Analytical Treatment of Stabilization

C. Figueira de Morisson Faria†, A. Fring‡ and R. Schrader‡
†Max-Born-Institut, Rudower Chaussee 6, D-12474 Berlin, Germany
‡Institut für Theoretische Physik, Freie Universität Berlin,
Arnimallee 14, D-14195 Berlin, Germany

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Abstract

We present a summarizing account of a series of investigations whose central topic is to address the question whether atomic stabilization exists in an analytical way. We provide new aspects on several issues of the matter in the theoretical context when the dynamics is described by the Stark Hamiltonian. The main outcome of these studies is that the governing parameters for this phenomenon are the total classical momentum transfer and the total classical displacement. Whenever these two quantities vanish, asymptotically weak stabilization does exist. For all other situations we did not find any evidence for stabilization. We found no evidence that strong stabilization might occur. Our results agree qualitatively with the existing experimental findings.

1 Introduction

Due to the breakdown of standard perturbation theory, the understanding of the physics of an atom in a strong (intensities larger than $3.5 \times 10^{16} W cm^{-2}$ for typical frequencies) laser field is still poorly understood to a very large extent. Hitherto the large majority of the obtained results is based on numerical treatments. In our investigations we aim at a rigorous analytical description of phenomena occurring in this regime. This proceeding will provide an account of a series of publications [1-4]. In many cases we will simply summarize and state some of the results and refer the reader for a detailed derivation to the original manuscripts, but we shall try to put an emphasis on new aspects for which we will supply an extensive discussion. Several of the presented arguments and results may not be found in [1-4].

Amongst the phenomena occurring in the high intensity regime in particular the one of so-called stabilization has recently caused some controversy, not only concerning its definition, but even its very existence altogether [5-26]. Roughly speaking stabilization means that atomic bound states become resistent to ionization in ultra-intense laser fields. A more precise definition may be found in section 2.2.

2 Physical Framework

The object of our investigations is an atom in the presence of a sufficiently intense[1] laser field, which may be described in the non-relativistic regime by the time-dependent Schrödinger equation in the dipole approximation

$$i \frac{\partial \left|\psi(t)\right\rangle}{\partial t} = H\left(t\right)\left|\psi(t)\right\rangle . \qquad (1)$$

We use atomic units throughout. The time dependent external electric field will be treated classically

[0] e-mail addresses:
Faria@mbi-berlin.de,
Fring@physik.fu-berlin.de,
Schrader@physik.fu-berlin.de.

[1] Sufficiently refers to the validity of a classical treatment of the laser field. A rigerous quantum electrodynamical treatment of ionization phenomena was recently initiated in [27].

and is assumed to be linearly polarized of the general form $E(t) = E_0 f(t)$, where E_0 denotes the field amplitude and $f(t)$ is some arbitrary function which equals zero for $t < 0$ and $t > \tau$, such that τ defines the pulse length. Depending on the context it is convenient to express the Hamiltonian in equation (1) in different gauges.

2.1 Gauge equivalent Hamiltonians

Taking $A_{j\leftarrow i}(t)$ to be a one parameter family of unitary operators, we may construct the gauge equivalent Hamiltonian $H_i(t)$ from $H_j(t)$ by the usual gauge transformation

$$H_i(t) = i\partial_t A_{j\leftarrow i}(t) A^{-1}_{j\leftarrow i}(t) + A_{j\leftarrow i}(t) H_j(t) A^{-1}_{j\leftarrow i}(t). \tag{2}$$

Choosing the most conventional gauge, the so-called length gauge, the Hamiltonian to describe the above mentioned physical situation is the Stark Hamiltonian

$$H^S_l(t) = H^0_l + V(\vec{\mathbf{x}}) + \mathbf{z}\cdot E(t)\,. \tag{3}$$

$V(\vec{\mathbf{x}})$ is the atomic potential and $H^0_l = \vec{\mathbf{p}}^2/2$ denotes the Hamilton operator of the free particle. We introduced here sub-and superscripts in order to keep track of the particular gauge we are in and to identify a specific Hamiltonian, respectively. In our conventions $\vec{\mathbf{p}}$ and $\vec{\mathbf{x}}$ denote operators, whilst $\vec{p}$ and $\vec{x}$ are elements in R^3. Other commonly used Hamiltonians are the one in the velocity gauge

$$H^S_v(t) = \frac{1}{2}(\vec{\mathbf{p}} - b(t)e_z)^2 + V(\vec{\mathbf{x}}) \tag{4}$$

and the one in the Kramers-Henneberger (KH) frame [28]

$$H^S_{KH}(t) = H^0_l + V(\vec{\mathbf{x}} - c(t)e_z). \tag{5}$$

Here e_z denotes the unit vector in the z-direction. These Hamiltonians may be obtained from each other by using

$$\begin{aligned} A_{v\leftarrow l}(t) &= e^{ib(t)\mathbf{z}}, &(6)\\ A_{v\leftarrow KH}(t) &= e^{-ia(t)}e^{ic(t)\mathbf{p}_z}, &(7)\\ A_{l\leftarrow KH}(t) &= e^{-ia(t)}e^{-ib(t)\mathbf{z}}e^{ic(t)\mathbf{p}_z}, &(8)\end{aligned}$$

