta), \zeta \in \mathbb{C} \backslash \operatorname{spec} H^0$, is uniquely defined by the conditions

$$\zeta = \frac{t(1-t)+b^2}{a^2}, \quad \operatorname{Re} t > 1/2. \tag{1.15}$$

In the following lemma, we collect the properties of G^0 which are needed below.

Lemma 1 *For any $z \in X$ there exists the limit*

$$\lim_{z' \to z} \left[G^0(z, z'; \zeta) - \frac{1}{2\pi} \log d(z, z') \right] =: q(\zeta). \tag{1.16}$$

This limit does not depend on z, in fact

$$q(\zeta) = -\frac{1}{4\pi} \left[\psi(t+b) + \psi(t-b) + 2C_E - \log 4a^2 \right]. \tag{1.17}$$

Here $\psi(z) = [\log \Gamma(z)]'$, and C_E is the Euler constant.

Proof According to [13; 2.3.1(2)] we have for $|1-z| < 1$

$$F(\alpha, \beta; \alpha+\beta; z) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(n!)^2} [k_n - \log(1-z)](1-z)^n, \tag{1.18}$$

where $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1)$, and

$$k_n = 2\psi(n+1) - \psi(\alpha+n) - \psi(\beta+n). \tag{1.19}$$

In view of (1.12), we have to perform the limit $\sigma \to 1$. Substituting (1.17) and (1.18) in (1.11) and taking into account that $C_E = -\psi(1)$, we get (1.16). $\square$

Since $\operatorname{Re} t(\zeta) \to -\infty$ as $\operatorname{Re} \zeta \to -\infty$, we obtain the following assertion from the well known asymptotics of $\psi(z)$.

Lemma 2 $\operatorname{Re} q(\zeta) \to -\infty$ *as* $\operatorname{Re} \zeta \to -\infty$.

Lemma 3 *For every $\varepsilon > 0$ and $\zeta \in \mathbb{C}$ with $\operatorname{Re} \zeta < 0$, there exist constants $c_1(\varepsilon, \zeta) = c_1 > 0$ and $\tilde{c}_1(\varepsilon, \zeta) = \tilde{c}_1 > 0$ such that*

$$|G^0(z, z'; \zeta)| \leq c_1 \exp(-\tilde{c}_1 d(z, z')), \tag{1.20}$$

whenever $d(z, z') \geq \varepsilon$.

Moreover, if ε is fixed then $c_1(\zeta) = o(1)$ and $\tilde{c}_1(\zeta) \to \infty$ as $\operatorname{Re} \zeta \to -\infty$.

Proof If $\operatorname{Re}\zeta < 0$, then from formula 2.12 (1) in [13] we get for $|z| < 1$

$$F(t+b, t-b; 2t; z) = \frac{\Gamma(2t)}{\Gamma(t+b)\Gamma(t-b)} \int_0^1 s^{t-b-1}(1-s)^{t+b-1}(1-zs)^{-t-b} ds. \tag{1.21}$$

The assumptions imply $\operatorname{Re} t > |b| + 1/2$ and $z \in \mathbb{R}$, $|z| < 1$. Then we estimate

$$\begin{aligned}
&\left| \int_0^1 s^{t-b-1}(1-s)^{t+b-1}(1-zs)^{-t-b} ds \right| \\
= &\left| \int_0^1 s^{t-b-1/2} \left(\frac{1-s}{1-zs} \right)^{t+b-1/2} (1-zs)^{-1/2}(s(1-s))^{-1/2} ds, \right| \\
\leq &\ (1-|z|)^{-1/2} \int_0^1 s^{\operatorname{Re} t - |b| - 1/2}(s(1-s))^{-1/2} ds,
\end{aligned}$$

and the last integral converges to 0, by dominated convergence. Substituting this in (1.11), we conclude the proof. $\square$

Lemma 4 *Let K be a compact subset of X and z_0 a fixed point of X. Then for every $\varepsilon > 0$ and $\zeta \in \mathbb{C}, \operatorname{Re}\zeta < 0$, there exist constants $c_2(K, z_0, \varepsilon, \zeta) =: c_2 > 0$ and $\tilde{c}_2(\varepsilon, \zeta) =: \tilde{c}_2 > 0$ such that*

$$\sup\{|G^0(z, z'; \zeta)| : z' \in K\} \leq c_2 \exp(-\tilde{c}_2 d(z, z_0)), \tag{1.22}$$

whenever $d(z, K) \geq \varepsilon$.

Moreover, if K, z_0, and ε remain fixed then $c_2(\zeta) = o(1)$ and $\tilde{c}_2(\zeta) \to \infty$ as $\operatorname{Re}\zeta \to -\infty$.

Proof Let $z, z' \in X$ such that $z' \in K, d(z, K) \geq \varepsilon$. Then $d(z, z') \geq \varepsilon$ and

$$d(z, z') \geq d(z, z_0) - d(z_0, z') \geq d(z, z_0) - k, \tag{1.23}$$

where $k = \sup\{d(z', z_0) : z' \in K\}$. Substituting (1.23) in (1.19) completes the proof. $\square$

We now recall that a bounded linear operator, L, in the space $L^2(X)$ is called a Carleman operator if L has an integral kernel $L(z,z')$ such that

$$\int\limits_X |L(z,z')|^2 d\sigma(z') < \infty$$

for almost every $z \in X$ [14, Thm. 11.6].

Lemma 5 *For any $\zeta \in \mathbb{C}\backslash \operatorname{spec} H^0$, the resolvent $R^0(\zeta)$ is a Carleman operator. Moreover, the integral*

$$\int\limits_X |G^0(z,z';\zeta)|^2 d\sigma(z') \tag{1.24}$$

does not depend on z.

Proof The last statement of the lemma is valid because, for a fixed ζ, the value $|G^0(z,z';\zeta)|$ depends on $d(z,z')$ only. Fix $z_0 \in X$ and denote by $B_\varepsilon(z_0)$ the metric ball around z_0 in X of radius ε.

Taking into account Lemma 1 we obtain with $\varepsilon = \varepsilon(z_0,\zeta)$

$$\int\limits_{B_\varepsilon} |G^0(z_0,z';\zeta)|^2 d\sigma(z') < \infty, \tag{1.25}$$

and with Lemma 3,

$$\int\limits_{B_1} |G^0(z_0,z';\zeta)|^2 d\sigma(z') < \infty.$$

It is known that the area of the circle B_n is equal to $2\pi(\cosh\frac{n}{a} - 1)$. Hence, area $(B_{n+1}\backslash B_n) = 0(e^n)$ as $n \to \infty$. According to Lemma 3 we can find a number ζ_0, $\operatorname{Re}\zeta_0 < 0$, such that

$$\int\limits_{B_{n+1}\backslash B_n} |G^0(z_0,z';\zeta_0)|^2 d\sigma(z') = 0(e^{-2n}), \tag{1.26}$$

implying

$$\int\limits_{X\backslash B_1} |G^0(z_0,z';\zeta_0)|^2 d\sigma(z') < \infty.$$

Therefore, $R^0(\zeta_0)$ is a Carleman operator. It is known that the space of Carleman operators in $L^2(X)$ is a right ideal in the algebra $\mathcal{L}(L^2(X))$ of all bounded linear operators in $L^2(X)$[14]. Using the Hilbert identity

$$R^0(\zeta) = R^0(\zeta_0) + (\zeta - \zeta_0)R^0(\zeta_0)R^0(\zeta) \tag{1.27}$$

we see that $R^0(\zeta)$ is a Carleman operator for any $\zeta \in \mathbb{C}\backslash \operatorname{spec} H^0$. □

