# On representations of star product algebras over cotangent spaces on Hermitian line bundles

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**Abstract**

For every formal power series $B$ of closed two-forms on a manifold $Q$ and every value of an ordering parameter $\kappa \in [0,1]$ we construct a concrete star product $\star^B_\kappa$ on the cotangent bundle $T^*Q$. The star product $\star^B_\kappa$ is associated to the symplectic form on $T^*Q$ given by the sum of the canonical symplectic form $\omega$ and the pull-back of $B$ to $T^*Q$. Deligne's characteristic class of $\star^B_\kappa$ is calculated and shown to coincide with the formal de Rham cohomology class of $B$ divided by $\lambda$. Therefore, every star product on $T^*Q$ corresponding to the canonical Poisson bracket is equivalent to some $\star^B_\kappa$. It turns out that every $\star^B_\kappa$ is strongly closed. In this paper we also construct and classify explicitly formal representations of the deformed algebra as well as operator representations given by a certain global symbol calculus for pseudodifferential operators on $Q$. Moreover, we show that the latter operator representations induce the formal representations by a certain Taylor expansion. We thereby obtain a compact formula for the WKB expansion.

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## Introduction

In recent years the theory of deformation quantization has gained more and more interest, not only because it provides a successfull approach to the quantization problem (see e.g. the work of BAYEN et al. [2], FEDOSOV [12] and KONTSEVICH [17]), but also because there are fascinating connections to analysis which culminate in a deformation theoretical proof of the Atiyah–Singer–Index Theorem (cf. FEDOSOV [12] and NEST, TSYGAN [18, 19]). The main object in the theory of deformation quantization is a formal deformation, the so-called star product $\star$, of the commutative algebra of all smooth complex-valued functions $\mathcal{C}^\infty(M)$ on a given Poisson manifold $(M, \{\ ,\ \})$, such that in first order of the deformation parameter $\lambda$ the $\star$-commutator equals a multiple of the Poisson bracket. More precisely, $\star$ comprises a $\mathbb{C}[[\lambda]]$-bilinear associative product

$$\star : \mathcal{C}^\infty(M)[[\lambda]] \times \mathcal{C}^\infty(M)[[\lambda]] \to \mathcal{C}^\infty(M)[[\lambda]], \quad (f,g) \mapsto f \star g = fg + \sum_{k=1}^{\infty} C_k(f,g)\lambda^k \qquad (0.1)$$

where the $C_k : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ are assumed to be bidifferential operators which satisfy $C_1(f,g) - C_1(g,f) = \mathrm{i}\{f,g\}$ and $C_k(1,g) = C_k(f,1) = 0$.

Although the existence and classification of star products on $M$ up to equivalence is by now fairly well-understood, there seem to be not very many results dealing with the problem of how to define and construct some kind of operator representations for $\star$. There are at least two principal choices. The first, more algebraic point of view is to stay within the category of modules over the ring $\mathbb{C}[[\lambda]]$ and regard representations as $\mathbb{C}[[\lambda]]$-linear morphisms of the algebra $(\mathcal{C}^\infty(M)[[\lambda]], \star)$ to the algebra of endomorphisms of some $\mathbb{C}[[\lambda]]$-module. The second, more analytic approach consists in linearly mapping $\mathcal{C}^\infty(M)$ to operators on a suitable function space such that the representation identity is satisfied 'asymptotically', a notion which depends on the topologies involved and which we shall make more precise further down.

The latter, analytic notion of representation has already been described in the work of FEDOSOV, where a certain subspace of $\mathcal{C}^\infty(M)[[\lambda]]$ is mapped to the space of trace-class operators on a complex Hilbert space (see e.g. his book [12, Chap. 7] for details). FEDOSOV also provides an existence proof for his asymptotic operator representations which requires a certain integrality condition for the symplectic form involved.

BORDEMANN and WALDMANN have set up a *formal* representation theory for deformation quantization (see [8] and Appendix A for a short outline). There, a $\mathbb{C}[[\lambda]]$-analogue of the GNS construction for $C^*$-algebras is introduced using $\mathbb{C}[[\lambda]]$-linear formally positive functionals $\mathcal{C}^\infty(M)[[\lambda]] \to \mathbb{C}[[\lambda]]$ which are shown to occur in a number of examples relevant to quantum theory.

The representation problem becomes more interesting in the class of cotangent bundles $\pi : T^*Q \to Q$ equipped with the canonical 2-form $\omega$. Here, the space $\mathcal{C}^\infty(T^*Q)[[\lambda]]$ is directly related to the space of symbols of pseudodifferential operators over the base manifold $Q$, and calls for a comparison of formal representations and concrete pseudodifferential operator realisations of star product algebras on $T^*Q$. To this end, PFLAUM [20, 21] has defined a family $\star_\kappa$ of star products on $T^*Q$ parametrized by a so-called ordering parameter $\kappa \in [0,1]$. This approach is based on a global symbol calculus using the Levi-Civita connection $\nabla$ of a fixed Riemannian metric on $Q$. The resulting star product $\star_0$, also called the *standard star product* associated to $\nabla$, arises as a particularly simple generalization of quantization by standard ordering for polynomial symbols on $\mathbb{R}^{2n}$ (cf. [14, Eq. (50.1)]). The products $\star_0$ and $\star_{1/2}$ have also been obtained by BORDEMANN–NEUMAIER–WALDMANN in [4, 5]. The starting point there is an arbitrary torsion-free connection $\nabla$ on $Q$ from which $\star_0$ is defined by means of a natural Fedosov–Weyl algebra construction. Moreover, $\star_{1/2}$ has been obtained from $\star_0$ by an explicit equivalence transformation $N_{1/2}$ which depends on a fixed volume density $\mu$ on $Q$. Additionally, formal GNS representations on the space $\mathcal{C}^\infty(Q)[[\lambda]]$

have been constructed. In Section 2 we will briefly recall the basic definitions and some necessary precise statements of the above results.

In the present article we study again star products over cotangent bundles $T^*Q$, but now equipped with a more general symplectic form of the form $\omega_B = \omega + \pi^* B$, where $B$ is a (formal) closed 2-form on the base manifold $Q$. Thus we deal with a space of two-forms $\omega_B$ inducing all of the second de Rham cohomology groups of $T^*Q$ and which includes the Hamiltonian mechanics inside a magnetic field described by $B$ (see e.g. [14, Chap. 3]). Hereby, the integral of $B$ over any closed two-dimensional surface in $Q$ can be interpreted as the magnetic monopole charge inside that surface.

Our first main objective is to give a concrete coordinate-free construction of star products on $T^*Q$ corresponding to the above symplectic forms, and to classify their equivalence classes.

Secondly, we are interested in an explicit construction of representations of the deformed algebras. Due to the possible nonexactness of $B$ we can no longer expect to represent them as formal or (pseudo-) differential operators on $\mathcal{C}^\infty(Q)[[\lambda]]$ resp. $\mathcal{C}^\infty(Q)$, but rather on the space of all smooth sections $\Gamma^\infty(L)$ of a complex line bundle $L$ over $Q$. Let us explain this in more detail. A *formal representation* of $(\mathcal{C}^\infty(T^*Q)[[\lambda]], \star)$ on the line bundle $L$ then is a $\mathbb{C}[[\lambda]]$-linear algebra homomorphism

$$\varrho : \mathcal{C}^\infty(T^*Q)[[\lambda]] \to \mathsf{End}\big(\Gamma^\infty(L)\big)[[\lambda]], \quad f \mapsto \sum_{k=0}^{\infty} \lambda^k \varrho_k(f), \quad f \in \mathcal{C}^\infty(T^*Q),$$

where the $\varrho_k(f)$ are differential operators on $\Gamma^\infty(L)$. Under an *operator representation* we understand a $\mathbb{C}$-linear assignment of a pseudodifferential operator $\mathsf{Op}_\hbar(a)$ on $\Gamma^\infty(L)$ for every $\hbar \in \mathbb{R}^+ := \{r \in \mathbb{R} \,|\, r > 0\}$ and every symbol $a \in \mathrm{S}^\infty(Q) \subset \mathcal{C}^\infty(T^*Q)$ such that for two symbols $a, b \in \mathrm{S}^\infty(Q)$ the asymptotic expansion

$$\mathsf{Op}_\hbar(a) \cdot \mathsf{Op}_\hbar(b) \sim \mathsf{Op}_\hbar(ab) + \sum_{k=1}^{\infty} \hbar^k \mathsf{Op}_\hbar(C_k(a,b))$$

holds. This means (cf. [16]) that there exists a decreasing sequence $m_j \to -\infty$ such that $\mathsf{Op}_\hbar(a) \cdot \mathsf{Op}_\hbar(b) - \big(\mathsf{Op}_\hbar(ab) + \sum_{k=1}^{j} \hbar^k \mathsf{Op}_\hbar(C_k(a,b))\big)$ is a pseudodifferential operator of order $m_j$ for all $j \in \mathbb{N}$. We now call the formal representation $\varrho$ *induced* by the operator representation $\mathsf{Op}_\hbar$, if for every $m \in \mathbb{N}$

$$\mathsf{Op}_\hbar(a) - \sum_{k=0}^{m} \hbar^k \varrho_k(a) = \hbar^{m+1} R_\hbar^m,$$

where $R_\hbar^m$ is a pseudodifferential operator which depends continuously on $\hbar$ in the weak operator topology. We will denote this situation by $\mathsf{Op}_\hbar(a) = \sum_{k=0}^{\infty} \hbar^k \varrho_k(a) \mod \hbar^\infty$ and can then interpret $\varrho(a)$ as a kind of Taylor expansion for $\mathsf{Op}_\hbar(a)$. Note that this definition differs from the one given by Fedosov: the pseudodifferential operators we expect from quantum mechanics such as Schrödinger operators will be in general unbounded and not of the trace-class.

We have obtained the following main results. Section 3 contains the heart of all later constructions. It consists in a quantization or in other words formal abelian deformation of the commutative group of all *fiber translating diffeomorphisms*. These are diffeomorphisms of $T^*Q$ of the form $\zeta_x \mapsto \zeta_x + A_0(x)$, where $A_0 \in \Gamma^\infty(T^*Q)$ is a one-form on $Q$ and $\zeta_x \in T_x^*Q$. The outcome will be an isomorphism $\mathcal{A}$ interpolating between different standard order (or more generally $\kappa$-ordered) star product algebras whose corresponding affine connections differ by the one-form $A_0$.

After fixing a torsion-free affine connection $\nabla$ and a volume density $\mu$ on $Q$ we provide in Section 4 for every value of the ordering parameter $\kappa \in [0,1]$ and for every formal power series of closed two-forms $B = B_0 + \sum_{k=1}^{\infty} B_k\,\lambda^k$ a star product $\star_\kappa^B$ corresponding to the above formal symplectic form $\omega_B = \omega + \pi^* B$. We call it the *$\kappa$-ordered star product* associated to $\nabla$, $\mu$ and $B$. Hereby we first create the star product $\star_\kappa$ out of the standard star product on $T^*Q$ by means of a global equivalence transformation $N_\kappa$. Then we use the above-mentioned quantized fiber translations along local potentials of $B$ to define local star products corresponding to $\omega_B$. It turns out that there are no obstructions to glue these local star products together, so we obtain a global deformation $\star_\kappa^B$ having the desired properties. Moreover, applying the results of BERTELSSON–CAHEN–GUTT [3], DELIGNE [10], FEDOSOV [12], GUTT–RAWNSLEY [13] and WEINSTEIN-XU [24] on the isomorphy classes of formal deformations over symplectic manifolds, we show that every star product on $(T^*Q, \omega)$ is equivalent to some $\star_\kappa^B$ the Deligne class of which is given by the de Rham class of $B$ divided by $\lambda$. Another nice property of the products $\star_\kappa^B$ is that they are strongly closed, i.e. that the integral over the Liouville form provides a formal *trace* for the algebra $\mathcal{C}^\infty(M)[[\lambda]]$.

In the following we construct *formal* representations for the star products $\star_\kappa^B$ explicitly. For exact $B = dA$ this can be done easily by means of the global quantized fiber translating map along $A$ (Section 5). Moreover, we arrive at a compact closed formula for the WKB-expansion, and show afterwards in Section 6 that $\star_0^B$ can be canonically represented on the space of half-densities where the 'magnetic field' $B$ is now a certain multiple of the trace of the curvature tensor. In Section 7 we succeed in classifying the isomorphy classes of representations associated to vector potentials of $B$ by de Rham cohomology classes modulo integral cohomology classes. The underlying one-forms induce additional terms in the covariant derivative on the trivial complex line bundle over $Q$ and thus may create nontrivial holonomy. In particular for vanishing $B$ this gives a possible interpretation of Aharonov–Bohm like effects as known from physics in deformation quantization.

For nonexact $B = \lambda B_1 + O(\lambda^2)$ we can explicitly construct formal representations of the deformed algebra on the space of all smooth sections of a complex line bundle $L$ over $Q$ with Chern class $[\frac{1}{2\pi} B_1]$ (Section 8). Hence, the two-form $B$ has to satisfy a certain integrality condition which is also known in geometric quantization.

In the rest of our paper we consider concrete pseudodifferential operator representations in the above defined sense. A considerable part of Section 10 is concerned about a global pseudodifferential and symbol calculus on manifolds which essentially goes back to the work of WIDOM [27, 26] (cf. also [22, 20]). We introduce a somewhat more general symbol calculus than there and construct for every $a \in \mathrm{S}^\infty(Q)$ and every value of the ordering parameter $\kappa \in [0,1]$ an operator $\mathsf{Op}_{\hbar,\kappa}(a)$ on $\Gamma^\infty(L)$ which can be interpreted as the *$\kappa$-ordered pseudodifferential operator* associated to $a$. As the main result of Section 10 we prove that the map $\mathsf{Op}_{\hbar,\kappa}$ comprises an operator representation for the star product $\star_\kappa^B$, and that the previously defined formal representations are in fact induced by $\mathsf{Op}_{\hbar,\kappa}$.

Finally, the algebras $\mathcal{C}^\infty(T^*Q)$, $\mathsf{End}_{\mathbb{C}[[\lambda]]}\big(\Gamma^\infty(L)[[\lambda]]\big)$ and $\Psi^\infty(L)$ all carry a $^*$-structure depending on the choice of the volume density $\mu$ and the Hermitian structure of $L$, which is given respectively by complex conjugation, formal adjoint, and (formal) operator adjoint. By spectral theoretical reasons and reasons of applicability in quantum mechanics one is particularly interested in star products and representations thereof which preserve the $^*$-structures or in other words which fulfill

$$\overline{f} \star \overline{g} = \overline{g \star f} \quad \text{and} \quad \varrho(\overline{f}) = \varrho(f)^* \tag{0.2}$$

for all $f, g \in \mathcal{C}^\infty(T^*Q)$. Now, this is the essential property of a star product of Weyl type as it is defined precisely in Section 2. Throughout our paper we have put much attention to this particularly important case and proved that for the value $\kappa = 1/2$ of the ordering parameter the star products $\star_\kappa^B$ and its representations have indeed the property (0.2). Again thinking of $C^*$-algebras we also asked the question in Section 11, whether the representations $\varrho_{1/2}$ and $\mathsf{Op}_{\hbar,1/2}$ are induced by a (formal) GNS construction. Surprisingly, we succeeded not only to interpret the formal representations of $\star^B_{1/2}$ as particular GNS representations, where each transversal section in $\Gamma^\infty(L)$ serves as a cyclic vector, but could also prove a kind of GNS theorem for $\mathsf{Op}_{\hbar,1/2}$.


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## 1 Preliminaries

In this section, which is more of a technical nature and for the convenience of the reader, we set up the notation and introduce some differential geometric material needed in the sequel.

Under $Q$ we will always understand a smooth $n$-dimensional manifold, the configuration space, and under $\pi : T^*Q \to Q$ its cotangent bundle. The canonical one-form on $T^*Q$ will be denoted by $\theta$, the induced symplectic form by $\omega = -d\theta$, and the Liouville form by $\Omega = \frac{1}{n!}(-1)^{[n/2]}\omega^n$. The equation $i_\xi\omega = -\theta$ defines a unique vector field $\xi$ on $T^*Q$ called the Liouville vector field. We regard $Q$ as naturally embedded in $T^*Q$ as its zero section and often write $\iota : Q \hookrightarrow T^*Q$ for that embedding. Sometimes we will make use of local coordinates $q^1, \dots, q^n$ on $Q$; the induced canonical coordinates for $T^*Q$ will be denoted by $q^1, \dots, q^n, p_1, \dots, p_n$. If not stated otherwise functions and tensor fields on $Q$ are assumed to be complex valued.