in (2). $\mathbf{p}_z$ is the component of the momentum operator in the z-direction. We have employed the important quantities

$$\begin{aligned} b(t) &= \int_0^t ds\, E(s)\,, &(9)\\ c(t) &= \int_0^t ds\, b(s)\,, &(10)\\ a(t) &= \frac{1}{2}\int_0^t ds\, b^2(s)\,, &(11)\end{aligned}$$

which are the classical momentum transfer, the classical displacement and the classical energy transfer, respectively. It will turn out, that in particular $b(\tau)$ and $c(\tau)$ are the crucial parameters for the description of the phenomenon we are going to discuss. The classical energy transfer $a(t)$ is not a crucial quantity since it enters all expressions only as a phase and will therefore cancel in all relevant physical expressions.

In our considerations we will also need the Hamiltonians

$$H^A_l(t) = H^0_l + V(\vec{\mathbf{x}})\,, \quad H^{GV}_l(t) = H^0_l + \mathbf{z}\cdot E(t) \tag{12}$$

which describe an electron in the atomic potential or in the electric field, respectively. Of course these Hamiltonians may also be transformed into the other gauges by (6)-(8). Notice that $H^{GV}_{KH}(t) = H^0_l$.

2.2 Definition of Stabilization

Since stabilization means different things to different authors and a universally accepted concept does not seem to exist yet, we will precisely state our definition of it. We will not discuss the behaviour of ionization rates as some authors do, but we shall consider exclusively ionization probabilities. Denoting by $\|\psi\|^2 = \langle\psi,\psi\rangle = \int |\psi(\vec{x})|^2 d^3x$ the usual Hilbert space norm, the ionization probability is defined as

$$\mathcal{P}(\psi) \;=\; 1 - \|P_+ S\psi\|^2 \quad . \tag{13}$$

We used the scattering matrix

$$S = \lim_{t_\pm \to \pm\infty} \exp(it_+ H_+)\cdot U(t_+, t_-)\cdot \exp(-it_- H_-)\,, \tag{14}$$

where $H_\pm = \lim_{t\to\pm\infty} H(t)$, ψ is a normalized bound state of H_-, $P_\pm$ is the projector onto the bound state space of $H_\pm$ and $U(t,t')$ is the time evolution operator from time t' to t associated[2] to $H(t)$. The time evolution operators may be transformed from one gauge to another by

$$U_i^a(t,t') = A_{j\leftarrow i}(t) U_j^a(t,t') A_{j\leftarrow i}^{-1}(t'). \qquad (15)$$

The ionization probability $\mathcal{P}(\psi)$ is a gauge invariant quantity [1]. Note that for the gauge equivalent Hamiltonians quoted above we have in general

$$\lim_{t\to\pm\infty} H_l^S(t) \neq \lim_{t\to\pm\infty} H_v^S(t) \neq \lim_{t\to\pm\infty} H_{KH}^S(t). \quad (16)$$

However, (recall that $b(0) = c(0) = 0$), equality in the first case holds whenever we have $b(\tau) = 0$ and in both cases when in addition $c(\tau) = 0$. We will encounter this condition of a particularly switched on and off pulse below once more as the necessary condition for the presence of what we refer to as asymptotically weak stabilization [3]. We would like to point out that this condition does not coincide necessarily with the notion of adiabatically switched on and off pulses, because we may achieve $b(\tau) = 0$ and $c(\tau) = 0$ of course also with a very rapid switch on and off. Since we are interested in the behaviour of the atomic bound states $|\psi(t=0)\rangle = |\psi\rangle$ of the Hamiltonian H_l^A we should commence the discussion in (13) in the length gauge (in this case we have $\lim_{t\to\pm\infty} H_l^S(t) = H_l^A$) such that in our situation

$$\mathcal{P}(\psi) = 1 - \left\| P_+ U_l^S(\tau, 0)\, \psi \right\|^2 \quad . \qquad (17)$$

Regarding the ionization probability as a function of the field amplitude E_0, stabilization means that

$$\frac{d\mathcal{P}(\psi)}{dE_0} \leq 0 \qquad \text{for} \quad \mathcal{P}(\psi) \neq 1 \qquad (18)$$

for $E_0 \in [0, \infty)$ on a finite interval. Hence the occurence of a saddle point does not qualify as stabilization. Also we would like to introduce some terminology in order to distinguish in (18) between the case of equality and strict inequality. If the former sign holds we call this behaviour *"weak stabilization"* and in the latter case *"strong stabilization"*. In case weak stabilization only occurs in the limit $E_0 \to \infty$, we shall refer to it as *"asymptotically weak stabilization"*.

[2] Associated is to be understood in the sense that the time evolution operator obeys the Schrödinger equation $i\partial_t U(t,t') = H(t)U(t,t')$.

[3] There are doubts expressed by experimentalists about the possibility to realise pulses having simultaneously $b(\tau) = 0$ and $c(\tau) = 0$ [29].

