Parallel Pure Spinors on Pseudo-Riemannian Manifolds

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Abstract

The aim of this paper is to investigate the relation between properties of projective pure spinors and the associated almost optical structures on pseudo-Riemannian spin manifolds. We prove that a projective pure spinor is parallel if and only if the associated almost optical structure is parallel. Furthermore we will calculate the curvature condition for the existence of a global parallel representative of such a projective spinor. Moreover, we specialize these results to metrics of signature (m, m) and $(m+1, m)$.

1 Introduction

We consider Clifford algebras, spin groups and spinors associated with the standard pseudo-Euclidean space $\mathbb{R}^{p,q} = (\mathbb{R}^n, \langle \cdot\,,\cdot \rangle_{p,q})$ of signature (p, q) and dimension $n = p+q$. Following E. Cartan (see [C 38]) a spinor is called pure if

$$\dim\{v \in \mathbb{C}^n \mid v \cdot u = 0\} = \left[\frac{n}{2}\right] =: m.$$

There is a well-known bijection between the set of projective pure spinors and the Grassmannian of isotropic m-dimensional subspaces of $\mathbb{C}^n$ (endowed with the bilinear extension of $\langle \cdot\,,\cdot \rangle_{p,q}$). On the other hand each totally isotropic subspace corresponds to an optical structure on $\mathbb{R}^{p,q}$. An optical structure on $\mathbb{R}^{p,q}$ consists of a pair (K, L) of subspaces $K \subset L$ of $\mathbb{R}^{p,q}$ and a complex structure J on L/K. It is called orthogonal if K is isotropic, $L = K^{\perp}$ and J is orthogonal with respect to the induced (indefinite) metric on L/K. We can globalize these objects to obtain projective spinors and orthogonal almost optical structures on pseudo-Riemannian manifolds. Optical structures on manifolds are studied e. g. in [T 85], [RT 87] and [N 96]. An almost optical structure on a pseudo-Riemannian manifold is called parallel if K and J are parallel. We will prove that a projective pure spinor is parallel if and only if the associated almost optical structure is parallel. In the Riemannian case this was already done by H. B. Lawson and M. L. Michelsohn in [LM 89]. Furthermore we will calculate the curvature condition for the existence of a global parallel representative of a parallel projective spinor. The question arises of the existence of pseudo-Riemannian manifolds with parallel pure spinors. 4-dimensional Lorentzian manifolds with non-trivial parallel (and, hence, parallel pure) spinors are well known (see [EK 62], [Bm 81]). If a 4-dimensional Lorentzian manifold admits a non-trivial parallel spinor then it is a pp-manifold. Furthermore, any pp-manifold admits locally defined non-trivial pure parallel spinors. Any simply-connected indecomposable solvable Lorentzian symmetric space of dimension $n \geq 3$ admits parallel pure spinors (see [Bm 98]). Further examples are non-trivial parallel real isotropic spinors on pseudo-Riemannian manifolds of signature $(4,3)$. The existence of such metrics was shown by R. L. Bryant in [Br 98]. In the last section we will generalize this result. We use Bryant's proof to show that any metric of signature (m,m) and $(m+1,m)$ with a real pure parallel spinor can be written in a certain normal form with respect to special local coordinates. Moreover, this proof shows that a generic metric defined on an open set $U \subset \mathbb{R}^{2m}$ (or $U \subset \mathbb{R}^{2m+1}$) satisfying this normal form admits exactly one real pure parallel spinor (up to constant multiples) and its holonomy group is isomorphic to the identity component of the stabilizer of a real isotropic spinor. The generic such metric is not Ricci-flat. Furthermore, we use the results on pure spinors and almost optical structures to give a description of pseudo-Riemannian manifolds of signature (m,m) and $(m+1,m)$ with a non-trivial real pure parallel spinor as foliation with m-dimensional isotropic leaves for which the distribution tangential to the leaves is parallel and the restriction of the Riemannian curvature tensor to this distribution is trace-free.

2 Spinor Representations

Let $\mathbb{R}^{p,q}$, $p+q>2$, be the pseudo-Euclidean space $(\mathbb{R}^{p+q}, \langle\cdot\,,\cdot\rangle_{p,q})$ where $\langle\cdot\,,\cdot\rangle_{p,q}$ is a scalar product of index p on $\mathbb{R}^n$, $n=p+q$, given by

$$\langle x,y\rangle_{p,q} = \sum_{\nu=1}^{n} \kappa_\nu x_\nu y_\nu \ , \quad \kappa_\nu \in \{1,-1\}\,. \tag{1}$$

Let $\mathcal{C}_{p,q}$ be the Clifford algebra of $(\mathbb{R}^n, -\langle\cdot\,,\cdot\rangle_{p,q})$ and denote by $\mathcal{C}^{\mathbb{C}}_{p,q}$ its complexification. It is generated by the standard basis $e_1,\ldots,e_n$ of $\mathbb{R}^n$ with relations $e_ie_j + e_je_i = -2\langle e_i,e_j\rangle_{p,q}$. If n is even, i.e. $n=2m$, then $\mathcal{C}^{\mathbb{C}}_{p,q}$ is isomorphic to the algebra $\mathbb{C}(2^m)$ of complex $2^m\times 2^m$ matrices. If n is odd, i.e. $n=2m+1$, then $\mathcal{C}^{\mathbb{C}}_{p,q}$ is isomorphic to $\mathbb{C}(2^m)\oplus\mathbb{C}(2^m)$. Here the following isomorphisms are used. Let E, T, g_1, g_2 denote the matrices

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad g_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We set

$$\tau(j) = \begin{cases} i & \text{if} \quad \kappa_j = -1 \\ 1 & \text{if} \quad \kappa_j = 1 \end{cases}.$$

Then in case $p+q=2m$ an isomorphism

$$\Phi_{p,q} : \mathcal{C}^{\mathbb{C}}_{p,q} \longrightarrow \mathbb{C}(2^m)$$

is given by

$$\begin{aligned} \Phi_{p,q}(e_{2j-1}) &= \tau(2j-1)\cdot E\otimes\ldots\otimes E\otimes g_1\otimes \underbrace{T\otimes\ldots\otimes T}_{(j-1)\times} \\ \Phi_{p,q}(e_{2j}) &= \tau(2j)\cdot E\otimes\ldots\otimes E\otimes g_2\otimes \underbrace{T\otimes\ldots\otimes T}_{(j-1)\times} \end{aligned} \tag{2}$$

for $j = 1, \ldots, m$. If $p + q = 2m + 1$ an isomorphism

$$\Phi_{p,q} : \mathcal{C}^{\mathbb{C}}_{p,q} \longrightarrow \mathbb{C}(2^m) \oplus \mathbb{C}(2^m)$$

is defined by

$$\begin{aligned} \Phi_{p,q}(e_j) &= (\Phi_{p,q}(e_j), \Phi_{p,q}(e_j)) \quad (1 \leq j \leq 2m) \\ \Phi_{p,q}(e_{2m+1}) &= \tau(2m+1)(iT \otimes \ldots \otimes T, -iT \otimes \ldots \otimes T)\,. \end{aligned} \tag{3}$$

Denote by $\bar{\Phi}_{p,q}$ the map $\Phi_{p,q}$ if $p + q = n$ is even and the projection of $\Phi_{p,q}$ onto the first component if $p + q = n$ is odd. If we restrict this map to $Spin(p,q) \subset \mathcal{C}_{p,q}$ we obtain a representation of $Spin(p,q)$ on $\Delta_{p,q} = \mathbb{C}^{2^m}$ called the spinor representation. $\mathcal{C}^{\mathbb{C}}_{p,q}$ acts on $\Delta_{p,q}$ and defines (because of $\mathbb{R}^n \subset \mathcal{C}^{\mathbb{C}}_{p,q}$) a multiplication

$$\begin{array}{rcl} \mathbb{R}^n \times \Delta_{p,q} & \longrightarrow & \Delta_{p,q} \\ (X, \psi) & \longmapsto & X \cdot \psi := \bar{\Phi}_{p,q}(X)(\psi) \end{array}$$

called Clifford multiplication. We identify the Lie algebra $\mathfrak{spin}(p,q)$ of $Spin(p,q)$ with $\mathrm{span}_{\mathbb{R}}\{e_k e_l | 1 \leq k < l \leq n\} \subset \mathcal{C}_{p,q}$. Let

$$\lambda : Pin(p,q) \longrightarrow O(p,q)$$

be the 2-fold covering. We obtain for the differential $\lambda_* : \mathfrak{spin}(p,q) \to \mathfrak{so}(p,q)$ of λ

$$\lambda_*(e_k e_l) = 2E_{kl}\ .$$

We denote by $Spin^+(p,q)$ the identity component of $Spin(p,q)$.