In this article $\mu$ will always be a volume density on $Q$, i.e. a smooth positive density $\mu \in \Gamma^\infty(|\bigwedge^n| T^*Q)$. The induced densities on $T_xQ$ and $T^*_xQ$, $x \in Q$ are denoted by $\mu_x$ resp. $\mu^*_x$. Moreover, $\nabla$ always means a torsion-free connection for $Q$. By $E \to Q$ we denote a complex vector bundle over $Q$ with Hermitian metric $\langle\cdot,\cdot\rangle^E$, and by $\nabla^E$ a metric connection on $E$.

The volume density $\mu$ and the affine connection $\nabla$ determine a one-form $\alpha_\mu \in \Gamma^\infty(T^*Q)$ by

$$\nabla_X \mu = \alpha_\mu(X)\,\mu, \qquad X \in \Gamma^\infty(TQ). \tag{1.1}$$

Similarly, the divergence $\mathsf{div}_\mu(X)$ of a smooth vector field $X$ with respect to $\mu$ can be defined by the equation

$$\mathcal{L}_X \mu = \mathsf{div}_\mu(X)\,\mu, \tag{1.2}$$

where $\mathcal{L}_X$ is the Lie derivative with respect to $X$. One checks easily that

$$\mathsf{div}_\mu(X) = \alpha_\mu(X) + \mathsf{tr}\,(Z \mapsto \nabla_Z X)\,.$$

This equation suggests to extend the divergence $\mathsf{div}_\mu$ to a differential operator $\mathsf{div}^E_\mu$ on the space $\Gamma^\infty(\bigvee TQ \otimes E)$ by setting

$$\begin{aligned}\mathsf{div}^E_\mu(X_1 \vee \cdots \vee X_k \otimes e) &= \sum_{l=1}^k X_1 \vee \cdots \overset{l}{\widehat{\cdots}} \cdots \vee X_k \otimes \big(\mathsf{div}_\mu(X_l)s + \nabla^E_{X_l}s\big) \\ &+ \sum_{\substack{j,l=1\\ j\neq l}}^k \big(\nabla_{X_j}X_l\big) \vee X_1 \vee \cdots \overset{l}{\widehat{\cdots}} \cdots \overset{j}{\widehat{\cdots}} \cdots \vee X_k \otimes e, \qquad X_1, \cdots, X_k \in \Gamma^\infty(TQ),\ e \in \Gamma^\infty(E).\end{aligned} \tag{1.3}$$

Hereby $\overset{l}{\widehat{\cdots}}$ means to leave out the $l$-th term. The for our purposes essential property of $\mathsf{div}^E_\mu$ is stated in the following lemma.

**Lemma 1.1** *Let $X$ be a smooth real vector field on $Q$. Then for any Hermitian vector bundle $E \to Q$ with metric connection $\nabla^E$ the (formal) adjoint of $\nabla^E_X$ with respect to the scalar product $\langle e, f\rangle_{E,\mu} = \int_Q \langle e, f\rangle^E \mu$, where $e, f \in \Gamma^\infty_{\mathrm{cpt}}(E)$, is given by*

$$\left(\nabla^E_X\right)^* e = -\nabla^E_X e - \mathsf{div}_\mu(X)e = -\mathsf{div}^E_\mu(X \otimes e). \tag{1.4}$$

Proof: By Stokes' Theorem we have

$$0 = \int_Q \mathcal{L}_X \left(\langle e, f\rangle^E \mu\right) = \int_Q \left(\left\langle \nabla^E_X e, f\right\rangle^E + \left\langle e, \nabla^E_X f\right\rangle^E + \langle e, f\rangle^E \,\mathsf{div}_\mu(X)\right)\mu,$$

hence the claim follows. □

A second differential operator on $\Gamma^\infty(\bigvee TQ\otimes E)$ needed in the sequel is given by the symmetrized covariant derivative $\mathsf{D} : \Gamma^\infty(\bigvee^k TQ \otimes E) \to \Gamma^\infty(\bigvee^{k+1} TQ \otimes E)$. It is defined by

$$\mathsf{D}T(X_1, ..., X_{k+1}) = \sum_{\sigma\in S_{k+1}} \left(\left(\nabla_{X_{\sigma(1)}} \otimes \mathsf{id} + \mathsf{id} \otimes \nabla^E_{X_{\sigma(1)}}\right) T\right) \left(X_{\sigma(2)}, \cdots, X_{\sigma(k+1)}\right), \tag{1.5}$$

where $T$ is a smooth section of $\bigvee^k TQ\otimes E$, and the $X_j$ are smooth vector fields on $Q$ (cf. Widom[27]).

Consider now the pulled back bundle $\pi^*E \to T^*Q$ then we denote by $\mathcal{P}^k(Q; E) \subset \Gamma^\infty(\pi^*E)$ the sections which are *polynomial of degree $k$ in the momenta*, i.e. those sections $\mathsf{p} : T^*Q \to \pi^*E$ having the property that for every $x \in Q$ the map $\mathsf{p}|_{T^*_xQ} : T^*_xQ \to E_x$ is a polynomial of degree $k$ with values in $E_x$. Moreover, let $\mathcal{P}(Q; E) = \bigoplus_{k=0}^\infty \mathcal{P}^k(Q; E)$. For the trivial bundle $E = Q \times \mathbb{C}$ we simply write $\mathcal{P}^k(Q)$ or $\mathcal{P}^k$ instead of $\mathcal{P}^k(Q; E)$ and note that $\mathsf{p} \in \mathcal{P}^k(Q) \subset C^\infty(T^*Q)$ if and only if $\mathcal{L}_\xi \mathsf{p} = k\mathsf{p}$.

To every contravariant symmetric tensor field $T \in \Gamma^\infty(\bigvee^k TQ)$ we assign a smooth function $J(T) \in \mathcal{P}^k$ by defining $J(T)(\zeta_x) = \frac{1}{k!}\langle \zeta_x \otimes \ldots \otimes \zeta_x, T_x\rangle$ for all $\zeta_x \in T^*_xQ$, $x \in Q$. Then $J : \Gamma^\infty(\bigvee TQ) \to \mathcal{P}(Q)$ turns out to be an isomorphism of $\mathbb{Z}$-graded commutative algebras.

Next let us denote by $\Delta^E_\mu : \Gamma^\infty(\pi^*E) \to \Gamma^\infty(\pi^*E)$ the generalized vector-valued Laplacian

$$\Delta^E_\mu = \sum_j \nabla^\pi_{(dq^j)^{\mathrm{v}}} \nabla^\pi_{(\partial_{q^j})^{\mathrm{h}}} + \nabla^\pi_{(\alpha_\mu)^{\mathrm{v}}}, \tag{1.6}$$

where $(\cdot)^{\mathrm{v}}$ resp. $(\cdot)^{\mathrm{h}}$ means the vertical resp. horizontal lift to $T^*Q$ induced by $\nabla$, and $\nabla^\pi$ is the pull-back of $\nabla^E$ to $\pi^*E$ via $\pi$. Note that the definition (1.6) is independent of the used chart.

**Lemma 1.2** *For $\mathsf{p} \in \mathcal{P}(Q; E)$ one has $\Delta^E_\mu \mathsf{p} = J\mathsf{div}^E_\mu J^{-1}\mathsf{p}$.*

Let us now define a few important differential operators on the cotangent bundle $T^*Q$. First assign to any covariant symmetric tensor field $\gamma \in \Gamma^\infty(\bigvee^k T^*Q)$ a fiberwise acting differential operator $\mathsf{F}(\gamma)$ on $T^*Q$ of order $k \in \mathbb{N}$ by defining

$$\left(\mathsf{F}(\gamma_1 \vee \cdots \vee \gamma_k) f\right)(\zeta_x) = \left.\frac{\partial^k}{\partial t_1 \cdots \partial t_k}\right|_{t_1=\ldots=t_k=0} f\left(\zeta_x + t_1\gamma_1(x) + \ldots + t_k\gamma_k(x)\right). \tag{1.7}$$

Hereby $\zeta_x \in T^*_xQ$, $\gamma_1, \ldots, \gamma_k \in \Gamma^\infty(T^*Q)$ and $f \in \mathcal{C}^\infty(T^*Q)$. Clearly (1.7) extends linearly to an injective algebra morphism from $\Gamma^\infty(\bigvee T^*Q)$ into the algebra of differential operators of $\mathcal{C}^\infty(T^*Q)$. Note that $\mathsf{F}(\gamma)$ commutes with $\mathsf{F}(\gamma')$ for any $\gamma, \gamma' \in \Gamma^\infty(\bigvee T^*Q)$.

In this article we will also make use of the Laplacian of a metric $G$ on $T^*Q$ which is naturally induced by the a priori chosen torsion-free covariant connection $\nabla$ on $Q$ (cf. [28] or [4, App. A]). The connection $\nabla$ induces a splitting of $T(T^*Q)$ into its horizontal and vertical subbundle. Now, if $X, Y \in \Gamma^\infty(TQ)$ and $\alpha, \beta \in \Gamma^\infty(T^*Q)$ are smooth sections, we denote by $X^{\mathrm{h}}, Y^{\mathrm{h}}$ the horizontal lifts of $X, Y$ and by $\alpha^{\mathrm{v}}, \beta^{\mathrm{v}}$ the vertical lifts of $\alpha, \beta$ to vector fields on $T^*Q$. Then $G$ is defined by $G(X^{\mathrm{h}}, Y^{\mathrm{h}}) := 0 =: G(\alpha^{\mathrm{v}}, \beta^{\mathrm{v}})$ and $G(X^{\mathrm{h}}, \alpha^{\mathrm{v}}) := 2\alpha(X) =: G(\alpha^{\mathrm{v}}, X^{\mathrm{h}})$. Obviously $G$ is an indefinite metric on $T^*Q$, so its Laplacian ('d'Alembertian') $\Delta$ is well-defined. One checks easily that in local canonical coordinates $\Delta$ has the form

$$\Delta = \sum_k \frac{\partial^2}{\partial q^k \partial p_k} + \sum_{j,k,l} p_l \left(\pi^* \Gamma^l_{jk}\right) \frac{\partial^2}{\partial p_j \partial p_k} + \sum_{j,k} \left(\pi^* \Gamma^j_{jk}\right) \frac{\partial}{\partial p_k}, \tag{1.8}$$

where $\Gamma^l_{jk}$ are the Christoffel symbols of $\nabla$. Consequently, the relation

$$\Delta^Q_\mu = \Delta + \mathsf{F}\left(\alpha_\mu\right) \tag{1.9}$$

holds, where $\Delta^Q_\mu$ is the generalized vector-valued Laplacian $\Delta^E_\mu$ associated to the trivial line bundle $E = Q \times \mathbb{C} \to Q$.

## 2 Deformation quantization

Let us now recall some essential notions and results from the theory of formal deformation quantization.

A star product $\star$ on a symplectic manifold $M$ is called of *Vey type*, if the order of every bidifferential operator $C_k$ in each of its variables is lower or equal to $k$. It is called of *Weyl type*, if $C_k(f,g) = (-1)^k C_k(g,f)$ and if in addition $C_k$ is real (resp. imaginary) for $k$ even (resp. odd). A star product of Weyl type statisfies in particular $\overline{f \star g} = \overline{g} \star \overline{f}$, i.e. complex conjugation is an algebra anti-automorphism. Hereby we have formally set $\overline{\lambda} = \lambda$. Two star products $\star$ and $\star'$ are called *equivalent* if there exists a formal power series $S = \mathsf{id} + \sum_{k=1}^\infty \lambda^k S_k$ of differential operators $S_k : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ such that $S1 = 1$ and $S(f \star g) = S(f) \star' S(g)$ for all $f, g \in \mathcal{C}^\infty(M)[[\lambda]]$.

In case $M$ is a cotangent bundle $T^*Q$ we impose some more structure on a reasonable star product, that means on a star product which should be relevant for quantum mechanics.

First one has a second type of 'ordering' for star products on $T^*Q$ besides the Weyl type: one calls a star product $\star$ of *standard order* or *normal order type*, if $(\pi^*u) \star f = (\pi^*u) f$ for all $u \in \mathcal{C}^\infty(Q)$ and $f \in \mathcal{C}^\infty(T^*Q)$.

Next let us consider the homogeneity operator $\mathsf{H} = \lambda \frac{\partial}{\partial \lambda} + \mathcal{L}_\xi$ on $\mathcal{C}^\infty(T^*Q)[[\lambda]]$. A star product on $T^*Q$ is said to be *homogeneous* (cf. [11]), if $\mathsf{H}$ is a derivation. This implies in particular the simple but important fact that $\mathcal{P}[\lambda]$ is a $\mathbb{C}[\lambda]$-algebra with respect to a homogeneous star product (cf. [4, Prop. 3.7]). Hence it is possible to substitute the formal parameter $\lambda$ by a real number $\hbar$. Speaking in analytical terms this means that in the homogeneous case the convergence problem for a star product can be trivially solved for functions polynomial in the momenta. Moreover, it has been shown in [5] that all homogeneous star products are strongly closed.

In [4] a homogeneous star product of standard order type $\star_0$ was constructed by a modified Fedosov procedure using the connection $\nabla$ on $Q$. Additionally, a canonical representation $\varrho_0$ of the deformed algebra by formal power series of differential operators on the space $\mathcal{C}^\infty(Q)[[\lambda]]$ of formal wave functions was constructed (cf. [4, Thm. 9]). Explicitly this representation is given by

$$
\begin{aligned}
\varrho_0(f)u = \iota^*(f \star_0 \pi^* u) &= \sum_{l=0}^{\infty} \sum_{j_1,\cdots,j_l} \frac{(-\mathrm{i}\lambda)^l}{l!} \iota^* \left( \frac{\partial^l f}{\partial p_{j_1} \cdots \partial p_{j_l}} \right) \frac{1}{l!} \left\langle \partial_{q^{j_1}} \otimes \cdots \otimes \partial_{q^{j_l}}, \mathsf{D}^l u \right\rangle \\
&= \sum_{l=0}^{\infty} \frac{(-\mathrm{i}\lambda)^l}{l!} \iota^* \left( \mathsf{F}\left(\mathsf{D}^l u\right)(f) \right) = \iota^* \mathsf{F}\left(\exp(-\mathrm{i}\lambda \mathsf{D}) u\right) f, \quad f \in \mathcal{C}^\infty(T^*Q)[[\lambda]],\ u \in \mathcal{C}^\infty(Q)[[\lambda]].
\end{aligned} \tag{2.1}
$$

After fixing a volume density $\mu \in \Gamma^\infty(|\bigwedge^n| T^*Q)$ on $Q$ one can pass for every value of an *ordering parameter* $\kappa \in [0,1]$ to a new star product $\star_\kappa$ on $\mathcal{C}^\infty(T^*Q)[[\lambda]]$ in the following way. Consider the one-form $\alpha_\mu$ and let $N_\kappa = N_\kappa(\alpha_\mu)$ be the operator

$$
N_\kappa(\alpha_\mu) = \exp\left(-\mathrm{i}\lambda\kappa \left(\Delta + \mathsf{F}\left(\alpha_\mu\right)\right)\right) = \exp\left(-\mathrm{i}\lambda\kappa\Delta_\mu^Q\right). \tag{2.2}
$$

Then one can define a scale of star products $\star_\kappa$ each equivalent to $\star_0$ by

$$
f \star_\kappa g = N_\kappa^{-1}\left(N_\kappa(f) \star_0 N_\kappa(g)\right). \tag{2.3}
$$

It turns out that all $\star_\kappa$ are homogeneous since $N_\kappa$ commutes with $\mathsf{H}$. Moreover, a representation $\varrho_\kappa$ of $\star_\kappa$ on $\mathcal{C}^\infty(Q)[[\lambda]]$ is given by $\varrho_\kappa(f) = \varrho_0(N_\kappa f)$. For the case $\kappa = 1/2$ one obtains the so-called *Weyl star product* $\star_{\mathrm{W}} = \star_{1/2}$ with corresponding *Schrödinger representation* $\varrho_{\mathrm{W}} = \varrho_{1/2}$ having the following two nice properties:

$$
\overline{f \star_{\mathrm{W}} g} = \overline{g} \star_{\mathrm{W}} \overline{f}, \tag{2.4}
$$

$$
\int_Q \overline{u}\, \varrho_{\mathrm{W}}(f) v\, \mu = \int_Q \overline{\varrho_{\mathrm{W}}(\overline{f}) u}\, v\, \mu, \qquad u, v \in \mathcal{C}^\infty_{\mathrm{cpt}}(Q)[[\lambda]]. \tag{2.5}
$$

In other words $\varrho_{\mathrm{W}}$ is a $^*$-representation of the (formal) $^*$-algebra $(\mathcal{C}^\infty(Q)[[\lambda]], \star_{\mathrm{W}})$ with complex conjugation as its $^*$-involution. In particular $\star_{\mathrm{W}}$ is of Weyl type. Finally, the representation $\varrho_{\mathrm{W}}$ coincides with the formal GNS representation (see Appendix A) with respect to the functional $f \mapsto \omega_\mu(f) = \int_Q \iota^* f\ \mu$ defined on the two-sided ideal $\mathcal{C}^\infty_Q(T^*Q)[[\lambda]]$, where $\mathcal{C}^\infty_Q(T^*Q) = \{f \in \mathcal{C}^\infty(T^*Q) \mid \operatorname{supp}(\iota^* f) \text{ is compact}\}$ (cf. [5, Prop. 4.2]).