3 Upper and lower bounds for the Ionization Probability

The outcome of every theoretical investigation will attach some sort of error to any physical quantity. In the minority of cases this error can be precisely stated, since it may either be the consequence of various qualitative assumptions based on some physical reasoning which are difficult to quantify or it may be of a more technical nature originating in the method used. For instance for the physical quantity we are interested in, the ionization probability $P(\psi)$, the most fundamental error is introduced by the assumptions for the validity of the main physical framework, that is the Schrödinger equation (1) (i.e. non-relativity, dipole approximation, classical treatment of the external field, neglect of the magnetic field, etc.). Examples for errors rooted in a particular method used are: When solving the Schrödinger equation numerically one is forced to discretise $H(t)$, insert the atom into a finite box and introduce absorbing mask functions at the boundary, etc. Also one is not able to project on all bound states or all states of the "discrete continuum"[4] and is forced to introduce a cut-off, whose effect is in our opinion not discussed in the literature. Some further examples are the errors resulting from the termination of a Floquet, Fourier expansion or perturbation series[5]. We would like the reader to keep these basic facts in mind, i.e. "exact" results do not exist and one is always dealing with some form of bounds, when judging about the method presented in this section. The essence of the method consists in treating bounds which restrict a physical quantity rather than looking at its actual

[4] See the conclusion for a discussion of this point.

[5] See section 6 and the conclusion for a discussion of this point.

value. One of the main virtues of this approach is that it may be carried out purely analytically. In different contexts it has turned out to be extremely fruitful, for instance in the proof of the stability of matter [30] and the stability of matter in a magnetic field [31].

We will provide rigorous analytic expressions for the upper and lower bound, $P_u(\psi)$ and $P_l(\psi)$, respectively, for the ionization probability in the sense that

$$P_l(\psi) \leq P(\psi) \leq P_u(\psi)\,. \tag{19}$$

Hence within the basic theoretical framework upper and lower bounds serve as sharp error bars. Surely one should treat these expressions with care and be aware of their limitations in the sense that about the actual shape of $P(\psi)$ no decisive conclusion can be drawn whenever $P_l(\psi)$ differs strongly from $P_u(\psi)$. However, it seems a reasonable assumption that the analytic expression of the bounds reflect qualitatively the behaviour of the precise ionization probability. Nonetheless, there exist certain type of questions in the present context which can be answered decisively with this method. Concerning the question of stabilization we may consider the bounds as functions of the field amplitude and can conclude that stabilization exists or does not exist once we find that $P_u(\psi)$ for increasing field amplitude tends to zero and $P_l(\psi)$ tends to one, respectively. Unfortunately, one does not always succeed in deriving analytic expressions which are of this restrictive form.

In [1] we obtained

$$\begin{aligned} P_l(\psi) = 1 - \Bigg\{ & \int_0^\tau \|(V(\vec{x}-c(t)e_z) - V(\vec{x}))\psi\| dt \\ & + \frac{2}{2E+b(\tau)^2} \|(V(\vec{x}-c(\tau)e_z) - V(\vec{x}))\psi\| \\ & + \frac{2|b(\tau)|}{2E+b(\tau)^2} \|p_z\psi\| \Bigg\}^2 , \end{aligned} \tag{20}$$

which is valid when $-E < b(\tau)^2/2$. Here E is the binding energy. With the same restriction on $b(\tau)$ we found as an upper bound

$$\begin{aligned} P_u(\psi) \;=\; \Bigg\{ & \int_0^\tau \|(V(\vec{x}-c(t)e_z) - V(\vec{x}))\psi\| dt \\ & + |c(\tau)|\, \|p_z\psi\| + \frac{2|b(\tau)|}{2E+b(\tau)^2} \|p_z\psi\| \Bigg\}^2 . \end{aligned} \tag{21}$$

By a slightly different analysis we also derived a bound valid without any additional restrictions

$$\begin{aligned} P_u(\psi) \;=\; \Bigg\{ & \int_0^\tau \|(V(\vec{x}-c(t)e_z) - V(\vec{x}))\psi\| dt \\ & + |c(\tau)|\, \|p_z\psi\| + |b(\tau)|\, \|z\psi\| \Bigg\}^2 . \end{aligned} \tag{22}$$

In [2] we applied these bounds to the Hydrogen atom, obtaining

$$\begin{aligned} P_l(\psi_{n00}) = 1 - \Bigg\{ & \frac{2}{n^{3/2}}\tau + \frac{4}{b(\tau)^2 - 1/n^2} \frac{1}{n^{3/2}} \\ & + \frac{1}{n\sqrt{3}} \frac{2|b(\tau)|}{b(\tau)^2 - 1/n^2} \Bigg\}^2 \end{aligned} \tag{23}$$

$$P_u(\psi_{n00}) = \left\{ \frac{2\tau}{n^{3/2}} + \frac{|c(\tau)|}{n\sqrt{3}} + \sqrt{\frac{5n^3+n}{6}}\,|b(\tau)| \right\}^2 \tag{24}$$

Our method allows in principle to consider any bound state, but initially we restricted ourselves to s-wave functions, keeping however the dependence on the principal quantum number n. The discussion of (21) and (22) in [1] was plagued by the requirement that the pulse duration should be fairly small. This limitation, which ensured that bounds for the ionization probability are between zero and one, was overcome in [2], since in there we had an additional parameter, i.e. n, at hand. As the expressions (23) and (24) show after a quick inspection, one may achieve that their values remain physical, even if one increases the pulse duration, but now together with n.