Lemma 6 *Let K be a compact subset of X and z_0 any point in X. Then for any $\zeta \in \mathbb{C}, \operatorname{Re}\zeta < 0$, there exist constants $c_3(K, z_0, \zeta) = c_3 > 0$ and $\tilde{c}_3(\zeta) = \tilde{c}_3 > 0$ such that*

$$\left[\int_K |G^0(z, z'; \zeta)|^2 d\sigma(z')\right]^{1/2} \leq c_3 \exp(-\tilde{c}_3 d(z, z_0)). \tag{1.28}$$

Moreover, $\tilde{c}_3(\zeta) \to \infty$ as $\operatorname{Re}\zeta \to -\infty$. If K and z_0 are fixed, then $c_3(\zeta) = 0(1)$ as $\operatorname{Re}\zeta \to -\infty$.

Proof This is an easy consequence of Lemmas 4 and 5. □

2. Γ-Equivariance

Let Γ be a group of isometries of the plane X. The field $\mathbf{B}$ is invariant with respect to Γ but the Hamiltonian H^0 is not. To obtain the invariance group of H^0 we must consider an extension of Γ, the so-called "magnetic translation group" [15], [3]. Let us recall the construction of this group.

Denote by U the standard unitary representation of Γ in $L^2(X)$:

$$U_\gamma f(z) = f(\gamma^{-1}z), \quad \gamma \in \Gamma, \quad f \in L^2(X). \tag{2.1}$$

If $\gamma^*\mathbf{A} \neq \mathbf{A}$ then $U_\gamma H^0 \neq H^0 U_\gamma$. Nevertheless, $d(\gamma^*\mathbf{A} - \mathbf{A}) = 0$ because $\gamma^*\mathbf{B} = \mathbf{B}$. Hence there exists a function $\omega_\gamma \in C^\infty(X)$ such that

$$d\omega_\gamma = \gamma^*\mathbf{A} - \mathbf{A}. \tag{2.2}$$

Fix for every $\gamma \in \Gamma$ such a function ω_γ; for $\gamma = 1$ we put $\omega_1 = 0$. Introduce the unitary operator

$$W_\gamma f := \exp(i\omega_\gamma) f, \quad f \in L^2(X), \tag{2.3}$$

and define $T^0_\gamma = W_\gamma U_\gamma$. Then $T^0_\gamma H^0 = H^0 T^0_\gamma$ for each $\gamma \in \Gamma$, by (1.10). Unfortunately, the correspondence $\gamma \mapsto T^0_\gamma$ is not a unitary representation of Γ in $L^2(X)$ but only a projective representation in the sense that

$$T^0_\beta T^0_\gamma = \Theta(\beta,\gamma) T^0_{\beta\gamma}, \quad \beta,\gamma \in \Gamma, \tag{2.4}$$

where $\Theta(\beta,\gamma) \in \mathbb{C}, |\Theta(\beta,\gamma)| = 1$. The family $\Theta(\beta,\gamma)$ has the property

$$\Theta(\gamma_1,\gamma_2)\Theta(\gamma_1\gamma_2,\gamma_3) = \Theta(\gamma_1,\gamma_2\gamma_3)\Theta(\gamma_2,\gamma_3), \tag{2.5}$$

i.e. this family is a 2-cocycle of the group Γ with coefficients in $\mathbb{U}(1)$. This cocycle determines a group extension of Γ by $\mathbb{U}(1)$,

$$1 \to \mathbb{U}(1) \to M(\Gamma,\Theta) \to \Gamma \to 1; \tag{2.6}$$

the group $M(\Gamma,\Theta)$ is called the *magnetic translation group*. An explicit construction of $M(\Gamma,\Theta)$ is the following: $M(\Gamma,\Theta) = \Gamma \times \mathbb{U}(1)$ with multiplication defined by

$$(\gamma_1,\zeta_1)(\gamma_2,\zeta_2) = (\gamma_1\gamma_2, \Theta(\gamma_1,\gamma_2)\zeta_1\zeta_2). \tag{2.7}$$

Denote by $[\gamma,\zeta]$ the unitary operator ζT^0_γ; the correspondence $(\gamma,\zeta) \mapsto [\gamma,\zeta]$ is then a faithful unitary representation of the group $M(\Gamma,\Theta)$ in $L^2(X)$; we shall denote this representation by T. H^0 is invariant with respect to T; we will refer to this fact as the *gauge-periodicity* of H^0.

We need the following lemma.

Lemma 7 *Let L be a linear integral operator in $L^2(X)$ with kernel $L(z,z')$, z, $z' \in X$. The operator L is invariant with respect to T if and only if for any $\gamma \in \Gamma$ the following relation holds a.e.:*

$$\exp(i\omega_\gamma(z))L(\gamma^{-1}z,z') = \exp(i\omega_\gamma(\gamma z'))L(z,\gamma z'), \tag{2.8}$$

or, equivalently,

$$L(z,z') = \exp[i(\omega_\gamma(z) - \omega_\gamma(z'))]L(\gamma^{-1}z,\gamma^{-1}z'). \tag{2.9}$$

Proof Let $f \in \mathcal{D}(L), \gamma \in \Gamma$, then we have

$$[\gamma,1]Lf(z) = \exp(i\omega_\gamma(z)) \int_X L(\gamma^{-1}z,z')f(z')d\sigma(z'), \tag{2.10}$$

$$\begin{aligned} L[\gamma,1]f(z) &= \int\limits_X L(z,z')\exp(i\omega_\gamma(z'))f(\gamma^{-1}z')d\sigma(z') = \\ &= \int\limits_X L(z,\gamma z')\exp(i\omega_\gamma(\gamma z'))f(z')d\sigma(z'). \end{aligned} \tag{2.11}$$

Comparing (2.10) and (2.11), we get (2.8). □

From now on, we impose the following requirements on the group Γ:

(Γ1) Γ acts properly discontinuously on X,

(Γ2) the orbit space $\Gamma\backslash X$ is compact.

Fix once and for all a fundamental domain $\mathcal{F}$ of Γ, i.e. a subset $\mathcal{F}\subset X$ such that: (a) $\overline{\mathcal{F}} = \overline{\mathrm{Int}\mathcal{F}}$, (b) $\overline{\mathcal{F}}$ is a compact set, (c) the restriction to $\mathcal{F}$ of the canonical projection $X\to\Gamma\backslash X$ is a bijective mapping.