# 3 Quantization of fiber translating diffeomorphisms

In this section we will provide a method for quantizing fiber translating diffeomorphisms. To this end we first compute explicitly the $\star_\kappa$-product of a homogeneous function of degree 0 on $T^*Q$ with an arbitrary function.

**Lemma 3.1** *Let $\gamma, \gamma' \in \Gamma^\infty(\bigvee TQ)$. Then*

$$
\left[\Delta + \mathsf{F}\left(\gamma'\right), \mathsf{F}\left(\gamma\right)\right] = \mathsf{F}\left(\mathsf{D}\gamma\right), \tag{3.1}
$$

$$
N_\kappa^{-1} \circ \mathsf{F}\left(\gamma\right) \circ N_\kappa = \mathsf{F}\left(\exp(\mathrm{i}\kappa\lambda \mathsf{D})\gamma\right). \tag{3.2}
$$

PROOF: For example by computation in local coordinate it is easy to see that (3.1) is true, if $\gamma$ has symmetric degree 0 or 1. Then (3.1) follows for arbitrary symmetric degree, since $\mathsf{F}$ is an algebra morphism and $\mathsf{D}$ a derivation. Eq. (3.2) is an immediate consequence of (3.1). □

**Proposition 3.2** *Let $f \in \mathcal{C}^\infty(T^*Q)[[\lambda]]$ and $u \in \mathcal{C}^\infty(Q)[[\lambda]]$. Then for all $\kappa \in [0,1]$ the equalities*

$$\pi^* u \star_\kappa f \;=\; \mathsf{F}\left(\exp(\mathrm{i}\kappa\lambda \mathsf{D})u\right) f, \tag{3.3}$$

$$f \star_\kappa \pi^* u \;=\; \mathsf{F}\left(\exp(-\mathrm{i}(1-\kappa)\lambda \mathsf{D})u\right) f \tag{3.4}$$

*hold and consequently also*

$$\mathrm{ad}_\kappa(\pi^* u) = N_\kappa^{-1} \mathrm{ad}_0(\pi^* u) N_\kappa = \mathsf{F}\left(\frac{\exp(\mathrm{i}\kappa\lambda\mathsf{D}) - \exp(-\mathrm{i}(1-\kappa)\lambda\mathsf{D})}{\mathsf{D}} du\right). \tag{3.5}$$

Proof: Due to the equivalence of $\star_\kappa$ and $\star_0$ via $N_\kappa$ and the fact that $N_\kappa \circ \pi^* = \pi^*$, it suffices by (3.2) to proof (3.3) and (3.4) for the case $\kappa = 0$. Assuming $\kappa = 0$ the relation (3.3) is trivially fulfilled since $\star_0$ is of standard order type. To prove (3.4) we can restrict to $f, \pi^* u \in \mathcal{C}^\infty(T^*Q)$ without $\lambda$-powers. Since $\star_0$ is homogeneous we have in order $\lambda^l$ at least $l$ vertical derivatives acting all on $f$ since $\pi^* u$ is homogeneous of degree 0. On the other hand $\star_0$ is of Vey type (see [4, Lem. 5]) whence there are at most and thus *exactly* $l$ derivatives in fiber direction acting on $f$. Moreover, the coefficient functions of the bidifferential operator in order $\lambda^l$ can thus only depend on $Q$ due to the homogeneity. Thus they are determined by their values at the zero section which are known from (2.1) proving the proposition. □

Note that the products (3.3) and (3.4) do not depend on $\alpha_\mu$ and thus not on the choice of the volume density $\mu$. Moreover, the above expressions for the commutators are well-defined formal power series in $\lambda$ with differential operators as coefficients. One observes that the star product $\star_1$ is of *anti-standard order type* in an obvious sense.

Next one makes the simple but crucial observation that $\mathrm{ad}_\kappa(\pi^* u)$ only depends on the one-form $du$. Thus Eq. (3.5) still makes sense as formal series of differential operators, if $du$ is replaced by an arbitrary formal one-form $A = A_0 + \lambda A_1 + O(\lambda^2) \in \Gamma^\infty(T^*Q)[[\lambda]]$. This motivates to define an operator $\delta_\kappa[A] : \mathcal{C}^\infty(T^*Q)[[\lambda]] \to \mathcal{C}^\infty(T^*Q)[[\lambda]]$ by

$$\delta_\kappa[A] = \mathsf{F}\left(\frac{\exp(\mathrm{i}\kappa\lambda\mathsf{D}) - \exp(-\mathrm{i}(1-\kappa)\lambda\mathsf{D})}{\mathsf{D}} A\right). \tag{3.6}$$

Now, the following result is immediate.

**Proposition 3.3** *The operator $\delta_\kappa[A]$ is a derivation of $\star_\kappa$ if and only if $dA = 0$, and an inner derivation if and only if $A$ is exact, that means if and only if $A = du$ with $u \in \mathcal{C}^\infty(Q)[[\lambda]]$. Additionally the relation*

$$\delta_\kappa[A] = N_\kappa^{-1} \circ \delta_0[A] \circ N_\kappa \tag{3.7}$$

*is true. Finally, one has in lowest order an expansion of the form $\delta_\kappa[A] = \mathrm{i}\lambda \mathsf{F}(A_0) + O(\lambda^2)$.*

In deformation quantization the Heisenberg equation of motion for a given Hamiltonian $H$ induces a one-parameter group of star product automorphisms provided the corresponding classical Hamiltonian vector field of $H$ has a complete flow (see e.g. [5, App. B]). Although the operators $\delta_\kappa[A]$ are derivations or in other words infinitesimals of star product automorphisms only in the case $dA = 0$, we consider time evolution equations induced by these maps, i.e. differential equations of the form

$$\frac{d}{dt} F(t) = \frac{\mathrm{i}}{\lambda} \delta_\kappa[A]\, F(t), \qquad F(0) = f, \tag{3.8}$$

where $f \in \mathcal{C}^\infty(T^*Q)[[\lambda]]$ is an initial value. First of all note that the form $A_0$ in the expansion $A = A_0 + \lambda A_1 + O(\lambda^2)$ induces a one-parameter group of diffeomorphisms $\phi_t : T^*Q \to T^*Q$ by

$$\phi_t(\zeta_x) = \zeta_x - t A_0(x). \tag{3.9}$$

Hereby we assume $A_0$ to be real. Then $\phi_t$ is the flow of the vector field determined by the differential operator $-\mathsf{F}(A_0)$. Note that $\phi_t$ is symplectic if and only if $dA_0 = 0$. Moreover, $\phi_t$ is a fiber translation, and

$$\phi_t^* \circ \mathsf{F}(\gamma) = \mathsf{F}(\gamma) \circ \phi_t^* \tag{3.10}$$

holds for all $\gamma \in \Gamma^\infty(\bigvee T^*Q)[[\lambda]]$. Using this fact we obtain the following result.

**Theorem 3.4** *Let $A \in \Gamma^\infty(T^*Q)[[\lambda]]$ with real $A_0$. Then for every value of the ordering parameter $\kappa$ and every initial value $f \in \mathcal{C}^\infty(T^*Q)[[\lambda]]$ Eq. (3.8) has a unique solution $F : \mathbb{R} \to \mathcal{C}^\infty(T^*Q)[[\lambda]]$. It is given by $F(t) = \mathcal{A}_\kappa(t)f$, where the operator $\mathcal{A}(t) = \mathcal{A}_\kappa(t) : \mathcal{C}^\infty(T^*Q)[[\lambda]] \to \mathcal{C}^\infty(T^*Q)[[\lambda]]$ is defined by*

$$\mathcal{A}_\kappa(t) = \phi_t^* \circ \exp\left(-t\mathsf{F}\left(\frac{\exp(\mathrm{i}\kappa\lambda\mathsf{D}) - \exp(-\mathrm{i}(1-\kappa)\lambda\mathsf{D})}{\mathrm{i}\lambda\mathsf{D}}(A) - A_0\right)\right), \tag{3.11}$$

*and has the following properties.*

*i.) $\mathcal{A}_\kappa(t)$ commutes with $\mathsf{F}(\gamma)$ and the pull-back $\psi^*$ for all $\gamma \in \Gamma^\infty(\bigvee TQ)$ and all fiber translating diffeomorphisms $\psi$.*

*ii.) For all times $t \in \mathbb{R}$ one has*

$$\mathcal{A}_\kappa(t) = N_\kappa^{-1} \circ \mathcal{A}_0(t) \circ N_\kappa. \tag{3.12}$$

*iii.) If $A' \in \Gamma^\infty(T^*Q)[[\lambda]]$ is another one-form fulfilling $A'_0 = \overline{A'_0}$, then the equality*

$$\mathcal{A}(t) \circ \mathcal{A}'(t') = \tilde{\mathcal{A}}(1) = \mathcal{A}'(t') \circ \mathcal{A}(t) \tag{3.13}$$

*holds for all $t, t' \in \mathbb{R}$, where $\mathcal{A}'$ resp. $\tilde{\mathcal{A}}$ denotes the evolution operator for the one-form $A'$ resp. $tA + t'A'$.*

*iv.) The map $t \mapsto \mathcal{A}_\kappa(t)$ comprises a one-parameter group of $\star_\kappa$-automorphisms if and only if $dA = 0$.*

*v.) If $A$ has the form $A = \lambda A_1$, then $\mathcal{A}(t)$ commutes with $\mathsf{H}$. In case $A = \overline{A}$, then $\mathcal{A}_{\mathrm{W}}(t) := \mathcal{A}_{1/2}(t)$ commutes with complex conjugation.*

Proof: It is a well-known fact that the Heisenberg equation (3.8) has a unique solution for any initial value if the flow of the vector field corresponding to the zeroth order is complete (see e.g. [5]). Thus it remains to show that (3.11) is indeed a solution. But this is easily obtained by differentiation. From (3.11) the first part follows directly as well as part ii.) which is no surprise due to (3.7). Part iii.) is computed using the above stated properties of fiber translating diffeomorphisms as well as the fact that all involved operators commute. Part iv.) is known in the general case (see e.g. [5]) but can also be verified explicitly using (3.11). Moreover if $A = \lambda A_1$ then $\frac{\mathrm{i}}{\lambda}\delta_\kappa[A]$ commutes with $\mathsf{H}$ and so does $\mathcal{A}_\kappa(t)$ which can be seen from (3.8) or explicitly from (3.11). Moreover (3.11) clearly implies that for $A = \overline{A}$ and $\kappa = 1/2$ the operator $\mathcal{A}_{\mathrm{W}}(t) = \mathcal{A}_{1/2}(t)$ commutes with complex conjugation. □

Note that in general $\mathcal{A}$ is not homogeneous, i.e. that it does not commute with $\mathsf{H}$. But nevertheless one has the relation $\mathcal{A} \circ \pi^* = \pi^*$ for arbitrary $A$, see also Lemma 4.3.

The above formulas suggest that we call $\mathcal{A}_\kappa(t)$ the *quantization* of the fiber translating diffeomorphism $\phi_t$ for the value $\kappa$ of the ordering parameter.

**Example 3.5** Let $X \in \Gamma^\infty(TQ)$ be a vector field. Then $J(X)$ is a function on $T^*Q$ linear in momentum. Hence one computes for $A \in \Gamma^\infty(T^*Q)[[\lambda]]$ that $\delta_\kappa[A]\, J(X) = \mathrm{i}\lambda\pi^*(A(X))$ and

$$\mathcal{A}_\kappa(t)J(X) = J(X) - t\pi^*(A(X)). \tag{3.14}$$

As we will need it later in this article let us finally prove the following factorization property for $N_\kappa(\alpha_\mu)$.

**Lemma 3.6** *Let $\alpha \in \Gamma^\infty(T^*Q)$ and $t \in \mathbb{R}$. Then*

$$N_\kappa(t\alpha) = \exp\left(-\mathrm{i}\kappa\lambda(\Delta + t\mathsf{F}(\alpha))\right) = \exp\left(t\mathsf{F}\left(\frac{\exp(-\mathrm{i}\kappa\lambda\mathsf{D}) - \mathsf{id}}{\mathsf{D}}\,\alpha\right)\right)\exp\left(-\mathrm{i}\kappa\lambda\Delta\right). \tag{3.15}$$

Proof: By a straightforward computation one finds

$$\frac{d}{dt}\exp\left(-\mathrm{i}\kappa\lambda(\Delta + t\mathsf{F}(\alpha))\right) = \exp\left(-\mathrm{i}\kappa\lambda(\Delta + t\mathsf{F}(\alpha))\right)\left(\frac{\mathsf{id} - \exp\left(\mathsf{ad}\left(\mathrm{i}\kappa\lambda(\Delta + t\mathsf{F}(\alpha))\right)\right)}{\mathsf{ad}\left(\Delta + t\mathsf{F}(\alpha)\right)}\left(\mathsf{F}(\alpha)\right)\right).$$

Using (3.1) and the fact that $\mathsf{F}(\gamma)$ commutes with $\mathsf{F}(\gamma')$ one concludes that the left-hand side of (3.15) satisfies the differential equation

$$\frac{d}{dt}\exp\left(-\mathrm{i}\kappa\lambda(\Delta + t\mathsf{F}(\alpha))\right) = \mathsf{F}\left(\frac{\exp\left(-\mathrm{i}\kappa\lambda\mathsf{D}\right) - \mathsf{id}}{\mathsf{D}}\,\alpha\right)\exp\left(-\mathrm{i}\kappa\lambda(\Delta + t\mathsf{F}(\alpha))\right).$$

Now, the right-hand side of (3.15) solves this differential equation with the correct initial condition. Hence the claim follows. □

# 4 Star products for cotangent bundles with magnetic fields

In this section we construct new star products from $\star_\kappa$ reflecting the presence of a *magnetic field* on the configuration space. Hereby we will describe the magnetic field by a formal power series $B \in \Gamma^\infty(\bigwedge^2 T^*Q)[[\lambda]]$ of *closed* two-forms on $Q$. If $B$ is exact, then $A \in \Gamma^\infty(T^*Q)[[\lambda]]$ with $B = dA$ is called a *vector potential* for $B$. Whenever $B$ is only closed but not exact we shall speak of a *magnetic monopole*. As before we only consider vector potentials and magnetic fields such that (at least) in order 0 the forms $A$ and $B$ are real.