In particular we investigated the effect resulting from different pulse shapes, since it is widely claimed in the literature that a necessary condition for the existence of stabilization is an adiabatically smooth turn on (sometimes also off) of the laser field. For definiteness we assumed the laser light to be of the general form $E(t) = E_0 \sin(\omega t) g(t)$. Besides other pulses we investigated in particular the ones which are widely used in the literature, where the enveloping function is either trapezoidal

$$g(t) = \begin{cases} \frac{t}{T} & \text{for} \quad 0 \le t \le T \\ 1 & \text{for} \quad T < t < (\tau - T) \\ \frac{(\tau - t)}{T} & \text{for} \quad (\tau - T) \le t \le \tau \end{cases} \tag{25}$$

or of sine-squared shape

$$\tilde{g}(t) = \begin{cases} \sin^2\left(\frac{\pi t}{2T}\right) & \text{for} \quad 0 \le t \le T \\ 1 & \text{for} \quad T < t < (\tau - T) \\ \sin^2\left(\frac{\pi(\tau - t)}{2T}\right) & \text{for} \quad (\tau - T) \le t \le \tau . \end{cases} \tag{26}$$

We found no evidence for stabilization for an *adiabatically smoothly switched on field*, provided that $b(\tau) \neq 0$. The latter restriction emerges in our analysis as a technical requirement, but it will turn out that it is of a deeper physical nature.[6]

4 Ionization Probability in the ultra-extreme Intensity Limit

Since we are interested in very high intensities we expect to be able to draw some conclusions from the expressions for the ionization probability in which the field amplitude is taken to its ultra-extreme limit, i.e. infinity. In particular we may decide whether asymptotic stabilization exists. Despite the fact that in this regime one should commence with a relativistic treatment, our physical framework, that is the Schrödinger equation (1) remains self-consistent, and should certainly represent the overall behaviour. In [3] we rigorously take this limit under certain general assumptions on the atomic potential[7] and the laser field, which include almost all physical situations discussed in the literature. Whenever $b(\tau)$ and $c(\tau)$ vanish simultaneously we found

$$\lim_{|E_0| \to \infty} \mathcal{P}(\psi) = \left\| e^{-i\tau H_l^0} \psi \right\| \le 1 \tag{27}$$

whereas in all other cases we obtained

$$\lim_{|E_0| \to \infty} \mathcal{P}(\psi) = 1 \quad . \tag{28}$$

This means that in the former case we have asymptotically weak stabilization. It should be noted that in this analysis the pulse shape is kept fixed, such that adiabaticity can not be guaranteed anymore. Hence, weak stabilization is found for a situation in which it is generally not expected to occur. It would be very interesting to perform similar computations as in [3] in which the pulse shape is varied in order to keep also adiabaticity and study whether the effect will become enhanced in any way. Furthermore, in this case the time evolution operator coincides with the one of the free particle H_l^0

$$\lim_{|E_0| \to \infty} \left\| \left(U_l^S(\tau, 0) - \exp -i\tau H_l^0 \right) \psi \right\| = 0. \tag{29}$$

Of course this type of argument does not allow to draw any decisive conclusions concerning strong stabilization.

5 Gordon-Volkov Perturbation Theory

In the high intensity regime for the radiation fields, the basic assumption for the validity of conventional perturbation theory breaks down, i.e. that the absolute value of the potential is large in comparison with the absolute value of the field. However, there is a replacement for this, the so-called Gordon-Volkov (GV) perturbation theory [32]. Since the basic idea is simple, it makes this approach very attractive. Instead

[6] Already in (16) we observed that $b(\tau) = 0$ is somewhat special.

[7] We made the general assumptions that $V(\vec{x})$ is a real measurable function on R^3 and that each $\varepsilon > 0$ one may decompose V as $V = V_1 + V_2$ where V_1 is in $L^2(R^3)$ (i.e. square integrable) with compact support and V_2 is in $L^\infty(R^3)$ with $\|V_2\|_\infty = \text{ess} \sup_{\vec{x} \in R^3} |V_2(\vec{x})| \le \varepsilon$. Furthermore we assumed that H_l^A has no positive bound states. Such potentials are Kato small. In particular the Coulomb potential is Kato small.

of constructing the power series, either for the fields or for the time evolution operator, out of the solution for the Schrödinger equation involving the Hamiltonian H_l^A and regarding $\mathbf{z}E(t)$ as the perturbation, one constructs the series out of solutions involving the Hamiltonian H_l^{GV} and treats the potential V as the perturbation.