Remark 2.1 The representations $\bar{\Phi}_{p,q}$ of $\mathcal{C}^{\mathbb{C}}_{p,q}$ with (p,q) fixed but different $(p+q)$-tupels $(\kappa_1, \ldots, \kappa_{p+q})$ are equivalent. More exactly, the following holds. Consider $\mathbb{R}^{p,q}$ with two different scalar products $\langle \cdot , \cdot \rangle_{p,q,1}$, $\langle \cdot , \cdot \rangle_{p,q,2}$ as in (1) and associated $(p+q)$-tupels $(\kappa^1_1, \ldots, \kappa^1_{p+q})$ and $(\kappa^2_1, \ldots, \kappa^2_{p+q})$. Let $f : (\mathbb{R}^{p,q}, \langle \cdot , \cdot \rangle_{p,q,1}) \to (\mathbb{R}^{p,q}, \langle \cdot , \cdot \rangle_{p,q,2})$ be an orientation preserving isometry such that

$$f(e_\alpha) = \pm e_\beta, \quad \alpha, \beta \in \{1, \ldots, p+q\}$$

holds. We can extend f to an isomorphism f from the complexified Clifford algebra $\mathcal{C}^{\mathbb{C}}_{p,q,1}$ of $(\mathbb{R}^{p,q}, -\langle \cdot , \cdot \rangle_{p,q,1})$ to the complexified Clifford algebra $\mathcal{C}^{\mathbb{C}}_{p,q,2}$ of $(\mathbb{R}^{p,q}, -\langle \cdot , \cdot \rangle_{p,q,2})$. Let $\bar{\Phi}_{p,q,1}$ and $\bar{\Phi}_{p,q,2}$ be the above defined representations of $\mathcal{C}^{\mathbb{C}}_{p,q,1}$ and $\mathcal{C}^{\mathbb{C}}_{p,q,2}$. We will see that there is an isomorphism $\Phi : \Delta_{p,q} \to \Delta_{p,q}$ such that

$$\bar{\Phi}_{p,q,1}(a) \circ \Phi(u) = \Phi \circ \bar{\Phi}_{p,q,2}(f(a))(u) \tag{4}$$

holds for all $a \in \mathcal{C}_{p,q,1}$ and $u \in \Delta_{p,q}$. This is obvious if $p+q$ is even, since in this case there exists (up to equivalence) only one irreducible representation of the complexified Clifford algebra. In the odd-dimensional case the complexified Clifford algebra has two inequivalent irreducible representations. They can be distinguished by the action of the volume element (see [LM 89]). Since f is orientation preserving we obtain from (3)

$$\bar{\Phi}_{p,q,1}(e_1 \cdot \ldots \cdot e_{p+q}) = \bar{\Phi}_{p,q,2}(f(e_1 \cdot \ldots \cdot e_{p+q}))\,.$$

Consequently, $\bar{\Phi}_{p,q,1}$ and $\Phi_{p,q,2} \circ f$ are equivalent representations of $\mathcal{C}^{\mathbb{C}}_{p,q,1}$ and (4) follows.

3 Pure Spinors in $\Delta_{p,q}$

As above we denote $p+q=n$ and $m=\left[\frac{n}{2}\right]$. We extend the Clifford multiplication $\mathbb{R}^n \times \Delta_{p,q} \to \Delta_{p,q}$ to a complex bilinear map $\mathbb{C}^n \times \Delta_{p,q} \to \Delta_{p,q}$. Given an element $u \in \Delta_{p,q}$ we define

$$V_u := \{X \in \mathbb{C}^n \mid X \cdot u = 0\}, \qquad U_u := \{X \in \mathbb{R}^n \mid X \cdot u = 0\}.$$

Then V_u is totally isotropic with respect to the complex linear extension $\langle\cdot\,,\cdot\rangle_{p,q}^{\mathbb{C}}$ of $\langle\cdot\,,\cdot\rangle_{p,q}$. In particular, U_u is totally isotropic with respect to $\langle\cdot\,,\cdot\rangle_{p,q}$.

Definition 3.1 *A spinor $u \in \Delta_{p,q}$ is said to be pure if $\dim_{\mathbb{C}} V_u = m$. In this case $\dim_{\mathbb{R}} U_u$ is called the real index of u.*

Complex lines in $\Delta_{p,q}$ are called projective spinors. The set of all projective spinors is denoted by $P\Delta_{p,q}$. Analogously, $P\Delta_{p,q}^{+}$ and $P\Delta_{p,q}^{-}$ are defined. Note, that if $u \in \Delta_{p,q}$ is pure, than cu is pure with the same real index for all $c \neq 0$, $c \in \mathbb{C}$. Hence, a projective spinor $\mathbb{C}u$ is called pure with real index l if u is pure with real index l. It is well known (see [C 38]) that the map

$$\mathbb{C}u \longmapsto V_u$$

restricted to the set of pure projective spinors is a bijection onto the Grassmannian of isotropic subspaces of dimension m of $\mathbb{C}^n$.

Definition 3.2 *Let $(E, \langle\cdot\,,\cdot\rangle)$ be an n-dimensional pseudo-Euclidean space. If $n=2m$ then an orthogonal optical structure is a pair $O=(U,J)$ consisting of a totally isotropic subspace $U \subset E$ and an orthogonal complex structure J on $U^{\perp}/U$ (i.e. orthogonal with respect to the scalar product induced by $\langle\cdot\,,\cdot\rangle$ on $U^{\perp}/U$). If $n=2m+1$ then an orthogonal optical structure is a triple $O=(U,H,J)$ where $U \subset E$ is totally isotropic, H is a subspace of codimension one of $U^{\perp}/U$ and J is an orthogonal complex structure on H.*

Next we shall recall the one-to-one correspondence between totally isotropic (with repect to $\langle\cdot\,,\cdot\rangle_{p,q}^{\mathbb{C}}$) subspaces of dimension m of $\mathbb{C}^n$ and orthogonal optical structures on $\mathbb{R}^{p,q}$. Let $V \subset \mathbb{C}^n$ be totally isotropic. Then V defines a subspace $U \subset \mathbb{R}^{p,q}$ by $U^{\mathbb{C}} = V \cap \bar{V}$. Now we choose a complement W of $U^{\mathbb{C}}$ in V, i.e. $V = U^{\mathbb{C}} \oplus W$. Then W defines a subspace $W_0 \subset \mathbb{R}^{p,q}$ by $W_0^{\mathbb{C}} = W \oplus \bar{W}$. Then we have

$$U^{\perp} = \begin{cases} U \oplus W_0 & \text{if} \quad n=2m \\ U \oplus W_0 \oplus \mathbb{R}x_0 & \text{if} \quad n=2m+1 \end{cases} \tag{5}$$

for some $x_0 \in W_0^{\perp}$ where the sum is orthogonal. Let $\bar{J}$ be that complex structure on W_0 for which W is the $(-i)$-eigenspace of $\bar{J}^{\mathbb{C}}$. Since W is totally isotropic, $\bar{J}$ is orthogonal. $\bar{J}$ defines an orthogonal complex structure J on $U^{\perp}/U$. Conversely, if (U,J) is an orthogonal optical structure and $U^{\perp} = U \oplus W_0$, $W_0 \subset U^{\perp}$, then let W be the $(-i)$-eigenspace of the induced orthogonal structure $\bar{J}^{\mathbb{C}}$ on $W_0^{\mathbb{C}}$ and define an m-dimensional totally isotropic subspace V by $V = U^{\mathbb{C}} \oplus W$. When we compose the bijection from the set of pure projective spinors onto the Grassmannian of isotropic subspaces of dimension m in $\mathbb{C}^n$ explained above with this bijection onto the set of orthogonal optical structures on $\mathbb{R}^{p,q}$ we obtain the following proposition. Note that for an arbitrary fixed spinor $u \in \Delta_{p,q}$ the Clifford multiplication $\mathbb{R}^{p,q} \times \Delta_{p,q}$ induces a map

$$\begin{aligned} U_u^{\perp}/U_u &\longrightarrow \Delta_{p,q} \\ [X] &\longmapsto X \cdot u \quad (X \in U_u^{\perp}) . \end{aligned}$$

Proposition 3.1 *Let $\mathbb{C}u \in P\Delta_{p,q}$ be a pure projective spinor of real index l and set $U := U_u$.*
If $n = 2m$ we can define an orthogonal complex structure J on $U^{\perp}/U$ by

$$(J[X]) \cdot u = i[X] \cdot u \tag{6}$$

for all $X \in U^{\perp}$. In particular, $O = (U, J)$ is an orthogonal optical structure on $\mathbb{R}^{p,q}$.
If $n = 2m + 1$ there exists a unique element $\xi \in U^{\perp}/U$ such that

$$\begin{aligned} \xi \cdot u = iu \qquad &\textit{if} \qquad l \equiv p \mod 2 \\ \xi \cdot u = u \qquad &\textit{if} \qquad l \equiv q \mod 2 \end{aligned} \tag{7}$$

and we can define an orthogonal complex structure J on $\xi^{\perp} \subset U^{\perp}/U$ by

$$(J[X]) \cdot u = i[X] \cdot u$$

for all $X \in U^{\perp}$ with $[X] \perp \xi$. In particular, $O = (U, \xi^{\perp}, J)$ is an orthogonal optical structure on $\mathbb{R}^{p,q}$.

