Existence and Regularity for higher dimensional H-systems

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August 1998

1 Introduction

In this paper we are concerned with the existence and regularity of solutions of the degenerate nonlinear elliptic systems known as H-systems. For a given real valued function H defined on (a subset of) $\mathbb{R}^{n+1}$, the associated H-system on a subdomain of $\mathbb{R}^n$ (we will generally take the domain to be B, the unit ball) is given by

$$D_{x_i}(|Du|^{n-2}D_{x_i}u) = \sqrt{n^n}(H\circ u)u_{x_1}\times\cdots\times u_{x_n} \tag{1.1}$$

for a map u from B to $\mathbb{R}^{n+1}$ (obviously for (1.1) to make sense classically we look for $u \in C^2(B,\mathbb{R}^{n+1})$; as we discuss in Section 2, it also makes sense to look for a weak solution $u \in W^{1,n}(B,\mathbb{R}^{n+1})$ to (1.1) under suitable restrictions on H). Here we use the summation convention, and the cross product $w_1\times\cdots\times w_n : \mathbb{R}^{n+1}\oplus\cdots\oplus\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ is defined by the property that $w\cdot w_1\times\cdots\times w_n = \det W$ for all vectors $w\in\mathbb{R}^{n+1}$, where W is the $(n+1)\times(n+1)$ matrix whose first row is $(w^1,\cdots,w^{n+1})$ and whose jth row is $(w^1_{j-1},\cdots,w^{n+1}_{j-1})$ for $2\le j\le n+1$.

Analytically, it is natural to consider boundary value problems associated to (1.1), for example Dirichlet boundary conditions

$$u|_{\partial B} = \varphi \tag{1.2}$$

for a suitably regular prescribed φ. We denote the Dirichlet problem associated with H and φ (viz. (1.1), (1.2)) by $\mathcal{D}(H,\varphi)$.

One of the main reasons for considering (1.1) is that, if u fulfills certain additional conditions, then it represents a hypersurface in $\mathbb{R}^{n+1}$ whose mean curvature at the point $u(x)$, for $x\in B$, is given by $H\circ u(x)$.

Specifically a map $u\colon B\to\mathbb{R}^{n+1}$ is called *conformal* if

$$u_{x_i}\cdot u_{x_j} = \lambda^2(x)\delta_{ij} \quad \text{on } B \tag{1.3}$$

for some real-valued function λ. In the case $n=2$, the map u satisfies a *Plateau boundary condition* (for Γ) if

$$u|_{\partial B} \quad \text{is a homeomorphism from } \partial B \text{ to } \Gamma \tag{1.4}$$

for a given rectifiable Jordan curve Γ in $\mathbb{R}^3$. A solution u to (1.1), (1.3), (1.4) solves the Plateau problem for H and Γ, which we will denote by $\mathcal{P}(H,\Gamma)$; the solution solves $\mathcal{P}(H,\Gamma)$ classically if $u\in C^0(\overline{B},\mathbb{R}^3)\cap C^2(B,\mathbb{R}^3)$, and has mean curvature $H\circ u(x)$ at every regular point $u(x)$. The problem $\mathcal{P}(H,\Gamma)$ is thus a generalization of the classical Plateau problem for minimal surfaces (i.e. the case $H\equiv 0$) first solved by Douglas and by Radó in the early 1930's: we refer the reader to the monograph [DHKW] for details and literature concerning this case, and assume that H does not vanish identically in the rest of this discussion.

The first existence results were obtained by Heinz [He], and further existence results were obtained by many authors, including Werner [Wr], Hildebrandt [Hi1], [Hi2], Wente [W], Gulliver and Spruck [GS1], [GS2] and Steffen [St1], [St2]. In particular we note the so-called Wente-type existence theorems, such as [W, Theorem 6.2] (in the case of constant H) and [St1, Theorem 6.2] (for H not a priori constant, and under more general conditions), where smallness of H in a suitable sense (namely when compared to an appropriate power of the minimal area of a surface spanning Γ) guarantees a solution of $\mathcal{P}(H,\Gamma)$. Similar results for the Dirichlet problem $\mathcal{D}(H,\varphi)$ are given in [St1, Theorem 6.2].

In higher dimensions the formulation of the Plateau problem $\mathcal{P}(H,\Gamma)$ depends crucially upon the chosen generalization of the boundary condition (1.4), and in particular on the boundary Γ.

In the setting of geometric measure theory one can take Γ to be an integer multiplicity, rectifiable current of dimension $n+1$, and the Plateau problem $\mathcal{P}(H,\Gamma)$ is to find an n-dimensional integer multiplicity rectifiable current T with $\partial T = \Gamma$ such that the weak version of (1.1) is satisfied for T, i.e.

$$\int_M (\operatorname{div}_M Y + H\, Y \cdot \nu_T)\, d\mu_T = 0 \tag{1.5}$$

for all test vectorfields $Y \in C^1_c(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ with $\operatorname{spt}(Y) \cap \operatorname{spt}\Gamma = \varnothing$; here μ_T is n-dimensional Hausdorff measure weighted by the multiplicity function of T, ν_T is the unit normal vector field on T, and M is the supporting set of T in $\mathbb{R}^{n+1}$ (cf. [Si, Section 16.5]). Existence results, again in terms of Wente-type theorems, were proven by Duzaar and Fuchs [DF1], [DF3] and the first author [Du2].

The general strategy in both settings (the 2-dimensional parametric setting and the higher dimensional geometric measure theoretic setting) is similar. The first step is to construct a suitable energy whose critical points are (at least formally) the desired solutions of the Plateau problem $\mathcal{P}(H,\Gamma)$. The next step is to show that the minimum of this energy is in fact achieved, and that it is achieved by a surface or a current in the desired class. This energy is composed of two terms, the first of which is the (n-)Dirichlet integral, the second of which is an appropriately weighted (depending on H) volume term. The volume term is not lower semicontinuous with respect to weak convergence in any space which is appropriate to these settings, so it is necessary to control the volume in terms of the Dirichlet energy term. This is done by applying appropriate isoperimetric inequalities.

This same broad strategy is followed in the current paper to obtain existence results for the Dirichlet problem $\mathcal{D}(H,\varphi)$ in higher dimensions. In Section 3 we give a variational formulation of the problem in the space $W^{1,n}(B,\mathbb{R}^{n+1})$; the aim is to realize the solutions of $\mathcal{D}(H,\varphi)$ as minimizers of an appropriate subclass of $W^{1,n}(B,\mathbb{R}^{n+1})$. Since weak $W^{1,n}$ convergence does not preserve homology, we are unable to directly adapt the methods of [DS3] to our situation (in the setting of geometric measure theory, these authors obtained existence results for solutions of the Plateau problem with the image being contained in Riemannian manifold of arbitrary dimension). This motivates the definitions of spherical currents and of homologically n-aspherical domains (Definition 3.1), which allows a reasonable definition of the H-volume enclosed by two maps in $W^{1,n}(B,A)$ for $A \subset \mathbb{R}^{n+1}$ (Definition 3.4), and hence of the energy functional to be minimized.

In order to control the H-volume by the Dirichlet integral, we need an estimate of how much of the volume and surface area can be lost under passage to the weak limit in our chosen subclass. This is accomplished in Lemma 4.1. Such 'bubbling phenomena' are an important feature of many nonlinear elliptic and parabolic problems, in particular in the area of harmonic maps: see for example [SU], and recent papers concerning the heat-flow for harmonic maps, such as [Q] and [DT].

Once this is accomplished, we need to adapt the notions of isoperimetric conditions from [St1] and later works to our situation. Having done this, in Section 5 we are able to prove existence results under various assumptions on H and on the support of a given extension of our Dirichlet boundary data. Our results include as a special case (see Corollary 5.3) previous results for constant H obtained by Duzaar and Fuchs [DF2] and Mou and Yang [MY]; in [MY] the authors also obtain existence results for unstable solutions of higher-dimensional H-systems for suitably restricted, constant H.

In Section 6 we consider the regularity of the solutions whose existence is guaranteed by the theorems of Section 5. In the geometric measure theory setting for the Plateau problem $\mathcal{P}(H,\Gamma)$ discussed above, optimal regularity results were obtained by the first author [Du2] and by Duzaar and Steffen [DS1], [DS2]. The authors established that the (energy minimizing) solutions of $\mathcal{P}(H,\Gamma)$ are classical hypersurfaces smooth up to the boundary for $n \leq 6$ and have a singular set which is closed, disjoint from the support of the boundary and of Hausdorff dimension at most $n-7$ for $n \geq 7$. Due to our setting, we are able to obtain more satisfactory results (Theorem 6.1); in particular, our solutions to $\mathcal{D}(H,\varphi)$ are Hölder continuous, and are $C^{1,\alpha}$ under reasonable additional smoothness assumptions on H.

We close this introduction with a few remarks on notation. We will denote p-dimensional Lebesgue measure by $\mathcal{L}^p$. The symbol α_p is used to to denote $\mathcal{L}^p(B^p)$, where B^p is the unit ball in $\mathbb{R}^p$, and we denote by γ_p the optimal isoperimetric constant in $\mathbb{R}^p$, i.e. the smallest constant such that (cf. [Fe, 4.5.9 (31)])

$$\mathbf{M}(Q) \leq \gamma_p \mathbf{M}(\partial Q)^{\frac{p}{p-1}} \tag{1.6}$$

holds for all integer multiplicity rectifiable p-currents in $\mathbb{R}^p$ (note that $\gamma_p = p^{\frac{-p}{p-1}}\alpha_p^{\frac{-1}{p-1}}$). We will denote the standard volume form on $\mathbb{R}^{n+1}$ by Ω.

2 The variational problem

We begin by giving a variational formulation of the H-system (1.1). We wish to consider, for $u \in W^{1,n}(B,\mathbb{R}^{n+1})$, an energy of the form

$$\mathbf{E}_H(u) := \mathbf{D}(u) + n\mathbf{V}_H(u) \tag{2.1}$$

with $\mathbf{D}(u) = \frac{1}{\sqrt{n^n}}\int_B |Du|^n\, dx$ and $\mathbf{V}_H$ a functional which will be precisely specified later, and which will be seen to be a signed volume weighted by H, in an appropriate sense. For the moment, the only requirement we make of $\mathbf{V}_H$ is that the following *homotopy formula* is valid:

$$\mathbf{V}_H(u_t) - \mathbf{V}_H(u) = \int_B \int_0^t (H \circ U)\langle \Omega \circ U, U_t \wedge U_{x_1} \wedge \cdots \wedge U_{x_n}\rangle\, dt\, dx \tag{2.2}$$

for variations $U(t,x) = u_t(x)$ of $u(x) = u_0(x)$.

A variation U is termed *sufficiently regular* in $\mathbb{R}^{n+1}$ if $u_t \in W^{1,n}(B,\mathbb{R}^{n+1})$ for sufficiently small t, the initial velocity field $\zeta = \frac{d}{ds}\big|_{s=0} u_s$ belongs to $W^{1,n}(B,\mathbb{R}^{n+1}) \cap L^\infty(B,\mathbb{R}^{n+1})$, and differentiation under the integral with respect to t is valid at $t=0$ for $\mathbf{D}(u_t)$ and $\mathbf{V}_H(u_t)-\mathbf{V}_H(u)$.