The starting point in this analysis is the Du Hamel formula, which gives a relation between two time evolution operators $U_i^a(t,t')$ and $U_j^b(t,t')$ associated to two different Hamiltonians $H_i^a(t)$ and $H_j^b(t)$, respectively

$$U_i^a(t,t') = U_j^b(t,t') - i\int_{t'}^{t} ds\ U_i^a\,(t,s)\,H_{i,j}^{a,b}(s)U_j^b(s,t'). \tag{30}$$

Here we use the notation $H_{i,j}^{a,b}(s) = H_i^a(s) - H_j^b(s)$. The formal iteration of (30) yields the perturbative series

$$U_i^a(t,t') = \sum_{n=0}^{\infty} U_{i,j}^{a,b}(n|t,t') \quad . \tag{31}$$

We introduced in an obvious notation the quantity $U_{i,j}^{a,b}(n|t,t')$ relating to the time evolution operator order by order in perturbation theory, i.e. $U_{i,j}^{a,b}(0|t,t') = U_j^b(t,t')$, $U_{i,j}^{a,b}(1|t,t') = i\int_{t'}^{t} ds\ U_j^b\,(t,s)\,H_{i,j}^{a,b}(s)U_j^b(s,t')$, etc. It should be noted that the perturbative series is gauge invariant in each order, since $U_{i,j}^{a,b}(n|t,t')$ is a gauge invariant quantity by itself. Mixing however expansions for different choices of the Hamiltonians, i.e. a and b or different gauges i and j will not guarantee this property in general. A rather unnatural choice (for instance with regard to the possible convergence of the series) would be $i \neq j$. Taking therefore $i = j$ and in addition $a = S$ and $b = GV$ we obtain

$$U_i^S(t,t') = U_i^{GV}(t,t') + U_{i,i}^{S,GV}(1|t,t') + \ldots \tag{32}$$

In this case we need $H_{l,l}^{S,GV}(t) = H_{v,v}^{S,GV}(t) = V(\vec{\mathbf{x}})$, $H_{KH,KH}^{S,GV}(t) = V(\vec{\mathbf{x}} - c(t)e_z)$ and the Gordon-Volkov time evolution operator, which in the KH-gauge equals the free-particle evolution operator in the length gauge

$$\begin{aligned} U_{KH}^{GV}(t,t') &= A_{l\leftarrow KH}^{-1}(t)U_l^{GV}(t,t')A_{l\leftarrow KH}(t') \\ &= U_l^0(t,t'). \end{aligned} \tag{33}$$

The expressions for the Gordon-Volkov time evolution operator in the length and velocity gauge may then simply be obtained from (33) by the application of (7)-(8) according to (15). The choice $i = j$ together with $a = S$ and $b = A$ in (30) yields the usual perturbation series, which is well known from the low intensity regime. One may also decide for a rather strange procedure and take the latter choice in the first iterative step and terminate the series after the second iterative step in which one makes the former choice. In that case one obtains

$$\begin{aligned} U_i^S(t,t') &= U_i^A(t,t') \\ &-i\int_{t'}^{t} ds\ U_i^A\,(t,s)\,H_{i,i}^{S,A}(s)U_i^{GV}(s,t') + \mathcal{O}(n^2). \end{aligned} \tag{34}$$

For $i = l$ or $i = v$, this procedure is sometimes referred to as the Keldysh- [33] or Faisal-Reiss [34] approximation, respectively. As we demonstrated this method is of course not "non-perturbative", as sometimes wrongly stated in the literature.

There are some exact results which may be derived from the perturbative expression, one concerning the ultra-extreme intensity limit of the previous section and the other the ultra-extreme high frequency limit. Both results are simple consequences of the Riemann-Lebesgue theorem[8]. We obtain

$$\lim_{\omega\to\infty} A_{i\leftarrow j}(t) = 1 \tag{35}$$

such that with (33)

$$\lim_{\omega\to\infty} U_{KH}^{GV}(t,t') = U_l^0(t,t') = e^{-i(t-t')H_l^0}. \tag{36}$$

Since the atomic potential is independent of E_0 we obtain with (36) that the entire Gordon-Volkov series

[8] If $g(x) \in L_1(-\infty,\infty)$ (i.e. $|g(x)|$ is integrable) then $\lim_{t\to\pm\infty} \int_{-\infty}^{\infty} g(x)e^{-itx}dx = 0$.

(31) is independent of the field amplitude as well, such that

$$\frac{d}{dE_0}\left(\lim_{\omega\to\infty}\mathcal{P}(\psi)\right)=0 \quad . \tag{37}$$

We have therefore weak stabilization in this ultra-extreme high frequency limit for all systems for which (31) makes sense and for which the laser field is of the form $E(t)=E_0\sin(\omega t)g(t)$. One should keep in mind that the use of the Stark Hamiltonian in (1) assumes the validity of the dipole approximation, such that the limit $\omega\to\infty$ (a slightly milder assumption was used in the seminal paper [6] which formulates the high frequency approach) only makes formally sense. In order to describe real physics in this frequency regime one should actually also take multipole terms into account. This is a further example of the errors we mentioned at the beginning of section 3.