Consider now $\mathbb{R}^{p,q}$ with $p+q = 2m$ (or $p+q = 2m+1$). Let (U, J) (or (U, H, J)) be an orthogonal optical structure on $\mathbb{R}^{p,q}$. Then there exists a uniquely defined projective pure spinor $\mathbb{C}u \in P\Delta_{p,q}$ with

(i) $U = U_u$

(ii) $(J[X]) \cdot u = i[X] \cdot u$ *for all* $[X] \in U^{\perp}/U$
(or for all $[X] \in H \subset U^{\perp}/U$ if $p + q = 2m + 1$).

A. Trautmann and W. Kopczyński proved in [Ko 94] and [KoT 92] the following fact.

Proposition 3.2 *Consider the spinor module $\Delta_{p,q}$ and a number $l \leq \min(p, q)$.*
If $n = p + q = 2m + 1$ then $Spin(p, q)$ acts transitively on the set of projective pure spinors of real index l.
If $n = p + q = 2m$ then $Spin(p, q)$ acts transitively on the set of projective pure spinors of real index l in $P\Delta^+_{p,q}$ (and similarly in $P\Delta^-_{p,q}$).

In particular, this proposition together with the $Pin(p, q)$-invariance of $\Delta^+_{p,q} \cup \Delta^-_{p,q}$ implies the well-known fact, that in case $n = 2m$ pure spinors are contained either in $\Delta^+_{p,q}$ or in $\Delta^-_{p,q}$.

4 Parallel Pure Spinors on Pseudo-Riemannian Spin Manifolds

Now we want to globalize the results obtained in the previous section. Let (M,g) be a pseudo-Riemannian spin manifold of signature (p,q) and dimension $n = p+q$. We fix an n-tuple $(\kappa_1, \dots, \kappa_n)$, $\kappa_i = \pm 1, \#\{i \mid \kappa_i = -1\} = p$. Then a local frame $s_1, \dots, s_n$ is said to be orthonormal if $\langle s_i, s_j\rangle = 0$ if $i \neq j$ and $\langle s_i, s_i\rangle = \kappa_i$ for $i,j = 1,\dots,n$. We fix a *Spin*-structure Q of (M,g), i.e. a $Spin(p,q)$-reduction of the bundle P of positively oriented orthonormal frames. Let S be the spinor bundle. A section $\psi \in \Gamma(S)$ is called pure if it is pure at every point $x \in M$. We define projective spinor bundles $PS = Q \times_{Spin(p,q)} P\Delta_{p,q}$, and $PS^{\pm} = Q \times_{Spin(p,q)} P\Delta^{\pm}_{p,q}$. Clearly, $PS = P \times_{SO(p,q)} P\Delta_{p,q}$, and $PS^{\pm} = P \times_{SO(p,q)} P\Delta^{\pm}_{p,q}$ and, thus, these bundles are defined, even if there is no *Spin*-structure on M. Sections of these bundles are called projective spinors. Furthermore, we can globalize the notion of optical structures. If $n = 2m$ then an orthogonal almost optical structure on (M,g) is a pair $O = (U,J)$ consisting of a totally isotropic smooth distribution $U \subset TM$ and an orthogonal almost complex structure J on $U^{\perp}/U$ (i.e. orthogonal with respect to the scalar product $\bar{g}$ induced by g on $U^{\perp}/U$). If $n = 2m+1$ then an orthogonal almost optical structure is a triple $O = (U,H,J)$ where $U \subset TM$ is a totally isotropic smooth distribution, H is a section in the bundle of subspaces of codimension one of $U^{\perp}/U$ and J is an orthogonal almost complex structure on H.

In the following we will say that a distribution $V \subset TM$ is parallel if $\nabla_X v \in \Gamma(V)$ holds for all $v \in \Gamma(V)$ and $X \in \mathfrak{X}(M)$.

Lemma 4.1 *Let $U \subset TM$ be a parallel distribution. Then $U^{\perp}$ is also parallel. In particular ∇ induces a covariant derivative ∇ on $U^{\perp}/U$ by $\nabla[Y] = [\nabla Y]$ for all $Y \in \Gamma(U^{\perp})$. If, in addition, U is totally isotropic, then the scalar product $\bar{g}$ induced by g on $U^{\perp}/U$ is parallel with respect to ∇.*

Definition 4.1 *An orthogonal almost optical structure (U,J) on an even-dimensional pseudo-Riemannian manifold is called a parallel optical structure if U and J are parallel (J with respect to the induced covariant derivative on $U^{\perp}/U$). An orthogonal almost optical structure (U,H,J) on an odd-dimensional pseudo-Riemannian manifold (M,g) is called a parallel optical structure if the following conditions are satisfied.*

1. *U is parallel.*
2. *H is parallel, i.e. $\nabla_X h \in \Gamma(H)$ for all $X \in \mathfrak{X}(M)$ and $h \in \Gamma(H)$ with respect to the induced covariant derivative on $U^{\perp}/U$.*
3. *J is parallel on H.*

We denote the almost optical structure associated with the projective pure spinor $\Psi \in \Gamma(PS)$ by (U_Ψ, J_Ψ) if n is even and by (U_Ψ, H_Ψ, J_Ψ) if n is odd.

The next aim is the geometrical description of pseudo-Riemannian spin manifolds admitting parallel pure spinors or parallel projective pure spinors.

Definition 4.2 *A projective pure spinor $\Psi \in \Gamma(PS)$ is called parallel if it is parallel as 1-dimensional bundle, i.e. if any local representative ψ of Ψ satisfies $\nabla^S\psi = \mu_\psi \otimes \psi$ for some linear function $\mu_\psi : TM \to \mathbb{C}$.*

We note that a parallel spinor $\psi \in \Gamma(S)$ on a connected manifold is pure if it is pure at one point. Indeed, since ψ is parallel we can find by a well-known fact of holonomy theory for any $x \in M$ a local section q in Q around x such that $\psi = [q,u]$ for a constant $u \in \Delta_{p,q}$.

Theorem 4.1 *Let $\Psi \in \Gamma(PS)$ be a projective pure spinor and (U_Ψ, J_Ψ) (or (U_Ψ, H_Ψ, J_Ψ)) the associated almost optical structure on (M,g). Then Ψ is parallel if and only if (U_Ψ, J_Ψ) (or (U_Ψ, H_Ψ, J_Ψ)) is a parallel optical structure.*

Proof.

First let Ψ be a parallel projective pure spinor of real index l and ψ a local representative of Ψ. If Z is a section of U_Ψ then we have

$$0 = \nabla^S_X(Z \cdot \psi) = \nabla_X Z \cdot \psi + \mu_\psi(X) Z \cdot \psi = (\nabla_X Z) \cdot \psi \,.$$

Hence, $\nabla_X Z \in \Gamma(U_\Psi)$. Now let n be odd. From Proposition 3.1 we know that ψ defines a section $\xi \in \Gamma(U^\perp/U)$ by

$$\begin{aligned} \xi \cdot \psi = i\psi \qquad &\text{if} \qquad l \equiv p \mod 2 \\ \xi \cdot \psi = \psi \qquad &\text{if} \qquad l \equiv q \mod 2. \end{aligned}$$

Since Ψ is parallel we have

$$(\nabla \xi) \cdot \psi = \nabla(\xi \cdot \psi) - \xi \cdot \nabla\psi = \nabla(c\psi) - \mu_\psi \xi \cdot \psi = c\mu_\psi\psi - c\mu_\psi\psi = 0$$

where $c = i$ if $l \equiv p \mod 2$ and $c = 1$ if $l \equiv q \mod 2$. Hence $\nabla\xi = 0$ and, in particular, H_Ψ is parallel.

Furthermore, since Ψ is parallel we obtain by differentiating $J[X] \cdot \psi = i[X] \cdot \psi$ the equation

$$(\nabla_Z(J[X]) + \mu_\psi(Z) J[X]) \cdot \psi = i\nabla_Z[X] \cdot \psi + i\mu_\psi(Z)[X] \cdot \psi$$

for all $X \in \Gamma(U_\Psi^\perp)$ (if n is even) or $[X] \in \Gamma(H_\Psi)$ (if n is odd) and all $Z \in TM$. Consequently, $(\nabla J[X]) \cdot \psi = 0$ and, thus, $\nabla J = 0$.