To construct a gauge periodic point perturbation of H^0 we choose a finite subset $K\subset\mathcal{F}$ and denote by Λ the orbit of $K:\Lambda=\Gamma\cdot K$. The definition of Λ implies that each element $\lambda\in\Lambda$ has a unique representation of the form $\lambda=\gamma x$, where $\gamma\in\Gamma$ and $x\in K$. Define a unitary representation T^d of $M(\Gamma,\Theta)$ in the discrete space $l^2(\Lambda)$ by the rule

$$T^d_{(\gamma,\zeta)}\varphi(\lambda) = \zeta\exp(i\omega_\gamma(\lambda))\varphi(\gamma^{-1}\lambda) \tag{2.12}$$

where $(\gamma,\zeta)\in M(\Gamma,\Theta), \varphi\in l^2(\Lambda)$. We denote the operator $T^d_{(\gamma,\zeta)}$ by $[\gamma,\zeta]$, too.

The proof of the following lemma is similar to that of Lemma 7 and is omitted.

Lemma 8 *Let L be a densely defined closed linear operator in the space $l^2(\Lambda)$ having in the standard basis of this space the matrix $(L(\lambda,\mu))_{\lambda,\mu\in\Lambda}$. The operator L is T^d-invariant if and only if for any $\gamma\in\Gamma$ the following relation holds:*

$$\exp(i\omega_\gamma(\lambda))L(\gamma^{-1}\lambda,\mu) = \exp(i\omega_\gamma(\gamma\mu))L(\lambda,\gamma\mu), \tag{2.13}$$

or, equivalently,

$$L(\lambda,\mu) = \exp[i(\omega_\gamma(\lambda)-\omega_\gamma(\mu))]L(\gamma^{-1}\lambda,\gamma^{-1}\mu). \tag{2.14}$$

The following lemma is significant for the sequel.

Lemma 9 *There exist constants $c_\Lambda > 0$ and $\tilde{c}_\Lambda > 0$ such that for any $\lambda_0 \in \Lambda$ and $r \in \mathbb{R}, r > 0$, we have*

$$\#\{\lambda \in \Lambda : d(\lambda, \lambda_0) \leq r\} \leq c_\Lambda \exp(\tilde{c}_\Lambda r). \tag{2.15}$$

Proof Let d_Γ be the minimal word length metric with respect to a fixed finite set of generators in Γ. It is known (see [16]) that for some constants $k > 0, \tilde{k} > 0$ we have

$$\#\{\gamma \in \Gamma : d_\Gamma(\gamma, \gamma_0) \leq r\} \leq k \exp(\tilde{k} r), \tag{2.16}$$

where $\gamma_0 \in \Gamma$ is arbitrary. Moreover, there exists a constant $k_1 > 0$ such that

$$d_\Gamma(\gamma_1, \gamma_2) \leq k_1(\inf\{d(\gamma_1 z, \gamma_2 z') : z, z' \in \mathcal{F}\} + 1). \tag{2.17}$$

Let λ_0 be a point of Λ; then $\lambda_0 = \gamma_0 x_0$, for some $\gamma_0 \in \Gamma, x_0 \in K$. Hence for $r > 0$

$$\begin{aligned} \# \quad & \{\lambda \in \Lambda : d(\lambda, \lambda_0) \leq r\} = \#\{(\gamma, x) \in \Gamma \times K : d(\gamma x, \gamma_0 x_0) \leq r\} \\ \leq \quad & \sum_{x \in K} \#\{\gamma \in \Gamma : d(\gamma x, \gamma_0 x_0) \leq r\}. \end{aligned} \tag{2.18}$$

Now $d(\gamma x, \gamma_0 x_0) \leq r$ implies $d_\Gamma(\gamma, \gamma_0) \leq k_1(r+1)$, by (2.17), so the proof follows from (2.16). $\square$

3. Gauge periodic point perturbations

We construct a point perturbation of the operator H^0 in the sense of [17]. Since $\mathcal{D}(H^0) \subset C(X)$ we may define the domain

$$\mathcal{D}(S) := \{f \in \mathcal{D}(H^0) : f(\lambda) = 0 \quad \text{for } \lambda \in \Lambda\}, \tag{3.1}$$

and the operator S as the restriction of H^0 to $\mathcal{D}(S)$; clearly, S is a symmetric operator in $L^2(X)$. A self-adjoint extension H of S is said to be a *point perturbation of H^0 supported on* Λ if $\mathcal{D}(H) \cap \mathcal{D}(H^0) = \mathcal{D}(S)$. It is an important fact that the point perturbations of H^0 can be described by means of the Krein resolvent formula [4, 5, 17]. To do so, we must find a Hilbert space $\mathcal{G}$ isomorphic to each deficiency subspace of S, and two holomorphic functions

$$B : \mathbb{C} \backslash \operatorname{spec} H^0 \to \mathcal{L}(\mathcal{G}, L^2(X)),$$

$$Q : \mathbb{C} \backslash \operatorname{spec} H^0 \to \mathcal{L}(\mathcal{G}, \mathcal{G}),$$

satisfying some conditions which are called Krein's $(\Gamma)-$ and $(Q)-$ condition [17]; the functions B and Q are then called the Krein Γ- and Q-function, respectively. Fixing a Γ-function and a Q-function, we determine a one-to-one correspondence between point perturbations H of H^0 and (not necessarily bounded) self-adjoint operators A in $\mathcal{G}$. This correspondence is given by the *Krein resolvent formula* alluded to above:

$$(H-\zeta)^{-1} = (H^0-\zeta)^{-1} - B(\zeta)[Q(\zeta)+A]^{-1}B^*(\overline{\zeta}). \tag{3.2}$$

We denote by H_A the point perturbation H that corresponds to A via (3.2); the resolvent of H_A will be denoted by $R_A(\zeta)$.

Now we give some explicit description of the Krein Γ- and Q-functions, using Theorem 4 and Proposition 4 from [18] (the proofs of these statements are given in [18] in the case where X is a domain in Euclidean space, but these proofs remain valid in the case of Riemannian manifolds X, too). Denote by $\mathcal{G}$ the space $l^2(\Lambda)$ and by $Q(\zeta)$ the infinite matrix (cf. Lemma 1 for the notation)

$$Q(\lambda,\mu;\zeta) = \begin{cases} G^0(\lambda,\mu;\zeta), & \lambda,\mu \in \Lambda, \quad \lambda \neq \mu, \\ q(\zeta), & \lambda,\mu \in \Lambda, \quad \lambda = \mu. \end{cases} \tag{3.3}$$

Lemma 10 (1) *There exist constants $c_4(\zeta) = c_4 > 0$ and $\tilde{c}_4(\zeta) = \tilde{c}_4 > 0$ such that for $\operatorname{Re}\zeta < 0$ we have*

$$|Q(\lambda,\mu;\zeta)| \leq c_4(\zeta)\exp(-\tilde{c}_4(\zeta)d(\lambda,\mu)), \tag{3.4}$$

whenever $\lambda \neq \mu$. Moreover, $c_4(\zeta) = o(1)$ and $\tilde{c}_4(\zeta) \to \infty$ as $\operatorname{Re}\zeta \to -\infty$.
(2) $|Q(\lambda,\lambda;\zeta)| \to \infty$ *as* $\operatorname{Re}\zeta \to -\infty$.