Lemma 2.1 (*first variation*) *For sufficiently regular variations u_t in $W^{1,n}(B,\mathbb{R}^{n+1})$ with initial velocity field ζ in $W^{1,n}(B,\mathbb{R}^{n+1}) \cap L^\infty$ we have*

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} & \mathbf{E}_H(u_t) \\ &= n\int_B \left[\frac{1}{\sqrt{n^n}}|Du|^{n-2}Du\cdot D\zeta + (H\circ u)\zeta\cdot u_{x_1}\times u_{x_2}\times\cdots\times u_{x_n}\right]dx.\end{aligned}$$

Proof: Formal differentiation of $\mathbf{D}(u_t)$ yields the integrand $\frac{n}{\sqrt{n^n}}|Du|^{n-2}Du\cdot D\zeta$, and formal differentiation of (2.2) gives the integrand $(H\circ u)\langle \Omega\circ u, \zeta\wedge u_{x_1}\wedge\cdots\wedge u_{x_n}\rangle = (H\circ u)\zeta\cdot u_{x_1}\times\cdots\times u_{x_n}$. ■

The integral $\delta\,\mathbf{E}_H(u;\zeta)$ is termed the *first variation of the energy* $\mathbf{E}_H$ in the direction ζ.

As a direct consequence we have

Corollary 2.2 *A map $u\in W^{1,n}(B,\mathbb{R}^{n+1})$ is a weak solution of the H-surface equation if and only if $\delta\,\mathbf{E}(u;\zeta)=0$ for all vector fields $\zeta\in W_0^{1,n}(B,\mathbb{R}^{n+1})\cap L^\infty$.* ■

This means that the weak H-surface equation, i.e.

$$D_{x_i}(|Du|^{n-2}D_{x_i}u) = \sqrt{n^n}(H\circ u)u_{x_1}\times\cdots\times u_{x_n} \quad \text{in } B\,, \tag{2.3}$$

is precisely the Euler equation associated to the energy functional $\mathbf{E}_H$.

An important class of variations for our purposes are those of the form

$$u_t(x) = \Phi^Y(t\eta(x), u(x)), \tag{2.4}$$

for $Y\in C_c^1(\mathbb{R}^{n+1},\mathbb{R}^{n+1})$ a smooth vector field in $\mathbb{R}^{n+1}$, Φ^Y the flow associated to Y and η a sufficiently smooth function defined on $\overline{B}$ (generally $\eta\in C^1(\overline{B},\mathbb{R})$). The initial field is then $\eta(Y\circ u)$ (cf. [Du1, Section 2], [DS3, Lemma 1.3], [DS4, Section 2]).

The following variational equality and inequality follow in direct analogy to the proof of [DS4, Prop. 2.3 (ii)].

Lemma 2.3 *(i) Assume that $u\in W^{1,n}(B,\mathbb{R}^{n+1})$ is $\mathbf{E}_H$-minimizing with respect to the variation u_t given by (2.4) for each $Y\in C_c^1(\mathbb{R}^{n+1},\mathbb{R}^{n+1})$ and each $\eta\in C_c^1(B,\mathbb{R})$. Then u is a solution to the weak H-surface equation (2.3).*

(ii) Let $A\subset\mathbb{R}^{n+1}$ be the closure of a domain with C^2 boundary. Suppose further that u is $\mathbf{E}_H$-minimizing for one-sided variations u_t, $0\le t\ll 1$, for $\eta\ge 0$ and $Y(a)=0$ or $Y(a)$ directed strictly inwards at each $a\in\partial A$. Then u satisfies the inequality

$$\delta\,\mathbf{E}_H(x;\zeta) = n\int_B \left[\frac{1}{\sqrt{n^n}}|Du|^{n-2}Du\cdot D\zeta + (H\circ u)\zeta\cdot u_{x_1}\times\cdots\times u_{x_n}\right]dx \ge 0 \tag{2.5}$$

for all vector fields $\zeta\in W_0^{1,n}(B,\mathbb{R}^{n+1})\cap L^\infty(B,\mathbb{R}^{n+1})$ with $\zeta\cdot(\widetilde{\nu}\circ u)\ge 0$ almost everywhere on $u^{-1}V$ for some neighbourhood V of ∂A in $\mathbb{R}^{n+1}$ und some C^1-extension $\widetilde{\nu}$ of the (inwardly pointing) unit normal vector field ν on ∂A to $\mathbb{R}^{n+1}$. ■

Proposition 2.4 *Let $A \subset \mathbb{R}^{n+1}$ be the closure of a domain with C^2-boundary, ν be the (inwardly pointing) unit normal on ∂A, and $\mathcal{K}_{\partial A}(a)$ be the minimum of the principal curvatures of ∂A at the point a (with respect to ν). Let $u \in W^{1,n}(B,A)$ satisfy the inequality (2.5). Then we have:*

(i) There exist a nonnegative Radon measure λ on B which is absolutely continuous with respect to $\mathcal{L}^n$ and which is concentrated on the coincidence set $u^{-1}\partial A$, such that:

$$\delta \mathbf{E}_H(u;\zeta) = \int_{u^{-1}\partial A} \zeta \cdot (\nu \circ u)\, d\lambda \tag{2.6}$$

for each $\zeta \in W_0^{1,n}(B,\mathbb{R}^{n+1}) \cap L^\infty(B,\mathbb{R}^{n+1})$;

(ii) If $|H| \le \mathcal{K}_{\partial A}$ on ∂A, we have $\lambda = 0$; more generally

$$\lambda \le \mathcal{L}^n \llcorner \frac{n}{\sqrt{n^n}} |Du|^n \Big(|H \circ u| - \mathcal{K}_{\partial A} \circ u \Big)_+ \quad \text{on } u^{-1}\partial A; \tag{2.7}$$

(iii) If $|H(a)| < \mathcal{K}_{\partial A}(a)$ for some $a \in \partial A$ and if $u|_{\partial B}$ omits some neighbourhood of a, then there exists a neighbourhood V of a in $\mathbb{R}^{n+1}$ such that $u(B) \cap V = \varnothing$.

Proof: We write $d(p) = \operatorname{dist}(p, \partial A)$ for $p \in \mathbb{R}^{n+1}$, and extend the (inwardly pointing) unit normal vector field ν to a C^1-vector field, again denoted by ν, such that ν coincides with $\operatorname{grad} d$ on a neighbourhood of ∂A.

We firstly consider the case that A is compact. In this case, $\zeta = \eta(\nu \circ u)$ is admissable in (2.5) if $0 \le \eta \in C_c^1(B,\mathbb{R})$. Applying the Riesz representation theorem we deduce the existence of a nonnegative Radon measure λ on B such that

$$\delta\, \mathbf{E}_H(u, \eta(\nu \circ u)) = \int_B \eta\, d\lambda \tag{2.8}$$

holds for all $\eta \in C_c^1(B,\mathbb{R})$.

We now choose $\vartheta \in C^\infty(\mathbb{R},\mathbb{R})$ nonincreasing with $\vartheta \equiv 1$ on $(-\infty, \frac{1}{2}]$, $\vartheta \equiv 0$ on $(1,\infty)$ and define $\vartheta_\varepsilon(t) = \vartheta(\frac{t}{\varepsilon})$ for $\varepsilon > 0$. We consider $\zeta_\varepsilon = \eta(\vartheta_\varepsilon \circ d \circ u)(\nu \circ u)$ with $\eta \ge 0$ as before. Then $\zeta = \zeta_\varepsilon$ on the preimage under u of a neighbourhood of ∂A, so that $\zeta - \zeta_\varepsilon$ and $\zeta_\varepsilon - \zeta$ are both admissable in the variational inequality. This means

$$\delta\, \mathbf{E}_H(u;\zeta_\varepsilon) = \delta \mathbf{E}_H(u;\zeta) \ge 0. \tag{2.9}$$

For ε sufficiently small we estimate

$$u_{x_i} \cdot (\zeta_\varepsilon)_{x_i} \le (\vartheta_\varepsilon \circ d \circ u)[\eta_{x_i} u_{x_i} \cdot (\nu \circ u) + \eta u_{x_i} \cdot ((D\nu) \circ u) u_{x_i}].$$

Applying this in (2.5), noting that $u_{x_i} \cdot (\nu \circ u) = 0$ almost everywhere on $u^{-1}\partial A$ and letting ε approach 0 we have

$$\begin{aligned} 0 \le \tfrac{1}{n} \delta\, \mathbf{E}_H(u;\zeta) \quad &\le \quad \int_{u^{-1}\partial A} \Bigg[\frac{1}{\sqrt{n^n}} |Du|^{n-2} u_{x_i} \cdot ((D\nu) \circ u\, u_{x_i}) \\ &\qquad\qquad + (H \circ u)(\nu \circ u) \cdot u_{x_1} \times \cdots \times u_{x_n} \Bigg] \eta\, dx. \end{aligned}$$

Since $u_{x_i} \cdot ((D\nu) \circ u u_{x_i}) = -b_{\partial A} \circ u(u_{x_i}, u_{x_i})$ almost everywhere on $u^{-1}\partial A$, where $b_{\partial A}$ denotes the second fundamental form of ∂A in $\mathbb{R}^{n+1}$ relative to the outwardly pointing normal on ∂A, we have

$$\begin{aligned}\frac{1}{n}\delta\,\mathbf{E}_H(u,\zeta) &\leq \int_{u^{-1}\partial A} \frac{1}{\sqrt{n^n}}|Du|^{n-2}\left[|H\circ u|\,|Du|^2 - \sum_{i=1}^n b_{\partial A}\circ u(u_{x_i},u_{x_i})\right]\eta\,dx \\ &\leq \int_{u^{-1}\partial A} \frac{1}{\sqrt{n^n}}|Du|^n\Big(|H\circ u| - \mathcal{K}_{\partial A}\circ u\Big)\eta\,dx.\end{aligned}$$

Combining this with (2.9) und (2.8) shows

$$\int_B \eta\,d\lambda \leq \frac{n}{\sqrt{n^n}}\int_{u^{-1}\partial A} |Du|^n\Big(|H\circ u| - \mathcal{K}_{\partial A}\circ u\Big)\eta\,dx,$$

which yields the claimed estimate on the Radon measure λ, i.e.

$$\lambda \leq \mathcal{L}^n \llcorner \frac{n}{\sqrt{n^n}}|Du|^n\Big(|H\circ u| - \mathcal{K}_{\partial A}\circ u\Big)_+ \quad \text{on } u^{-1}\partial A.$$

This completes the proof of (ii).

To show (i) we begin by noting that (ii) immediately yields the absolute continuity of λ with respect to $\mathcal{L}^n$, and further that $\lambda(B \setminus u^{-1}\partial A) = 0$. It is easy to see by approximation that (2.8) holds for all $\eta \in W_0^{1,n}(B,\mathbb{R}) \cap L^\infty(B,\mathbb{R}^{n+1})$. In the case of a general vector field $\zeta \in W_0^{1,n}(B,\mathbb{R}^{n+1}) \cap L^\infty(B,\mathbb{R}^{n+1})$, we decompose $\zeta = \zeta^\perp + \zeta^\top$, where $\zeta^\perp = \eta(\nu\circ u)$ with $\eta = \zeta\cdot(\nu\circ u) \in W_0^{1,n}(B,\mathbb{R}^{n+1}) \cap L^\infty(B,\mathbb{R}^{n+1})$. We apply (2.8) to conclude

$$\delta\,\mathbf{E}_H(u;\zeta^\perp) = \delta\,\mathbf{E}_H(u,(\zeta\cdot\nu\circ u)\nu\circ u) = \int_{u^{-1}\partial A} \zeta\cdot(\nu\circ u)\,d\lambda. \tag{2.10}$$

Further we have that $\zeta^\top\cdot(\nu\circ u) = 0$ almost everywhere on the preimage of a neighbourhood of ∂A under u, i.e. $\zeta^\top$ and $-\zeta^\top$ are both admissable in (2.5), and hence $\delta\,\mathbf{E}_H(u;\zeta^\top) = 0$. Combining this with (2.10), we have shown (i).

In the case of arbitrary A, one replaces $\nu\circ u$ in the above discussion by $(\psi_k\circ u)(\nu\circ u)$ with $\psi_k \in C_c^1(\mathbb{R}^{n+1},[0,1])$, such that the ψ_k's tend to the identity on $\mathbb{R}^{n+1}$. One then argues directly analogously to the case $n = 2$ ([DS4, Proposition 2.4]) to show that the associated Radon measures λ_k approach a limit measure λ which satisfies (i) and (ii).