Now let $\{O_j\}_{j\in I}$ be an open cover of $Q$ by contractible sets and $\{A^j\}_{j\in I}$ a family of local vector potentials for $B$, i.e. $A^j \in \Gamma^\infty(T^*O_j)[[\lambda]]$ and $dA^j = B|_{O_j}$ for all $j \in I$. Furthermore let $\mathcal{A}^j$ be the local time evolution operator which is induced by $A^j$ on $C^\infty(T^*O_j)[[\lambda]]$ at time $t = 1$ according to Theorem 3.4. Note that the operator $\mathcal{A}^j$ depends on $\kappa$. Now we define an associative product $\star_\kappa^j$ on $C^\infty(T^*O_j)[[\lambda]]$ by

$$f \star_\kappa^j g := \mathcal{A}^j\left(\left(\mathcal{A}^j\right)^{-1}(f) \star_\kappa \left(\mathcal{A}^j\right)^{-1}(g)\right), \qquad f, g \in C^\infty(T^*O_j)[[\lambda]]. \tag{4.1}$$

In case $O_j \cap O_k$ is not empty, Theorem 3.4 (iii) entails that $\left(\mathcal{A}^k\right)^{-1}\mathcal{A}^j$ is the evolution operator for $t = 1$ of the closed one-form $A^j|_{T^*(O_j\cap O_k)} - A^k|_{T^*(O_j\cap O_k)}$. Hence by Theorem 3.4 (iv) one concludes that $\left(\mathcal{A}^k\right)^{-1}\mathcal{A}^j$ is an automorphism of $(C^\infty(T^*(O_j \cap O_k))[[\lambda]], \star_\kappa)$. But then $f \star_\kappa^j g = f \star_\kappa^k g$ follows for all $f, g \in C^\infty(T^*(O_j \cap O_k))[[\lambda]]$. Therefore one can define a product $\star_\kappa^B$ on $C^\infty(T^*Q)[[\lambda]]$ by setting for all $j \in I$

$$f \star_\kappa^B g\big|_{T^*O_j} = f\big|_{T^*O_j} \star_\kappa^j g\big|_{T^*O_j}, \qquad f, g \in C^\infty(T^*Q). \tag{4.2}$$

The product $\star_\kappa^B$ now does not depend on the particular choice of the covering $\{O_j\}_{j\in I}$ and its local vector potentials $A^j$ but only on $B$, as the above argument shows.

**Theorem 4.1** *Let $B \in \Gamma^\infty(\bigwedge^2 T^*Q)[[\lambda]]$ be a formal series of closed two-forms $dB = 0$. Then $\star_\kappa^B$ has the following properties.*

*i.) $\star_\kappa^B$ is a star product on $\mathcal{C}^\infty(T^*Q)[[\lambda]]$ with respect to the symplectic form $\omega_B = \omega + \pi^* B_0$.*

*ii.) $\star_0^B$ is of standard order type.*

*iii.) One has $f \star_\kappa^B g = N_\kappa^{-1}\left(N_\kappa(f) \star_0^B N_\kappa(g)\right)$ for all $f, g \in \mathcal{C}^\infty(T^*Q)[[\lambda]]$.*

*iv.) If $B = \lambda B_1$ then $\star_\kappa^B$ is homogeneous; if $B = \overline{B}$ then $\star_{\mathrm{W}}^B := \star_{1/2}^B$ satisfies $\overline{f \star_{\mathrm{W}}^B g} = \overline{g} \star_{\mathrm{W}}^B \overline{f}$ for all $f, g$.*

PROOF: It suffices to prove the theorem locally that means to show it under the assumption $Q = O_j$. Now, in order 0 in $\lambda$ the evolution operator $\mathcal{A}^j$ is equal to the diffeomorphism $\phi_j : \zeta_x \mapsto \zeta_x - A^j(x)$ of $T^*O_j$. But as $\phi_j^*\omega = \omega + \pi^* B_0$, this entails (i). Since part (ii) is a local statement it is sufficient to consider functions having their support in some $\pi^{-1}(O_j)$. Then $\pi^*u \star_0{}^B f = \mathcal{A}^j(((\mathcal{A}^j)^{-1}\pi^*u) \star_0 ((\mathcal{A}^j)^{-1}f)) = \mathcal{A}^j(\pi^*u(\mathcal{A}^j)^{-1}f) = \pi^*uf$ since $\star_0$ is of standard order type and $\mathcal{A}^j \circ \pi^* = \pi^*$. Eq. (3.12) entails (iii). Due to theorem 3.4 (iv) the homogeneity of $\star_\kappa^{\lambda B_1}$ is obvious as well as the fact that complex conjugation is an anti-automorphism of $\star_{\mathrm{W}}^B$ for $B = \overline{B}$. $\square$

If $B$ is exact, then of course one can use any global one-form $A$ with $B = dA$ to define $\mathcal{A}_\kappa(1)$ and $\star_\kappa^B$ directly via Eq. (4.1). The crucial point however lies in the fact that $\star_\kappa^B$ can still be constructed, if $B$ is only closed but not necessarily exact.

Let us now state an immediate consequence from the preceding theorem concerning the star exponential which will ease some later calculations. The star exponential $\mathrm{Exp}(tH)$ of a function $H \in \mathcal{C}^\infty(T^*Q)[[\lambda]]$ with $t \in \mathbb{R}$ is defined as the unique solution of the differential equation $\frac{d}{dt}\mathrm{Exp}(tH) = H * \mathrm{Exp}(tH)$ with initial condition $\mathrm{Exp}(0) = 1$ (cf. [6, Lem. 2.2]).

**Corollary 4.2** *i.) For all $u, v \in \mathcal{C}^\infty(Q)[[\lambda]]$ one has*

$$(\pi^*u) \star_\kappa^B (\pi^*v) = \pi^*(uv). \tag{4.3}$$

*ii.) The star exponential $\mathrm{Exp}(\pi^*u)$ with respect to $\star_\kappa^B$ coincides with $\mathrm{e}^{\pi^*u}$.*

Next we compute Deligne's characteristic class of the star product $\star_\kappa^B$. This characteristic class has been introduced in [10] (see also [13] for a detailed exposition) and classifies in a functorial way the isomorphy classes of star products on a symplectic manifold $(M, \omega)$. It lies in the affine space $\frac{[\omega]}{\lambda} + H^2_{\mathrm{dR}}(M)[[\lambda]]$ and can be calculated by methods of Čech cohomology.

To relate the formal form $B$ to the charactersistic class of $\star_\kappa^B$ we first need the following lemma. Note that it involves the symmetric degree derivation $\deg_s$ which is defined by $\deg_s \gamma = k\gamma$ for all $\gamma \in \Gamma^\infty(\bigvee^k T^*Q)$.

**Lemma 4.3** *Let $\gamma \in \Gamma^\infty(\bigvee T^*Q)[[\lambda]]$ then*

$$[\mathsf{H}, \mathsf{F}(\gamma)] = \mathsf{F}\left(\left(\lambda \frac{\partial}{\partial \lambda} - \deg_s\right)\gamma\right) \tag{4.4}$$

*and for $A = \lambda A_1 + O(\lambda^2) \in \Gamma^\infty(T^*Q)[[\lambda]]$ we have*

$$\mathcal{A}_\kappa(t) \circ \mathsf{H} \circ \mathcal{A}_\kappa(-t) = \mathsf{H} + t\mathsf{F}\left(\frac{\exp(\mathrm{i}\kappa\lambda \mathsf{D}) - \exp(-\mathrm{i}(1-\kappa)\lambda\mathsf{D})}{\mathrm{i}\lambda\mathsf{D}} \sum_{r=2}^{\infty}(r-1)\lambda^r A_r\right) \tag{4.5}$$

PROOF: Since (4.4) is a relation between algebra morphisms and derivations it is sufficient to prove it for the case where $\gamma$ is a function or one-form on $Q$, which is a simple computation. But then (4.5) follows immediately from (3.11) and (4.4). $\square$

We consider the case $B = \lambda B_1 + O(\lambda^2) \in \Gamma^\infty(\bigwedge^2 T^*Q)[[\lambda]]$ in order to get only the Poisson bracket of $\omega$.

**Theorem 4.4** *The Deligne characteristic class of $\star_\kappa^B$ is $\frac{1}{\lambda}[\pi^* B] \in H^2_{\mathrm{dR}}(T^*Q)[[\lambda]]$ and hence for all equivalence classes of star products for $(T^*Q, \omega)$ there is a representative $\star_\kappa^B$.*

PROOF: The Deligne class can be computed in terms of Čech cohomology as e.g in [13, Sect. 5, 6]. The preceeding lemma enables us to find an explicit formula for the local $\lambda$-Euler derivations for the star products $\star_\kappa^B$, namely the conjugation of $\mathsf{H}$ by the local $\mathcal{A}_\kappa(1)$ which is the local equivalence transformation from $\star_\kappa$ to $\star_\kappa^B$. Hence the computation in [13] can be carried out explicitly and we find $\frac{1}{\lambda}[B]$ as Deligne class of $\star_\kappa^B$. Since the de Rham cohomology of $T^*Q$ is canonically isomorphic via $\pi^*$ to $H_{\mathrm{dR}}(Q)$ all equivalence classes are obtained this way. $\square$

As a first application of our explicit formulas relating $\star_\kappa^B$ and $\star_\kappa$ we will prove that all star products $\star_\kappa^B$ are strongly closed, that means integration over $T^*Q$ with respect to the corresponding volume form $\Omega_B = \frac{(-1)^{[n/2]}}{n!}\omega_B \wedge \cdots \wedge \omega_B$ is a trace functional for these star products (see [9] for definitions). Note that $\Omega_B = \Omega$ since $\pi^* B$ is a horizontal two-form.

**Lemma 4.5** *Let $A = A_0 + \lambda A_1 + O(\lambda^2) \in \Gamma^\infty(T^*Q)[[\lambda]]$ be a formal one-form with $A_0 = \overline{A_0}$ and let $\mathcal{A}_\kappa$ be the corresponding time development operator at $t = 1$. Then for all $f \in C^\infty_{\mathrm{cpt}}(T^*Q)[[\lambda]]$*

$$\int_{T^*Q} f\Omega = \int_{T^*Q} \left(\mathcal{A}_\kappa^{-1} f\right)\Omega. \tag{4.6}$$

PROOF: The fiber translating diffeomorphism induced by $A_0$ in $\mathcal{A}_\kappa$ is volume preserving since $\Omega_B = \Omega$. Thus we have only to deal with the differential operator part of $\mathcal{A}_\kappa$ as in (3.11) which is the identity in lowest order of $\lambda$ and a sum of homogeneous differential operator of negative degree in higher orders of $\lambda$. Due to [5, Lem. 8.1] these higher orders do not contribute to the integral. This proves the lemma. $\square$

**Proposition 4.6** *Let $B = B_0 + \lambda B_1 + O(\lambda^2) \in \Gamma^\infty(\bigwedge^2 T^*Q)[[\lambda]]$ be a closed two-form with real $B_0$. Then for every $\kappa \in [0,1]$ the star product $\star_\kappa^B$ is strongly closed.*

PROOF: We have to show that the integral over $T^*Q$ with respect to $\Omega$ of the commutator of two functions $f, g \in C^\infty_{\mathrm{cpt}}(T^*Q)[[\lambda]]$ vanishes. By a partition of unity argument we may assume that the supports of these functions are contained in a contractible open set $O$. Then choose a local formal one-form $A$ such that $dA = B|O$. Due to the definition of $\star_\kappa^B$ and Lemma. 4.5 we have

$$\int_{T^*Q} (f \star_\kappa^B g - g \star_\kappa^B f)\Omega = \int_{T^*O} \left((\mathcal{A}_\kappa^{-1} f) \star_\kappa (\mathcal{A}_\kappa^{-1} g) - (\mathcal{A}_\kappa^{-1} g) \star_\kappa (\mathcal{A}_\kappa^{-1} f)\right)\Omega = 0$$

since $\star_\kappa$ is homogeneous and thus strongly closed due to [5, Thm. 8.5]. This proves the proposition. $\square$

## 5 Global representations for $\star_\kappa^B$

After having constructed the product $\star_\kappa^B$ of observables for the case, where a magnetic field is present, we are now interested in representations of these algebras. It will turn out that, though the algebra structure depends only on $B$, the representations have to be constructed by explicit use of the vector potential $A$. Interesting physical effects like the Aharonov-Bohm effect are only visible by comparing the representations for varying vector potential $A$ but fixed $dA = B$.

Motivated by our investigations in [5, Sect. 7] we will consider first the case where $A \in \Gamma^\infty(T^*Q)[[\lambda]]$ is globally given with real $A_0$ and $B = dA$. Then one can embed the graph

$$L_{A_0} := \mathrm{graph}(A_0) = \{\zeta_x \in T^*Q \mid \zeta_x = A_0(x)\} \tag{5.1}$$

into $T^*Q$ via $\iota_{A_0} : L_{A_0} \hookrightarrow T^*Q$. Recall that $L_{A_0}$ is Lagrangian if and only if $dA_0 = 0$. Furthermore the relation $\phi_{-t}(\iota(Q)) = \iota_{tA_0}(L_{tA_0})$ holds for all $t \in \mathbb{R}$. Hence there exists an induced diffeomorphism $\Phi_t : Q \to L_{tA_0}$ fulfilling

$$\phi_{-t} \circ \iota = \iota_{tA_0} \circ \Phi_t. \tag{5.2}$$

We denote the pull-back $(\Phi_t^{-1})^*$ by $U_t : \mathcal{C}^\infty(Q)[[\lambda]] \to \mathcal{C}^\infty(L_{tA_0})[[\lambda]]$ and clearly have $U_t^{-1} = \Phi_t^*$. Using $U = U_1$ we can now define a representation $\tilde{\varrho}_\kappa^A$ of the algebra $\left(\mathcal{C}^\infty(T^*Q)[[\lambda]], \star_\kappa^B\right)$ on the space $\mathcal{C}^\infty(L_{A_0})[[\lambda]]$ of formal wave functions on $L_{A_0}$ by setting

$$\tilde{\varrho}_\kappa^A(f)u := \left(U \circ \varrho_\kappa\left(\mathcal{A}^{-1}f\right) \circ U^{-1}\right) u, \qquad u \in \mathcal{C}^\infty(L_{A_0}), \tag{5.3}$$

where $\mathcal{A} = \mathcal{A}_\kappa(1)$ is again the time development with respect to $A$. Obviously $\tilde{\varrho}_\kappa^A$ is a representation with respect to $\star_\kappa^B$. Moreover, $\tilde{\varrho}_\kappa^A$ is equivalent to the representation $\varrho_\kappa^A$ on $\mathcal{C}^\infty(Q)[[\lambda]]$ given by

$$\varrho_\kappa^A(f) := \varrho_\kappa\left(\mathcal{A}^{-1}f\right). \tag{5.4}$$

The operator $U$ then serves as an intertwiner between $\tilde{\varrho}_\kappa^A$ and $\varrho_\kappa^A$ and one has $\varrho_\kappa^A(f) = \varrho_0^A(N_\kappa f)$.

In order to compute $\varrho_0^A$ and thus the other representations $\varrho_\kappa^A$ and $\tilde{\varrho}_\kappa^A$ explicitly we first prove the following lemma.

**Lemma 5.1** *Let $\gamma \in \Gamma^\infty(\bigvee T^*Q)[[\lambda]]$ be a formal series of symmetric forms of degree $\geq 1$, and let $\gamma$ act on $\Gamma^\infty(\bigvee T^*Q)[[\lambda]]$ by left-multiplication. Then for all $c \in \mathbb{C}[[\lambda]]$ and $t \in \mathbb{R}$ the operator $\exp(c\mathsf{D} + t\gamma)$ satisfies the factorization property*

$$\exp(c\mathsf{D} + t\gamma) = \exp\left(t\frac{\exp(c\mathsf{D}) - \mathsf{id}}{c\mathsf{D}}\gamma\right)\exp(c\mathsf{D}). \tag{5.5}$$

PROOF: Since $c\mathsf{D} + t\gamma$ raises the symmetric degree at least by one, the operator $\exp(c\mathsf{D} + t\gamma)$ is well-defined as formal series in the symmetric degree. By commutativity of $\vee$ and the derivation property of $\mathsf{D}$ one has $[c\mathsf{D} + t\gamma, \gamma'] = c\mathsf{D}\gamma'$. Now, proceeding analogously as in the proof of Lemma 3.6 one finds

$$\frac{d}{dt}\exp(c\mathsf{D} + t\gamma) = \left(\frac{\exp(c\mathsf{D}) - \mathsf{id}}{c\mathsf{D}}\gamma\right)\exp(c\mathsf{D} + t\gamma).$$

The right-hand side of (5.5) now solves this differential equation, hence the claim follows. □

Next let us compute $\varrho_\kappa^A$ explicitly for the case where $A = \lambda A_1 + \lambda^2 A_2 + O(\lambda^3)$ starts in order one. This condition simplifies computations since in this case no diffeomorphism part is present. Nevertheless, it is the most important situation for the following constructions.