Concerning the ultra-extreme intensity limit, we consider the transition amplitude between two bound states $\psi_i(\vec{x})$, $\psi_j(\vec{x})$ of the Hamiltonian H_l^A perturbatively

$$\begin{aligned}&\left\langle\psi_i,U_l^S(\tau,0)\psi_j\right\rangle \\ &=\left\langle\psi_i,A_{KH\leftarrow l}(\tau)U_{KH}^S(\tau,0)\psi_j\right\rangle\\ &=\left\langle\psi_i,A_{KH\leftarrow l}(\tau)U_{KH}^{GV}(\tau,0)\psi_j\right\rangle\\ &+\left\langle\psi_i,A_{KH\leftarrow l}(\tau)U_{KH,KH}^{S,GV}(1|\tau,0)\psi_j\right\rangle+\ldots\end{aligned} \tag{38}$$

Recall that $a(0)=b(0)=c(0)=0$, such that $A_{l\leftarrow KH}(0)=1$. Using now (33) it is clear that to zeroth order we obtain

$$\left\langle\psi_i(\vec{x}),e^{-i\tau H_l^0}\psi_j(\vec{x})\right\rangle \quad , \tag{39}$$

when $b(\tau)=c(\tau)=0$. In all other cases we may bring this term into a form suitable for the application of the Riemann-Lebesgue theorem, such that the zeroth order matrix element always vanishes in the ultra-extreme intensity limit. For the higher order terms the argument is analogous with the difference that the condition $b(\tau)=c(\tau)=0$ does not have the consequence that these expressions become independent of E_0, since also terms like $b(t),c(t)$ for $0<t<\tau$ appear. Hence by the application of the Riemann-Lebesgue theorem all higher order terms vanish in the limit $E_0\to\infty$. If we now sum over all bound states i in (38) we obtain the results of section 4 (27) and (28).

6 1-dimensional $\delta-$potential

In [4] we applied the GV-perturbation theory to the one-dimensional delta potential with coupling constant α

$$V(\mathbf{x})=-\alpha\,\delta(\mathbf{x}) \ . \tag{40}$$

In the momentum space representation this potential becomes

$$V(p,p')=\langle p\,|V|\,p'\rangle=-\frac{\alpha}{2\pi} \tag{41}$$

and the wave function for its only bound-state is well known to be

$$\psi(p,t=0)=\sqrt{\frac{2}{\pi}}\frac{\alpha^{\frac{3}{2}}}{\alpha^2+p^2}. \tag{42}$$

The very fact that this potential possesses one bound state only with bound state energy $-\alpha^2/2$ makes it a very attractive theoretical atomic toy potential (e.g. [35, 21, 23, 36]). Also with regard to the GV-perturbation theory one expects intuitively a good convergence.

We construct the exact time-dependent wave function:

$$\psi(p,t)=\psi_{GV}(p,t)+\Psi(p,t) \tag{43}$$

with

$$\begin{aligned}\Psi(p,t) &= i\frac{\alpha}{2\pi}\int_0^t ds e^{-ia(t)}e^{ic_{ts}(p-b(t))}\\ &\quad\times e^{-\frac{i}{2}(p-b(t))^2(t-s)}\psi_I(s) \qquad (44)\\ \psi_I(t) &= e^{ia(t)}\int_{-\infty}^{\infty}dp\,\psi(p,t) \quad . \qquad (45)\end{aligned}$$

Integrating (43) with respect to p we obtain a Volterra equation of the second kind in t

$$\psi_I(t)=\int_{-\infty}^{\infty}dp\,\psi_{GV}(p,t)+\sqrt{\frac{i\alpha^2}{2\pi}}\int_0^t ds\psi_I(s)\,\frac{e^{i\frac{c_{ts}^2}{2(t-s)}}}{\sqrt{t-s}}. \tag{46}$$

The virtue of this equation is that the error of its solution, even when obtained by an iterative procedure, is completely controllable. The iteration of the Volterra equation yields

$$\psi(t) = \int_{-\infty}^{\infty} dp\, \psi_{GV}(p,t) + \sum_{n=1}^{\infty} \psi_n(t) \tag{47}$$

with $\psi_n(t)$ denoting the function order by order. We derived [4] an upper bound for the absolute value of this function

$$|\psi_n(t)| \leq \sqrt{8\alpha\pi}\frac{1}{n\Gamma\left(\frac{n}{2}\right)}\left(\alpha\sqrt{t/2}\right)^n . \tag{48}$$

It is known from the theory of integral equations [37] that this is sufficient to prove that the series converges for <u>all</u> values of α, and in addition we were even able to sum up the whole series

$$\begin{aligned}\sum_{n=1}^{\infty}|\psi_n(t)| &= \sqrt{2\alpha\pi}\left(2\exp(\alpha^2 t/2) - 1\right. \\ &\quad \left. - U_{\frac{1}{2},\frac{1}{2}}\left(\alpha^2 t/2\right)/\sqrt{\pi}\right) \end{aligned}\tag{49}$$

such that we can compute the maximal relative error after the zeroth order in GV perturbation theory

$$\mu = 2\sqrt{\pi}\left|\frac{\left(2\exp(\alpha^2 t/2) - 1 - U_{\frac{1}{2},\frac{1}{2}}\left(\alpha^2 t/2\right)/\sqrt{\pi}\right)}{\left(U_{\frac{1}{2},\frac{1}{2}}(\Phi_-) + U_{\frac{1}{2},\frac{1}{2}}(\Phi_+)\right)}\right| . \tag{50}$$

Here $U_{\frac{1}{2},\frac{1}{2}}(z)$ is the confluent hypergeometric function (see for instance [38]) and

$$\Phi_\pm := \tau\alpha^2\left(\pm\gamma + \frac{i}{2}\left(1-\gamma^2\right)\right), \gamma := \frac{c(\tau)}{\tau\alpha}. \tag{51}$$

This analysis allows us to determine the error which is introduced by the termination of the GV-series. Usually this is done without any justification about the precision and the only reasoning provided is very often solely the comparison with the next order term. This is however not enough as one knows from simple rest term estimations in a Taylor expansion.