Proof This follows immediately from Lemmas 2 and 3. □

Lemma 11 *There exist a number $E_1 \in \mathbb{R}$ such that for any $\zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta < E_1$, the matrix $Q(\zeta)$ determines a bounded linear operator in $l^2(\Lambda)$ (this operator is denoted by $Q(\zeta)$ as well).*

Proof This is an immediate consequence of Lemmas 9, 10, and A2 (in the appendix). □

For every $\zeta \in \mathbb{C} \backslash \operatorname{spec} H^0$ and $\lambda \in \Lambda$ we denote by $g_\lambda(\zeta)$ the function on X that takes each point $z \in X$ to $G^0(z, \lambda; \zeta)$. It follows from Lemma 5 that $g_\lambda(\zeta) \in L^2(X)$.

Lemma 12 *There exist $\zeta \in \mathbb{C} \backslash \operatorname{spec} H^0$ such that the Gram matrix*

$$(< g_\lambda(\zeta) | g_\mu(\zeta) >)_{\lambda, \mu \in \Lambda}$$

determines a bounded operator in $l^2(\Lambda)$.

Proof Let $\operatorname{Im} \zeta \neq 0$ and $\operatorname{Re} \zeta < E_1$, where E_1 is taken from Lemma 11. By the Hilbert resolvent identity, we have for $\lambda \neq \mu$

$$< g_\lambda(\zeta) | g_\mu(\zeta) >= (\overline{\zeta} - \zeta)^{-1} [Q(\lambda, \mu; \overline{\zeta}) - Q(\lambda, \mu; \zeta)]. \tag{3.5}$$

Since the diagonal elements of the matrices $Q(\zeta)$ and $< g_\lambda(\zeta) | g_\mu(\zeta) >$ are constants at any fixed ζ, the proof follows from Lemma 11. $\square$

Now we state the main result of this section.

Theorem 1 1. *For any $\zeta \in \mathbb{C} \backslash \operatorname{spec} H^0$ the family $(g_\lambda(\zeta))_{\lambda \in \Lambda}$ is a Riesz basis for its own closed linear hull in $L^2(X)$.*

If $B(\zeta) : l^2(\Lambda) \to L^2(X)$ is defined by

$$B(\zeta)\varphi = \sum_{\lambda \in \Lambda} \varphi(\lambda) g_\lambda(\zeta), \quad \varphi \in l^2(\Lambda), \tag{3.6}$$

then $B(\zeta)$ is a Krein Γ-function of the pair (S, H^0).

2. *There exists $E_0 \in \mathbb{R}$ such that for any $\zeta \in \mathbb{C}$ with $\operatorname{Re} < E_0$ the matrix $Q(\zeta)$ determines a Krein Q-function of the pair (S, H^0). Hence for any $f \in L^2(X)$ we have*

$$R_A(\zeta) f = R^0(\zeta) f - \sum_{\lambda \in \Lambda} \left(\sum_{\mu \in \Lambda} [Q(\zeta) + A]^{-1}(\lambda, \mu) \right) < g_\mu(\zeta) | f > g_\lambda(\zeta). \tag{3.7}$$

Proof In view of Lemmas 11 and 12 the theorem follows immediately from Theorem 4 and Proposition 4 of [18]. $\square$

We are interested in T-invariant point perturbations, H_A, only. The following proposition provides a necessary and sufficient condition for T-invariance.

Proposition 1 *The operator H_A is T-invariant if and only if the operator A is T^d-invariant.*

To prove this proposition we need the following lemma.

Lemma 13 *For any $\zeta \in \mathbb{C}\backslash \operatorname{spec} H^0$ and $\gamma \in \Gamma$*

$$[\gamma, 1]B(\zeta) = B(\zeta)[\gamma, 1]. \tag{3.8}$$

In other words, $B(\zeta)$ is an interwining operator for the representations T and T^d.

Proof From (2.8) we have

$$[\gamma, 1]g_\lambda(\zeta) = \exp(i\omega_\gamma(\gamma\lambda))g_{\gamma\lambda}(\zeta), \tag{3.9}$$

hence for $\varphi \in l^2(\Lambda)$ we get

$$\begin{aligned}
[\gamma, 1]B(\zeta)\varphi &= \sum_{\lambda\in\Lambda} \varphi(\lambda)[\gamma, 1]g_\lambda(\zeta) \\
&= \sum_{\lambda\in\Lambda} \varphi(\lambda)\exp(i\omega_\gamma(\gamma\lambda))g_{\gamma\lambda}(\zeta) \\
&= \sum_{\lambda\in\Lambda} \varphi(\gamma^{-1}\lambda)\exp(i\omega_\gamma(\lambda))g_\lambda(\zeta) = B(\zeta)[\gamma, 1]\varphi.
\end{aligned}$$

$\square$

Proof of Proposition 1: Take $E_0 \in \mathbb{R}$ from Theorem 1, then for $\zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta < E_0$ the operator $Q(\zeta)$ is T^d-invariant, by Lemmas 7 and 8. Consequently, $[Q(\zeta) + A]^{-1}$ is T^d-invariant if and only if A is T^d-invariant. Hence the proposition follows from Lemma 13 and from the fact that $R^0(\zeta)$ is T-invariant. $\square$

In what follows we consider only self-adjoint extensions, H_A, that are invariant with respect to the representation T. For applications in physics, the most interesting case arises if A is a diagonal matrix in the standard basis of the space $l^2(\Lambda)$ [4], [6], [7]; only these operators appear as the limits of Hamiltonians with short-range potentials [4]. In this case, the invariance property of A implies that there are only finitely many values among the diagonal elements of A. From now on we restrict ourselves to this class of operators. It follows from Lemma A2 and Lemma 8 that these operators are bounded. Moreover, Theorem A1 in the appendix implies the following assertion.

Theorem 2 *There is a number $E_A \in \mathbb{R}$ with the following properties:*
(1) for any $\zeta \in \mathbb{C}$ with $\mathrm{Re}\,\zeta < E_A$, the operator $Q(\zeta) + A$ has a bounded inverse.
(2) If $\mathrm{Re}\,\zeta < E_A$ then there are constants $c_5(\zeta) = c_5 > 0$ and $\tilde{c}_5(\zeta) = \tilde{c}_5 > 0$ such that for any $\lambda, \mu \in \Lambda$

$$|[Q(\zeta) + A]^{-1}(\lambda, \mu)| \leq c_5 \exp(-\tilde{c}_5 d(\lambda, \mu)). \tag{3.10}$$

Moreover, $c_5(\zeta) = 0(1)$ and $\tilde{c}_5(\zeta) \to \infty$ as $\mathrm{Re}\,\zeta \to -\infty$.