**Theorem 5.2** *Let $A = \lambda A_1 + \lambda^2 A_2 + O(\lambda^3) \in \Gamma^\infty(T^*Q)[[\lambda]]$ be a formal series of one-forms starting in order $\lambda$ and $B = dA$ the corresponding magnetic field. Then one has for all $f \in \mathcal{C}^\infty(T^*Q)[[\lambda]]$ and $u \in \mathcal{C}^\infty(Q)[[\lambda]]$*

$$\varrho_\kappa^A(f)u = \sum_{l=0}^{\infty} \frac{(-\mathrm{i}\lambda)^l}{l!} \sum_{j_1,\cdots,j_l} \iota^* \left( \frac{\partial^l N_\kappa f}{\partial p_{j_1} \cdots \partial p_{j_l}} \right) \frac{1}{l!} \left\langle \partial_{q^{j_1}} \otimes \cdots \otimes \partial_{q^{j_l}}, \left( \mathsf{D} + \frac{\mathrm{i}}{\lambda} A \right)^l u \right\rangle. \qquad (5.6)$$

Proof: Using the relation $\varrho_\kappa^A(f) = \varrho_0^A(N_\kappa f)$ it suffices to show Eq. (5.6) only for the case $\kappa = 0$, so let us fix $\kappa = 0$ for the rest of the proof. Now, the right-hand side of (5.6) is independent of the particular chosen canonical coordinate system, so it defines a global object indeed. Moreover, both sides are sheaf morphisms in $u$, so it is sufficient to show (5.6) over the domain $U \subset Q$ of a chart on $Q$, and to assume $f \in \mathcal{C}^\infty(T^*U)[[\lambda]]$ and $u \in \mathcal{C}^\infty(U)[[\lambda]]$. To achieve this consider first the space

$$\mathcal{T}(U) := \left( \prod_{k=0}^{\infty} \mathcal{C}^\infty(T^*U) \otimes_{\mathcal{C}^\infty(U)} \Gamma^\infty \left( \bigvee^k T^*U \right) \right) [[\lambda]].$$

The space $\mathcal{T}(U)$ carries a natural commutative product $m$ given by the pointwise product of functions in the first and the $\vee$-product in the second tensor factor. Using canonical coordinates over $T^*U$ one can deform $m$ to a product $\star_{\mathrm{loc}}$ by

$$F \star_{\mathrm{loc}} G := m \circ \exp \left( -\mathrm{i}\lambda \sum_j \partial_{p_j} \otimes i_{\mathrm{s}}(\partial_{q^j}) \right) (F \otimes G), \qquad F, G \in \mathcal{T}(U).$$

Hereby $i_{\mathrm{s}}(\partial_{q^i})G$ denotes the insertion of $\partial_{q^i}$ in the symmetric part of $G$. Since $\partial_{p_j}$ and $i_{\mathrm{s}}(\partial_{q^j})$ are commuting $\mathcal{C}^\infty(U)$-linear derivations of $\mathcal{T}(U)$, $\star_{\mathrm{loc}}$ is indeed an associative product deforming $m$. Let us denote by $\sigma : \mathcal{T}(U) \to \mathcal{C}^\infty(T^*U)[[\lambda]]$ the projection on the factor of symmetric degree zero. Denoting the right-hand side of (5.6) shortly by $\pi(f)\, u$ we obtain due to Lemma 5.1 and the particular form of $\star_{\mathrm{loc}}$

$$\begin{aligned} \pi(f)\, u &= \iota^* \sigma \left( f \star_{\mathrm{loc}} \exp \left( \mathsf{D} + \frac{\mathrm{i}}{\lambda} A \right) u \right) = \iota^* \sigma \left( f \star_{\mathrm{loc}} \left( \exp \left( \frac{\exp \mathsf{D} - \mathsf{id}}{\mathsf{D}} \frac{\mathrm{i}}{\lambda} A \right) \exp(\mathsf{D}) u \right) \right) \\ &= \iota^* \sigma \left( f \star_{\mathrm{loc}} \exp \left( \frac{\exp \mathsf{D} - \mathsf{id}}{\mathsf{D}} \frac{\mathrm{i}}{\lambda} A \right) \star_{\mathrm{loc}} \exp(\mathsf{D}) u \right) \\ &= \iota^* \sigma \left( \sigma \left( f \star_{\mathrm{loc}} \exp \left( \frac{\exp \mathsf{D} - \mathsf{id}}{\mathsf{D}} \frac{\mathrm{i}}{\lambda} A \right) \right) \star_{\mathrm{loc}} \exp(\mathsf{D}) u \right). \end{aligned}$$

Now compute by Taylor expansion in the right factor

$$\sigma \left( f \star_{\mathrm{loc}} \frac{\exp \mathsf{D} - \mathsf{id}}{\mathsf{D}} \frac{\mathrm{i}}{\lambda} A \right) = \mathsf{F} \left( \frac{\mathsf{id} - \exp(-\mathrm{i}\lambda \mathsf{D})}{\mathrm{i}\lambda \mathsf{D}} A \right) f.$$

Since $\partial_{p_j} \frac{\exp \mathsf{D} - \mathsf{id}}{\mathsf{D}} \frac{\mathrm{i}}{\lambda} A = 0$ we can use $\star_{\mathrm{loc}}$-products instead of $\vee$-products in the exponential function. Hence, due to Theorem 3.4 and the fact that $A$ starts in order $\lambda$ we obtain

$$\sigma \left( f \star_{\mathrm{loc}} \exp \left( \frac{\exp \mathsf{D} - \mathsf{id}}{\mathsf{D}} \frac{\mathrm{i}}{\lambda} A \right) \right) = \exp \left( \mathsf{F} \left( \frac{\mathsf{id} - \exp(-\mathrm{i}\lambda \mathsf{D})}{\mathrm{i}\lambda \mathsf{D}} A \right) \right) f = \mathcal{A}_0^{-1} f.$$

Using (2.1) this finally leads to

$$\pi(f)\, u = \iota^* \sigma \left( \mathcal{A}_0^{-1} f \star_{\mathrm{loc}} \exp(\mathsf{D}) u \right) = \varrho_0(\mathcal{A}_0^{-1} f) u = \varrho_0^A(f) u.$$

$\square$

The interpretation of these formulas is that we replaced the covariant derivative $\nabla_X$ acting as Lie derivative $\mathcal{L}_X$ on functions $u \in \mathcal{C}^\infty(Q)$ by the covariant derivative $\mathcal{L}_X + \frac{\mathrm{i}}{\lambda} A(X)$ where $A(X)$ acts as left-multiplication on $u$. Thus we endowed the trivial $\mathbb{C}$-bundle whose sections are the functions on $Q$ with a non-trivial covariant derivative by adding a *formal connection one-form* $\frac{\mathrm{i}}{\lambda} A$ which corresponds physically to the 'minimal coupling' description and thus justifies our interpretation of $A$ as 'vector potential'.

Now we consider the Weyl ordered star product $\star_{\mathrm{W}}^B$ for a real and exact magnetic field $B = \overline{B} = dA$ with a real vector potential $A \in \Gamma^\infty(T^*Q)[[\lambda]]$. In this case we already mentioned that $\star_{\mathrm{W}}^B$ satisfies $\overline{f \star_{\mathrm{W}}^B g} = \overline{g} \star_{\mathrm{W}}^B \overline{f}$ for all $f, g \in \mathcal{C}^\infty(T^*Q)[[\lambda]]$. Whence it is natural to seek for positive $\mathbb{C}[[\lambda]]$-linear functionals on the $^*$-algebra $\left(\mathcal{C}^\infty(T^*Q)[[\lambda]], \star_{\mathrm{W}}^B\right)$ such that the induced GNS representation is $\varrho_{\mathrm{W}}^A$ resp. $\tilde{\varrho}_{\mathrm{W}}^A$. Analogously to [5, Sect. 7] and guided by the general idea of [5, Prop. 5.1] we denote the subspace of those functions $f \in \mathcal{C}^\infty(T^*Q)$ such that $\mathrm{supp}(\iota_{A_0}^* f)$ is compact in $L_{A_0}$ by $\mathcal{C}^\infty_{L_{A_0}}(T^*Q)$. Since $\star_{\mathrm{W}}^B$ is a differential star product, $\mathcal{C}^\infty_{L_{A_0}}(T^*Q)[[\lambda]]$ is a two-sided ideal which is stable under complex conjugation. Denoting by $\mathcal{A}_{\mathrm{W}}$ the time evolution operator with respect to $A$ at time $t = 1$ the relation

$$\mathcal{A}_{\mathrm{W}} \left(\mathcal{C}^\infty_Q(T^*Q)[[\lambda]]\right) = \mathcal{C}^\infty_{L_{A_0}}(T^*Q)[[\lambda]]$$

holds. Hence, the $\mathbb{C}[[\lambda]]$-linear functional

$$\omega_A = \omega_\mu \circ \mathcal{A}_{\mathrm{W}}^{-1} : \mathcal{C}^\infty_{L_{A_0}}(T^*Q)[[\lambda]] \to \mathbb{C}[[\lambda]]$$

is well-defined. Moreover, it is positive with respect to $\star_{\mathrm{W}}^B$, since for all $f \in \mathcal{C}^\infty_{L_{A_0}}(T^*Q)[[\lambda]]$

$$\omega_A(\overline{f} \star_{\mathrm{W}}^B f) = \omega_\mu \left(\mathcal{A}_{\mathrm{W}}^{-1}(\overline{f} \star_{\mathrm{W}}^B f)\right) = \omega_\mu \left(\overline{\mathcal{A}_{\mathrm{W}}^{-1} f} \star_{\mathrm{W}} \mathcal{A}_{\mathrm{W}}^{-1} f\right) \geq 0$$

due to the reality of $\mathcal{A}_{\mathrm{W}}$ and the definition of $\star_{\mathrm{W}}^B$. Explicitly, $\omega_A$ is given by

$$\omega_A(f) = \int_{L_{A_0}} \iota_{A_0}^* \exp\left(\mathsf{F}\left(\frac{\sinh\left(\frac{\mathrm{i}\lambda}{2}\mathsf{D}\right)}{\frac{\mathrm{i}\lambda}{2}\mathsf{D}}(A) - A_0\right)\right)(f)\, \mu_{A_0} \tag{5.7}$$

according to (3.12) where the positive density $\mu_{A_0} \in \Gamma^\infty(|\bigwedge^n| T^*L_{A_0})$ is given by $\mu_{A_0} = \left(\Phi^{-1}\right)^* \mu$ with $\Phi$ as in (5.2) at $t = +1$. Clearly $\omega_A(\overline{f} \star_{\mathrm{W}}^B f) = 0$ if and only if $\mathcal{A}_{\mathrm{W}}^{-1} f \in \mathcal{J}_\mu$ is an element of the Gel'fand ideal $\mathcal{J}_\mu$ of $\omega_\mu$. Thus we have the following proposition as slight generalization of [5, Lem. 7.3]. Note that due to [8, Cor. 1] the GNS representation extends to the whole algebra.

**Proposition 5.3** *For real $A \in \Gamma^\infty(T^*Q)[[\lambda]]$ the Gel'fand ideal $\mathcal{J}_A$ of $\omega_A$ is given by $\mathcal{A}_{\mathrm{W}}(\mathcal{J}_\mu)$. The GNS representation pre-Hilbert space $\mathcal{C}^\infty_{L_{A_0}}(T^*Q)[[\lambda]]/\mathcal{J}_A$ is canonically isometric to $\mathcal{C}^\infty_{\mathrm{cpt}}(L_{A_0})[[\lambda]]$ endowed with the $\mathbb{C}[[\lambda]]$-valued Hermitian product*

$$\langle u, v\rangle_A = \int_{L_{A_0}} \overline{u}\, v\, \mu_{A_0}, \qquad u, v \in \mathcal{C}^\infty_{\mathrm{cpt}}(L_{A_0})[[\lambda]]. \tag{5.8}$$

*A $\mathbb{C}[[\lambda]]$-linear unitary intertwiner is given by $\psi_f \mapsto {\Phi^{-1}}^* \iota^* N_{1/2}(\alpha_\mu) \mathcal{A}_{\mathrm{W}}^{-1}(f)$, where $\psi_f$ denotes the equivalence class of $f \in \mathcal{C}^\infty_{L_{A_0}}(T^*Q)[[\lambda]]$ in the GNS pre-Hilbert space. The intertwiner has inverse $v \mapsto \psi_{\pi^*\Phi^* v}$. Moreover, the GNS representation induced by $\omega_A$ on $\mathcal{C}^\infty_{\mathrm{cpt}}(L_{A_0})[[\lambda]]$ coincides with $\tilde{\varrho}_{\mathrm{W}}^A$ which on $\mathcal{C}^\infty_{\mathrm{cpt}}(Q)[[\lambda]]$ is unitarily equivalent to $\varrho_{\mathrm{W}}^A$.*

**Corollary 5.4** *For all $\kappa \in [0,1]$ the scalar product $\left\langle u, \varrho_\kappa^A(f)v\right\rangle_A$ is given by*

$$\left\langle u, \varrho_\kappa^A(f)v\right\rangle_A = \int_{L_{A_0}} \overline{u}\,\varrho_\kappa^A(f)v\,\mu_{A_0} = \int_{L_{A_0}} \overline{\varrho_\kappa^A\left(N_{1-2\kappa}\overline{f}\right)u}\,v\,\mu_{A_0} = \left\langle \varrho_\kappa^A(N_{1-2\kappa}\overline{f})u, v\right\rangle_A. \tag{5.9}$$

As an application of the above considerations and formulas we will briefly examine the WKB expansion scheme as already discussed in [5, Sect. 7] and [7]. The starting point is a real-valued Hamiltonian function $H \in \mathcal{C}^\infty(T^*Q)$ and a projectable Lagrangean submanifold $\iota_{A_0} : L_{A_0} \hookrightarrow T^*Q$ determined by the graph of a closed real one-form $A_0 \in \Gamma^\infty(T^*Q)$ such that the Hamilton–Jacobi equation

$$H \circ A_0 = E \quad \text{resp.} \quad H|_{L_{A_0}} = E \tag{5.10}$$

is satisfied for some energy value $E \in \mathbb{R}$. Then we have shown in [5, Thm. 7.4] that the eigenvalue problem

$$\varrho_{\mathrm{W}}\left(\mathcal{A}_{\mathrm{W}}^{-1} H\right) u = Eu \tag{5.11}$$

for an eigenfunction $u \in \mathcal{C}^\infty_{\mathrm{cpt}}(Q)[[\lambda]]$ or better an eigendistribution $u \in \mathcal{C}^\infty_{\mathrm{cpt}}(Q)'[[\lambda]]$ results in the usual transport equations of the WKB expansion by simple expansion of (5.11) in powers of $\lambda$. Here $\mathcal{A}_{\mathrm{W}}$ is the time development operator corresponding to $A_0$ at time $t = +1$ which is in this case an automorphism of $\star_{\mathrm{W}}$ since $dA_0 = 0$. In [5] we factored $\mathcal{A}_{\mathrm{W}}^{-1}$ as $(\phi^{-1})^* \circ T_{-1}$ with some formal series of differential operators $T_{-1}$ which were given by a recursion formula using iterated integrals. Now we can determine $T_{-1}$ explicitly using Theorem 3.4 resulting in

$$\mathcal{A}_{\mathrm{W}}^{-1} = \left(\phi^{-1}\right)^* \circ \exp\left(\mathsf{F}\left(\frac{\sinh\left(\frac{\mathrm{i}\lambda}{2}\mathsf{D}\right)}{\frac{\mathrm{i}\lambda}{2}}(A_0) - A_0\right)\right). \tag{5.12}$$

Thus the expansion of $T_{-1}$ in powers of $\lambda$ is explicitly known and can be used for a more explicit WKB expansion. Note that the result of [5, Lem. 7.5] implying $\mathcal{A}_{\mathrm{W}}^{-1}H = (\phi^{-1})^* H$ for $H \in \mathcal{P}(Q)[[\lambda]]$ at most quadratic in the momenta follows now directly form (5.12) by 'counting degrees'.