Now assume that the almost optical structure defined by Ψ is parallel. If $n = 2m+1$ we fix a field L of lines in $U_\Psi^\perp$ which represents $H_\Psi^\perp \subset U_\Psi^\perp/U_\Psi$. By Equation (5) there exists a complementary distribution W_0 of U_Ψ (if $n = 2m$) or of $U_\Psi \oplus L$ (if $n = 2m+1$) in $U_\psi^\perp$ such that

$$W = \{X + iY \mid X, Y \in W_0 \,, J_\Psi[X] = [Y]\}$$

is a complement of $U_\Psi^{\mathbb{C}}$ in $V_\Psi = \{v \in T^{\mathbb{C}}M \mid v \cdot \psi = 0\}$. We have to prove that $\nabla^S_Z\psi$ is a complex multiple of ψ for all $Z \in T_mM$. If $\nabla^S_Z\psi = 0$ this is obvious. Now consider $m \in M$ and $Z \in T_mM$ with $\nabla^S_Z\psi \neq 0$. Let $X, Y \in \Gamma(W_0)$ be such that $X + iY \in \Gamma(W)$, i.e. $[Y] = J_\Psi[X]$. Then $\nabla J_\Psi = 0$ implies

$\nabla_Z[Y] = J_\Psi[\nabla_Z X]$. In particular we have

$$i\nabla_Z X \cdot \psi = \nabla_Z Y \cdot \psi. \tag{8}$$

On the other hand we obtain differentiating $(X+iY)\cdot\psi=0$ the equation

$$(\nabla_Z(X+iY)\cdot\psi+(X+iY)\cdot\nabla_Z\psi=0$$

and, hence, by (8) we obtain $(X+iY)\cdot\nabla^S_Z\psi=0$. Consequently, in $m\in M$

$$V_\Psi=U^{\mathbb{C}}\oplus W\subset V:=\{v\in T^{\mathbb{C}}_m\mid v\cdot\nabla^S_Z\psi=0\}$$

holds. Since $\nabla^S_Z\psi\neq 0$ this shows that $\nabla^S_Z\psi$ is pure and $V=V_\Psi=U^{\mathbb{C}}\oplus W$. Hence, it is defined by the same almost optical structure in T_mM as $\psi(m)$ and is therefore a complex multiple of $\psi(m)$. □

Obviously, a projective spinor is parallel if there exists a globally defined parallel representative. Now we will answer the question under which conditions we can find a global parallel representative of a parallel pure projective spinor on a simply connected manifold. It turns out that, similarly as on Riemannian manifolds, this is the case if and only if some curvature conditions are satisfied.

Theorem 4.2 *Let (M,g) be a pseudo-Riemannian spin manifold of dimension n. If (M,g) admits a parallel pure spinor then there exists a parallel orthogonal optical structure (U,J) (if n is even) or (U,H,J) (if n is odd) on (M,g) which satisfies the following conditions:*

1. *The trace of the restriction of $R(X,Y)$ to U equals 0:*

$$\operatorname{tr} R(X,Y)\mid_U=0\quad\textit{for all } X,Y\in\mathfrak{X}(M).$$

(Note that the Riemannian curvature tensor R satisfies $R(X,Y)U\subset U$ for all $X,Y\in\mathfrak{X}(M)$).

2. *The curvature R^∇ of the induced covariant derivative ∇ on $U^\perp/U$ satisfies*

$$\operatorname{tr} J\circ R^\nabla(X,Y)=0\quad\textit{for all } X,Y\in\mathfrak{X}(M).$$

3. *The image of the Ricci map is contained in U.*

Conversely, if (M,g) is a simply connected pseudo-Riemannian spin manifold which admits a parallel optical structure satisfying conditions 1. and 2., then there exists a parallel pure spinor on (M,g).

Proof.
The existence of a parallel orthogonal optical structure (U,J) (if n is even) or (U,H,J) (if n is odd) on (M,g) is equivalent to the existence of a pure parallel projective spinor Ψ. We view Ψ as a 1-dimensional bundle over M. Since Ψ is parallel, ∇^S is a covariant derivative on Ψ. If there exists a global parallel section in Ψ, then the curvature of this covariant derivative equals 0. Conversely, if M is simply connected the vanishing of this curvature implies the existence of a

global parallel section in Ψ. Hence we have to prove, that the curvature of ∇^S on Ψ vanishes if and only if the associated optical structure satisfies Conditions *1.* and *2.*

If $n = 2m$ we may assume $\Psi \in \Gamma(PS^+)$. We denote the dimension of U by l. We choose $\kappa_{2j-1} = -1$, $\kappa_{2j} = 1$ for $j = 1, \ldots, l$ and $\kappa_{2l+2j-1} = \kappa_{2l+2j}$ for $j = 1, ..., m-l$. By Proposition 3.2 we can find around each $m \in M$ a local section q of Q such that $\Psi = \mathbb{C} \cdot [q, u(1, 1, \ldots, 1)]$. Let the frame $s_1, \ldots, s_n$ be the projection of q. Then U is spanned by $s_{2i-1} + s_{2i}$ $(i = 1, \ldots, l)$ and $U^\perp/U$ by $[s_j]$ for $j > l$. Furthermore, we have $J[s_{2k-1}] = -[s_{2k}]$ for $k = l+1, \ldots, m$. Now we calculate the curvature R^S of ∇^S on Ψ. Consider the local section $[q, u(1, 1, \ldots, 1)]$ of Ψ. Then we have

$$\begin{aligned} R^S(X,Y)\psi &= \nabla^S_X \nabla^S_Y \psi - \nabla^S_Y \nabla^S_X \psi - \nabla^S_{[X,Y]}\psi \\ &= \frac{1}{2} \sum_{i<j} \kappa_i \kappa_j R(X, Y, s_i, s_j) s_i \cdot s_j \cdot \psi \\ &= \frac{1}{2} \sum_{i<j} \kappa_i \kappa_j R(X, Y, s_i, s_j)[q, e_i \cdot e_j \cdot u(1, 1, \ldots, 1)]\,. \end{aligned}$$

Since $R^S(X,Y)\psi = f\psi$ for a certain complex function f we obtain using (3)

$$R^S(X,Y)\psi = \Big(\frac{1}{2} \sum_{j=1}^{l} R(X, Y, s_{2j-1}, s_{2j}) + \frac{i}{2} \sum_{j=l+1}^{m} \kappa_{2j} R(X, Y, s_{2j-1}, s_{2j})\Big)\, \psi\,.$$

Hence, the curvature R^S vanishes on Ψ if and only if

$$\sum_{j=1}^{l} R(X, Y, s_{2j-1}, s_{2j}) = 0 \tag{9}$$

and

$$\sum_{j=l+1}^{m} \kappa_{2j} R(X, Y, s_{2j-1}, s_{2j}) = 0\,. \tag{10}$$

On the other hand we have for $1 \leq j \leq l$

$$\begin{aligned} R(X,Y)(s_{2j-1} + s_{2j}) &= \sum_{k=1}^{l} (-R(X, Y, s_{2j-1}, s_{2k-1}) s_{2k-1} + R(X, Y, s_{2j-1}, s_{2k}) s_{2k} \\ &\qquad - R(X, Y, s_{2j}, s_{2k-1}) s_{2k-1} + R(X, Y, s_{2j}, s_{2k}) s_{2k}) \\ &= R(X, Y, s_{2j-1}, s_{2j})(s_{2j-1} + s_{2j}) + \sum_{k \neq j} r_k (s_{2k-1} + s_{2k})\,, \end{aligned}$$

where $r_k \in \mathbb{R}$. Hence

$$\operatorname{tr} R(X,Y)|_U = \sum_{j=1}^{l} R(X, Y, s_{2j-1}, s_{2j}). \tag{11}$$

Furthermore, if $l + 1 \leq j \leq m$ then we have

$$\begin{aligned} R(X, Y, s_{2j-1}, s_{2j}) &= g(R(X,Y) s_{2j-1}, s_{2j}) \\ &= \bar{g}([R(X,Y) s_{2j-1}], [s_{2j}]) \\ &= -\bar{g}(R^\nabla(X,Y)[s_{2j-1}], J[s_{2j-1}]) \\ &= \bar{g}(J \circ R^\nabla(X,Y)[s_{2j-1}], [s_{2j-1}]) \end{aligned}$$

since J is orthogonal and

$$\begin{aligned} R(X,Y,s_{2j-1},s_{2j}) &= \bar{g}(R^{\nabla}(X,Y)[s_{2j-1}],[s_{2j}]) \\ &= \bar{g}(R^{\nabla}(X,Y)(J[s_{2j}]),[s_{2j}]) \\ &= \bar{g}(J\circ R^{\nabla}(X,Y)[s_{2j}],[s_{2j}]) \end{aligned}$$

since J is parallel.