Corollary 1 *The operator H_A is semibounded from below.*

4. The main result

First we recall the notion of the twisted group C^*-algebra $C^*(\Gamma, \Theta)$ of the pair (Γ, Θ) [19], [20], [21]. Let

$$C_0(\Gamma) = \{a : \Gamma \to \mathbb{C} : a \,\text{has finite support }\}. \tag{4.1}$$

Define an associative multiplication in $C_0(\Gamma)$ by the rule

$$(a \cdot b)(\gamma) = \sum_{\beta\in\Gamma} \Theta(\gamma\beta^{-1}, \beta)^{-1} a(\gamma\beta)^{-1} b(\beta), \tag{4.2}$$

and a $*$-operation by

$$a^*(\gamma) = \Theta(\gamma^{-1}, \gamma)\Theta(1, 1)\overline{a(\gamma^{-1})}. \tag{4.3}$$

There is an injective $*$-homomorphism, I, of $C_0(\Gamma)$ into the operator algebra $L(l^2(\Gamma))$ that takes each $a \in C_0(\Gamma)$ to $\tilde{a} = Ia$, where

$$\tilde{a}(\varphi)(\gamma) = \sum_{\beta\in\Gamma} \Theta(\gamma\beta^{-1}, \beta)^{-1} a(\gamma\beta^{-1})\varphi(\beta). \tag{4.4}$$

The twisted group C^*-algebra $C^*(\Gamma,\Theta)$ is defined as the completion of $C_0(\Gamma)$ with respect to the norm $\|a\| := \|Ia\|_{l^2(\Gamma)}$. The algebra $C^*(\Gamma,\Theta)$ has a standard trace, τ, defined by

$$\tau(a) = a(1). \tag{4.5}$$

Now denote by ρ_γ, for $\gamma \in \Gamma$, the operator in $l^2(\Gamma)$ defined by

$$(\rho_\gamma \varphi)(\beta) = \Theta(\beta,\gamma)\varphi(\beta\gamma). \tag{4.6}$$

As usual, let δ_γ denote the element of $l^2(\Gamma)$ with $\delta_\gamma(\beta) = \delta_{\beta\gamma}$.

Lemma 14 (1) *For any $\beta,\gamma \in \Gamma$ we have*

$$\rho_\beta\rho_\gamma = \Theta(\beta,\gamma)\rho_{\beta\gamma}, \tag{4.7}$$

i.e. ρ is a unitary projective representation of Γ in $l^2(\Gamma)$.
(2) *For any $\varphi \in l^2(\Gamma)$,*

$$\varphi(\gamma) = \Theta(1,\gamma)^{-1}(\rho_\gamma\varphi)(1). \tag{4.8}$$

(3)

$$\delta_\gamma = \Theta(\gamma,\gamma^{-1})^{-1}\rho_{\gamma^{-1}}\delta_1. \tag{4.9}$$

(4) *For any $a \in C_0(\Gamma)$ and $\gamma \in \Gamma$*

$$\rho_\gamma\tilde{a} = \tilde{a}\rho_\gamma\ . \tag{4.10}$$

Proof (1)

$$\begin{aligned}\rho_\beta\rho_\gamma\varphi(\alpha) &= \Theta(\alpha,\beta)\rho_\gamma\varphi(\alpha\beta) = \Theta(\alpha,\beta)\Theta(\alpha\beta,\gamma)\varphi(\alpha\beta\gamma)\\ &= \Theta(\beta,\gamma)\Theta(\alpha,\beta\gamma)\varphi(\alpha\beta\gamma) = \Theta(\beta,\gamma)\rho_{\beta\gamma}\varphi(\alpha).\end{aligned}$$

(2)

$$\rho_\gamma\varphi(1) = \Theta(1,\gamma)\varphi(\gamma).$$

(3)

$$\rho_{\gamma^{-1}}\delta_1(\beta) = \Theta(\beta,\gamma^{-1})\delta_1(\beta\gamma^{-1}) = \Theta(\gamma,\gamma^{-1})\delta_\gamma(\beta).$$

(4)

$$\begin{aligned}(\tilde{a}\rho_\gamma\varphi)(\alpha) &= \sum_\beta \theta(\alpha\beta^{-1},\beta)^{-1}a(\alpha\beta^{-1})(\rho_\gamma\varphi)(\beta)\\ &= \sum_\beta \Theta(\alpha\beta^{-1},\beta)^{-1}\Theta(\beta,\gamma)a(\alpha\beta)^{-1}\varphi(\beta\gamma), \qquad (4.11)\end{aligned}$$

$$\begin{aligned}(\rho_\gamma \tilde{a}\varphi)(\alpha) &= \Theta(\alpha,\gamma)\tilde{a}\varphi(\alpha\gamma)\\ &= \Theta(\alpha,\gamma)\sum_\beta \Theta(\alpha\gamma\beta^{-1},\beta)^{-1}a(\alpha\gamma\beta^{-1})\varphi(\beta)\\ &= \Theta(\alpha,\gamma)\sum_\beta \Theta(\alpha\beta^{-1},\beta\gamma)^{-1}a(\alpha\beta^{-1}\varphi(\beta\gamma). \qquad (4.12)\end{aligned}$$

Using (2.5) we obtain

$$\Theta(\alpha\beta^{-1},\beta)\Theta(\alpha,\gamma) = \Theta(\alpha\beta^{-1},\beta\gamma)\Theta(\beta,\gamma), \qquad (4.13)$$

and statement (4) is proved. □

Now we define a "canonical" isomorphism

$$\Phi : L^2(X) \to l^2(\Gamma)\otimes L^2(F) = l^2(\Gamma, L^2(F)) \qquad (4.14)$$

by the rule

$$(\Phi f)(\gamma) = r_F([\gamma,1]f = r_F(T^0_\gamma f), \qquad (4.15a)$$

where r_F denotes the restriction to $F : r_F f = f|F$, as in [2].

We also record the explicit form of the inverse:

$$\Phi^{-1}f = \sum_{\gamma\in\Gamma} \chi_\gamma T^{0*}_\gamma e_F(f(\gamma)), \qquad (4.15b)$$

where $e_F : L^2(F) \to L^2(X)$ denotes extension by zero and χ_γ is the characteristic function of $\gamma^{-1}F$. We extend ρ to a projective unitary representation, $\tilde{\rho}$, in $l^2(\Gamma)\otimes L^2(F)$ by the formula

$$\tilde{\rho}_\gamma = \rho_\gamma \otimes 1. \qquad (4.16)$$

Then the action of Γ on $l^2(\Gamma, L^2(F))$ is, again, given by (4.6).

Let $\mathcal{K}$ be the algebra of compact operators in the space $L^2(F)$. We denote the tensor product $C^*(\Gamma,\Theta)\otimes\mathcal{K}$ by $\mathcal{A}$. $\mathcal{K}$ has a natural trace, tr_F, which gives, with (4.5), the canonical trace

$$\tilde{\tau} := \tau \otimes \mathrm{tr}_F$$

on $\mathcal{A}$. The isomorphism Φ defines a canonical embedding $I_\mathcal{K}$ of $\mathcal{A}$ in the C^*-algebra $\mathcal{L}(L^2(X)) = \mathcal{L}(l^2(\Gamma)\otimes L^2(F))$. Denote by $\tilde{\mathcal{A}}$ the image $I_\mathcal{K}(\mathcal{A})$.