# 6 Standard and Weyl representations on half-densities

As another application we will discuss how to carry over the representations $\varrho_\kappa$ to representations on half-densities, i.e. on sections of the bundle $\Gamma^\infty(|\bigwedge^n|^{\frac{1}{2}}\, T^*Q)$. Since the covariant derivative $\nabla$ acts on half-densities as well, we define the standard order representation $\pi_0$ of $\mathcal{C}^\infty(T^*Q)[[\lambda]]$ on $\Gamma^\infty(|\bigwedge^n|^{\frac{1}{2}}\, T^*Q)[[\lambda]]$ by

$$\pi_0(f)\nu := \sum_{l=0}^\infty \frac{(-\mathrm{i}\lambda)^l}{l!} \sum_{j_1,\cdots,j_l} \iota^*\left(\frac{\partial^l f}{\partial p_{j_1}\cdots\partial p_{j_l}}\right)\frac{1}{l!}\left\langle \partial_{q^{j_1}}\otimes\cdots\otimes\partial_{q^{j_l}}, \mathsf{D}^l\nu\right\rangle, \tag{6.1}$$

where $\mathsf{D}^l\nu$ is viewed as a smooth section of $\bigvee^l T^*Q \otimes |\bigwedge^n|^{\frac{1}{2}}\, T^*Q$. Clearly (6.1) is globally defined, and we will show in the sequel that this ad hoc quantization rule is in fact a representation of a $\star_0^B$-product algebra for a particular magnetic field $B$.

In order to compare $\pi_0$ with $\varrho_0$ we choose the square root $\mu^{\frac{1}{2}}$ of the positive density $\mu$ as a trivializing section for the half-density bundle. Note that $\nabla_X \mu^{\frac{1}{2}} = \frac{1}{2}\alpha_\mu(X)\,\mu^{\frac{1}{2}}$ for all $X \in \Gamma^\infty(TQ)$. Now one calculates easily

$$\mathsf{D}(\gamma\otimes\mu^{1/2}) = \left(\mathsf{D}\gamma + \frac{1}{2}\,\alpha_\mu \vee \gamma\right)\otimes\mu^{\frac{1}{2}}, \qquad \gamma \in \Gamma^\infty(\textstyle\bigvee^l T^*Q). \tag{6.2}$$

This implies the following result.

**Theorem 6.1** *Let $f \in C^\infty(T^*Q)[[\lambda]]$ and $\nu = v\mu^{\frac{1}{2}} \in \Gamma^\infty(|\bigwedge^n|^{\frac{1}{2}} T^*Q)[[\lambda]]$ with $v \in C^\infty(Q)[[\lambda]]$. If one sets $A = -\frac{\mathrm{i}\lambda}{2}\alpha_\mu \in \Gamma^\infty(T^*Q)[[\lambda]]$, the relation*

$$\pi_0(f)\nu = \left(\sum_{l=0}^{\infty} \frac{(-\mathrm{i}\lambda)^l}{l!} \sum_{j_1,\dots,j_l} \iota^* \left(\frac{\partial^l f}{\partial p_{j_1}\cdots\partial p_{j_l}}\right) \frac{1}{l!} \left\langle \partial_{q^{j_1}} \otimes \cdots \otimes \partial_{q^{j_l}}, \left(\mathsf{D} + \frac{1}{2}\alpha_\mu\right)^l v \right\rangle\right) \mu^{\frac{1}{2}} \quad (6.3)$$

*holds, which implies $\pi_0(f)\nu = \left(\varrho_0^A(f)v\right)\mu^{\frac{1}{2}}$.*

Hence, the (non-canonical) trivialization of $\Gamma^\infty(|\bigwedge^n|^{\frac{1}{2}} T^*Q)[[\lambda]]$ induced by $\mu$ intertwines the (canonical) representation $\pi_0$ of the (canonical) star product $\star_0^B$ with the (non-canonical) representation $\varrho_0^A$ associated to the purely imaginary vector potential $A$ and $B = dA$. Hereby *canonical* means depending only on $\nabla$, whereas *non-canonical* means depending on $\mu$ as well. Note that by definition of $\alpha_\mu$ the relation $\mathrm{tr}R = -d\alpha_\mu$ is true, where $\mathrm{tr}R$ denotes the trace of the curvature tensor $R$ of $\nabla$ (cf. [5, Eqn. (21)]). So the 'magnetic field' fulfills

$$B = \frac{\mathrm{i}\lambda}{2}\mathrm{tr}R, \quad (6.4)$$

hence $\star_0^B$ does not depend on the particular choice of $\mu$ indeed, but only on $\nabla$. Thus we will denote $\star_0^B$ for this particular $B$ also by $\bullet_0$.

**Corollary 6.2** *For $B = \frac{\mathrm{i}\lambda}{2}\mathrm{tr}R$ the star product $\bullet_0 = \star_0^B$ has a canonical representation of standard order type on half-densities $\Gamma^\infty(|\bigwedge^n|^{\frac{1}{2}} T^*Q)[[\lambda]]$. Explicitly it is given by (6.3). Moreover, $\bullet_0$ is homogeneous and equivalent to $\star_0$ via the time evolution operator $\mathcal{A}_0(1)$ corresponding to $A = -\frac{\mathrm{i}\lambda}{2}\alpha_\mu$. The two star products $\star_0$ and $\bullet_0$ of standard order type coincide if and only if the connection $\nabla$ is unimodular, i.e. if $\mathrm{tr}R = 0$. In this case $\pi_0$ is a representation of $\star_0$ on half-densities.*

Note that in general $\star_0 \neq \bullet_0$ unless $\nabla$ is unimodular. Nevertheless the unimodular case is the most important one, since e.g. any Levi-Civita connection to a given metric on $Q$ is unimodular.

The space of complex-valued smooth half-densities with compact support $\Gamma^\infty_{\mathrm{cpt}}(|\bigwedge^n|^{\frac{1}{2}} T^*Q)$ carries a natural pre-Hilbert space structure via the Hermitian product

$$\langle \nu, \rho\rangle = \int_Q \overline{\nu}\rho\,, \qquad \nu, \rho \in \Gamma^\infty_{\mathrm{cpt}}(|\textstyle\bigwedge^n|^{\frac{1}{2}} T^*Q). \quad (6.5)$$

It naturally extends to a $\mathbb{C}[[\lambda]]$-valued Hermitian product on $\Gamma^\infty_{\mathrm{cpt}}(|\bigwedge^n|^{\frac{1}{2}} T^*Q)[[\lambda]]$. Therefore it is reasonable to ask for the formal adjoint of $\pi_0(f)$. One can now calculate $\langle \pi_0(f)\nu, \rho\rangle$ either by intrinsic partial integration analogously to [5, Lem. 4.1] or by using a particular trivialization as we shall do in the following.

**Proposition 6.3** *For $f \in C^\infty(T^*Q)[[\lambda]]$ one has*

$$\left(N_{1/2}(\alpha_\mu)\right)^2 \overline{\mathcal{A}_0^{-1} f} = \mathcal{A}_0^{-1}\left(N_{1/2}(0)\right)^2 \overline{f}, \quad (6.6)$$

*and for any two half-densities $\nu, \rho \in \Gamma^\infty_{\mathrm{cpt}}(|\bigwedge^n|^{\frac{1}{2}} T^*Q)[[\lambda]]$ one has*

$$\langle \nu, \pi_0(f)\rho\rangle = \left\langle \pi_0\left(\left(N_{1/2}(0)\right)^2 \overline{f}\right)\nu, \rho\right\rangle. \quad (6.7)$$

PROOF: The first part is a straightforward computation using Lemma 3.6 and (3.1). Then the second part follows from [5, Lem. 4.1] and (6.6) using the trivialization induced by $\mu$. □

Thus $\pi_0$ is not a unitary representation, but due to (6.7) we can define a coresponding Weyl star product $\bullet_{\mathrm{W}}$ together with a representation by setting

$$f \bullet_{\mathrm{W}} g = \left(N_{1/2}(0)\right)^{-1} \left(\left(N_{1/2}(0) f\right) \bullet_0 \left(N_{1/2}(0) g\right)\right) \tag{6.8}$$

and $\pi_{\mathrm{W}}(f) = \pi_0\big(N_{1/2}(0) f\big)$. Hence we obtain the formula

$$\pi_{\mathrm{W}}(f)\nu = \sum_{l=0}^{\infty} \frac{(-\mathrm{i}\lambda)^l}{l!} \sum_{j_1,\cdots,j_l} \iota^* \left(\frac{\partial^l N_{1/2}(0) f}{\partial p_{j_1} \cdots \partial p_{j_l}}\right) \frac{1}{l!} \left\langle \partial_{q^{j_1}} \otimes \cdots \otimes \partial_{q^{j_l}}, \mathsf{D}^l \nu \right\rangle . \tag{6.9}$$

Thus $\pi_{\mathrm{W}}$ comprises a representation with respect to $\bullet_{\mathrm{W}}$ on the space of half-densities such that

$$\langle \nu, \pi_{\mathrm{W}}(f)\rho \rangle = \left\langle \pi_{\mathrm{W}}(\overline{f})\nu, \rho \right\rangle, \qquad \nu, \rho \in \Gamma^\infty_{\mathrm{cpt}}(|{\textstyle\bigwedge}^n|^{\frac{1}{2}}\, T^*Q)[[\lambda]]. \tag{6.10}$$

Note that $\bullet_{\mathrm{W}}$ is defined by use of the connection $\nabla$ alone and does not depend on the choice of $\mu$.

**Proposition 6.4** *The star product $\bullet_{\mathrm{W}}$ is a homogeneous star product of Weyl type, and*

$$\overline{f \bullet_{\mathrm{W}} g} = \overline{g} \bullet_{\mathrm{W}} \overline{f}, \qquad f, g \in \mathcal{C}^\infty(T^*Q)[[\lambda]]. \tag{6.11}$$

*Moreover, $\bullet_{\mathrm{W}}$ is equivalent to the product $\star^B_{\mathrm{W}}$ with $B = \frac{\mathrm{i}\lambda}{2}\mathrm{tr}R$ as well as to the product $\star_{\mathrm{W}}$. An algebra isomorphism from $\star^B_{\mathrm{W}}$ to $\bullet_{\mathrm{W}}$ is given by $\left(N_{1/2}(\alpha_\mu)\right)^{-1} N_{1/2}(0)$, one from $\star_{\mathrm{W}}$ to $\bullet_{\mathrm{W}}$ by*

$$\mathcal{B}_\mu := \mathcal{A}_{\mathrm{W}}^{-1} \left(N_{1/2}(\alpha_\mu)\right)^{-1} N_{1/2}(0) = \exp\left(\mathsf{F}\left(\frac{\cosh\left(\frac{\mathrm{i}\lambda}{2}\mathsf{D}\right) - \mathsf{id}}{\mathsf{D}} \alpha_\mu\right)\right). \tag{6.12}$$

Proof: The equivalence of $\bullet_{\mathrm{W}}$ and $\star^B_{\mathrm{W}}$ resp. $\star_{\mathrm{W}}$ can be shown by a straightforward computation using the definitions. The explicit formula for $\mathcal{B}_\mu$ follows from Lemma 3.6, the fact that $\overline{N}_{1/2}(\alpha_\mu) = \left(N_{1/2}(\alpha_\mu)\right)^{-1}$, and the explicit form of $\mathcal{A}_{\mathrm{W}}$ according to (3.11). Finally (6.11) follows from (6.12) and (2.4) implying that $\bullet_{\mathrm{W}}$ is of Weyl type since the whole construction depends only on combination $\mathrm{i}\lambda$. $\square$

**Remark 6.5** If the connection $\nabla$ is unimodular, e.g. the Levi-Civita connection of a Riemannian metric on $Q$, we always have $\star^B_{\mathrm{W}} = \star_{\mathrm{W}}$ for any choice of $\mu$. Now one can ask for a covariantly constant density $\mu_0$ or in other words for a density $\mu_0$ fulfilling $\alpha_{\mu_0} = 0$. Note that for a non unimodular $\nabla$ such a density cannot exist. Now, if $\nabla\mu_0 = 0$ and $Q$ is connected, then $\mu_0$ is uniquely determined up to a (positive) scalar multiple. Making the Ansatz $\mu_0 = \mathrm{e}^{-\varphi}\mu$ in the unimodular case we find that $\mu_0$ is covariantly constant if and only if $\alpha_\mu$ is not only closed but exact with $\alpha_\mu = d\varphi$. In particular we succeed in finding a covariantly constant density for any given unimodular connection in case $H^1_{\mathrm{dR}}(Q) = \{0\}$. Now having found such a density one has even $\bullet_{\mathrm{W}} = \star_{\mathrm{W}}$ and thus $\pi_{\mathrm{W}}$ is a representation for $\star_{\mathrm{W}}$. Note furthermore that even in this case $\bullet_{\mathrm{W}}$ does in general *not* coincide with the homogeneous Fedosov star product $\star_{\mathrm{F}}$ of Weyl type constructed in [4, Sect. 3].

Finally we would like to show that the representation $\pi_{\mathrm{W}}$ is a GNS representation. Let $\nu \in \Gamma^\infty(|{\textstyle\bigwedge}^n|^{\frac{1}{2}}\, T^*Q)$ be an arbitrary nowhere vanishing half-density and $\mu = \overline{\nu}\nu$ its square which comprises a positive density. Define $\alpha_\mu$, etc. with respect to this $\mu$. Then consider the $\mathbb{C}[[\lambda]]$-linear functional

$$\omega_\nu : \mathcal{C}^\infty_Q(T^*Q)[[\lambda]] \to \mathbb{C}[[\lambda]],\ f \mapsto \omega_\nu(f) = \int_Q \iota^* \mathcal{B}_\mu(f)\, \mu. \tag{6.13}$$

Clearly it is well-defined and local since $\mathcal{B}_\mu$ is a formal series of differential operators. Moreover, $\omega_\nu(f) = \omega_\mu(\mathcal{B}_\mu(f))$, which allows us to apply the general result on the 'pull back' of GNS representations [5, Prop. 5.1] since due to Proposition 6.4 the map $\mathcal{B}_\mu$ is a *real* algebra isomorphism between $\bullet_{\mathrm{W}}$ and $\star_{\mathrm{W}}$. Note that, though $\omega_\nu$ is only defined on a twosided ideal, the GNS representation again extends to the whole algebra due to [8, Cor. 1].

**Proposition 6.6** *Let $\nu \in \Gamma^\infty(|\bigwedge^n|^{\frac{1}{2}} T^*Q)$ be a nowhere vanishing half-density. Then $\omega_\nu$ is a positive $\mathbb{C}[[\lambda]]$-linear functional with Gel'fand ideal*

$$\mathcal{J}_\nu = \left\{ f \in \mathcal{C}^\infty_Q(T^*Q)[[\lambda]] \;\middle|\; \iota^* N_{1/2}(\alpha_\mu)\mathcal{B}_\mu(f) = 0 \right\}. \tag{6.14}$$

*The GNS pre-Hilbert space $\mathcal{C}^\infty_Q(T^*Q)[[\lambda]]/\mathcal{J}_\nu$ is canonically isometric to $\Gamma^\infty_{\mathrm{cpt}}(|\bigwedge^n|^{\frac{1}{2}} T^*Q)[[\lambda]]$ via*

$$\psi_f \mapsto \left(\iota^* N_{1/2}(\alpha_\mu)\mathcal{B}_\mu f\right)\nu \qquad \textit{and its inverse} \qquad v\nu \mapsto \psi_{\pi^* v}, \tag{6.15}$$

*where $f \in \mathcal{C}^\infty_Q(T^*Q)[[\lambda]]$ and $v \in \mathcal{C}^\infty_{\mathrm{cpt}}(Q)[[\lambda]]$. The induced GNS representation of $\mathcal{C}^\infty(T^*Q)[[\lambda]]$ on $\Gamma^\infty_{\mathrm{cpt}}(|\bigwedge^n|^{\frac{1}{2}} T^*Q)[[\lambda]]$ coincides with $\pi_{\mathrm{W}}$.*

PROOF: This can be either checked explicitly by using the defining formula for $\mathcal{B}_\mu$ or by using [5, Prop. 5.1]. $\square$

## 7 Aharonov-Bohm representations

Now we shall consider the question of equivalence classes of representations $\varrho^A_\kappa$ for different vector potentials but fixed magnetic field $B = dA$. So let $A, A' \in \Gamma^\infty(T^*Q)[[\lambda]]$ both fulfill $B = dA = dA'$ but not necessarily $A = A'$. Then we call the representations $\varrho^A_\kappa$ and $\varrho^{A'}_\kappa$ *equivalent*, if there exists a bijective $\mathbb{C}[[\lambda]]$-linear *intertwining* operator $U : \mathcal{C}^\infty(Q)[[\lambda]] \to \mathcal{C}^\infty(Q)[[\lambda]]$, i.e. $U$ satisfies for all $f \in \mathcal{C}^\infty(T^*Q)[[\lambda]]$ the relation

$$U^{-1} \varrho^{A'}_\kappa(f)\, U = \varrho^A_\kappa(f). \tag{7.1}$$

In the case, where $A$ and $A'$ are real and $\star^B_\kappa$ is the Weyl product $\star^B_{\mathrm{W}}$, the representations $\varrho^A_{\mathrm{W}}$ and $\varrho^{A'}_{\mathrm{W}}$ are called *unitarily equivalent*, if in addition $U$ is unitary with respect to the Hermitian product of Prop. 5.3.