Hence

$$2R(X,Y,s_{2j-1},s_{2j}) = \bar{g}(J\circ R^{\nabla}(X,Y)[s_{2j-1}],[s_{2j-1}]) + \bar{g}(J\circ R^{\nabla}(X,Y)[s_{2j}],[s_{2j}])$$

and we conclude

$$\begin{aligned} 2\sum_{j=l+1}^{m} \kappa_{2j} R(X,Y,s_{2j-1},s_{2j}) &= \\ &= \sum_{j=l+1}^{m} \kappa_{2j}(\bar{g}(J\circ R^{\nabla}(X,Y)[s_{2j-1}],[s_{2j-1}]) + \bar{g}(J\circ R^{\nabla}(X,Y)[s_{2j}],[s_{2j}]) \qquad (12) \\ &= \operatorname{tr} J\circ R^{\nabla}\,. \end{aligned}$$

Comparing (9) and (10) with (11) and (12) we see that R^S vanishes on Ψ if and only if Conditions *1.* and *2.* are satisfied. It remains to prove that the existence of a pure parallel spinor ψ on (M,g) implies Condition *3.* for the orthogonal optical structure associated with $\mathbb{C}\psi$. But this follows from the well-known fact that $\mathrm{Ric}(X)\cdot\psi = 0$ holds for any parallel spinor ψ. $\square$

Example 4.1 *Consider the Lorentzian metric $g = 2dx_1dx_2 + f dx_2^2 + \sum_{i=3}^{n} dx_i^2$ on an open set U of $\mathbb{R}^n$, where f is a real function depending on $x_2,\dots,x_n$. This is a pp-metric and every n-dimensional pp-manifold is locally isometric to such a space (see [S 74]). (U,g) admits pure parallel spinors. For the generic choice of f the Lie algebra of the holonomy group of g equals*

$$\mathfrak{n} = \mathbb{R}^{n-2} = \left\{ \left(\begin{array}{ccc} 0 & 0 & X^{\top} \\ 0 & 0 & 0 \\ 0 & 2X & 0 \end{array} \right) \;\middle|\; X\in\mathbb{R}^{n-2} \right\}$$

and the space of parallel spinors is spanned by $\{[q,u(\varepsilon_m,\dots,\varepsilon_2,1)] \mid \varepsilon_i = \pm 1\}$ for a certain section q of the Spin-structure Q of (U,g).

Proof. Consider the frame

$$X_1 = \sqrt{2}\partial_1,\; X_2 = \frac{f}{\sqrt{2}}\partial_1 - \sqrt{2}\partial_2,\; X_i = \partial_i \;(i=3,\dots,n)$$

and its dual frame

$$\omega^1 = \frac{1}{\sqrt{2}}dx_1 + \frac{f}{2\sqrt{2}}dx_2,\; \omega^2 = -\frac{1}{\sqrt{2}}dx_2,\; \omega^i = dx_i \;(i=3,\dots,n).$$

Then the matrix θ of connection forms $\omega^i_j = \omega^i(\nabla X_j)$ equals

$$\theta = (\omega^i_j) = \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 2F^\top & 0 \end{pmatrix}$$

where $F = \frac{1}{2\sqrt{2}}(\frac{\partial f}{\partial x_3}dx_2, \ldots, \frac{\partial f}{\partial x_n}dx_2)$. Hence, the sections $[q, u(\varepsilon_m, \ldots, \varepsilon_2, 1)]$ where q projects to the orthonormal frame $\frac{1}{2}(X_1 + X_2),\ \frac{1}{2}(X_1 - X_2),\ X_3, \ldots, X_n$ are parallel. In particular, the Lie algebra $\mathfrak{h}$ of the holonomy group is contained in the image under λ of the stabilizer of all $u(\varepsilon_m, \ldots, \varepsilon_2, 1)$. This image equals $\mathfrak{n}$, where the matrices of $\mathfrak{so}(n)$ are written with respect to the basis $e_1 + e_2,\ e_1 - e_2,\ e_3, \ldots, e_n$.

Now fix a point $x \in U$. By the Ambrose-Singer theorem all linear maps $(d\theta + \theta \wedge \theta)(\partial_\alpha, \partial_2)(x)$ are contained in $\mathfrak{h}$. One calculates

$$(d\theta + \theta \wedge \theta)(\partial_\alpha, \partial_2) = \begin{pmatrix} 0 & 0 & \Phi \\ 0 & 0 & 0 \\ 0 & 2\Phi^\top & 0 \end{pmatrix}$$

where

$$\Phi = \frac{1}{2\sqrt{2}}(\frac{\partial^2 f}{\partial x_\alpha \partial x_3}, \ldots, \frac{\partial^2 f}{\partial x_\alpha \partial x_n}).$$

Consequently, for the generic choice of f we obtain $\mathfrak{h} = \mathfrak{n}$ and the space of parallel spinors is spanned by the sections $[q, u(\varepsilon_m, \ldots, \varepsilon_2, 1)]$.

For instance, all $[q, u(\varepsilon_m, \ldots, \varepsilon_2, 1)]$ are pure parallel spinors. $\square$

5 The Case of Signature (m, m) or $(m+1, m)$

Here we consider n-dimensional pseudo-Riemannian manifolds of signature (m, m) or $(m+1, m)$. Let $\langle \cdot, \cdot \rangle_{m,m}$ and $\langle \cdot, \cdot \rangle_{m+1,m}$ be those scalar products on $\mathbb{R}^{2m}$ and $\mathbb{R}^{2m+1}$, respectively, which satisfy

$$\kappa_j = \begin{cases} -1 & \text{if } j \text{ is odd} \\ 1 & \text{if } j \text{ is even} \end{cases} \tag{13}$$

for all $1 \leq j \leq n$. From (2) and (3) we see that the restriction of $\Phi_{m,m}$ to $\mathcal{C}_{m,m}$ is an isomorphism onto $\mathbb{R}(m)$ and the restriction of $\Phi_{m+1,m}$ to $\mathcal{C}_{m+1,m}$ is an isomorphism onto $\mathbb{R}(m) \oplus \mathbb{R}(m)$. In particular, the spinor representations $\Delta_{m,m}$ and $\Delta_{m+1,m}$ are real. Therefore, $\Delta_{m,m} = \mathbb{R}^{2^m}$ and $\Delta_{m+1,m} = \mathbb{R}^{2^m}$ in this section. Elements of these spinor modules are called real spinors. Real pure spinors are exactly those pure spinors in the complexified spinor modules which have real index m.

We introduce another basis of $\mathbb{R}^{m,m}$ which is more convenient for some of the following calculations. We define

$$f_i^+ = e_{2i-1} + e_{2i}, \qquad f_i^- = e_{2i-1} - e_{2i}, \qquad (1 \leq i \leq m)$$

and calculate

$$
\begin{aligned}
\Phi_{m,m}(f_i^+) &= E\otimes\ldots\otimes E\otimes\begin{pmatrix} 0 & -2\\ 0 & 0\end{pmatrix}\otimes\underbrace{T\otimes\ldots\otimes T}_{(i-1)\times}\\
\Phi_{m,m}(f_i^-) &= E\otimes\ldots\otimes E\otimes\begin{pmatrix} 0 & 0\\ -2 & 0\end{pmatrix}\otimes\underbrace{T\otimes\ldots\otimes T}_{(i-1)\times}.
\end{aligned}
\tag{14}
$$

Next we assign to each real spinor in $\Delta_{m,m}$ or $\Delta_{m+1,m}$ a certain m-form. Let $(\cdot\,,\cdot)$ be the standard inner product of $\mathbb{R}^{2^m}$. We define by

$$b(u,v)=(u,e_1e_3e_5\ldots e_{2m-1}\cdot v)$$

a $Spin^+(m,m)$-invariant bilinear form b on $\Delta_{m,m}$ which satisfies

$$b(u,X\cdot v)+(-1)^m b(X\cdot u,v)=0$$

for all $X\in\mathbb{R}^{m,m}, u,v\in\Delta_{m,m}$ and a by

$$b'(u,v)=(u,e_1e_3e_5\ldots e_{2m+1}\cdot v)$$

a $Spin^+(m+1,m)$-invariant bilinear bilinear form b' on $\Delta_{m+1,m}$ which satisfies

$$b'(u,X\cdot v)+(-1)^{m+1}b'(X\cdot u,v)=0$$

for all $X\in\mathbb{R}^{m+1,m}, u,v\in\Delta_{m+1,m}$.

For $u\in\Delta_{m,m}$ and $u'\in\Delta_{m+1,m}$ we define m-forms b_u and $b'_{u'}$ by

$$
\begin{aligned}
b_u(X_1,\ldots,X_m) &:= \frac{1}{m!}\sum_{\sigma\in S_m}\operatorname{sgn}\sigma\cdot b(X_{\sigma(1)}X_{\sigma(2)}\ldots X_{\sigma(m)}u,u)\\
b'_{u'}(X_1,\ldots,X_m) &:= \frac{1}{m!}\sum_{\sigma\in S_m}\operatorname{sgn}\sigma\cdot b'(X_{\sigma(1)}X_{\sigma(2)}\ldots X_{\sigma(m)}u',u')
\end{aligned}
$$

Proposition 5.1 *1. If $u\in\Delta_{m,m}$ is a real pure spinor then b_u is decomposable. Analogously, if $u\in\Delta_{m+1,m}$ is a real pure spinor then b'_u is decomposable.*

2. We have

$$b_{u(1,1,\ldots,1)}=b'_{u(1,1,\ldots,1)}=(-1)^{\frac{m(m-1)}{2}}(\sigma_1-\sigma_2)\wedge(\sigma_3-\sigma_4)\wedge\ldots\wedge(\sigma_{2m-1}-\sigma_{2m}),$$

where $\sigma_1,\sigma_2,\ldots,\sigma_n$ is a dual basis of $e_1,e_2,\ldots,e_n$.