Lemma 15 *(Cf.* [2],[21]). *For any $\gamma \in \Gamma$ we have*

$$\Phi[\gamma, 1]\Phi^{-1} = \tilde{\rho}_\gamma. \tag{4.17}$$

Proof Let $f \in L^2(X)$, then

$$(\tilde{\rho}_\gamma \Phi f)(\beta) = \Theta(\beta, \gamma)\Phi f(\beta\gamma) = r_F(\Theta(\beta, \gamma)[\beta\gamma, 1]f), \tag{4.18}$$

$$(\Phi[\gamma, 1]f)(\beta) = r_F([\beta, 1][\gamma, 1]f).$$

Taking into account the multiplication rule (2.7), we get the result. □

Now we denote by

$$\mathcal{M}(\Gamma, \Theta) := \{B \in \mathcal{L}(l^2(\Gamma) \otimes L^2(F)) : B\tilde{\rho}_\gamma = \tilde{\rho}_\gamma B \text{ for } \gamma \in \Gamma\} \tag{4.19}$$

the commutant of $(\tilde{\rho}_\gamma)_{\gamma\in\Gamma}$. From Lemmas 14, (4) and 15 we obtain

$$\tilde{\mathcal{A}} \subset \mathcal{M}(\Gamma, \Theta). \tag{4.20}$$

Besides, Lemma 15 implies that

$$R_A(\zeta) \in \mathcal{M}(\Gamma, \Theta), \quad \zeta \in \mathbb{C} \backslash \operatorname{spec} H_A, \tag{4.21}$$

where R_A is given by (3.7) (recall that we consider only T-invariant operators H_A).

Now, following [2] we define the Fourier coefficients for $B \in \mathcal{M}(\Gamma, \Theta)$. For any $\gamma \in \Gamma$ the Fourier coefficient $\hat{B}(\gamma)$ is the operator in $L^2(F)$ given by

$$\hat{B}(\gamma)(u) = \tilde{\rho}_\gamma B(\delta_1 \otimes u)(1), \quad u \in L^2(\mathcal{F}). \tag{4.22}$$

Lemma 16 *(Cf.* [2], [21]).
(1) *For any $B \in \mathcal{M}(\Gamma, \Theta)$ and $f \in l^2(\Gamma, L^2(F))$ we have*

$$Bf(\gamma) = \Theta(1, \gamma)^{-1} \sum_\beta \Theta(\beta, \beta^{-1})^{-1} \Theta(\gamma, \beta^{-1}) \hat{B}(\gamma\beta^{-1})(f(\beta)). \tag{4.23}$$

(2) *Let $\gamma \in \Gamma, B \in \mathcal{L}(L^2(F))$, then*

$$(\widehat{\tilde{\delta}_\gamma \otimes B})(\beta) = \Theta(1, \beta)\Theta(\beta, 1)^{-1}(\delta_\gamma(\beta) \otimes B). \tag{4.24}$$

(3) *For $B \in \mathcal{M}(\Gamma, \Theta)$, we have*

$$\|B\| \le \sum_{\gamma\in\Gamma} \|\hat{B}(\gamma)\|. \tag{4.25}$$

Proof (1) Let $f \in l^2(\Gamma, L^2(F))$. Then, from (4.9)

$$\begin{aligned} f &= \sum_{\beta\in\Gamma} \delta_\beta \otimes f(\beta) \\ &= \sum_{\beta\in\Gamma} \Theta(\beta,\beta^{-1})^{-1}\tilde{\rho}_{\beta^{-1}}(\delta_1 \otimes f(\beta)). \end{aligned} \tag{4.26}$$

Hence

$$Bf = \sum_\beta \Theta(\beta,\beta^{-1})^{-1}\tilde{\rho}_{\beta^{-1}} B(\delta_1 \otimes f(\beta)), \tag{4.27}$$

$$(Bf)(1) = \sum_\beta \Theta(\beta,\beta^{-1})^{-1}\tilde{\rho}_{\beta^{-1}} B(\delta_1 \otimes f(\beta))(1). \tag{4.28}$$

In view of Lemma 14, (2) and (1), we have

$$\begin{aligned} (Bf)(\gamma) &= \Theta(1,\gamma)^{-1}\sum_\beta \Theta(\beta,\beta^{-1})^{-1}\tilde{\rho}_\gamma\tilde{\rho}_{\beta^{-1}} B(\delta_1 \otimes f(\beta))(1) \\ &= \Theta(1,\gamma)^{-1}\sum_\beta \Theta(\beta,\beta^{-1})^{-1}\Theta(\gamma,\beta^{-1})\tilde{\rho}_{\gamma\beta^{-1}} B(\delta_1 \otimes f(\beta))(1) \\ &= \Theta(1,\gamma)^{-1}\sum_\beta \Theta(\beta,\beta^{-1})^{-1}\Theta(\gamma,\beta^{-1})\hat{B}(\gamma\beta^{-1})(f(\beta)), \end{aligned}$$

and (4.23) is proved.
(2) By (4.22),

$$\begin{aligned} (\widehat{\tilde{\delta}_\gamma \otimes B})(\beta)(u) &= \tilde{\rho}_\beta(\tilde{\delta}_\gamma \otimes B)(\delta_1 \otimes u)(1) \\ &= \Theta(1,\beta)(\tilde{\delta}_\gamma \otimes B)(\delta_1 \otimes u)(\beta) \\ &= \Theta(1,\beta)\tilde{\delta}_\gamma(\delta_1)(\beta) \otimes B(u). \end{aligned} \tag{4.29}$$

On the other hand, by (4.4),

$$\begin{aligned} \tilde{\delta}_\gamma(\delta_1)(\beta) &= \sum_{\alpha\in\Gamma} \Theta(\beta\alpha^{-1},\alpha)^{-1}\delta_\gamma(\beta\alpha^{-1})\delta_1(\alpha) \\ &= \Theta(\beta,1)^{-1}\delta_\gamma(\beta). \end{aligned} \tag{4.30}$$

Substituting (4.30) in (4.29), we obtain (4.24).
(3) The proof is similar to the proof of Lemma 3 in [2]. □

Lemma 17 [2] *Let $B \in \mathcal{M}(\Gamma,\Theta)$. If for all $\gamma \in \Gamma$, the operator $\hat{B}(\gamma)$ is compact and $\sum_{\gamma\in\Gamma}\|\hat{B}(\gamma)\| < \infty$, then $B \in \tilde{\mathcal{A}}$.*

Proof Cf. the proof of the Corollary of Lemma 3 in [2]. □

Now let us state the main result of the paper.