In the same way (iii) can be proven by direct analogy with the case $n = 2$: we refer the reader to [DS4, Proposition 2.4]. ■

Remark 2.5 If we assume that u is a *conformal solution* of the variational inequality, (i.e. (1.3) holds), then $\mathcal{K}_{\partial A}$ can be replaced by the mean curvature $H_{\partial A}$ in the assumptions.

3 The volume functional

Given $u \in W^{1,n}(B,\mathbb{R}^{n+1})$ we can define the **associated n-current** J_u in $\mathbb{R}^{n+1}$ via integration of n-forms over u, i.e.

$$J_u(\beta) = \int_B u^\# \beta = \int_B \langle \beta\circ u, u_{x_1}\wedge\cdots\wedge u_{x_n}\rangle\,dx \quad \text{for } \beta \in \mathcal{D}^n(\mathbb{R}^{n+1}); \tag{3.1}$$

here $\mathcal{D}^k(\mathbb{R}^{n+1})$ denotes the space of smooth, compactly supported k-forms on $\mathbb{R}^{n+1}$. It is straightforward to see that J_u is an n-current of finite mass (where the mass of a k-current T on $\mathbb{R}^{n+1}$ is defined by $\mathbf{M}(T) := \sup\{T(\beta) : \beta \in \mathcal{D}^k(\mathbb{R}^{n+1}),\ ||\beta||_\infty \leq 1\}$), since

$$\mathbf{M}(J_u) \leq \int_B |u_{x_1} \wedge \cdots \wedge u_{x_n}|\, dx \leq \frac{1}{\sqrt{n^n}} \int_B |Du|^n\, dx = \mathbf{D}(u). \tag{3.2}$$

Using a Lusin-type approximation argument for mappings in $W^{1,n}$ (cf. [EG, 6.6.3]) we can argue similarly to the case $n = 2$ (cf. [DS4, Section 3]) to see that J_u is a (locally) rectifiable n-current in $\mathbb{R}^{n+1}$. If v is another surface in $W^{1,n}(B, \mathbb{R}^{n+1})$, then $(J_u - J_v)(\beta)$ is determined by integration of $u^\#\beta - v^\#\beta$ over $G = \{x \in B : u(x) \neq v(x)\}$, as $Du = Dv$ $\mathcal{L}^n$-almost everywhere on $B \setminus G$. Thus we can refine (3.2) to

$$\mathbf{M}(J_u - J_v) \leq \mathbf{D}_G(u) + \mathbf{D}_G(v), \quad \text{if } u = v \quad \text{on } B \setminus G, \tag{3.3}$$

where

$$\mathbf{D}_U(u) = \frac{1}{\sqrt{n^n}} \int_U |Du|^n\, dx \tag{3.4}$$

for $\mathcal{L}^n$-measurable $U \subset B$.

In general the boundary ∂T of a k-current T, $k \geq 1$, is defined by $\partial T(\alpha) = T(d\alpha)$ for $\alpha \in \mathcal{D}^{k-1}(\mathbb{R}^{n+1})$. For u, $v \in W^{1,n}(B, \mathbb{R}^{n+1})$ with $u - v \in W_0^{1,n}(B, \mathbb{R}^{n+1})$ we calculate directly that $J_u - J_v$ is a closed n-current, i.e. $\partial(J_u - J_v) = 0$. First we see that for u, $v \in C^2(\overline{B}, \mathbb{R}^{n+1})$ with $u = v$ on ∂B we have

$$\begin{aligned} \partial J_u(\alpha) &= J_u(d\alpha) = \int_B u^\#\, d\alpha = \int_B d(u^\#\alpha) \\ &= \int_{\partial B} u^\#\alpha = \int_{\partial B} v^\#\alpha = \partial J_v(\alpha). \end{aligned}$$

In the general case we approximate u by $u_i \in C^2(\overline{B}, \mathbb{R}^{n+1})$ and $v - u$ by $w_i \in C^\infty_{\mathrm{kpt}}(B, \mathbb{R}^{n+1})$, the approximations being in the $W^{1,n}$-norm. We see that $u_i + w_i$ approaches v in $W^{1,n}$, and since $u_i = u_i + w_i$ on ∂B we have $\partial(J_{u_i} - J_{u_i+w_i}) = 0$. Letting i tend to infinity we see $\partial(J_u - J_v) = 0$, which is the desired conclusion.

In the following we take A to be a closed subset of $\mathbb{R}^{n+1}$ – the *obstacle* – and $u_0 \in W^{1,n}(B, A)$ to be a fixed reference surface. We let

$$\mathcal{S}(u_0, A) = \{u \in W^{1,n}(B, A) \ : \ u - u_0 \in W_0^{1,n}(B, \mathbb{R}^{n+1})\} \tag{3.5}$$

denote the class of *admissable surfaces*. The idea behind the geometric definition of the H-volume $\mathbf{V}_H(u, v)$ which is enclosed by two surfaces u, $v \in \mathcal{S}(u_0, A)$ is to consider an $(n+1)$-current Q in $\mathbb{R}^{n+1}$ with $\partial Q = J_u - J_v$, and to integrate $H\Omega$ over Q. Such currents have a relatively simple structure; they are representable by an $L^1(\mathbb{R}^{n+1}, \mathbb{Z})$-function i_Q , such that for all $\gamma \in \mathcal{D}^{n+1}(\mathbb{R}^{n+1})$ there holds

$$Q(\gamma) = \int_{\mathbb{R}^{n+1}} i_Q \gamma.$$

One can consider i_Q to be a set with integer multiplicities and finite absolute volume; in this context the condition $\partial Q = J_u - J_v$ means that u and v paramaterize the boundary of this set with multiplicities in the dual sense of Stokes' theorem, i.e.

$$\int_{\mathbb{R}^{n+1}} i_Q\, d\beta = \int_B u^\#\beta - \int_B v^\#\beta \quad \text{for all } \beta \in \mathcal{D}^n(\mathbb{R}^{n+1}).$$

Since ∂Q is finite we can conclude that i_Q is a BV-function on $\mathbb{R}^{n+1}$, which is a strong motivation for defining the H-volume by

$$\mathbf{V}_H(u,v) = \int_{\mathbb{R}^{n+1}} i_Q H\Omega. \tag{3.6}$$

In order to make this a well-defined functional, we need to clarify the questions of existence and uniqueness for Q. One could try to finesse the question of existence by considering the variational problem restricted to those $u \in \mathcal{S}(u_0, A)$ for which $J_u - J_{u_0}$ is homologically trivial in A, i.e. $J_u - J_{u_0}$ is the boundary of an $(n+1)$-current Q with support in A. However simple examples show that such a homological property is not a priori preserved under passage to a weak limit: see [DS4, Section 1]. It is thus reasonable to impose the restriction that $J_u - J_v$ be homologically trivial in A for all u, $v \in \mathcal{S}(u_0, A)$; this amounts to the condition that certain n-currents are boundaries in A, as made precise in the following definition.

Definition 3.1 An n-current T on $\mathbb{R}^{n+1}$ with support in A is called:

(i) **spherical** in A when it can be written in the form $T = f_\# [\![S^n]\!]$ for a map $f \in W^{1,n}(S^n, A)$, i.e.

$$T(\beta) = \int_{S^n} f^\# \beta \quad \text{for} \quad \beta \in \mathcal{D}^n(\mathbb{R}^{n+1}); \text{and}$$

(ii) **homologically trivial** in A when it is the boundary of of a rectifiable $(n+1)$-current with support in A.

If (ii) holds for every spherical n-current with support in A, we say that A is **homologically n-aspherical** in $\mathbb{R}^{n+1}$. ■

If $T = f_\# [\![S^n]\!]$ is homologically trivial in A, then there is an $(n+1)$-current Q in $\mathbb{R}^{n+1}$ with $\partial Q = T$, $\mathbf{M}(Q) < \infty$ and $\operatorname{spt} Q \subset A$. By the constancy theorem [Fe, 4.1.7, 4.1.31] we have that Q is uniquely determined up to real multiples of $[\![\mathbb{R}^{n+1}]\!]$, i.e. Q is *unique*. Further it follows from the general theory of rectifiable currents [Fe, Chapter 4] that we can take Q to be an integer multiplicity rectifiable current. The following lemma shows that, under mild regularity assumptions on A, every spherical n-current T in A can be approximated by smooth maps from S^n to A, and that if the approximating maps are all homologically trivial (when viewed as spherical n-currents), then so is T.

Lemma 3.2 *Let A be a uniform Lipschitz (respectively C^1) neighbourhood retract in $\mathbb{R}^{n+1}$ and let $f \in W^{1,n}(S^n, A)$.*

(i) Given $\varepsilon > 0$ there exists $g \in W^{1,n}(S^n, A)$ such that $\|g - f\|_{W^{1,n}} < \varepsilon$, $g = f$ outside a subset of S^n of measure less than ε, and g is Lipschitz continuous (respectively C^1).

(ii) For given s and r with $0 < s \le \infty$, $0 < r < \infty$ let $\mathbf{M}(f_\# [\![S^n]\!]) < s$, and let $g_\# [\![S^n]\!]$ be the boundary of a rectifiable $(n+1)$-current with mass not greater than r and with support in A for all Lipschitz continuous (respectively C^1) $g : S^n \to A$ with $\mathbf{M}(g_\# [\![S^n]\!]) < s$. Then $f_\# [\![S^n]\!]$ is homologically trivial in A.

Proof: (i) By following the proof of [EG, Theorem 6.6.3, Step 2] we can find, for a given $\lambda > 0$, Lipschitz maps $g_\lambda : S^n \to \mathbb{R}^{n+1}$, such that $\|g_\lambda - f\|_{W^{1,n}} \to 0$ as $\lambda \to \infty$ and $g_\lambda = f$ outside a set $E_\lambda \subset S^n$ with $\lambda^n |E_\lambda| \to 0$ as $\lambda \to \infty$. Further from Step 4 of the same proof we see that $\mathrm{Lip}(g_\lambda) \leq C\lambda$ for C depending only on n. An elementary calculation shows that, for $|E_\lambda| < |S^n|$, no ball of radius $\pi \sqrt[n]{\frac{|E_\lambda|}{|S^n|}}$ can be enclosed in E_λ. Hence given $w \in E_\lambda$ we can find $w' \in S^n \setminus E_\lambda$ with $g_\lambda(w') = f(w')$, and $|w - w'| \leq \pi \sqrt[n]{\frac{|E_\lambda|}{|S^n|}}$. We thus have

$$|g_\lambda(w) - g_\lambda(w')| \leq C\lambda\pi \sqrt[n]{\frac{|E_\lambda|}{|S^n|}}.$$

Since $\lim_{\lambda\to\infty} \lambda^n |E_\lambda| = 0$ we see that, for λ sufficiently large, $g_\lambda(S^n)$ is contained in a uniform neighbourhood $V_\rho(A)$ which admits a Lipschitz retraction $\pi : V_\rho(A) \to A$. We set $g = \pi \circ g_\lambda$ for such λ. Then $g \in \mathrm{Lip}(S^n, A)$, $g = f$ on $S^n \setminus E_\lambda$ and

$$\|g - f\|_{W^{1,n},S^n} \leq \|g\|_{W^{1,n},E_\lambda} + \|f\|_{W^{1,n},E_\lambda}.$$

The last term vanishes as $\lambda \to \infty$ (due to absolute continuity of the integral, and since $|E_\lambda| \to 0$). Further we have (again, as $\lambda \to \infty$),

$$\begin{aligned} \|Dg\|^n_{L^n,E_\lambda} &\leq (\mathrm{Lip}\, g)^n |E_\lambda| \leq (\mathrm{Lip}\,\pi)^n C^n \lambda^n |E_\lambda| \to 0 \quad \text{and} \\ \|g\|_{L^n,E_\lambda} &\leq \|\pi \circ g_\lambda - \pi \circ f\|_{L^n,E_\lambda} + \|f\|_{L^n,E_\lambda} \\ &\leq (\mathrm{Lip}\,\pi)^n \|g_\lambda - f\|_{L^n,E_\lambda} + \|f\|_{L^n,E_\lambda}, \end{aligned}$$

which also converges to 0 as λ tends to ∞. Hence for λ sufficiently large we have $|E_\lambda| < \varepsilon$ and also $\|g - f\|_{W^{1,n},S^n} < \varepsilon$, which completes the proof in the Lipschitz case.