Proof.

It follows from the definition of b and B' that

$$
\begin{aligned}
b(u(\varepsilon_m,\ldots,\varepsilon_1),u(\delta_m,\ldots,\delta_1))\neq 0 \quad &\text{iff}\quad \varepsilon_j=-\delta_j \text{ for all } j=1,\ldots,m\\
b'(u(\varepsilon_m,\ldots,\varepsilon_1),u(\delta_m,\ldots,\delta_1))\neq 0 \quad &\text{iff}\quad \varepsilon_j=-\delta_j \text{ for all } j=1,\ldots,m.
\end{aligned}
$$

By (14) for $f_{i_j}\in\{f_1^+,\ldots,f_m^+,f_1^-,\ldots,f_m^-\}$

$$f_{i_1}\cdot\ldots\cdot f_{i_m}u(1,1,\ldots,1)=cu(-1,-1,\ldots,-1)\quad c\neq 0$$

holds if and only if $\{f_{i_1},\ldots,f_{i_m}\}=\{f_1^-,\ldots,f_m^-\}$.

□

Remark 5.1 *In the following we only list results for signature $(m+1,m)$. One obtains analogous facts for signature (m,m) if one omits the last basis vector, the last coordinate etc.*

Next we will compute the isotropy group $H' \subset Spin(m+1,m)$ of $u(1,1,\ldots,1)$. Since $-1 \notin H'$ the map $\lambda|_{H'} : H' \to \lambda(H')$ is an isomorphism. We denote $\lambda(H') \subset SO(m+1,m)$ by H. Note that H and H' are not connected, since e.g. $-e_1e_2 \in H'$ but $-e_1e_2 \notin Spin^+(m+1,m)$. Denote by H_+ and H'_+ the intersections $H \cap SO^+(m+1,m)$ and $H' \cap Spin^+(m+1,m)$, respectively.

Proposition 5.2 *We have*

$$H = \left\{ \begin{pmatrix} A & B & X \\ 0 & (A^\top)^{-1} & 0 \\ 0 & Y & 1 \end{pmatrix} \;\middle|\; \begin{array}{c} 2B^\top(A^\top)^{-1} + 2A^{-1}B + Y^\top Y = 0 \\ 2A^{-1}X + Y^\top = 0 \\ \det A = \pm 1 \end{array} \right\}$$

and

$$H_+ = \left\{ \begin{pmatrix} A & B & X \\ 0 & (A^\top)^{-1} & 0 \\ 0 & Y & 1 \end{pmatrix} \in H \;\middle|\; \det A = 1 \right\}$$

with respect to the basis $f_1^+,\ldots,f_m^+,f_1^-,\ldots,f_m^-,e_{2m+1}$ of $\mathbb{R}^{m+1,m}$. H_+ is the semi-direct product of $SL(m,\mathbb{R})$ and the nilpotent group

$$N = \left\{ \begin{pmatrix} E_m & B & X \\ 0 & E_m & 0 \\ 0 & -2X^\top & 1 \end{pmatrix} \;\middle|\; B \in \mathbb{R}(m),\ B^\top + B + 2XX^\top = 0,\ X \in \mathbb{R}^m \right\}. \quad (15)$$

In particular, H_+ is connected.

Proof. An element $h' \in Spin(m+1,m)$ preserves the line $\mathbb{C}\,u(1,1,\ldots,1)$ if and only if $\lambda(h')$ preserves the optical structure defined by $\mathbb{C}\,u(1,1,\ldots,1)$, i.e. if it preserves $U_{u(1,\ldots,1)}$. Hence we have $h \in H$ if and only if $h \in SO(m+1,m)$, h preserves the optical structure defined by $\mathbb{C}\,u(1,1,\ldots,1)$ and it preserves $b'_{u(1,\ldots,1)}$ up to sign. Thus, $h \in H$ if and only if $h \in SO(m+1,m)$, $h(\mathrm{span}\{f_1^+,\ldots,f_m^+\}) = \mathrm{span}\{f_1^+,\ldots,f_m^+\}$ and $h \cdot b'_{u(1,\ldots,1)} = \pm b'_{u(1,\ldots,1)}$ holds, where $h \cdot b'_{u(1,\ldots,1)} = b'_{u(1,\ldots,1)}$ if and only if $h \in H_+$. □

Corollary 5.1 *$\lambda_* : \mathfrak{spin}(m+1,m) \to \mathfrak{so}(m+1,m)$ maps $\mathfrak{h}'$ isomorphically onto*

$$\begin{aligned} \lambda_*\mathfrak{h}' &= \left\{ \begin{pmatrix} A & B & x \\ 0 & -A^\top & 0 \\ 0 & -2x^\top & 0 \end{pmatrix} \mid x \in \mathbb{R}^n,\ A,B \in \mathbb{R}(m),\ \mathrm{tr}\,A = 0,\ B^\top = -B \right\} \\ &\subset \mathfrak{so}(m+1,m) \end{aligned}$$

where $\mathfrak{so}(m+1,m)$ is described with respect to the basis $f_1^+,\ldots,f_m^+,f_1^-,\ldots,f_m^-,e_{2m+1}$ of $\mathbb{R}^{m+1,m}$.

Proposition 5.3 *The Lie algebra $\mathfrak{h}'$ of the isotropy group of $u(1,1,\dots,1) \in \Delta_{m+1,m}$ with respect to the $Spin(m+1,m)$-action is spanned as a subalgebra of $\mathfrak{spin}(m+1,m) = \mathrm{span}_{\mathbb{R}}\{e_ie_j \mid 1 \le i < j \le 2m+1\}$ by*

$$\begin{array}{rl} f_i^+ f_j^- & (1 \le i,j \le m,\, i \neq j) \\ f_i^+ f_j^+ & (1 \le i < j \le m) \\ f_i^+ e_{2m+1} & (1 \le i \le m) \\ e_1e_2 - e_{2i-1}e_{2i} & (1 < i \le m). \end{array}$$

Proof. By (14) all the elements of $\mathfrak{spin}(m+1,m)$ given above annihilate $u(1,1,\dots,1)$. On the other hand $f_i^- f_j^-$ $(1 \le i,j \le m,\, i \neq j)$, $f_i^- e_{2m+1}$ $(1 \le i \le m)$ and $e_1e_2 + e_3e_4$ span a complement of $\mathrm{span}\{f_i^+ f_j^-, f_i^+ f_j^+, f_i^+ e_{2m+1}, e_1e_2 - e_{2i-1}e_{2i}\}$ in $\mathfrak{spin}(m+1,m)$ and by (14) $f_i^- f_j^- u(1,1,\dots,1)$, $f_i^- e_{2m+1}u(1,1,\dots,1)$, $(e_1e_2 + e_3e_4)u(1,1,\dots,1)$ are linearly independent. $\square$

Now let (M,g) be a pseudo-Riemannian spin manifold of signature $(m+1,m)$ or (m,m) with *Spin*-structure Q. Here we consider the real spinor bundle S associated to Q with fibre $\Delta_{m,m} = \mathbb{R}^{2^m}$ or $\Delta_{m+1,m} = \mathbb{R}^{2^m}$. Sections in S are called real spinors. If (M,g) is space and time oriented the bilinear forms b and b' on $\Delta_{m,m}$ and $\Delta_{m+1,m}$ define bilinear forms on S.

Proposition 5.4 *Let (M,g) be a pseudo-Riemannian spin manifold of signature $(m+1,m)$ or (m,m) which admits a real pure parallel spinor field $\psi \in \Gamma(S)$. Then M is a foliation with m-dimensional totally isotropic leaves such that for the distribution $U \subset TM$ which is tangential to the leaves the following holds.*

1. *U is parallel.*
2. *The restriction of the Riemannian curvature tensor R of (M,g) to U satisfies*

$$\mathrm{tr}\, R(X,Y)|_U = 0$$

   for all $X, Y \in \mathfrak{X}(M)$.

3. *The image of the Ricci map is contained in U.*

If , in addition, (M,g) is space and time oriented then

4. *there exists a nowhere vanishing parallel decomposable m-form ω on M with $\omega(X_1,\dots,X_m) = 0$ if there is an $i \in \{1,\dots,m\}$ with $X_i \in U$.*

Vice versa, if (M,g) is a simply connected pseudo-Riemannian spin manifold of signature (m,m) or $(m+1,m)$ which is a foliation with m-dimensional totally isotropic leaves satisfying Conditions 1. and 2. then (M,g) admits a real pure parallel spinor field.

Proof.