Theorem 3 *Let A be a T^d-invariant self-adjoint operator in $l^2(\Lambda)$ with diagonal matrix. Then for any $\zeta \in \mathbb{C}\backslash \operatorname{spec} H_A$ the resolvent $R_A(\zeta)$ belongs to $\tilde{\mathcal{A}}$.*

Proof First note that it is enough to prove that $R_A(E) \in \tilde{\mathcal{A}}$ for E ranging over some semi-axis $(-\infty, E_0)$. Indeed, let this property be satisfied but $R_A(\zeta) \notin \tilde{\mathcal{A}}$ for some $\zeta_0 \in \mathbb{C}\backslash \operatorname{spec} H_A$. By the Hahn-Banach Theorem, there exists a continuous linear functional $\Psi \in \mathcal{L}(L^2(X))'$ such that $\Psi(B) = 0$ for each $B \in \tilde{\mathcal{A}}$ and $\Psi(R_A(\zeta_0)) \neq 0$. But this contradicts the analyticity of the function $\zeta \mapsto \Psi(R_A(\zeta))$.

Now note that $R^0(\zeta) \in \tilde{\mathcal{A}}$ for any $\zeta \in \mathbb{C}\backslash \operatorname{spec} H^0$. In fact, the proof given in [2], [3] for the fact that $e^{-tH_0} \in \tilde{\mathcal{A}}$ for each $t \geq 0$ carries over to the case at hand without major changes. Using the Laplace transform, we show that $R^0(E) \in \tilde{\mathcal{A}}$ for any $E < 0$ and hence for each $\zeta \in \mathbb{C}\backslash \operatorname{spec} H^0$.

Thus, it remains to prove that $V(E) := R^0(E) - R_A(E) \in \tilde{\mathcal{A}}$ if E ranges over some semi-axis $(-\infty, E_0)$.

By Theorem 1,2) we can find a number $E_0 \in \mathbb{R}$ such that for any $f \in L^2(X)$ we have

$$V(\zeta)f = \sum_{\lambda \in \Lambda} \left(\sum_{\mu \in \Lambda} M(\lambda, \mu; \zeta) < g_\mu(\bar{\zeta}) | f > \right) g_\lambda(\zeta), \tag{4.31}$$

whenever $\operatorname{Re} \zeta < E_0$. Here we have written

$$M(\lambda, \mu; \zeta) = [Q(\zeta) + A]^{-1}(\lambda, \mu). \tag{4.32}$$

Moreover, by (3.10) there are constants c_0 and $\tilde{c}_0(\zeta)$ such that

$$|M(\lambda, \mu; \zeta)| \leq c_0 \exp(-\tilde{c}_0(\zeta) d(\lambda, \mu)); \tag{4.33}$$

c_0 is independent of ζ, and $\tilde{c}_0(\zeta) \to \infty$ as $\operatorname{Re} \zeta \to -\infty$. So we can suppose that

$$\tilde{c}_0(\zeta) > 3\tilde{c}_\Lambda, \tag{4.34}$$

where $\tilde{c}_\Lambda$ is the constant from Lemma 9.

To see that $V(\zeta)$ is in $\tilde{\mathcal{A}}$ it is enough, in view of the definition and Lemma 17, to show that

$$\widehat{V(\zeta)}(\gamma) \text{ is compact in } L^2(F), \tag{4.35a}$$

and

$$\sum_{\gamma\in\Gamma}\|\widehat{V(\zeta)}(\gamma)\|<\infty. \tag{4.35b}$$

We compute with (4.22), (4.31), (4.16), and (4.6), for $v\in L^2(F)$,

$$\widehat{V(\zeta)}(\gamma)=\Theta(1,\gamma)\exp(i\omega\gamma)\sum_{\lambda,\mu\in\Lambda}M(\lambda,\mu;\bar\zeta)<g_\mu(\bar\zeta),\Phi^{-1}(\delta_1^v)>r_F(g_\lambda(\zeta)\circ\gamma^{-1}). \tag{4.36}$$

Using (4.15b) and Lemma 6 we obtain

$$\begin{aligned}|<g_\mu(\bar\zeta),\Phi^{-1}(\delta_1^v)>| &= |<g_\mu(\bar\zeta),e_F(v)>|\\ &= |\int_F G^0(z',\lambda;\bar\zeta)v(z')d\sigma(z')|\\ &\le c_3e^{-\tilde c_3(\zeta)d(\lambda,\kappa_0)}\|v\|_{L^2(F)}\,,\end{aligned} \tag{4.37a}$$

where we may, again, assume that

$$\tilde c_3(\zeta)>3\tilde c_\Lambda. \tag{4.38}$$

Finally, we see with Lemmas 6 and 7 that

$$\begin{aligned}\|r_F(g_\lambda(\zeta)\circ\gamma^{-1}\|_{L^2(F)} &\le \left[\int_F|G^0(\lambda,\gamma^{-1}(z');\zeta)|^2d\sigma(z')\right]^{1/2}\\ &\le c_3e^{-\tilde c_3(\zeta)d(\gamma\lambda,\kappa_0)}.\end{aligned} \tag{4.39}$$

Now we write $\lambda=\alpha\kappa,\mu=\beta\kappa'$ with $\kappa,\kappa'\in K$ and $\kappa,\beta\in\Gamma$, and further $\beta=:\alpha\beta',\gamma':=\gamma\alpha$ and find

$$\begin{aligned}&\sum_{\kappa,\kappa'\in K}\sum_{\alpha\in\Gamma}\sum_{\beta',\gamma'\in\Gamma}e^{-3\tilde c_\Lambda(d(\alpha\kappa,\alpha\beta'\kappa')+d(\alpha\kappa,\kappa_0)+d(\gamma\alpha\kappa,\kappa_0))}\\ =&\sum_{\substack{\kappa,\kappa'\in K\\ \alpha\in\Gamma}}e^{-3\tilde c_\Lambda d(\alpha\kappa,\kappa_0)}\sum_{\beta',\gamma'\in\Gamma}e^{-3\tilde c_\Lambda(d(\beta'\kappa',\kappa)+d(\gamma'\kappa,\kappa_0))}\\ \le& c_4<\infty,\end{aligned} \tag{4.40}$$

by Lemma 9 and A1. Now we use (4.38) and (4.39) in (4.36) to see, similarly, that the sum is norm convergent; since all summands are operators of rank one, the compactness of $\widehat{V(\zeta)}(\gamma)$ follows. The summation (4.40) proves (4.35b) and the theorem is proved. □

Corollary 2 *Let $E_1, E_2 \in \mathbb{R} \backslash \operatorname{spec} H_A$ and $E_1 \leq E_2$. Then the spectral projector $P_{[E_1,E_2]}$ of the operator H_A belongs to $\tilde{\mathcal{A}}$.*

Proof Indeed, we may write for $E \notin \operatorname{spec} H_A$: $P_{[E_1,E_2]} = \psi(R_A(E))$ where ψ is a continuous function with compact support. □

Fix now a number $E' \in \mathbb{R}$ such that $E' < \inf \operatorname{spec} H_A$, and consider the function

$$N(E) = \begin{cases} \tau P_{[E',E]}, & E \geq E' \\ 0, & E < E'. \end{cases}$$

Then this function is independent of the choice of E'. The values of $N(E)$ are constant on each gap of the spectrum of H_A. Therefore these values label in a natural way the gaps of H_A [22].