In the C^1-case we can argue completely analogously to the situation for $n = 2$ ([DS4, Lemma 3.2]).

(ii) Consider $f \in W^{1,n}(S^n, A)$ with $\mathbf{M}(f_\#[\![S^n]\!]) < s$. Then given $\varepsilon = \frac{1}{k}$, there exist Lipschitz maps $g_k : S^n \to A$ with $g_k = f$ on $S^n \setminus E_k$, $|E_k| < \frac{1}{k}$ and $\|f - g_k\|_{W^{1,n},S^n} < \frac{1}{k}$. The strong convergence of g_k to f means, in particular, that $\mathbf{M}(g_{k\#}[\![S^n]\!]) \to \mathbf{M}(f_\#[\![S^n]\!])$ as $k \to \infty$, i.e. $\mathbf{M}(g_{k\#}[\![S^n]\!]) < s$ for k sufficiently large. The assumptions then guarantee the existence of rectifiable $(n+1)$-currents Q_k with support in A, mass not greater than r and $\partial Q_k = g_{k\#}[\![S^n]\!]$. The BV-compactness theorem (see e.g. [EG, Theorem 5.2.4]) then ensures (after passage to a subsequence) the existence of a rectifiable $(n+1)$-current Q such that $Q_k \to Q$ (weakly). The lower semicontinuity of $\mathbf{M}$ then implies $\mathbf{M}(Q) \leq r$, and thus further $\partial Q = \lim_{k\to\infty} \partial Q_k = \lim_{k\to\infty} g_{k\#}[\![S^n]\!] = f_\#[\![S^n]\!]$ (the last step due to the strong convergence of g_k to f). ■

Corollary 3.3 *For all u, $v \in W^{1,n}(B, A)$ with $u - v \in W^{1,n}_0(B, \mathbb{R}^{n+1})$, $J_u - J_v$ is a spherical n-current.*

Proof: We compose u with stereographic projection from the south pole of S^n and v with that from the north pole, in order to obtain a map $f \in W^{1,n}(S^n, A)$ with $f_\#[\![S^n]\!] = J_u - J_v$. ■

Definition 3.4 Let u, $v \in W^{1,n}(B, A)$ with $u - v \in W^{1,n}_0(B, \mathbb{R}^{n+1})$. If $J_u - J_v$ is homologically trivial in A we define the H-**volume** enclosed by u and v by

$$\mathbf{V}_H(u, v) = I_{u,v}(H\Omega) = \int_{\mathbb{R}^{n+1}} i_{u,v} H\Omega.$$

Here $I_{u,v}$ is the (unique) rectifiable $(n+1)$-current Q in $\mathbb{R}^{n+1}$ which is associated to the n-current $T = J_u - J_v$, i.e. $\operatorname{spt} Q \subset A$, $\mathbf{M}(Q) < \infty$ and $\partial Q = T$, and $i_{u,v}$ denotes the multiplicity function of $I_{u,v}$. ■

We now need to show that the H-volume has the properties which we require in order to be able to apply the results of Section 2 concerning our variational equalities and inequalities. This is accomplished in the following lemma.

Lemma 3.5 *Let u, $v \in W^{1,n}(B, A)$ be as in Definition 3.4, so that $\mathbf{V}_H(u, v)$ is defined.*

(i) Assume that $A \subset \mathbb{R}^{n+1}$ has a uniform Lipschitz neighbourhood retraction π, $\widetilde{u} \in W^{1,n}(B, A)$, $u - \widetilde{u} \in W_0^{1,n}(B, \mathbb{R}^{n+1})$ and $\|u - \widetilde{u}\|_{L^\infty}$ is smaller than a certain positive constant which only depends on A. Then $\mathbf{V}_H(\widetilde{u}, v)$ and $\mathbf{V}_H(\widetilde{u}, u)$ are also well-defined, and satisfy

$$\mathbf{V}_H(\widetilde{u}, u) + \mathbf{V}_H(u, v) = \mathbf{V}_H(\widetilde{u}, v)$$

$$|\mathbf{V}_H(\widetilde{u}, u)| \le \sup_{\mathbb{R}^{n+1}} |H| \, \|u - \widetilde{u}\|_{L^\infty} (\operatorname{Lip} \pi)^{n+1} [\mathbf{D}_G(u) + \mathbf{D}_G(\widetilde{u})],$$

where $G = \{x \in B : \widetilde{u}(x) \ne u(x)\}$.

(ii) Let Φ_t^Y be the flow of a vector field $Y \in C_c^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ with $\Phi_t^Y(A) \subset A$ for small $t > 0$, $0 \le \eta \in C_c^1(B, \mathbb{R})$ and $u_t(x) = U(t, x)$, where $U(s, x) = \Phi^Y(s\eta(x), u(x))$. Then $\mathbf{V}_H(u_t, v)$ and $\mathbf{V}_H(u_t, u)$ are defined for sufficiently small $t > 0$, and we have

$$\begin{aligned} \mathbf{V}_H(u_t, v) - \mathbf{V}_H(u, v) &= \mathbf{V}_H(u_t, u) \\ &= \int_B \int_0^t (H \circ U) \langle \Omega \circ U, U_s \wedge U_{x_1} \wedge \cdots \wedge U_{x_n} \rangle \, ds \, dx. \end{aligned}$$

Proof: (i) Using the affine homotopy $U(s, x) = (1 - s)u(x) + s\widetilde{u}(x)$ we can define the $(n+1)$-current Q in $\mathbb{R}^{n+1}$ by

$$Q(\gamma) = \int_B \int_0^1 \langle \gamma \circ U, U_s \wedge U_{x_1} \wedge \cdots \wedge U_{x_n} \rangle \, ds \, dx \tag{3.7}$$

for $\gamma \in \mathcal{D}^{n+1}(\mathbb{R}^{n+1})$. The homotopy formula [Fe, 4.1.9] and the constraint $u - \widetilde{u} \in W_0^{1,n}(B, \mathbb{R}^{n+1})$ then imply $\partial Q = J_{\tilde{u}} - J_u$. From (3.7) we see

$$\mathbf{M}(Q) = \|u - \widetilde{u}\|_{L^\infty} \frac{1}{2} [\mathbf{D}_G(u) + \mathbf{D}_G(\widetilde{u})].$$

For $\|u - \widetilde{u}\|_{L^\infty}$ sufficiently small, $\pi_\# Q$ is thus an integer multiplicity rectifiable $(n+1)$-current with support in A, boundary $\partial(\pi_\# Q) = \pi_\# \partial Q = J_{\tilde{u}} - J_u$ and mass

$$\mathbf{M}(\pi_\# Q) \le (\operatorname{Lip} \pi)^{n+1} \mathbf{M}(Q), \tag{3.8}$$

which allows us to conclude $\pi_\# Q = I_{\tilde{u},u}$, and $I_{\tilde{u},v} = I_{\tilde{u},u} + I_{u,v}$. This means that the H-volume satisfies the identity

$$\mathbf{V}_H(\widetilde{u}, v) - \mathbf{V}_H(u, v) = \pi_\# Q(H\Omega) = \mathbf{V}_H(\widetilde{u}, u).$$

The conclusions of (i) now follow from (3.7) and (3.8) after approximating $H\Omega$ by smooth $\gamma \in \mathcal{D}^{n+1}(\mathbb{R}^{n+1})$ with $|\gamma| \leq |H|$.

The proof of part (ii) involves only minor modifications of the case $n = 2$; we omit the details, and refer the reader to [DS4, Lemma 3.6 (ii)]. ∎

Part (ii) of the above lemma shows that the homotopy formula (2.2) is valid for the variations considered in (ii) for the H-volume as defined by $\mathbf{V}_H(u) = \mathbf{V}_H(u, u_0)$, where $u_0 \in W^{1,n}(B, A)$ is a fixed reference surface, and u and u_0 satisfy the conditions of Definition 3.4. Thus all the conclusions of Section 2 are valid for the H-volume as defined in (3.6).

4 A general regularity theorem

In this section we apply the direct method of the calculus of variations to prove a general existence theorem for weak solutions of the Dirichlet problem $\mathcal{D}(H, u_0)$. We minimize the energy functional $\mathbf{E}_H(u) = \mathbf{D}(u) + n\mathbf{V}_H(u, u_0)$ in a suitable subclass of $\mathcal{S}(u_0, A)$.

The n-Dirichlet integral $\mathbf{D}(\cdot)$ is lower semicontinuous in the topology of weak convergence for $\mathcal{S}(u_0, A)$ in $W^{1,n}(B, \mathbb{R}^{n+1})$; however the H-volume $\mathbf{V}_H(\,\cdot\,, u_0)$ is not. This is because a sequence $\{u_i\}$ in $\mathcal{S}(u_0, A)$ converging weakly to u may involve a large part of the volume and the surface area of u_i being parametrized over a small subset of B in such a manner that the $\mathcal{L}^n$-measure converges to zero as $i \to \infty$. Geometrically this can be viewed as the *bubbling off* of a certain amount of the volume and the surface area in the limit. This bubbling phenomenon also means that the homology type will not be a priori preserved in the weak limit.

The following lemma (cf. [DS4, Lemma 4.1] in the 2–dimensional case) gives an analytical description of the bubbling.

Lemma 4.1 *Suppose that $u_i \rightharpoonup u$ weakly in $W^{1,n}(B, \mathbb{R}^m)$ and $u_i|_{\partial B} \to u|_{\partial B}$ uniformly in $L^\infty(\partial B, \mathbb{R}^m)$. Then given $\varepsilon > 0$ there exist $R > 0$, a measurable set G, $G \subset B$ and maps $\widetilde{u}_i \in W^{1,n}(B, \mathbb{R}^m)$, such that after passage to a subsequence:*

(i) $\widetilde{u}_i = u$ *on* $B \setminus G$ *with* $\mathcal{L}^n(G) < \varepsilon$;

(ii) $\widetilde{u}_i|_{\partial B} = u|_{\partial B}$;

(iii) $\widetilde{u}_i(x) = u_i(x)$ *if* $|u_i(x)| \geq R$ *or* $|u_i(x) - u(x)| \geq 1$;

(iv) $\lim\limits_{i\to\infty} \|\widetilde{u}_i - u_i\|_{L^\infty(B,\mathbb{R}^m)} = 0$;

(v) $\widetilde{u}_i \rightharpoonup u$ *weakly in* $W^{1,n}(B, \mathbb{R}^m)$ *as* $i \to \infty$;

(vi) $\limsup\limits_{i\to\infty} [\mathbf{D}_G(\widetilde{u}_i) + \mathbf{D}_G(u)] \leq \varepsilon + \liminf\limits_{i\to\infty} [\mathbf{D}(u_i) - \mathbf{D}(u)]$;

(vii) if the u_i assume values in a closed subset A of $\mathbb{R}^m$ which admits neighbourhood retractions which have Lipschitz constant arbitrarily close to 1 on neighbourhoods of compact subsets, then the $\tilde{x}_n$ can be chosen to have also values in A.