This theorem is a corollary of Theorem 4.2. Since ψ has real index m, the associated almost optical structure degenerates to an m-dimensional distribution $U \subset TM$. The existence of an m-form satisfying Condition *4.* follows from Proposition 5.1. $\square$

The next aim is to find metrics of signature $(m+1, m)$ and (m, m) with real pure parallel spinors. In the special case of dimension 7 this problem was solved by R. L. Bryant. He considers real isotropic (with respect to $\langle \cdot , \cdot \rangle_\Delta$)) spinors on pseudo-Riemannian manifolds of signature (4,3). In [K 98] it was shown that these spinors are exactly the real pure ones. In [Br 98] R. L. Bryant proved that any metric of signature (4,3) with a real isotropic parallel spinor can be written in a certain normal form with respect to special local coordinates. Moreover, a generic metric defined on an open set $U \subset \mathbb{R}^7$ satisfying this normal form admits exactly one real isotropic parallel spinor (up to constant multiples) and its holonomy group is isomorphic to the identity component of the stabilizer of a real isotropic spinor. Furthermore, the generic such metric is not Ricci-flat. We can copy Bryant's proof to obtain the following generalization.

Theorem 5.1 *Let (M, h) be a pseudo-Riemannian spin manifold of signature $(m+1, m)$ with a real pure parallel spinor. Then we can introduce local coordinates (x, y) or (x, y, z), $x = (x_1, \ldots, x_m)$, $y = (y^1, \ldots, y^m)$, $z \in \mathbb{R}$ around each point of M such that*

$$h = -dz^2 - 4\sum_{i=1}^{m} dx_i dy^i - 4\sum_{i,j=1}^{m} g_{ij} dy^i dy^j \tag{16}$$

where g_{ij} are arbitrary functions depending on x, y and z and satisfying

$$\sum_{i=1}^{m} \frac{\partial g_{ik}}{\partial x_i} = 0 \text{ for } k = 1, 2, \ldots, m. \tag{17}$$

Conversely, if we define by (16) and (17) a metric h on a connected open set $U \subset \mathbb{R}^{m+1,m}$ then (U, h) is spin and admits a real pure parallel spinor. For the generic choice of g_{ij} this metric has exactly one real pure parallel spinor (up to constant multiples) and the holonomy group is isomorphic to the image under λ of the identity component of the stabilizer of a real pure spinor. The Ricci curvature of h equals

$$\mathrm{Ric} = 2\sum_{k,l=1}^{m} \Big(-\frac{\partial^2 g_{kl}}{\partial z^2} - \sum_{\alpha=1}^{m} \frac{\partial^2 g_{kl}}{\partial x_\alpha \partial y^\alpha} + \sum_{\alpha,\beta=1}^{m} (g_{\alpha\beta} \frac{\partial^2 g_{kl}}{\partial x_\alpha \partial x_\beta} - \frac{\partial g_{\beta l}}{\partial x_\alpha} \frac{\partial g_{k\alpha}}{\partial x_\beta}) \Big) dy^k dy^l. \tag{18}$$

In particular, h is not necessarily Ricci-flat.

As already mentioned for signature (m, m) analogous statements are true where we have to omit in (16) and (18) the terms containing the z-coordinate. We recall the main points of the proof . We will use the Einstein rule for summation.

Let P be the bundle of positively oriented orthonormal frames of (M, h). P is isomorphic to the bundle P' of those positively oriented frames for which

$$h = \begin{pmatrix} 0 & -2E_m & 0 \\ -2E_m & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Denote by P^* the bundle of coframes of P'.

Let ψ be a real pure parallel spinor on (M,h). This spinor defines H-reductions P_H of P, P'_H of P' and P^*_H of P^*. Let $(\omega_+,\omega_-,\omega^{2m+1})$ be a local section in P^*_H, where $\omega_+=(\omega^1_+,\dots,\omega^m_+)$ and $\omega_-=(\omega^1_-,\dots,\omega^m_-)$. Since $\nabla^S\psi=0$ the structure equations can be written as

$$(d\omega_+,d\omega_-,d\omega^{2m+1})^\top=-\begin{pmatrix}\Phi & \Psi & \Theta\\ 0 & -\Phi^\top & 0\\ 0 & -2\Theta^\top & 0\end{pmatrix}\wedge(\omega_+,\omega_-,\omega^{2m+1})^\top, \tag{19}$$

where Φ and Ψ are $(m\times m)$-matrices of 1-forms with $\operatorname{tr}\Phi=0$ and $\Psi=-\Psi^\top$ and Θ is a vector of m 1-forms.

According to Proposition 5.4 ψ defines an m-dimensional distribution U. The annihilator $\mathrm{Ann}(U^\perp)\subset T^*M$ of $U^\perp$ is invariant under the action of H. In fact, H_+ acts as $SL(m,\mathbb{R})$ on $\mathrm{Ann}(U^\perp)$. If as above $(\omega_+,\omega_-,\omega^{2m+1})$ is a local section in P^*_H then $\mathrm{Ann}(U^\perp)$ is spanned by $\omega^1_-,\dots,\omega^m_-$. By (19) the system $\{\omega^1_-,\dots,\omega^m_-\}$ is integrable. Since, furthermore, $\omega=\omega^1_-\wedge\ldots\wedge\omega^m_-$ is a parallel and, hence, a closed m-form we can choose functions $y^1,\dots,y^m$ in a neighborhood of $x\in M$ such that $\mathrm{span}\{\omega^1_-,\dots,\omega^m_-\}=\mathrm{span}\{dy^1,\dots,dy^m\}$ and $\omega=dy^1\wedge\ldots\wedge dy^m$. Since H_+ acts as $SL(m,\mathbb{R})$ on $\mathrm{Ann}(U^\perp)$ it acts transitively on bases of $\mathrm{Ann}(U^\perp)$ with determinant ω. Hence, there exists a section $(\tilde\omega_+,\tilde\omega_-,\tilde\omega^{2m+1})$ in P^*_H with $\tilde\omega_-=dy:=(dy_1,\cdots,dy_m)$. Consequently, the bundle

$$P^*_N:=\{(\omega_+,\omega_-,\omega^{2m+1})\in P^*_H\mid\omega_-=dy\}$$

defined in a neighborhood of x is a (local) reduction of P^* to the kernel of the projection of H_+ to $SL(m,\mathbb{R})$, i.e. to N. Now we consider a local section $(\omega_+,dy,\omega^{2m+1})$ in P^*_N. By (19)

$d\omega^{2m+1}=-2\Theta^\top\wedge dy$ holds.

Hence we can choose a function z defined in a neighborhood of $x\in M$ such that $d\omega^{2m+1}=dz+p_\alpha dy^\alpha$ for some functions $p_1,\dots,p_m$.

By (15) we can find an element $\nu\in N$ such that

$$\nu\cdot(\omega_+,dy,\omega^{2m+1})^\top=(\omega'_+,dy,dz)^\top.$$

Hence, we can define the bundle

$$P^*_{\mathbb{R}^k}:=\{\ (\omega_+,dy,\omega^{2m+1})\in P^*_H\mid\omega^{2m+1}=dz\}$$

on this neighborhood. $P^*_{\mathbb{R}^k}$ is a (local) reduction of P^*_N with structure group

$$\mathbb{R}^k=\left\{\begin{pmatrix}E_m & B & 0\\ 0 & E_m & 0\\ 0 & 0 & 1\end{pmatrix}\ \middle|\ B\in\mathbb{R}(m),\ B^\top+B=0\right\}\subset N$$

where $k=(m-1)(m-2)/2$. Now let (ω_+,dy,dz) be a local section in $P^*_{\mathbb{R}^k}$. From (19) we know

$$\begin{array}{rcl} 0 = & ddy & = -\Phi^\top \wedge dy^\top \\ 0 = & ddz & = -2\Theta^\top \wedge dy^\top . \end{array}$$

Hence, the entries of Φ and Θ are contained in $\mathrm{span}\{dy^1, \ldots, dy^m\}$. Again by (19) we obtain $d\omega_+ = dy \wedge Q$ for a certain $(m \times m)$-matrix Q of 1-forms. Consequently, we can choose functions $dx_1, \ldots, dx_m$ such that

$$\omega_+ = dx + \tilde{g} \cdot dy$$

for $x := (x_1, \ldots, x_m)$ and a $(m \times m)$-matrix $\tilde{g}$ of functions depending on x, y and z. Moreover, multiplying $(\omega_+, dy, dz)^\top$ by a suitable element of $\mathbb{R}^k \subset N$ we obtain a coframe $(\eta_+, \eta_-, \eta_{2m+1}) = (dx + g \cdot dy, dy, dz)$ such that g is symmetric. We conclude

$$h = -4\,(\eta_+^1 \eta_-^1 + \ldots + \eta_+^{2m} \eta_-^{2m}) - \eta_{2m+1}^2,$$

where $\eta_+ = (\eta_+^1, \ldots, \eta_+^{2m})$ and $\eta_- = (\eta_-^1, \ldots, \eta_-^{2m})$ and, hence,

$$h = -dz^2 - 4dx_i dy^i - 4g_{ij} dy^i dy^j , \tag{20}$$

where $g = (g_{ij})_{i,j=1,\ldots,m}$.