The ionization probability turns out to be

$$\begin{aligned}\mathcal{P}(\psi) &= 1 - |q(\tau)|^2 &(52)\\ &= 1 - |\langle\psi, \psi_{GV}(\tau)\rangle + \langle\psi, \Psi(\tau)\rangle|^2 &(53)\end{aligned}$$

with

$$\begin{aligned}\langle\psi, \psi_{GV}(\tau)\rangle &= \frac{2}{\pi}\alpha^3 e^{-ia(\tau)} \\ &\times \int_{-\infty}^{\infty} dp \frac{\exp\left(-i\tau\frac{p^2}{2} - ic(\tau)p\right)}{\left(\alpha^2 + (p+b(\tau))^2\right)(\alpha^2+p^2)}\end{aligned}\tag{54}$$

and

$$\begin{aligned}\langle\psi, \Psi(\tau)\rangle &= ie^{-ia(\tau)}\sqrt{\frac{\alpha^5}{2\pi^3}}\int_0^\tau ds\psi_I(s) \\ &\times \int_{-\infty}^{\infty} dp\frac{e^{ic_{\tau s}(p-b(\tau))}e^{-\frac{i}{2}(p-b(\tau))^2(\tau-s)}}{(\alpha^2+p^2)} .\end{aligned}\tag{55}$$

$|q(\tau)|^2$ has of course the interpretation as survival probability. We use the abreviation $c_{tt'} := c(t) - c(t')$. These expressions constitute explicit examples for the general statements made at the end of section 5. Taking the ultra-extreme intensity limit we obtain as a consequence of the Riemann-Lebesgue theorem

$$\lim_{E_0\to\infty}|q(\tau)|^2 = \begin{cases}|\langle\psi, \psi_{GV}(\tau)\rangle|^2 & \text{for } b(\tau) = c(\tau) = 0 \\ 0 & \text{otherwise}\end{cases} \tag{56}$$

This means we have asymptotically weak stabilization and the result is in agreement with the one in section 4. However, the assumption made on the potential in [3] does not include the δ-potential[9], such that (56) is not only obtained by an alternative method but also covers an additional case. As we already observed, a different physical behaviour is obtained depending on the values of $b(\tau)$ and $c(\tau)$ and it is therefore instructive to treat several cases separately: To the lowest order we obtained

i) $b(\tau) = 0$, $c(\tau) = 0$

$$\mathcal{P}(\psi) = 1 - \frac{4}{\pi}\left|U_{\frac{1}{2},-\frac{1}{2}}\left(i\tau\frac{\alpha^2}{2}\right)\right|^2 \tag{57}$$

[9] The δ-potential is not a Kato small potential.

ii) $b(\tau)=0,\ c(\tau)\neq 0$

$$q(\tau)=\varphi_{-}\,U_{\frac{1}{2},\frac{1}{2}}\left(\Phi_{+}\right)+\varphi_{+}\,U_{\frac{1}{2},\frac{1}{2}}\left(\Phi_{-}\right)+i\tau\alpha^{2}U_{\frac{1}{2},\frac{3}{2}}\left(\frac{i\tau\alpha^{2}}{2}\right) \quad (58)$$

with $\varphi_{\pm}=\frac{1}{2}\left(1\pm\alpha c(\tau)-i\tau\alpha^{2}\right)$. Equation (58) is only valid for $|\gamma|<1$.

iii) $b(\tau)\neq 0,\ c(\tau)\neq 0$

$$q(\tau)=\frac{\tilde{\varphi}_{+}}{\sqrt{\pi}}\left[U_{\frac{1}{2},\frac{1}{2}}\left(\Phi_{+}\right)+U_{\frac{1}{2},\frac{1}{2}}(\tilde{\Phi}_{-})\right]+\frac{\tilde{\varphi}_{-}}{\sqrt{\pi}}\left[U_{\frac{1}{2},\frac{1}{2}}\left(\Phi_{-}\right)+U_{\frac{1}{2},\frac{1}{2}}(\tilde{\Phi}_{+})\right] \quad (59)$$

with

$$\tilde{\varphi}_{\pm} = \frac{\alpha^{2}}{b(\tau)^{2}\pm 2\alpha b(\tau)i} \quad (60)$$

$$\tilde{\Phi}_{\pm} = \tau\alpha^{2}\left(\pm\tilde{\gamma}+\frac{i}{2}\left(1-\tilde{\gamma}^{2}\right)\right) \quad (61)$$

Equation (59) is only valid for $|\tilde{\gamma}|=\left|\frac{c(\tau)}{\tau\alpha}+\frac{b(\tau)}{\alpha}\right|<1$.