Proof: Using Rellich's theorem and Egoroff's theorem in turn we can find $R > 3$, $\frac{1}{2} \geq \delta_n \downarrow 0$ and $G \subset B$ measurable with $\mathcal{L}^n(G) < \varepsilon$ und $\mathbf{D}_G(u) < \varepsilon'$ (ε' will be determined later) such that after passage to a subsequence, we have $\| u|_{\partial B}\|_{L^\infty} \leq \frac{1}{3}R$, $\sup_{B\setminus G} |u| \leq \frac{1}{3}R$, $\sup_{B\setminus G} |u_i - u| \leq \delta_i$

and $\|u_i|_{\partial B} - u|_{\partial B}\|_{L^\infty} \leq \delta_i$. We choose $\eta \in C^1(\mathbb{R})$ with $\eta = 1$ on $(-\infty, \frac{1}{3}R]$, $\eta = 0$ on $[\frac{2}{3}R, \infty)$, $0 \leq -\eta' \leq \frac{4}{R}$ on $\mathbb{R}$, and define ϑ_i by $\vartheta_i(t) = 1$ for $t \leq \delta_i$, $\vartheta_i(t) = \left(\frac{1}{t} - 1\right) / \left(\frac{1}{\delta_i} - 1\right)$ for $\delta_i \leq t \leq 1$ and $\vartheta_i(t) = 0$ for $t \geq 1$.

We further define

$$\widetilde{u}_i = u_i + (\eta \circ |u|)\,(\vartheta_i \circ |u_i - u|)\,(u - u_i); \tag{4.1}$$

note that $\vartheta_i \circ |u_i - u|$ and $\eta \circ |u|$ both take the value 1 on ∂B. Parts (i) und (ii) then follow directly, due to our choice of G, η and ϑ_i.

We note that if $|u_i(x)| \geq R$, then $|u_i(x) - u(x)| \geq \frac{R}{3} > 1$ or $|u(x)| \geq \frac{2}{3}R$. For $|u_i(x) - u(x)| \geq \frac{R}{3} > 1$, the definition of ϑ_i ensures $\vartheta_i(|u_i(x) - u(x)|) = 0$. If $|u(x)| \geq \frac{2}{3}R$ we have $\eta(|u(x)|) = 0$. These combine to show (iii). Since $0 \leq \eta \leq 1$ and $\sup_{t \geq 0} \vartheta_i(t)t \leq \delta_i \to 0$ as $i \to \infty$, we have also established (iv).

In order to show (vi) we differentiate (4.1) to obtain

$$\begin{aligned} D\widetilde{u}_i &= Du_i + (\eta \circ |u|) D\left[(\vartheta_i \circ |u_i - u|)(u - u_i)\right] \\ &\qquad + (\eta' \circ |u|)\left(\frac{u}{|u|} \cdot Du\right)(\vartheta_i \circ |u_i - u|)(u - u_i), \end{aligned} \tag{4.2}$$

(with the interpretation $\frac{u}{|u|} \cdot Du = 0$ for $u = 0$). Using the identity $t\vartheta_i'(t) + \vartheta_i(t) = -\frac{\delta_i}{1-\delta_i}$ for $t > \delta_i$ we have

$$\begin{aligned} D\widetilde{u}_i &= (1 - \eta \circ |u|)Du_i + (\eta \circ |u|)Du \\ &\qquad + (\eta' \circ |u|)\left(\frac{u}{|u|} \cdot Du\right)(u - u_i) \quad \text{if } |u - u_i| \leq \delta_i, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} D\widetilde{u}_i &= \Big[1 - (\eta \circ |u|)(\vartheta_i \circ |u_i - u|)\Big] P^\perp Du_i + \left[1 + (\eta \circ |u|)\frac{\delta_i}{1 - \delta_i}\right] PDu_i \\ &\quad + (\eta \circ |u|)(\vartheta_i \circ |u_i - u|)P^\perp Du - (\eta \circ |u|)\frac{\delta_i}{1 - \delta_i} PDu \\ &\quad + (\eta' \circ |u|)\left(\frac{u}{|u|} \cdot Du\right)(\vartheta_i \circ |u_i - u|)(u - u_i) \quad \text{if } |u - u_i| > \delta_i\,, \end{aligned} \tag{4.4}$$

where P denotes the field of rank 1 orthogonal projections

$$P\,:\, \mathbb{R}^m \ni \xi \to |u_i - u|^{-2}((u_i - u) \cdot \xi)(u_i - u)$$

(with $P^\perp = \mathrm{id} - P$). For almost all $x \in G$ with $|u_i(x) - u(x)| \leq \delta_i$ we therefore have, via (4.3)

$$|D\widetilde{u}_i| \leq |Du_i| + |Du| + \frac{4}{R}\delta_i |Du|,$$

and via (4.4) for $|u_i(x) - u(x)| > \delta_i$ we have

$$|D\widetilde{u}_i| \leq \left[|P^\perp Du_i|^2 + \frac{1}{(1 - \delta_i)^2}|PDu_i|^2\right]^{\frac{1}{2}} + |P^\perp Du| + \frac{\delta_i}{1 - \delta_i}|PDu| + \frac{4}{R}\delta_i|Du|,$$

i.e. we have (almost everywhere on G)

$$|D\widetilde{u}_i| \leq \frac{1}{1 - \delta_i}|Du_i| + 2|Du|.$$

After applying Young's inequality we have, for $\lambda > 0$,

$$\mathbf{D}_G(\widetilde{u}_i) \leq \left(\frac{1}{(1-\delta_i)^n} + \lambda\right) \mathbf{D}_G(u_i) + \frac{4^n}{\lambda} \mathbf{D}_G(u); \tag{4.5}$$

letting $i \to \infty$ and noting $\delta_i \to 0$, this becomes

$$\begin{aligned}
&\limsup_{i\to\infty} [\mathbf{D}_G(\widetilde{u}_i) + \mathbf{D}_G(u)] \\
&\quad \leq \; (1+\lambda) \limsup_{i\to\infty} [\mathbf{D}_G(u_i) - \mathbf{D}_G(u)] + \left(2 + \frac{4^n}{\lambda}\right) \mathbf{D}_G(u) \\
&\quad \leq \; (1+\lambda) \limsup_{i\to\infty} [\mathbf{D}(u_i) - \mathbf{D}(u)] + \left(2 + \frac{4^n}{\lambda}\right) \mathbf{D}_G(u);
\end{aligned} \tag{4.6}$$

in the last inequality, we use the fact that $\limsup_{i\to\infty} \mathbf{D}_{B\setminus G}(u_i) \geq \mathbf{D}_{B\setminus G}(u)$ (note $u_i \rightharpoonup u$ in $W^{1,n}(B, \mathbb{R}^m)$).

We now fix $\lambda > 0$, such that $\lambda \sup_i \mathbf{D}(u_i) \leq \frac{1}{2}\varepsilon$, and then ε' such that $\mathbf{D}_G(u) < \varepsilon'$ and $\left(2 + \frac{4^n}{\lambda}\right) \varepsilon' < \frac{1}{2}\varepsilon$. Part (vi) then follows from (4.6) after passing to a subsequence such that we can replace lim sup by lim inf in (4.6).

From (vi)we have $\sup_i \mathbf{D}_G(\widetilde{u}_i) < \infty$. Furthermore (cf. (i)) $\widetilde{u}_i = u$ on $B \setminus G$, i.e. $\sup_i \mathbf{D}(\widetilde{u}_i) < \infty$. Combining this with the weak convergence of u_i to u and with part (vi), this shows (v). To see (vii) we apply the above construction with $\frac{1}{2}\varepsilon$ in place of ε. Then $\widetilde{u}_i(x) = u_i(x) \in A$ if $|u_i(x)| \geq R$. Further by (iv) we have $\|\widetilde{u}_i - u_i\|_{L^\infty(B,\mathbb{R}^m)} = \delta_i \to 0$ as $i \to \infty$, so $\widetilde{u}_i(x)$ lies either in A or in a uniform δ_i-tubular neighbourhood of $\{a \in A : |a| \leq R\}$, which we denote by U_{δ_i}. Given this, we can find a Lipschitz neighbourhood rectraction $\pi : V \to A$ such that $U_{\delta_i} \subset V$ and $\mathrm{Lip}(\pi|_{U_{\delta_i}})$ is arbitrarily close to 1, for i sufficiently large. Then (i)–(vi) also follow if we replace $\widetilde{u}$ by $\pi \circ \widetilde{u}_i$. ∎

We can interpret $\liminf_{i\to\infty} [\mathbf{D}(u_i) - \mathbf{D}(u)]$ as the n-Dirichlet integral of the bubble which separates under the passage to the weak limit $u_i \rightharpoonup u$. In order to establish lower semicontinuity for the energy functional $\mathbf{E}_H(u) = \mathbf{D}(u) + n\mathbf{V}_H(u, u_0)$ with respect to weak convergence in $\mathcal{S}(u_0, A)$ we need to control the H–volume jump $\limsup_{i\to\infty} n\, |\mathbf{V}_H(u_i, u_0) - \mathbf{V}_H(u, u_0)|$. This will be accomplished by passing from u_i to $\widetilde{u}_i$ and by using a suitable isoperimetric condition, which will be defined below. We first recall the standard definition of an (unrestricted) isoperimetric condition (cf. [St1, (3.7)], [DS3, Definition 3.1]).

Definition 4.2 (1) Consider $0 < s \leq \infty$, $0 \leq c < \infty$ and $A \subset \mathbb{R}^{n+1}$.

(i) An **(unrestricted) isoperimetric condition** of type c, s is valid for H and A if

$$n|\langle Q, H\Omega\rangle| = n \left| \int_A i_Q H\Omega \right| \leq c\, \mathbf{M}(\partial Q) \tag{4.7}$$

for all integer multiplicity rectifiable $\mathbb{R}^{n+1}$-currents Q with $\operatorname{spt} Q \subset A$, and $\mathbf{M}(\partial Q) \leq s$; here i_Q is the multiplicity function of Q.

(2) Suppose that every spherical n-current T with support in A and $\mathbf{M}(T) \leq s$ is uniquely homologically trivial in A, i.e. there exists an integer multiplicity rectifiable $(n+1)$-current with

$\operatorname{spt} Q \subset A$, $\mathbf{M}(Q) < \infty$ and $\partial Q = T$. Further assume $0 \le c < \infty$. We say that H satisfies a **spherical isoperimetric condition** of type c, s on A, if we have

$$n|\langle Q, H\Omega\rangle| = n\left|\int_A i_Q H\Omega\right| \le c\,\mathbf{M}(T) \tag{4.8}$$

for all T, Q as above.

Remark 4.3 (1) If $A = \mathbb{R}^{n+1}$ (or, more generally, A is homologically n-aspherical), then an unrestricted isoperimetric inequality of type c, s implies a spherical isoperimetric condition of type c, s.

(2) If H satisfies a spherical isoperimetric condition of type c, s on A we can conclude from Lemma 3.3 (ii) and Definition 3.4 that the H-volume $\mathbf{V}_H(u,v)$ is defined for all $u, v \in W^{1,n}(B,A)$ with $u - v \in W_0^{1,n}(B, \mathbb{R}^{n+1})$, and further that we have the estimate

$$n|\mathbf{V}_H(u,v)| \le c\,\mathbf{M}(J_u - J_v) \tag{4.9}$$

In the following theoren we apply this isoperimetric condition to obtain existence results.