Now let $g_{ij} = g_{ji}$, $(i, j = 1, \ldots, m)$ be functions on a connected open set U in $\mathbb{R}^{2m+1}$ endowed with coordinates (x, y, z), $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$ and define a metric h by (20). Then we can calculate the matrix θ of connection coefficients with respect to the coframe $(\eta_+, \eta_-, \eta_{2m+1}) = (dx + g \cdot dy, dy, dz)$ and its dual frame

$$\begin{array}{lcll} X_i^+ & = & \partial x_i & i = 1, \ldots, n \\ X_i^- & = & -g_{\alpha i} \partial x_\alpha + \partial y_i & i = 1, \ldots, n \\ \partial z . & & & \end{array}$$

We obtain

$$\begin{array}{rcl} \theta & = & \begin{pmatrix} (\eta_i^+(\nabla X_j^+))_{i,j=1,\ldots,m} & (\eta_i^+(\nabla X_j^-))_{i,j=1,\ldots,m} & \eta_i^+(\nabla \partial z)_{i=1,\ldots,m} \\ (\eta_i^-(\nabla X_j^+))_{i,j=1,\ldots,m} & (\eta_i^-(\nabla X_j^-))_{i,j=1,\ldots,m} & \eta_i^-(\nabla \partial z)_{i=1,\ldots,m} \\ (\eta_{2m+1}(\nabla X_j^+))_{j=1,\ldots,m} & (\eta_{2m+1}(\nabla X_j^-))_{j=1,\ldots,m} & \eta_{2m+1}(\nabla \partial z) \end{pmatrix} \\ & = & \begin{pmatrix} \Phi & \Psi & \Theta \\ 0 & -\Phi^\top & 0 \\ 0 & -2\Theta^\top & 0 \end{pmatrix} , \end{array}$$

where

$$\begin{array}{rcl} \Phi_{ij} & = & \dfrac{\partial g_{ki}}{\partial x_j} dy_k \\ \Psi_{ij} & = & \Big(g_{\alpha i} \dfrac{\partial g_{kj}}{\partial x_\alpha} - g_{\alpha j} \dfrac{\partial g_{ki}}{\partial x_\alpha} + \dfrac{\partial g_{ki}}{\partial y_j} - \dfrac{\partial g_{kj}}{\partial y_i} \Big) dy_k \\ \Theta_i & = & \dfrac{\partial g_{ki}}{\partial z} dy_k . \end{array} \tag{21}$$

Hence, the values of θ are contained in $\mathfrak{h}$ if and only if (17) is satisfied. If this is the case then $[q, u(1,1,\ldots,1)]$ with $\lambda(q) = (X_1^+, \ldots, X_m^+, X_1^-, \ldots, X_m^-, \partial z)$ is a parallel pure spinor. In particular, the holonomy group of the metric h defined by (16) and (17) is contained in H. Moreover, since (U,h) is space and time oriented it must be contained in H_+. Now we will prove that for a generic choice of $g = (g_{ij})$ the Lie algebra of the holonomy group equals $\mathfrak{h}$. In particular, in this case the holonomy group equals H_+, since H_+ is connected.

Let $R = d\theta + \theta \wedge \theta$ be the matrix of the components of the Riemannian curvature tensor, i.e.

$$
\begin{aligned}
R &= \begin{pmatrix} (\eta_i^+(R(\cdot,\cdot)X_j^+)) & (\eta_i^+(R(\cdot,\cdot)X_j^-)) & \eta_i^+(R(\cdot,\cdot)\partial z) \\ (\eta_i^-(R(\cdot,\cdot)X_j^+)) & (\eta_i^-(R(\cdot,\cdot)X_j^-)) & \eta_i^-(R(\cdot,\cdot)\partial z) \\ (\eta_{2m+1}(R(\cdot,\cdot)X_j^+)) & (\eta_{2m+1}(R(\cdot,\cdot)X_j^-)) & \eta_{2m+1}(R(\cdot,\cdot)\partial z) \end{pmatrix} \\
&= \begin{pmatrix} R_\Phi & R_\Psi & R_\Theta \\ 0 & -R_\Phi^\top & 0 \\ 0 & -2R_\Theta^\top & 0 \end{pmatrix},
\end{aligned}
$$

where

$$R_\Phi = d\Phi + \Phi \wedge \Phi, \mathbb{R}_\Psi = d\Psi + \Phi \wedge \Psi - \Psi \wedge \Phi^\top - 2\Theta \wedge \Theta^\top, \mathbb{R}_\Theta = d\Theta + \Phi \wedge \Theta.$$

We may assume that $0 \in U$. By the Ambrose-Singer holonomy theorem it suffices to prove that all matrices of the form $R(X,Y)$, $X, Y \in T_0U$, generate $\mathfrak{h}$ as a Lie algebra. Using (21) we obtain

$$
\begin{aligned}
R_{\Phi\, ij} &= \frac{\partial^2 g_{ki}}{\partial x_l \partial x_j} dx_l \wedge dy_k + \frac{\partial^2 g_{ki}}{\partial z \partial x_j} dz \wedge dy_k + \varphi_{ij} \\
R_{\Theta\, i} &= \frac{\partial^2 g_{ki}}{\partial x_l \partial z} dx_l \wedge dy_k + \frac{\partial^2 g_{ki}}{\partial z^2} dz \wedge dy_k + \theta_i \\
R_{\Psi\, ij} &= \Big(\frac{\partial g_{\alpha i}}{\partial x_l} \frac{\partial g_{kj}}{\partial x_\alpha} - \frac{\partial g_{\alpha j}}{\partial x_l} \frac{\partial g_{ki}}{\partial x_\alpha} + g_{\alpha i} \frac{\partial^2 g_{kj}}{\partial x_l \partial x_\alpha} - g_{\alpha j} \frac{\partial^2 g_{ki}}{\partial x_l \partial x_\alpha} \\
&\qquad + \frac{\partial^2 g_{ki}}{\partial x_l \partial y_j} - \frac{\partial^2 g_{kj}}{\partial x_l \partial y_i} \Big) dx_l \wedge dy_k \\
&\quad + \Big(\frac{\partial g_{\alpha i}}{\partial z} \frac{\partial g_{kj}}{\partial x_\alpha} - \frac{\partial g_{\alpha j}}{\partial z} \frac{\partial g_{ki}}{\partial x_\alpha} + g_{\alpha i} \frac{\partial^2 g_{kj}}{\partial z \partial x_\alpha} - g_{\alpha j} \frac{\partial^2 g_{ki}}{\partial z \partial x_\alpha} \\
&\qquad + \frac{\partial^2 g_{ki}}{\partial z \partial y_j} - \frac{\partial^2 g_{kj}}{\partial z \partial y_i} \Big) dz \wedge dy_k + \psi_{ij}
\end{aligned}
$$

where $\varphi_{ij}, \theta_i, \psi_{ij} \in \mathrm{span}\{dy_k \wedge dy_l \mid k, l = 1, \ldots, m\}$. Hence, if we choose

$$g_{ij} = \frac{1}{2} c_{ij}^{kl} x_k x_l + \frac{1}{2} c_{ij} z^2, \quad i, j = 1, \ldots, m$$

where c_{ij} and c_{ij}^{kl} are constants satisfying $c_{ij} = c_{ji}$, $c_{ij}^{kl} = c_{ij}^{lk} = c_{ji}^{kl}$ and the trace condition $c_{kj}^{kl} = 0$, then

$$\frac{\partial g_{ij}}{\partial x_j} = 0$$

holds for all $i = 1, \ldots, m$ and we have in $0 \in U$

$$R(\partial x_l, \partial y_k) = \begin{pmatrix} C_l^k & 0 & 0 \\ 0 & -(C_l^k)^\top & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{where } C_l^k = (c_{lj}^{ki})_{i,j=1,\ldots,m};$$

$$R(\partial z, \partial y_k) = \begin{pmatrix} 0 & 0 & C_k \\ 0 & 0 & 0 \\ 0 & -2C_k^\top & 0 \end{pmatrix}, \qquad \text{where } C_k = (c_{k1}, \ldots, c_{km})^\top.$$

Hence, if the constants c_{ij} and c_{ij}^{kl} are taken sufficiently generically, the space $\{R(X,Y) \mid X, Y \in T_0U\}$ contains all matrices of the form

$$\begin{pmatrix} A & 0 & X \\ 0 & -A^\top & 0 \\ 0 & -2X^\top & 0 \end{pmatrix}$$

with $\operatorname{tr} A = 0$. The assertion now follows from the fact that these matrices generate the Lie algebra $\mathfrak{h}$.

Moreover, it follows, that there is an open, dense condition on the 2-jet of the functions g_{ij} which ensures that the corresponding metric will have holonomy H_+.

Equation (18) can be verified by direct calculation. □

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