The restrictions on the parameters γ and $\tilde{\gamma}$ originate in the limited validity of the integral representation for the confluent hypergeometric functions, which is employed here. To obtain all higher orders we have to compute

$$\langle\psi,\Psi\left(\tau\right)\rangle=\frac{i\alpha^{\frac{3}{2}}e^{-ia(\tau)}}{2\pi\sqrt{2}} \int_{0}^{\tau} ds\psi_{I}\left(s\right)e^{\frac{ic_{\tau s}}{2(\tau-s)}}\left(U_{\frac{1}{2},\frac{1}{2}}\left(\Phi'_{-}\right)+U_{\frac{1}{2},\frac{1}{2}}\left(\Phi'_{+}\right)\right) \quad (62)$$

with

$$\Phi'_{\pm} = \pm\alpha\left(c_{ts}-\left(t-s\right)b\left(t\right)\right)+\frac{i}{2}\left(t-s\right)\left(\alpha^{2}-\left(\frac{c_{ts}}{t-s}-b\left(t\right)\right)^{2}\right). \quad (63)$$

Hence the problem to compute the ionization probability has been reduced to solving (46) and subsequently evaluate (62). Surely this is not possible to perform entirely in an analytical way, but the initial problem has now been reduced to a numerical task, whose error is well under control. In [4] we carried out this analysis for a pulse involving the trapezoidal enveloping function. We do not find any evidence for strong stabilization even for $b(\tau)=0$ and $c(\tau)=0$, however, asymptotically weak stabilization exists for the latter case.

7 Conclusions

The main outcome of our investigations is that the governing parameters for the behaviour of an atom in an intense laser field are the total classical momentum transfer $b(\tau)$ and the total classical displacement $c(\tau)$[10]. Whenever both these two quantities vanish, asymptotically weak stabilization does exist. Also the authors of [26] found asymptotically weak stabilization for this situation (see Fig. 3 therein). For all other cases we did not find any evidence for stabilization.

Since our findings apparently differ from many theoretical results other authors obtained by alternative methods we would like comment on possible resolutions for this discrepancy:

Introducing a cut-off in the number of bound states will produce an upper bound for the ionization probability, since in this way one effectively enlarges the continuum. In case one finds stabilization for such a bound one could confidently conclude that this effect indeed exists. Since for lower intensities one can certainly expect this bound to be relatively close to the real value, whereas for higher intensities this bound should decrease even more.

The introduction of a cut-off (e.g. equation (3) in [12]) in the "discrete continuum" (besides the fact that this is an ill-defined concept) yields a lower bound for the real ionization probability, which is expected to be relatively accurate for low intensities but very far from the real value for high intensities.

[10] Some authors claim [39] that for a pulse of the form (25) $b(T)$ and $c(T)$ should be relevant parameters. Besides the fact that for a smoothly differentiable pulse (e.g. (26)) these quantities are not precisely defined they do not emerge in our analysis as significant.

Hence keeping the cut-off fixed and interpreting the result obtained in this way as "exact", one has certainly introduced an artificial mechanism to "create" stabilization.

A further approach is based on the Fourier expansion of the Hamiltonian in the KH-gauge and thereafter simply keeping the lowest terms. The way this, in principle legitimate, method is carried out sometimes makes conceptual assumptions which are in clear conflict to our main physical framework. For instance the basic Hamiltonian used in [22] (equation (1) therein) is in our notation $H^S_{KH}(t)$ for an instantaneously switched on monochromatic laser pulse, i.e. $E(t) = E_0 \cos(\omega t)$. However, the authors of [22] claim stabilization to exist for pulses with a smooth adiabatic turn on and off (gaußian enveloping function). Clearly equation (6) in [22] breaks the gauge invariance discussed in section 2.1. of our manuscript, such that the authors consider an entirely different system. In other words the Hamiltonian (6) in [22] is not gauge equivalent to $H^S_l(t)$ for a laser pulse with gaußian enveloping function. The potential in $H^S_{KH}(t)$ is shifted by $c(t)$ and not $E(t)$. From our point of view the findings of the authors are not surprising since they artificially impose that $c(\tau) = 0$, (we also find asymptotically weak stabilization in this case) such that one should solve the inverse problem in this case to find out which pulse is really considered there. However, even under these assumptions we would still not find strong stabilization. The same procedure is used for instance in [23, 24].

Concerning the investigations which do not find any evidence for stabilization at all, we would like to make the following comment on the use of the GV-perturbation theory. In the last reference of [7] a pulse with instantaneous switch on was used in this context, i.e. $E(t) = E_0 \cos(\omega t)$, and an analysis up to first order GV-perturbation theory was carried out. Typical parameters in [7] were $\alpha = 1/2, E_0 = 5, \omega = 1.5$ and the pulse length was 2 cycles, that is $\tau \sim 8$. For these parameters we obtain for the relative maximal error $\mu \approx 8.44$, such that we do not expect the GV-perturbation series to be a good approximation up to this order and statements made in this context should be treated with extreme care.

We would like to conclude with a remark on the existing experimental findings [40]. So far the experiments carried out only find evidence for asymptotically weak stabilization and confirmations for the existence of strong stabilization do not exist to our knowledge. We are therefore in complete qualitative (it is difficult to determine which values $b(\tau)$ and $c(\tau)$ have for the experimentally employed pulses (see also footnote 3 for this)) agreement with the existing experiments.

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