Theorem 4.4 *Let A be a closed subset of $\mathbb{R}^{n+1}$ which admits neighbourhood retractions which have Lipschitz constant arbitrarily close to 1 on neighbourhoods of compact subsets, let $H : A \to \mathbb{R}$ be a bounded, continuous function which satisfies a spherical isoperimetric condition of type c, s, and let $u_0 \in W^{1,n}(B,A)$ be a fixed reference surface for which the inequality $(1+\sigma)\mathbf{D}(u_0) \le s$ holds for some $1 < \sigma \le \infty$. Further let $\mathcal{S}(u_0, A; \sigma)$ denote the class of all surfaces $\widetilde{u} \in \mathcal{S}(u_0, A)$ with $\mathbf{D}(\widetilde{u}) \le \sigma\mathbf{D}(u_0)$. Then we have:*

(i) If $\sigma < \infty$ and $c \le 1$, or $\sigma = \infty$ and $c < 1$, then the variational problem

$$\mathbf{E}_H(\widetilde{u}) = \mathbf{D}(\widetilde{u}) + n\mathbf{V}_H(\widetilde{u}, u_0) \to \min \quad \text{in } \mathcal{S}(u_0, A; \sigma) \tag{4.10}$$

has a solution.

(ii) If

$$c \le \frac{\sigma - 1}{\sigma + 1} \quad \text{respectively } c < 1 \quad \text{if } \sigma = \infty, \tag{4.11}$$

then the variational problem (4.10) has a solution v with $\mathbf{D}(v) < \sigma\mathbf{D}(u_0)$; if we have strict inequality in (4.11), or if u_0 is itself not a solution to (4.10), then $\mathbf{D}(u) < \sigma\mathbf{D}(u_0)$ for every solution u to (4.10).

(iii) If A is the closure of a C^2-Domain in $\mathbb{R}^{n+1}$ and

$$|H| \le \mathcal{K}_{\partial A} \quad \text{pointwise on } \partial A, \tag{4.12}$$

then every minimum u of (4.10) with $\mathbf{D}(u) < \sigma\mathbf{D}(u_0)$ is a weak solution of the Dirichlet problem $\mathcal{D}(H, u_0)$ in A. If in addition $|H(a)| < \mathcal{K}_{\partial A}(a)$ in a given point $a \in \partial A$ and $u_0|_{\partial B}$ omits some neighbourhood of a, then there exists a neighbourhood V of a in $\mathbb{R}^{n+1}$ such that $u(B) \cap V = \varnothing$.

Proof: (i) From (3.3) we have, for $\widetilde{u} \in \mathcal{S}(u_0, A; \sigma)$

$$\mathbf{M}(J_{\tilde{u}} - J_{u_0}) \leq \mathbf{D}(\widetilde{u}) + \mathbf{D}(u_0) \leq (\sigma + 1)\mathbf{D}(u_0) \leq s, \tag{4.13}$$

so that $\mathbf{V}_H(\widetilde{u}, u_0)$ is defined for all $\widetilde{u} \in \mathcal{S}(u_0, A; \sigma)$. Using (4.9) and (4.13) we have

$$\mathbf{E}_H(\widetilde{u}) \geq \mathbf{D}(\widetilde{u}) - n|\mathbf{V}_H(\widetilde{u}, u_0)| \geq (1-c)\mathbf{D}(\widetilde{u}) - c\mathbf{D}(u_0), \tag{4.14}$$

i.e. $\mathbf{E}_H$ is bounded from below on $\mathcal{S}(u_0, A; \sigma)$. We now choose a minimizing sequence $(u_i)_{i\in\mathbb{N}}$ for (4.10) and note that (4.14) implies that $\sup_i \mathbf{D}(u_i) < \infty$ if $\sigma = \infty$ and $c < 1$; for finite σ this follows directly from the definition of $\mathcal{S}(u_0, A; \sigma)$. After passing to a subsequence we can assume that u_i converges to a map $u \in \mathcal{S}(u_0, A; \sigma)$ weakly in $W^{1,n}$ und pointwise almost everywhere. For given $\varepsilon > 0$ we apply Lemma 4.1 and obtain, after passage to a subsequence, surfaces $\widetilde{u}_i \in \mathcal{S}(u_0, A)$ with $\lim\limits_{i\to\infty} \|\widetilde{u}_i - u_i\|_{L^\infty(B,\mathbb{R}^{n+1})} = 0$. From Lemma 3.5, (i) we thus have that $\mathbf{V}_H(\widetilde{u}_i, u_0)$ and $\mathbf{V}_H(\widetilde{u}_i, u_i)$ are well-defined, and furthermore

$$\mathbf{V}_H(\widetilde{u}_i, u_0) - \mathbf{V}_H(u_i, u_0) = \mathbf{V}_H(\widetilde{u}_i, u_i) \to 0 \quad \text{bei } i \to \infty. \tag{4.15}$$

(The proof of Lemma 3.5, (i) shows that we do not need need to assume that A admits uniform Lipschitz neighbourhood retractions, since in the current situation, from Lemma 4.1, (iii) we have $\widetilde{u}_i(x) = u_i(x)$ for $|u_i(x)| \geq R$.)

Choosing $\varepsilon < \frac{1}{2}\mathbf{D}(u)$ we obtain via (3.3) and Lemma 4.1, (vi)

$$\mathbf{M}(J_{\tilde{u}_i} - J_u) \quad \leq \quad \mathbf{D}_G(\widetilde{u}_i) + \mathbf{D}_G(u) \leq 2\varepsilon + \mathbf{D}(u_i) - \mathbf{D}(u) < \sigma\mathbf{D}(u_0) \leq s. \tag{4.16}$$

for i sufficiently large (for $G \subset B$ given by Lemma 4.1). Thus we conclude from the spherical isoperimetric condition (note $c \leq 1$), Remark 4.3 and (4.16) the inequality

$$n|\mathbf{V}_H(\widetilde{u}_i, u)| \leq c\mathbf{M}(J_{\tilde{u}_i} - J_u) \leq 2\varepsilon + \mathbf{D}(u_i) - \mathbf{D}(u) \tag{4.17}$$

for i sufficiently large.

We next wish to show

$$\mathbf{V}_H(\widetilde{u}_i, u_0) = \mathbf{V}_H(\widetilde{u}_i, u) + \mathbf{V}_H(u, u_0). \tag{4.18}$$

To see this, note that (4.17) guarantees the existence of $\mathbf{V}_H(\widetilde{u}_i, u)$, that (4.15) ensures that $\mathbf{V}_H(\widetilde{u}_i, u_0)$ is well-defined, and that the existence of $\mathbf{V}_H(u, u_0)$ is guaranteed by the fact that $u \in \mathcal{S}(u_0, A; \sigma)$. Therefore we have the existence of rectifiable $(n+1)$-currents $I_{\tilde{u}_i,u}$, $I_{\tilde{u}_i,u_0}$ and I_{u,u_0} with support in A, all off which are uniquely determined by their boundaries $J_{\tilde{u}_i} - J_u$, $J_{\tilde{u}_i} - J_{u_0}$ and $J_u - J_{u_0}$. Thus we have

$$I_{\tilde{u}_i,u_0} = I_{\tilde{u}_i,u} + I_{u,u_0},$$

since the currents on both sides have the same boundary. This shows (4.18).

Using (4.15), (4.18) und (4.17) we have, for i sufficiently large,

$$\begin{aligned}
\mathbf{E}_H(u_i) &= \mathbf{D}(u_i) + n\mathbf{V}_H(u_i, u_0) \\
&= \mathbf{E}_H(u) - \mathbf{D}(u) + \mathbf{D}(u_i) + n\mathbf{V}_H(u_i, u_0) - n\mathbf{V}_H(u, u_0) \\
&= \mathbf{E}_H(u) + \mathbf{D}(u_i) - \mathbf{D}(u) + n\mathbf{V}_H(\widetilde{u}_i, u) - n\mathbf{V}_H(\widetilde{u}_i, u_i) \\
&\geq \mathbf{E}_H(u) - 2\varepsilon - n\mathbf{V}_H(\widetilde{u}_i, u_i) \\
&\geq \mathbf{E}_H(u) - 3\varepsilon.
\end{aligned}$$

This shows that u minimizes the H-energy in the class $\mathcal{S}(u_0, A; \sigma)$.

To see (ii), we note that $\mathbf{E}_H(u) \leq \mathbf{E}_H(u_0)$ for solutions of (4.10). Hence we have

$$\begin{aligned}
\mathbf{D}(u) &= \mathbf{E}_H(u) - n\mathbf{V}_H(u, u_0) \\
&\leq \mathbf{E}_H(u_0) - n\mathbf{V}_H(u, u_0) \\
&= \mathbf{D}(u_0) - n\mathbf{V}_H(u, u_0) \\
&\leq \mathbf{D}(u_0) + c[\mathbf{D}(u) + \mathbf{D}(u_0)] \\
&\leq [1 + c(1+\sigma)]\mathbf{D}(u_0) \\
&\leq \sigma\mathbf{D}(u_0),
\end{aligned}$$

where we have used in turn (3.3), the fact that $\mathbf{V}_H(u_0, u_0) = 0$, the isoperimetric condition, and (4.11). The strict inequality $\mathbf{D}(u) < \sigma\mathbf{D}(u_0)$ occurs in the following situations: when $\sigma = \infty$; or $c < \frac{\sigma+1}{\sigma-1}$ if $\sigma < \infty$; or in the case that u_0 is not a solution of (4.10), i.e. $\mathbf{E}(u) < \mathbf{E}(u_0)$. On the other hand, if u_0 solves (4.10), then $\mathbf{D}(u_0) < \sigma\mathbf{D}(u_0)$, since $\sigma > 1$.

Part (iii) follows from Lemma 3.5, part (ii) and the results from Section 2. ■

Remark 4.5 (1) In the case $A \neq \mathbb{R}^{n+1}$ it is not in fact necessary to assume that the integer multiplicity rectifiable $(n+1)$-currents $I_{\tilde{u},u_0}$ occuring in the proof of Theorem 4.4 have support in A. As long as we have that H is bounded and $\mathcal{L}^{n+1}$-measurable on some closed set $\tilde{A} \supset A$, we can weaken Definition 4.2(ii) by allowing $\operatorname{spt} Q \subset \tilde{A}$ (i.e. we only need to require that T is uniquely homologically trivial in $\tilde{A}$).

(2) A natural choice of reference surface u_0 is a minimizer of the n-Dirichlet integral relative to given boundary data, i.e. $\mathbf{D}(u_0) \leq \mathbf{D}(\widetilde{u})$ for all $\widetilde{u} \in \mathcal{S}(u_0, A)$. The existence of such a minimizer is guaranteed, for example, if we consider Dirichlet boundary data $\gamma \in C^0(\partial B, A)$ which admits an extension in $W^{1,n}(B, A)$. The above proof then goes through if we use $\mathcal{S}(\gamma, A) = \{\widetilde{u} \in W^{1,n}(B, A) : \widetilde{u}|_{\partial B} = \gamma\} \neq \varnothing$ in place of $\mathcal{S}(u_0, A)$, and $\mathcal{S}(\gamma, A; \sigma) = \{\widetilde{u} \in \mathcal{S}(\gamma, A) : \mathbf{D}(\widetilde{u}) \leq \sigma\mathbf{D}(u_0)\}$, where u_0 minimizes the n Dirichlet-integral in $\mathcal{S}(\gamma, A)$, in place of $\mathcal{S}(u_0, A; \sigma)$.

5 Geometric conditions sufficient for existence

In this section we combine the results of [DS3] concerning isoperimetric inequalities with Theorem 4.4 to obtain conditions on the Dirichlet boundary data $\varphi \in C^0(\partial B, A)$ and on the prescribed mean curvature H which are sufficient to ensure the existence of a (weak) solution of the Dirichlet problem $\mathcal{D}(H, \varphi)$. The first result is a Wente-type theorem. We consider Dirichlet boundary data $\varphi \in C^0(\partial B, A)$ which admits a $W^{1,n}(B, A)$-extension, and we denote by $u_0 \in W^{1,n}(B, A)$ the $\mathbf{D}$-minimizing map with $u_0|_{\partial B} = \varphi$ and set $d_\varphi = \mathbf{D}(u_0)$.