**Lemma 7.1** *An operator $U : \mathcal{C}^\infty(Q)[[\lambda]] \to \mathcal{C}^\infty(Q)[[\lambda]]$ is an intertwiner for $\varrho^A_\kappa$ and $\varrho^{A'}_\kappa$ if and only if for all $f \in \mathcal{C}^\infty(Q)$*

$$U^{-1} \varrho_0(f)\, U = \varrho_0^{A-A'}(f). \tag{7.2}$$

*In that case $U$ is the left multiplication by a function in $\mathcal{C}^\infty(Q)[[\lambda]]$ also denoted by $U$. Moreover $U$ is unique up to an invertible factor in $\mathbb{C}[[\lambda]]$.*

PROOF: The first part of the lemma is a simple consequence of (3.13), (5.3) and (5.4). For the second part check that $U$ commutes with all left multiplications, i.e. $Uuv = uUv$ for all $u, v \in \mathcal{C}^\infty(Q)[[\lambda]]$. Hence $U$ is a left multiplication by a function in $\mathcal{C}^\infty(Q)[[\lambda]]$ as well. $\square$

Due to this lemma the equivalence of two representations $\varrho^A_\kappa$ and $\varrho^{A'}_\kappa$ for a fixed value of the ordering parameter $\kappa$ entails equivalence of $\varrho^A_{\kappa'}$ and $\varrho^{A'}_{\kappa'}$ for any other value $\kappa'$ as well. Hence we will restrict our considerations for a moment only to the case where $\kappa = \frac{1}{2}$ and $B = 0$. The following lemma then is immediately checked using the fact that the corresponding time development operator is an *inner* automorphism, see e.g. [6, Lem. 2.3] and Corollary 4.2.

**Lemma 7.2** *Let $A = \lambda A_1 + \lambda^2 A_2 + O(\lambda^3) \in \Gamma^\infty(T^*Q)[[\lambda]]$ be an exact one-form with $A = dS$ and $S = \lambda S_1 + \lambda^2 S_2 + O(\lambda^3) \in \mathcal{C}^\infty(Q)[[\lambda]]$. Then $\varrho_{\mathrm{W}}$ and $\varrho_{\mathrm{W}}^A$ are equivalent, where an intertwiner $U : \mathcal{C}^\infty(Q)[[\lambda]] \to \mathcal{C}^\infty(Q)[[\lambda]]$ is given by the left multiplication with $\exp\left(\frac{\mathrm{i}}{\lambda}S\right)$. If $A$ is real and $S$ has also been chosen to be real, then the restriction of $U$ to $\mathcal{C}^\infty_{\mathrm{cpt}}(Q)[[\lambda]]$ is unitary.*

Now choose a good cover $\{O_j\}_{j\in I}$ of $Q$, i.e. assume that all finite intersections of the $O_j$ are contractible. Next let $\{S^j\}_{j\in I}$ be a formal Čech cochain integrating $A$, or in other words a family of formal functions $S^j = \lambda S^j_1 + \lambda^2 S^j_2 + O(\lambda^3) \in \mathcal{C}^\infty(O_j)[[\lambda]]$ fulfilling $dS^j = A|_{O_j}$ for all $j \in I$. Next let $U$ be an intertwiner from $\varrho_{\mathrm{W}}$ to $\varrho_{\mathrm{W}}^A$, and choose local logarithms $T^j \in \mathcal{C}^\infty(O_j)[[\lambda]]$ for $U$ that means $U\big|_{O_j} = \exp\left(2\pi\mathrm{i}\, T^j\right)$. Then $(T^j - T^k)\big|_{O_j\cap O_k} \in \mathbb{Z}$ holds for all nonempty intersections $O_j \cap O_k$. As the center of $\star_{\mathrm{W}}$ consists only of the constant functions, one can find by the preceding lemma $c^j \in \mathbb{C}$ such that $2\pi T^j = \frac{S^j}{\lambda} + c^j$. Therefore we may assume without restriction that

$$\frac{(S^j - S^k)}{\lambda} \in 2\pi\mathbb{Z} \tag{7.3}$$

for all pairs $j, k$ with $O_j \cap O_k \neq \emptyset$. Let us at this point abbreviate the notation by calling a formal closed $k$-form $\alpha = \lambda\alpha_1 + \lambda^2\alpha_2 + O(\lambda^3) \in \Gamma^\infty(\bigwedge^k T^*Q)[[\lambda]]$ *integral*, if it satisfies the following condition

(IC) $\quad \frac{1}{2\pi}\alpha_1$ defines an integral cohomology class and $\alpha_l$ is exact for all $l \geq 2$.

By the above the existence of an intertwiner $U$ then implies that (IC) holds for $A$.

Vice versa suppose now that $A$ is integral. Then one can find $S^j$ such that (7.3) holds again. Consequently one can define $U \in \mathcal{C}^\infty(Q)[[\lambda]]$ by requiring

$$U\big|_{O_j} = \exp\left(\frac{\mathrm{i}}{\lambda}S^j\right) = \iota^*\mathrm{Exp}\left(\frac{\mathrm{i}}{\lambda}\pi^*S^j\right) \tag{7.4}$$

for all $O_j$. By Corollary 4.2 and Lemma 7.2 the operator $U$ then is an intertwiner from $\varrho_{\mathrm{W}}$ to $\varrho_{\mathrm{W}}^A$. Altogether we thus obtain the following theorem which completely describes the equivalence classes of representations.

**Theorem 7.3** *Let $A, A' \in \Gamma^\infty(T^*Q)[[\lambda]]$ be vector potentials of the form $A = \lambda A_1 + \lambda^2 A_2 + O(\lambda^3)$ resp. $A' = \lambda A'_1 + \lambda^2 A'_2 + O(\lambda^3)$ fulfilling $B = dA = dA'$. Then the $\star_\kappa^B$-representations $\varrho_\kappa^A$ and $\varrho_\kappa^{A'}$ are equivalent if and only if the difference $A - A'$ is integral, whence the equivalence classes are parametrized by $\lambda H^1_{\mathrm{dR}}(Q)/H^1_{\mathrm{dR}}(Q,\mathbb{Z}) + \lambda^2 H^1_{\mathrm{dR}}(Q)[[\lambda]]$. In case, where in addition $A$ and $A'$ are real, the representations $\varrho_{\mathrm{W}}^A$ and $\varrho_{\mathrm{W}}^{A'}$ are even unitarily equivalent with an intertwiner $U$ given locally by $U|_{O_j} = \exp(\frac{\mathrm{i}}{\lambda}S^j)$, where $\{S^j\}_{j\in I}$ is a real Čech cochain integrating $A - A'$.*

In the case $B = 0$ we have a canonical reference representation namely $\varrho_\kappa$ which corresponds to $A = 0$. Hence we have also a canonical equivalence class of representations. Considering the particular case $A = \lambda A_1$ without higher order terms the resulting equivalence classes are parametrized by $H^1_{\mathrm{dR}}(Q)$ modulo integral classes. We shall call a representation $\varrho_\kappa^{\lambda A_1}$ an *Aharonov-Bohm* representation if it is not equivalent to the *vacuum* representation $\varrho_\kappa$ since in this case Aharonov-Bohm like effects are possible. Thus the well-known integrality condition for the vector potential can be found in deformation quantization as well. But note that the corresponding star product $\star_\kappa^B$ and thus the algebra of observables is unaffected.

# 8 Magnetic monopoles and non-trivial bundles

In this section we will investigate the representation theory for quantized algebras with product $\star_\kappa^B$, where the magnetic field $B$ is no longer exact but only closed. More precisely we will show under which conditions representations of $\star_\kappa^B$ can be constructed which correspond to the Schrödinger-like representations $\varrho_\kappa$. Throughout the whole section we consider only the case where $B = \lambda B_1 + \lambda^2 B_2 + O(\lambda^3)$ starts in order $\lambda$.

First of all we should mention that it is possible to construct *local* representations for all $\star_\kappa^B$-algebras in the following way: let $O \subset Q$ be a contractible open subset of $Q$ and choose by Poincaré's Lemma a local vector potential $A \in \Gamma^\infty(T^*O)[[\lambda]]$ fulfilling $B|_O = dA$. Then we can set

$$\varrho_\kappa^O(f)u = \varrho_\kappa^A(f|_{T^*O})u, \qquad f \in \mathcal{C}^\infty(T^*Q)[[\lambda]],\ u \in \mathcal{C}^\infty(O)[[\lambda]], \tag{8.1}$$

and thus obtain a $\star_\kappa^B$-representation of the whole algebra on $\mathcal{C}^\infty(O)[[\lambda]]$. Definitely, these representations are not quite satisfactory since they do not reflect the global nature of $B$, but by gluing them together appropriately we will obtain the global representations we are looking for. The following lemma which is proved by straightforward computation using Corollary 4.2 will provide the essential tool for the construction.

**Lemma 8.1** *Let $B = \lambda B_1 + \lambda^2 B_2 + O(\lambda^3) \in \Gamma^\infty(\bigwedge^2 T^*Q)[[\lambda]]$ be a closed two-form and $O_1, O_2 \subset Q$ two open subsets such that $O_1, O_2$ and the intersection $O_1 \cap O_2$ are contractible. Moreover, let $A^{(i)} = \lambda A_1^{(i)} + \lambda^2 A_2^{(i)} + O(\lambda^3) \in \Gamma^\infty(T^*O_i)[[\lambda]]$, $i = 1, 2$ be local vector potentials for $B$. Finally choose $S^{(12)} = \lambda S_1^{(12)} + \lambda^2 S_2^{(12)} + O(\lambda^3) \in \mathcal{C}^\infty(O_1 \cap O_2)[[\lambda]]$ such that $dS^{(12)} = (A^{(1)} - A^{(2)})|_{O_1 \cap O_2}$. Then the relation*

$$\varrho_\kappa^{O_1}(f)u = \mathrm{e}^{-\frac{\mathrm{i}}{\lambda}S^{(12)}}\,\varrho_\kappa^{O_2}(f)\,\mathrm{e}^{\frac{\mathrm{i}}{\lambda}S^{(12)}}u \tag{8.2}$$

*is fulfilled for all $f \in \mathcal{C}^\infty(T^*Q)[[\lambda]]$ and $u \in \mathcal{C}^\infty(O_1 \cap O_2)[[\lambda]]$.*

The lemma now suggests that one should view the local functions $u$ as coefficient functions for a particular *local trivialization* of a $\mathbb{C}[[\lambda]]$*-line bundle* having the $\mathrm{e}^{-\frac{\mathrm{i}}{\lambda}S^{(ij)}}$ as transition functions. We will not further develop such a theory of formal line bundles though this could be done easily using standard sheaf cohomological methods extended to modules over $\mathbb{C}[[\lambda]]$. Rather, we will construct an ordinary complex line bundle out of the given data together with a formal connection under the assumption that the magnetic field satisfies the integrality condition (IC), see e.g. [25, p. 97–107]. Then there exists a complex line bundle $\pi_L : L \to Q$ unique up to equivalence with Chern class $[\frac{1}{2\pi}B_1]$. Moreover, we can find a connection $\nabla^L$ on $L$ such that the curvature of $\nabla^L$ coincides with $\mathrm{i}B_1$ and vector potentials for all $B_k$ with $k \geq 2$, i.e. $B_k = dA_k$. Let $A = \sum_{k=2}^\infty \lambda^k A_k$ then the mapping

$$\nabla^A : \Gamma^\infty(L)[[\lambda]] \to \Gamma^\infty(T^*Q \otimes L)[[\lambda]] = \Omega^1(L)[[\lambda]] \quad s \mapsto \left(\nabla^L + \frac{\mathrm{i}}{\lambda}A\right)s \tag{8.3}$$

comprises a formal connection on $L$ which we call *adapted* to $B$. The equivalence classes of such $\nabla^A$ are parametrized by $\lambda H^1_{\mathrm{dR}}(Q)/H^1_{\mathrm{dR}}(Q,\mathbb{Z}) + \lambda^2 H^1_{\mathrm{dR}}(Q)[[\lambda]]$. As usual let $\mathsf{D}^L$ be the corresponding formal symmetrized covariant derivative defined as in (1.5) using $\nabla^A$.

**Theorem 8.2** *Let $B = \lambda B_1 + \lambda^2 B_2 + O(\lambda^3) \in \Gamma^\infty(\bigwedge^2 T^*Q)[[\lambda]]$ be a closed two-form which satisfies the integrality condition* (IC)*. Furthermore, let $\pi_L : L \to Q$ be a complex line bundle with Chern class $[\frac{1}{2\pi}B_1]$ and $A = \lambda^2 A_2 + O(\lambda^3) \in \Gamma^\infty(T^*Q)[[\lambda]]$ such that $dA = B - \lambda B_1$ whence $\nabla^A$ is a formal* *connection adapted to $B$. Denote for every $f \in \mathcal{C}^\infty(T^*Q)[[\lambda]]$ by $\eta^A_\kappa(f) : \Gamma^\infty(L)[[\lambda]] \to \Gamma^\infty(L)[[\lambda]]$ the operator fulfilling*

$$\eta^A_\kappa(f)s = \sum_{l=0}^{\infty} \frac{(-\mathrm{i}\lambda)^l}{l!} \sum_{j_1,\cdots,j_l} \iota^* \left( \frac{\partial^l N_\kappa f}{\partial p_{j_1} \cdots \partial p_{j_l}} \right) \frac{1}{l!} \left\langle \partial_{q^{j_1}} \otimes \cdots \otimes \partial_{q^{j_l}}, \left(\mathsf{D}^L\right)^l s \right\rangle, \qquad s \in \Gamma^\infty(L)[[\lambda]]. \tag{8.4}$$

*Then $\eta^A_\kappa$ comprises a representation of $\star^B_\kappa$ on $\Gamma^\infty(L)[[\lambda]]$. Using a local trivializing section $e_O : O \to L$, one has the local representation*

$$\eta^A_\kappa(f)s\big|_O = \left(\varrho^O_\kappa(f|_{T^*O})u\right) e_O, \tag{8.5}$$

*where $s|_O = u\, e_O$ with $u \in \Gamma^\infty(O)[[\lambda]]$ unique and where the local formal one-form $A'$ used for $\varrho^O_\kappa$ as in (8.1) is determined by $\nabla^A_X e_O = \frac{\mathrm{i}}{\lambda} A'(X) e_O$, i.e. the local vector potential satisfies $A' = \lambda A_1 + A$ with $dA_1 = B_1|_O$.*

Proof: Since all involved operators are differential, it suffices to prove the representation property locally. Hence one only has to prove (8.5) since $\varrho^O_\kappa$ is known to be a local representation. But this follows directly from the above characterization of $\mathsf{D}^L$ and Theorem 5.2. □

**Remark 8.3** Though the line bundle $L$ is unique up to equivalence for a given $B_1$, the equivalence classes $\lambda H^1_{\mathrm{dR}}(Q)/H^1_{\mathrm{dR}}(Q,\mathbb{Z}) + \lambda^2 H^1_{\mathrm{dR}}(Q)[[\lambda]]$ of the adapted connections $\nabla^A$ determine different non-equivalent representations which induce 'Aharonov-Bohm effects' analogously to Sect. 7.

In the case where $B_k = 0$ for $k \geq 2$, we can choose $A_k = 0$ leading to a representation which we shall denote by $\eta^L_\kappa$.

Finally let us consider the case, where in addition $B$ is real. Then $L$ carries a natural Hermitian structure $\langle\cdot,\cdot\rangle^L$ the equivalence classes of which are labeled by $H^1(Q, U(1))$. Thus we can endow the space $\Gamma^\infty_{\mathrm{cpt}}(L)[[\lambda]]$ with a pre-Hilbert space structure by the $\mathbb{C}[[\lambda]]$-valued Hermitian product

$$\langle s,t\rangle_{L,\mu} = \int_Q \langle s,t\rangle^L \mu, \qquad s,t \in \Gamma^\infty_{\mathrm{cpt}}(L)[[\lambda]]. \tag{8.6}$$

The following proposition now provides the adjoint of $\eta^A_\kappa(f)$ with respect to this Hermitian product.