Corollary 3 *(Gap Labelling Theorem). The value of the function $N(E)$ on a gap of* $\operatorname{spec} H_A$ *belongs to $\tau^*(K_0 C^*(\Gamma, \Theta))$, a countable set of real numbers (here $K_0\mathcal{B}$ denotes the K_0-group of a C^*-algebra $\mathcal{B}$).*

Recall that the pair (Γ, Θ) is said to have the *Kadison property* if there exist a constant $c_K > 0$ such that $\tau(P) \geq c_K$ for every nonzero self-adjoint projection P in $C^*(\Gamma, \Theta) \otimes \mathcal{K}$. It now follows as in [2], [16], [21]:

Corollary 4 *If the pair (Γ, Θ) has the Kadison property then the spectrum of H_A has band structure.*

Appendix

In this appendix, we provide some general results concerning a discrete metric space, Λ, with metric d. We suppose that the following condition on the "volume growth" of metric balls is fulfilled (cf. (2.15)):

There are constants $c_\Lambda > 0$ and $\tilde{c}_\Lambda > 0$ such that for any $\lambda_0 \in \Lambda$ and any $r \in (0, \infty)$ we have

$$\#\{\lambda \in \Lambda : d(\lambda, \lambda_0) \leq r\} \leq c_\Lambda \exp(\tilde{c}_\Lambda r). \tag{A1}$$

Lemma A1 *Let $\phi : \Lambda \to \mathbb{C}$ be a function such that*

$$|\phi(\lambda)| \leq c \exp[-(1+\delta)\tilde{c}_\Lambda d(\lambda, \mu)], \tag{A2}$$

where c and δ are positive constants and μ is any fixed element of Λ. *Then*

$$\sum_{\lambda\in\Lambda} |\phi(\lambda)| \leq c \cdot c_\Lambda \cdot \delta^{-1}. \tag{A3}$$

Proof See [23]. □

Lemma A2 *(Schur's test). Let* $(L(\lambda,\mu))_{\lambda,\mu\in\Lambda}$ *be an infinite matrix such that for some* $c' > 0$ *we have*

$$\sup_{\mu\in\Lambda} \sum_{\lambda\in\Lambda} |L(\lambda,\mu)| \leq c', \quad \sup_{\lambda\in\Lambda} \sum_{\mu\in\Lambda} |L(\lambda,\mu)| \leq c'. \tag{A4}$$

Then the matrix $L(\lambda,\mu)$ *determines a bounded linear operator* L *in the space* $l^2(\Lambda)$ *and*

$$\|L\| \leq c'. \tag{A5}$$

Proof See [23], [14]. □

Theorem A1 *Let* $(K_n)_{n\geq 0}$ *be a sequence of bounded linear operators in the space* $l^2(\Lambda)$*, having the matrices* $(K_n(\lambda,\mu))_{\lambda,\mu\in\Lambda}$ *with respect to the standard basis* $l^2(\Lambda)$*. Suppose that the following conditions are satisfied:*
(1) *if* $\lambda \neq \mu$, *then*

$$|K_n(\lambda,\mu)| \leq a\exp(-b_n d(\lambda,\mu)), \tag{A6}$$

where a is independent of n and $\lim_{n\to\infty} b_n = \infty$;
(2)

$$\inf_{\lambda\in\Lambda} |\, K_n(\lambda,\lambda)\,| \to \infty \text{ as } n\to\infty. \tag{A7}$$

Then, for any $\alpha \in (0,1)$ *there is* $n_0 \in \mathbb{N}$ *such that for every* $n \geq n_0$ *the operator* K_n *has a bounded inverse,* $L_n := K_n^{-1}$*. The matrix* $(L_n(\lambda,\mu))_{\lambda,\mu\in\Lambda}$ *of this operator admits the estimate*

$$|L_n(\lambda,\mu)| \leq 2c_n \exp(-\alpha b_n d(\lambda,\mu)), \tag{A8}$$

where

$$c_n = (\inf_{\lambda\in\Lambda} |K_n(\lambda,\lambda)|)^{-1}. \tag{A9}$$

Proof We introduce operators D_n, S_n by

$$D_n(\lambda,\mu) := K_n(\lambda,\mu)\delta_{\lambda\mu}\ , S_n(\lambda,\mu) := K_n(\lambda,\mu) - D_n(\lambda,\mu).$$

Moreover, we fix $\alpha \in (0,1)$ and determine $n_0 \in \mathbb{N}$ such that for $n \geq n_0$

$$\inf_{\lambda\in\Lambda} |K_n(\lambda,\lambda)| \geq 1 \quad \text{and} \qquad (1-\alpha)b_n \geq 2\tilde{c}_\Lambda. \tag{A10}$$

Then D_n^{-1} and S_n are bounded in $l^2(\Lambda)$ in view of Lemmas A1 and A2 and, clearly,

$$\begin{aligned} K_n &= D_n(1 + D_n^{-1}S_n), \\ L_n &= K_n^{-1} = (1 + D_n^{-1}S_n)^{-1}D_n^{-1} =: T_n D_n^{-1}. \end{aligned}$$

Thus, the theorem follows if we prove the estimate

$$|T_n(\lambda,\mu)| \leq 2\exp(-\alpha b_n d(\lambda,\mu)). \tag{A11}$$

Now

$$T_n(\lambda,\mu) = \sum_{j\geq 0}(-1)^j(D_n^{-1}S_n)^j(\lambda,\mu),$$

and it is enough to show that, with some constant $A \geq 1$,

$$|(D_n^{-1}S_n)^j(\lambda,\mu)| \leq (aAc_n)^j e^{-\alpha b_n d(\lambda,\mu)}, \tag{A12}$$

since $c_n \to 0$ as $n \to \infty$. This estimate is obvious for $j = 0, 1$; inductively, we find with (A5) and (A7)

$$\begin{aligned} |(D_n^{-1}S_n)^{j+1}(\lambda,\mu)| &= |\sum_{\nu\in\Lambda} K_n(\lambda,\lambda)^{-1}S_n(\lambda,\nu)(D_n^{-1}S_n)^j(\nu,\mu)| \\ &\leq \sum_{\nu\in\Lambda} c_n a e^{-b_n d(\lambda,\nu)}(Aac_n)^j e^{-\alpha b_n d(\nu,\mu)} \\ &\leq A^j(c_n a)^{j+1} e^{-\alpha b_n d(\lambda,\mu)} \sum_{\nu\in\Lambda} e^{-(1-\alpha)b_n d(\lambda,\nu)}. \end{aligned}$$

In view of (A8) and Lemma A1, the last sum has the bound c_Λ. The assertion (A10) for $j+1$ follows if we put $A := \max\{1, c_\Lambda\}$. ∎

Remark Theorem A1 strengthens Theorem 2.1 from [23]. The estimate there is insufficient for proving our results.

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