Theorem 5.1 *Let A be the closure of a C^2-domain in $\mathbb{R}^{n+1}$ such that the minimum of the principal curvatures $\mathcal{K}_{\partial A}$ (viewed with regard to the inward pointing normal) is positive at every point $a \in \partial A$. Further consider Dirichlet boundary data $\varphi \in C^0(\partial B, A)$ as above and $H : A \to \mathbb{R}$ bounded and continuous satisfying*

$$\sup_A |H| \leq \sqrt[n]{\frac{\alpha_{n+1}}{2d_\varphi}} \tag{5.1}$$

and

$$|H(a)| \leq \mathcal{K}_{\partial A}(a) \quad \textit{for } a \in \partial A. \tag{5.2}$$

Then there exits a weak solution $u \in W^{1,n}(B, A)$ to the Dirichlet problem $\mathcal{D}(H, \varphi)$, i.e.

$$D_{x_i}(|Du|^{n-2} D_{x_i} u) = H \circ u \cdot u_{x_1} \times \cdots \times u_{x_n} \quad \textit{in } B,$$

$$u|_{\partial B} = \varphi \quad \textit{on } \partial B.$$

Proof: We extend H via $H \equiv 0$ on $\mathbb{R}^{n+1} \setminus A$ to a bounded, measurable function. The isoperimetric inequality (1.6), applied to a closed rectifiable n-current T with support in A and mass not greater than s and the unique rectifiable $(n+1)$-current Q satisfying $\partial Q = T$ (recall the results of Section 3) implies

$$n|\langle Q, H\Omega\rangle| \leq n \sup_A |H| \cdot \mathbf{M}(Q) \leq n\,\gamma_{n+1} \sup_A |H| s^{\frac{1}{n}} \mathbf{M}(T), \tag{5.3}$$

i.e. H satisfies an isoperimetric condition of type $n\gamma_{n+1} \sup_A |H| s^{\frac{1}{n}}$, s on $\mathbb{R}^{n+1}$. Thus the conditions of Theorem 4.4, (i) (keeping in mind Remark 4.5, (i)) are therefore satisfied with $\sigma = \frac{s}{d_\varphi} - 1$, if $s > 2d_\varphi$ and $n\gamma_{n+1} \sup_A |H| s^{\frac{1}{n}} \leq 1$. If we further require

$$n\gamma_{n+1} \sup_A |H| s^{\frac{1}{n}} \leq \frac{\sigma - 1}{\sigma + 1} = \frac{s - 2d_\varphi}{s},$$

then we can apply (ii) of Theorem 4.4. Noting that the maximum of the function $s \mapsto \frac{s - 2d_\varphi}{s^{1+\frac{1}{n}}}$ on $(2d_\varphi, \infty)$ occurs for $s = 2(n+1)d_\varphi$, we obtain the sufficient condition

$$\sup_A |H| \leq \frac{1}{n\gamma_{n+1}} \frac{2(n+1)d_\varphi - 2d_\varphi}{[2(n+1)d_\varphi]^{1+\frac{1}{n}}} = \sqrt[n]{\frac{\alpha_{n+1}}{2d_\varphi}}.$$

The remaining conclusions follow from Theorem 4.4, (iii). ∎

We can exploit the fact that the functions i_{u,u_0} and i_Q introduced in Section 3 are actually in $BV(\mathbb{R}^{n+1}, \mathbb{Z})$, and hence in $L^{1+\frac{1}{n}}(\mathbb{R}^{n+1}, \mathbb{Z})$, to give a different set of sufficient conditions; cf. [St1, Theorem 6.1], [St2, Theorem 3.3].

Theorem 5.2 *Let A and φ be as in Theorem 5.1. Further let $H : A \to \mathbb{R}$ be a bounded continuous function satisfying*

$$\int_A |H|^{n+1}\, dx < \left(1 + \frac{1}{n}\right)^{n+1} \alpha_{n+1} \tag{5.4}$$

and

$$|H(a)| \leq \mathcal{K}_{\partial A}(a) \quad \textit{for } a \in \partial A.$$

Then there exits a weak solution $u \in W^{1,n}(B, A)$ to the Dirichlet problem $\mathcal{D}(H, \varphi)$.

Proof: As in the proof of Theorem 5.1 we extend H via $H \equiv 0$ on $\mathbb{R}^{n+1} \setminus A$ to a bounded, measurable function on $\mathbb{R}^{n+1}$. We use Hölder's inequality and [Fe, 4.5.9 (31)] in order to obtain, for a closed, rectifiable n-current T with support in A and its associated $(n+1)$-current Q satisfying $\partial Q = T$ and multiplicity function i_Q:

$$\begin{aligned} n|\langle Q, H\Omega\rangle| &= n\left|\int_{\mathbb{R}^{n+1}} i_Q H\Omega\right| \\ &\leq n\left(\int_{\mathbb{R}^{n+1}} |i_Q|^{\frac{n+1}{n}} dx\right)^{\frac{n}{n+1}} \left(\int_{\mathbb{R}^{n+1}} |H|^{n+1} dx\right)^{\frac{1}{n+1}} \\ &= \frac{n}{n+1}\alpha_{n+1}^{-\frac{1}{n+1}} \left(\int_A |H|^{n+1} dx\right)^{\frac{1}{n+1}} \mathbf{M}(T), \end{aligned}$$

i.e. H satisfies an isoperimetric condition of type c, ∞ for $c = \frac{n}{n+1}\alpha_{n+1}^{-\frac{1}{n+1}} \left(\int_A |H|^{n+1} dx\right)$. Hence the conditions of Theorem 4.4 (with $s = \sigma = \infty$) are therefore satisfied if $c < 1$; this is precisely (5.4). ■

The following corollary is immediate:

Corollary 5.3 *Let A, $\mathcal{K}_{\partial A}$ and φ be as above, and let H be a bounded, continuous function on A for which (5.2) and*

$$\sup_A |H| < \left(1 + \frac{1}{n}\right) \sqrt[n+1]{\frac{\alpha_{n+1}}{\mathcal{L}^{n+1}(A)}} \tag{5.5}$$

hold. Then there exits a weak solution $u \in W^{1,n}(B, A)$ to the Dirichlet problem $\mathcal{D}(H, \varphi)$.

In the case $A = \overline{B}_R(a) \subset \mathbb{R}^{n+1}$ conditions (5.5) and (5.2) simplify to

$$\sup_{B_R(a)} |H| < \frac{n+1}{n}\frac{1}{R}, \quad |H(a)| \leq \frac{1}{R} \quad \text{for } a \in \partial B_R(a),$$

i.e. Corollary 5.3 contains the results of [DF2, Satz 2.1] as a special case (cf. [MY, Theorem 4]), in the case of constant H.

Theorem 5.4 *Let A and φ be as in Theorem 5.1, and let $H : A \to \mathbb{R}$ be bounded and continuous, and satisfy*

$$\sup_{t>0} \left[\frac{t^{n+1}}{\alpha_{n+1}} \mathcal{L}^{n+1}\{a \in A : |H(a)| \geq t\}\right]^{\frac{1}{n+1}} =: c < 1 \tag{5.6}$$

in addition to (5.2). Then there exits a weak solution $u \in W^{1,n}(B, A)$ to the Dirichlet problem $\mathcal{D}(H, \varphi)$.

Proof: We extend H as before. Following the arguments of the proof of [St2, Proposition 5.1] and noting (5.5) we obtain an isoperimetric condition of type c, ∞ with $c < 1$, i.e. for every rectifiable n-current T with $\partial T = 0$ and $\operatorname{spt} T \subset A$ and the unique rectifiable $n+1$-current Q satisfying $\partial Q = T$ we have

$$n|\langle Q, H\Omega\rangle| \leq c \cdot \mathbf{M}(T).$$

Thus the conditions of Theorem 4.4 (with $s = \sigma = \infty$, $c < 1$ and $\widetilde{A} = \mathbb{R}^{n+1}$) are satisfied. ■

6 Regularity of Solutions

In this section we discuss the regularity of solutions to (4.10). We will call a domain $G \subset \mathbb{R}^{n+1}$ **locally convex up to Lipschitz transformations** if $G = \operatorname{int}(\overline{G})$ and if, for every point $a_0 \in \partial G$, we can find a neighbourhood U of a_0 and a bi-Lipschitz mapping f from the component of a_0 in $\overline{U} \cap \overline{G}$ to some closed convex set. The domain G is called **uniformly locally convex up to Lipschitz transformations** if there is a constant Λ independent of a_0, $0 < \Lambda \leq 1$, such that U und f can be chosen to satisfy

$$U \supset B_\Lambda(a_0), \;\; \operatorname{Lip}(f) \leq \Lambda^{-1}, \;\; \operatorname{Lip}(f^{-1}) \leq \Lambda^{-1} \tag{6.1}$$

(cf.[St1, Remark 3.9], and the comments thereafter).

Theorem 6.1 *Let A, H and u_0 satisfy the conditions of Theorem 4.4, with associated parameters σ, s and c. Further let A be the closure of a domain which is uniformly locally convex up to Lipschitz transformations. Then every solution u of (4.10) is Hölder-continuous inside B; further $u \in C^0(\overline{B}, \mathbb{R}^{n+1})$ if $u|_{\partial B} \in C^0(\partial B, \mathbb{R}^{n+1})$.*

Proof: Our goal is to prove that the inequality

$$\mathbf{D}_{B_\rho(x_0)}(u) \leq \mathbf{D}_{B_r(x_0)}(u) \left(\frac{\rho}{r}\right)^{n\alpha} \tag{6.2}$$

holds for all $x_0 \in B$ and $0 < \rho \leq r < \min\{r_0, 1 - |x_0|\}$. We can then apply Morrey's Dirichlet growth theorem [M, 3.5.2] to conclude the local Hölder continuity of u with exponent α.

To show (6.2) we begin by fixing $x_0 \in B$ und set $u(r, \omega) = u(x_0 + r\omega) = u_r(\omega)$ for $\omega \in S^{n-1}$ and $0 \leq r \leq 1 - |x_0|$. The function

$$\Phi(r) := \mathbf{D}_{B_r(x_0)}(u) = \frac{1}{\sqrt{n^n}} \int_0^r \int_{S^{n-1}} \left[\left| \frac{\partial u}{\partial \rho} \right|^2 + \frac{1}{\rho^2} |d_\omega u|^2 \right]^{\frac{n}{2}} \rho^{n-1} \, d\omega \, d\rho \tag{6.3}$$

is absolutely continuous on $[0, 1 - |x_0|]$, and for almost all r in this interval we have

$$\Psi(r) := \frac{1}{\sqrt{n^n}} \int_{S^{n-1}} |d_\omega u(r, \cdot)|^n \, d\omega \leq r\Phi'(r). \tag{6.4}$$

From now on we will only consider r such that (6.4) holds. Sobolev's embedding theorem then ensures

$$\operatorname*{osc}_{S^{n-1}} u(r, \cdot) = \sup_{\omega, \omega' \in S^{n-1}} |u(r, \omega) - u(r, \omega')| \leq c(n) \sqrt[n]{\Psi(r)}. \tag{6.5}$$

Our aim is to obtain an estimate for $\Psi(r)$. Denoting by $0 < \Lambda \leq 1$ the constant from (6.1), we consider the cases $\Psi(r) \geq \left(\frac{\Lambda}{2c(n)}\right)^n$ and $\Psi(r) < \left(\frac{\Lambda}{2c(n)}\right)^n$ separately. In the former case we have, using $\Phi(r) \leq \mathbf{D}(u)$,

$$\Phi(r) \leq \left(\frac{2c(n)}{\Lambda}\right)^n \Psi(r)\mathbf{D}(u) \leq \left(\frac{2c(n)}{\Lambda}\right)^n \sigma \mathbf{D}(u_0) r\Phi'(r). \tag{6.6}$$

In the latter case we have from (6.5) the inequality $\operatorname*{osc}_{S^{n-1}} u_r < \frac{\Lambda}{2}$, i.e. we can find a_1 with $a_1 \in u_r(S^{n-1}) = u(\partial B_r(x_0)) \subset B_{\frac{\Lambda}{2}}(a_1) \cap A$. If $B_{\frac{\Lambda}{2}}(a_1)$ is not contained in A then we can find

$a_0 \in \partial A$ with $\{ta_0 + (1-t)a_1 : 0 \le t \le 1\} \subset B_{\frac{\Lambda}{2}}(a_1) \cap B_\Lambda(a_0)$, and with f as in (6.1) we define $h \in W^{1,n}(B_r(x_0), \mathbb{R}^{n+1})$ to be the $\mathbf{D}_{B_r(x_0)}$-minimizing map with boundary values $f \circ u|_{\partial B_r(x_0)}$, and further define $w = f^{-1} \circ h \in W^{1,n}(B_r(x_0), \mathbb{R}^{n+1})$. These are well-defined, since $u(\partial B_r(x_0))$ s contained in the component of a_0 in $A \cap B_\Lambda(a_0)$, and hence $h(\partial B_r(x_0)) = f \circ u(\partial B_r(x_0))$ in the convex set $\mathrm{Im}(f)$, so that $h(\overline{B}_r(x_0)) \subset \mathrm{Im}(f)$. For w we have

$$w \in W^{1,n}(B_r(x_0), A), \quad u|_{B_r(x_0)} - w \in W_0^{1,n}(B_r(x_0), \mathbb{R}^{n+1}), \tag{6.7}$$

and

$$\mathbf{D}_{B_r(x_0)}(w) \le \Lambda^{-n} \mathbf{D}_{B_r(x_0)}(h) \le \Lambda^{-n} \mathbf{D}_{B_r(x_0)}(f \circ u) \le \Lambda^{-2n} \mathbf{D}_{B_r(x_0)}(u). \tag{6.8}$$

(here we have extended f to a map of all of A with the same Lipschitz constant Kirszbraun's theorem [Fe, 2.10.43]). If $B_{\frac{\Lambda}{2}}(a_1) \subset A$, we simply define $w := h$ to be the $\mathbf{D}_{B_r(x_0)}$-minimizing map with boundary data $u|_{\partial B_r(x_0)}$; in this case, too, we have (6.7) and (6.8).

The next step is to show the existence of $r_0 > 0$ such that the inequality

$$\mathbf{D}_{B_r(x_0)}(u) \le M_0 \, \mathbf{D}_{B_r(x_0)}(w) \tag{6.9}$$

holds for all $B_r(x_0) \subset B$ with $r \le r_0$, for a constant M_0 independent of r und x_0. M_0. We now define

$$\widetilde{u} = \begin{cases} u & \text{on } B \setminus B_r(x_0), \\ w & \text{on } B_r(x_0), \end{cases} \tag{6.10}$$

and note that $\widetilde{u} \in W^{1,n}(B, A)$ and $\widetilde{u} - u_0 \in W_0^{1,n}(B, \mathbb{R}^{n+1})$. If $\mathbf{D}(\widetilde{u}) > \sigma \mathbf{D}(u_0)$ then we have from (6.10), since $\mathbf{D}(u) \le \sigma \mathbf{D}(u_0)$,

$$\mathbf{D}_{B_r(x_0)}(u) < \mathbf{D}_{B_r(x_0)}(\widetilde{u}) = \mathbf{D}_{B_r(x_0)}(w),$$

and hence we have (6.9) with $M_0 = 1$. On the other hand if $\mathbf{D}(\widetilde{u}) \le \sigma \mathbf{D}(u_0)$ we can take $\widetilde{u}$ as a comparison surface for problem (4.11), which leads to $\mathbf{E}_H(u) \le \mathbf{E}_H(\widetilde{u})$, or equivalently, from (6.10),

$$\mathbf{D}_{B_r(x_0)}(u) \le \mathbf{D}_{B_r(x_0)}(w) + n[\mathbf{V}_H(\widetilde{u}, u_0) - \mathbf{V}_H(u, u_0)]. \tag{6.11}$$

We now consider the spherical n-current $J_{\tilde{u}} - J_u$. From (3.4), (6.10) and (6.8) we have

$$\mathbf{M}(J_{\tilde{u}} - J_u) \le \mathbf{D}_{B_r(x_0)}(w) + \mathbf{D}_{B_r(x_0)}(u) \le (\Lambda^{-2n} + 1) \mathbf{D}_{B_r(x_0)}(u). \tag{6.12}$$

Since $\mathbf{D}_{B_r(x_0)}(u)$ becomes arbitrarily small as $\mathcal{L}^n(B_r(x_0))$ converges to zero, we can find positive r_1 depending only on s such that $\mathbf{M}(J_{\tilde{u}} - J_u) \le s$ is for $r \le r_1$ (note that Λ depends only on A, and not on the parameters s, σ and c). This guarantees the existence of an integer rectifiable $(n+1)$-current $I_{\tilde{u},u}$ with support in A and boundary $J_{\tilde{u}} - J_u$. Denoting by $I_{\tilde{u},u_0}$, I_{u,u_0} the integer rectifiable $(n+1)$-currents with support in A with boundary $J_{\tilde{u}} - J_{u_0}$, $J_u - J_{u_0}$ respectively, we have $I_{\tilde{u},u} = I_{\tilde{u},u_0} - I_{u,u_0}$. This shows $\mathbf{V}_H(\widetilde{u}, u) = \mathbf{V}_H(\widetilde{u}, u_0) - \mathbf{V}_H(u, u_0)$, and hence from (6.11) we have

$$\mathbf{D}_{B_r(x_0)}(u) \le \mathbf{D}_{B_r(x_0)}(w) + n \mathbf{V}_H(\widetilde{u}, u), \tag{6.13}$$

if $0 < r \le r_1$.

Since H satisfies a spherical isoperimetric condition of type c, s we can use (4.9) and (6.12) to estimate $n|\mathbf{V}_H(\widetilde{u},u)|$ as follows:

$$n|\mathbf{V}_H(\widetilde{u},u)| \le c\,\mathbf{M}(J_{\tilde{u}} - J_u) \le c[\mathbf{D}_{B_r(x_0)}(w) + \mathbf{D}_{B_r(x_0)}(u)]. \tag{6.14}$$

From (6.14) and (6.13) we have, if $c < 1$:

$$\mathbf{D}_{B_r(x_0)}(u) \le \frac{1+c}{1-c}\mathbf{D}_{B_r(x_0)}(w),$$

and hence (6.9), with $M_0 = \frac{1+c}{1-c}$.

In the case $c = 1$ we use the isoperimetric inequality (1.6) and (6.12) to bound $|\mathbf{V}_H(\widetilde{u},u)|$ from above:

$$\begin{aligned} |\mathbf{V}_H(\widetilde{u},u)| &\le \|H\|_{L^\infty}\mathbf{M}(I_{\tilde{u},u}) \le \gamma_{n+1}\|H\|_{L^\infty}\mathbf{M}(J_{\tilde{u}} - J_u)^{1+\frac{1}{n}} \\ &\le \gamma_{n+1}\|H\|_{L^\infty}(\Lambda^{-2n}+1)^{1+\frac{1}{n}}(\mathbf{D}_{B_r(x_0)}(u))^{\frac{1}{n}}\,\mathbf{D}_{B_r(x_0)}(u). \end{aligned}$$

Thus given $\varepsilon > 0$ we can determine r_0, $0 < r_0 \le r_1$ such that $n|\mathbf{V}_H(\widetilde{u},u)| \le \varepsilon\,\mathbf{D}_{B_r(x_0)}(u)$. From (6.13) we thus have

$$\mathbf{D}_{B_r(x_0)}(u) \le \frac{1}{1-\varepsilon}\mathbf{D}_{B_r(x_0)}(w), \tag{6.15}$$

i.e. (6.9) is also valid in this case (in fact for M_0 arbitrarily close to 1, in that we can choose r_0 as small as we please).

We next define $p := \fint_{S^{n-1}} f\circ u(r,\omega)\,d\omega$ and

$$v(\rho,\omega) := \begin{cases} p & \text{for } \omega\in S^{n-1},\ 0\le\rho<\frac{r}{2}, \\ (2-\frac{2\rho}{r})p + \left(\frac{2\rho}{r}-1\right) f\circ u(r,\omega) & \text{for } \omega\in S^{n-1},\ \frac{r}{2}\le\rho\le r. \end{cases}$$

Using Poincaré's inequality we have

$$\begin{aligned} &\int_{\frac{r}{2}}^{r} \rho^{n-1}\int_{S^{n-1}} \left|\frac{\partial v}{\partial\rho}(\rho,\omega)\right|^n d\omega\,d\rho = \frac{2^n-1}{n}\int_{S^{n-1}} |f\circ u(r,\omega) - p|^n\,d\omega \\ &\le\ c(n)\int_{S^{n-1}} |d_\omega(f\circ u)(r,\omega)|^n\,d\omega \le c(n)\Lambda^{-n}\int_{S^{n-1}} |d_\omega u(r,\omega)|^n\,d\omega. \end{aligned}$$

For the tangential component we obtain

$$\int_{\frac{r}{2}}^{r}\frac{1}{\rho}\int_{S^{n-1}} |dv(\rho,\omega)|^n\,d\omega\,d\rho \le \Lambda^{-n}\log 2\int_{S^{n-1}} |d_\omega u(r,\omega)|^n\,d\omega.$$

Combining this with (6.4) we have

$$\mathbf{D}_{B_r(x_0)}(v) \le c(n)\Lambda^{-n}\int_{S^{n-1}} |du(r,\omega)|^n\,d\omega \le c(n)\Lambda^{-n}r\Phi'(r).$$

The $\mathbf{D}_{B_r(x_0)}(\cdot)$-minimality of h yields

$$\mathbf{D}_{B_r(x_0)}(h) \le \mathbf{D}_{B_r(x_0)}(v) \le c(n)\Lambda^{-n}r\Phi'(r),$$

and combining this with (6.9) and the definition of w (recall that w is either h or $f^{-1} \circ h$, depending on whether or not $B_{\frac{\Lambda}{2}}(a) \subset A$), we have

$$\begin{aligned} \Phi(r) &= \mathbf{D}_{B_r(x_0)}(u) \leq M_0 \mathbf{D}_{B_r(x_0)}(w) \\ &\leq M_0 \Lambda^{-n} \mathbf{D}_{B_r(x_0)}(h) \leq M_0 c(n) \Lambda^{-2n} r \Phi'(r). \end{aligned} \tag{6.16}$$

Setting $M_1 := \max\left\{ \left(\frac{2c(n)}{\Lambda}\right)^n, \, c(n) M_0 \Lambda^{-2n} \right\}$ we have from (6.6) and (6.16)

$$\Phi(r) \leq M_1 r \Phi'(r) \quad \text{for almost all } \; 0 < r \leq \min\{r_0, \, 1 - |x_0|\},$$

and hence, with $\alpha := (nM_1)^{-1}$

$$\Phi(\rho) \leq \left(\frac{\rho}{r}\right)^{n\alpha} \Phi(r) \quad \text{for } \; 0 < \rho \leq r \leq \min\{r_0, \, 1 - |x_0|\};$$

this yields (6.2) and hence, by the comments above, completes the proof of interior regularity.

If $u|_{\partial B} \in C^0(\partial B, \mathbb{R}^{n+1})$ we can generalize [HK, Lemma 3] directly to the current setting to obtain $u \in C^0(\overline{B}, \mathbb{R}^{n+1})$. ■

Higher regularity for solutions of the Dirichlet problem $\mathcal{D}(H, \varphi)$, e.g. $C^{1,\beta}$ for Lipschitz continuous H, follows from the arguments of [HL, Section 3].

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