**Proposition 8.4** *Let $B$ be in addition real and $L$ endowed with the Hermitian structure $\langle\cdot,\cdot\rangle^L$ and a formal Hermitian connection $\nabla^A$ obtained by a Hermitian connection $\nabla^L$ and real $A$. Then one has for $s, \tilde{s} \in \Gamma^\infty_{\mathrm{cpt}}(L)[[\lambda]]$ and $f \in \mathcal{C}^\infty(T^*Q)[[\lambda]]$*

$$\left\langle s, \eta^A_\kappa(f)\tilde{s}\right\rangle_{L,\mu} = \left\langle \eta^A_\kappa\left(N_{1-2\kappa}\overline{f}\right) s, \tilde{s}\right\rangle_{L,\mu}. \tag{8.7}$$

*Hence $\eta^A_{\mathrm{W}} = \eta^A_{1/2}$ is a $^*$-representation of $\star^B_{\mathrm{W}}$.*

Proof: By $\mathbb{C}[[\lambda]]$-linearity and a partition of unity we may assume that $s$ has support in some $O_j$ of a good cover of $Q$. Then the proposition follows directly from the definition of $\langle\cdot,\cdot\rangle^L$, the local expression (8.5) and Cor. 5.4. □

## 9 GNS representation for magnetic monopoles

In this section we consider only the Weyl ordered case that means the product $\star_{\mathrm{W}}^{B}$ for a real magnetic field $B = \lambda B_1 + \lambda^2 B_2 + O(\lambda^3)$ satisfying the integrality condition (IC). The complex Hermitian line bundle $\pi_L : L \to Q$ is given as in the previous section. Now we search for a positive linear functional such that the induced GNS representation coincides with the $^*$-representation $\eta_{\mathrm{W}}^{A}$. As we shall see a slight modification will be necessary to reconstruct $\eta_{\mathrm{W}}^{A}$ out of such a positive functional. The difficulty has its origin in the fact that in general the line bundle $L$ does not admit a global nowherevanishing section. Nevertheless $L$ possesses transversal sections $s_0 \in \Gamma^\infty(L)$. This follows from the general transversality arguments as e.g. in [15, Chap. 3, Thm. 2.1]. Now the following technical lemma should be well-known and is essential for our constructions:

**Lemma 9.1** *Let $\pi_L : L \to Q$ be a complex line bundle over $Q$ and let $\nabla$ be a connection for $L$ and $s_0 \in \Gamma^\infty(L)$ a transversal section. Let $s \in \Gamma^\infty_{\mathrm{cpt}}(L)$ be a section with compact support. Then there exists a function $v \in C^\infty_{\mathrm{cpt}}(Q)$ and a vector field $X \in \Gamma^\infty_{\mathrm{cpt}}(TQ)$ such that*

$$s = vs_0 + \nabla_X s_0 \tag{9.1}$$

*and* $\mathrm{supp} v \cup \mathrm{supp} X \subset \mathrm{supp} s$. *Hereby $v$ and $X$ are not unique.*

Proof: To avoid trivialities assume that $L$ is non-trivial and let $Z = s_0^{-1}(\{0\}) \neq \emptyset$ be the set of zeros of $s_0$. Choose a good cover $\{O_j\}_{j\in I}$ with corresponding local trivializations $e_j$ of $L$. Let $x \in Z \cap O_j$ for some $j$ and $s_0|_{O_j} = u_j e_j$ with $u_j(x) = 0$ but $du_j(x) \neq 0$ due to the transversality of $s_0$. Choose a smooth vector field $X^{(x)}$ such that $du_j(X^{(x)})|_x \neq 0$. This implies that $\nabla_{X^{(x)}} s_0|_x = du_j(X^{(x)}) e_j|_x \neq 0$ since $u_j(x) = 0$. By continuity there exists an open neighborhood $U_x$ of $x$ such that $\nabla_{X^{(x)}} s_0$ does not vanish in $U_x$. Together with $U_0 = Q \setminus Z$ the $\{U_x\}$, $x \in Z$ form an open cover of $Q$. Hence by compactness finitely many cover $\mathrm{supp} s$, say $U_0$ and $U_{x_1}, \ldots, U_{x_k}$. Let $\{\chi_0, \chi_1, \ldots, \chi_k\}$ be a smooth partition of unity on $U_0 \cup U_{x_1} \cup \cdots \cup U_{x_k}$ subordinate to these open sets. Then there exist functions $v_l \in C^\infty_{\mathrm{cpt}}(Q)$ for $l = 0, \ldots, k$ such that $\chi_0 s = v_0 s_0$, $\chi_l s = v_l \nabla_{X^{(x_l)}} s_0$ and $\mathrm{supp} v_l \subset \mathrm{supp} s \cap U_l$. Then we have globally $s = v_0 s_0 + \nabla_X s_0$ with $X = v_1 X^{(x_1)} + \cdots + v_k X^{(x_k)}$. This proves the claim. □

Due to this lemma we can reconstruct any section of $L$ with compact support out of a fixed transversal section $s_0$ if we allow for covariant differentiation. Now the idea is that with $\eta_{\mathrm{W}}^{A}$ we have indeed the possibility of covariant differentiation but this automatically raises the degree in $\lambda$. Hence we have to allow finitely many negative powers of $\lambda$ to get covariant differentiation of $s_0$ in the lowest order in $\lambda$. In other words we have to replace formal power series by formal Laurent series everywhere and thus obtain vector spaces over the field of formal Laurent series $\mathbb{C}((\lambda))$ instead of modules over $\mathbb{C}[[\lambda]]$. This does not cause any problems as can be seen directly from all constructions above or the general arguments given in [5, App. A].

Thus having fixed a transversal section $s_0 \in \Gamma^\infty(L)$ we define the $\mathbb{C}((\lambda))$-linear functional $\omega_{s_0} : \mathcal{C}^\infty_Q(T^*Q)((\lambda)) \to \mathbb{C}((\lambda))$ by

$$\omega_{s_0}(f) := \int_Q \left\langle s_0, \eta_{\mathrm{W}}^{A}(f) s_0 \right\rangle^L \mu \tag{9.2}$$

which is clearly well-defined for $f \in \mathcal{C}^\infty_Q(T^*Q)((\lambda))$. The positivity of $\omega_{s_0}$ is established easily.

**Lemma 9.2** *The $\mathbb{C}((\lambda))$-linear functional $\omega_{s_0}$ is positive and the Gel'fand ideal $\mathcal{J}_{s_0}$ is given by all $f$ fulfilling $\eta_{\mathrm{W}}^{A}(f) s_0 = 0$. Furthermore one has for $f, g \in \mathcal{C}^\infty_Q(T^*Q)((\lambda))$*

$$\omega_{s_0}(\overline{f} \star_{\mathrm{W}}^{B} g) = \int_Q \left\langle \eta_{\mathrm{W}}^{A}(f) s_0, \eta_{\mathrm{W}}^{A}(g) s_0 \right\rangle^L \mu. \tag{9.3}$$

PROOF: Clearly it suffices to prove (9.3). Due to the obvious $\mathbb{C}((\lambda))$-linearity we may consider $f, g \in C^\infty_Q(T^*Q)$ without $\lambda$-powers. In order to apply Prop. 8.4, where we need sections with compact support, we choose a smooth function $\phi : Q \to [0,1]$ with compact support such that $\phi$ is identical to 1 on $\operatorname{supp}(\iota^* f) \cup \operatorname{supp}(\iota^* g)$. Due to the representation property and (8.7)

$$
\begin{aligned}
\omega_{s_0}(\overline{f} \star^B_{\mathrm{W}} g) &= \int_Q \left\langle s_0, \eta^A_{\mathrm{W}}(\overline{f} \star^B_{\mathrm{W}} g) s_0 \right\rangle^L \mu = \int_Q \left\langle \phi s_0, \eta^A_{\mathrm{W}}(\overline{f} \star^B_{\mathrm{W}} g) \phi s_0 \right\rangle \mu \\
&= \int_Q \left\langle \eta^A_{\mathrm{W}}(f) \phi s_0, \eta^A_{\mathrm{W}}(g) \phi s_0 \right\rangle^L \mu = \int_Q \left\langle \eta^A_{\mathrm{W}}(f) s_0, \eta^A_{\mathrm{W}}(g) s_0 \right\rangle \mu
\end{aligned}
$$

which proves (9.3), thus the lemma. □

The following theorem shows that $\eta^A_{\mathrm{W}}$ can indeed be recovered as GNS representation induced by $\omega_{s_0}$. Note that the transversality as well as the usage of formal Laurent series is crucial.

**Theorem 9.3** *Let $s_0 \in \Gamma^\infty(L)$ be a transversal section and $\omega_{s_0}$ as in (9.2). Then the GNS representation space $\mathfrak{H}_{s_0} = C^\infty_Q(T^*Q)((\lambda)) / \mathcal{J}_{s_0}$ is canonically isometric to $\Gamma^\infty_{\mathrm{cpt}}(L)((\lambda))$ via*

$$
\mathfrak{H}_{s_0} \ni \psi_f \mapsto \eta^A_{\mathrm{W}}(f) s_0 \in \Gamma^\infty_{\mathrm{cpt}}(L)((\lambda)), \tag{9.4}
$$

*where $\Gamma^\infty_{\mathrm{cpt}}(L)((\lambda))$ is equipped with the Hermitian product (8.6). Moreover, the GNS representation on $\mathfrak{H}_{s_0}$ is unitarily equivalent to $\eta^A_{\mathrm{W}}$ under the unitary map (9.4) and extends to a representation of $C^\infty(T^*Q)((\lambda))$.*

PROOF: First notice that $\psi_f \mapsto \varrho^L_{\mathrm{W}}(f) s_0$ is well-defined and isometric since due to Lemma 9.2

$$
\langle \psi_f, \psi_g \rangle = \omega_{s_0}(\overline{f} \star^B_{\mathrm{W}} g) = \int_Q \left\langle \eta^A_{\mathrm{W}}(f) s_0, \eta^A_{\mathrm{W}}(g) s_0 \right\rangle^L \mu
$$

and thus (9.4) is injective. To prove surjectivity we apply Lemma 9.1: let first $s \in \Gamma^\infty_{\mathrm{cpt}}(L)((\lambda))$ be a section without $\lambda$ powers and extend the arguments by $\mathbb{C}((\lambda))$-linearity afterwards. Then due to the lemma there exists a function $v_0 \in C^\infty_{\mathrm{cpt}}(Q)$ and a vector field $X_0 \in \Gamma^\infty_{\mathrm{cpt}}(TQ)$ such that $s = v_0 s_0 + \nabla_{X_0} s_0$. Due to the explicit formula (8.4) for the standard order representation $\eta^A_0$ we find that $\eta^A_0(\pi^* v_0 + \frac{\mathrm{i}}{\lambda} J(X_0)) s_0$ coincides with $s$ up to higher orders in $\lambda$. By iterating this procedure we find a function $v \in C^\infty_{\mathrm{cpt}}(Q)[[\lambda]]$ and a vector field $X \in \Gamma^\infty_{\mathrm{cpt}}(TQ)[[\lambda]]$ such that $\eta^A_0(\pi^* v + \frac{\mathrm{i}}{\lambda} J(X)) s_0 = s$. This proves the surjectivity since $\eta^A_0$ can be replaced by $\eta^A_{\mathrm{W}}$ incorporating the bijection $N_{1/2}(\alpha_\mu)$. The other statements of the theorem follow immediatly. □

**Remark 9.4** In the context of formal GNS constructions the usage of the field of formal Laurent series was extensively discussed in [8]. Furthermore it may be desirable to use even an algebraically closed extension field for these constructions. Hereby the field of *formal completed Newton-Puiseux series* $\mathbb{C}\langle\langle\lambda\rangle\rangle$ is the natural candidate. We mention that anything can also be done in this framework due to the 'extension theorems' in [5, App. A].

## 10 Symbol calculus and star products

In the previous sections we have constructed representations $\varrho^A_\kappa$ of the deformed product $\star^B_\kappa$ on formal sections of line bundles $L$ over $Q$. We will show in the following that the formal representation $\varrho^A_\kappa$ can be interpreted as the asymptotic expansion of a certain pseudodifferential operator calculus on $Q$ with values in $L$. In other words, every $\varrho^A_\kappa$ comes from a particular global symbol calculus on $Q$. So finally we obtain an analytical interpretation for the formal representations $\varrho^A_\kappa$.

But before we go into the details of the construction we have to set up some notation and will recall some results from the theory of pseudodifferential operators on manifolds. For more information on this we refer the reader to HÖRMANDER [16].

Under $\mathrm{S}^m(Q)$, $m \in \mathbb{R}$ we understand the symbols on $T^*Q$ of Hörmander type, i.e. $\mathrm{S}^m(Q)$ consists of all smooth functions $a \in \mathcal{C}^\infty(T^*Q)$ such that uniformly on compact subsets $K \subset U$ of any coordinate patch $U \subset Q$

$$\left|\partial^\alpha_{q^\cdot}\, \partial^\beta_{p_\cdot}\, a(\zeta)\right| \leq C_{K,\alpha,\beta}\, \langle\zeta\rangle^{m-|\beta|}, \qquad \zeta \in T^*K, \quad \alpha,\beta \in \mathbb{N}^n. \tag{10.1}$$

Hereby $\langle\zeta\rangle = (1+\|p(\zeta)\|^2)^{1/2}$ and $C_{K,\alpha,\beta} > 0$ and we use the usual multi-index notation for $\alpha,\beta$. Note that this definition is independent of the used chart and provides a global characterization of symbols. As usual we set $\mathrm{S}^\infty(Q) = \cup_{m\in\mathbb{R}}\mathrm{S}^m(Q)$ and $\mathrm{S}^{-\infty}(Q) = \cap_{m\in\mathbb{R}}\mathrm{S}^m(Q)$. In case $E \to Q$ is a smooth vector bundle one defines $\mathrm{S}^m(Q;E)$ as the space of all sections $a \in \Gamma^\infty(\pi^*(E))$ such that for every section $e \in \Gamma^\infty(E^*)$ of the dual bundle of $E$ the relation $\langle e, a\rangle \in \mathrm{S}^m(Q)$ holds. The spaces $\mathcal{P}(Q;E)$ and $\mathcal{P}^k(Q;E)$ of fiberwise homogeneous resp. fiberwise homogeneous smooth sections of degree $k$ of $\pi^*(E)$ are then natural subspaces of $\mathrm{S}^\infty(Q;E)$ resp. of $\mathrm{S}^k(Q;E)$

For two smooth vector bundles $E \to Q$ and $F \to Q$ the space $\Psi^m(Q;E,F)$ of pseudodifferential operators from $E$ to $F$ of order $m$ consists of all continuous mappings $A : \Gamma^\infty_{\mathrm{cpt}}(Q,E) \to \Gamma^\infty_{\mathrm{cpt}}(Q,F)'$ such that the Schwartz kernel $K^A \in \Gamma^\infty_{\mathrm{cpt}}(Q\times Q, F\boxtimes E^*)'$ is a conormal distribution of order $m$ with respect to the diagonal of $Q\times Q$ (cf. [16, Def. 18.2.6.]). A global symbol calculus now provides an (almost) bijective correspondence between symbols and pseudodifferential operators over $Q$. The symbol calculus we introduce in the following generalizes the well-known ones of WIDOM [27] and SAFAROV [22], so we can be brief with proofs (cf. also [20, 23]).

By a *cut-off function* for the connection $\nabla$ on $Q$ we mean a smooth function $\chi : TQ \to [0,1]$ having support in an open neighboorhood $O \subset TQ$ of the zero section such that $(\pi,\exp)|_O$ is a diffeomorphism onto an open neighborhood $\widetilde{O}$ of the diagonal of $Q\times Q$. Additionally let us suppose that $\mathrm{supp}\chi \cap T_xQ$ is compact for every $x \in Q$. After shrinking $O$ we can assume that for every $\kappa \in [0,1]$ the map $\chi_\kappa : TQ \to [0,1]$ with

$$\chi_\kappa(v_x) = \begin{cases} \chi(-\kappa T_{\kappa v_x}\exp_x v_x)\chi((1-\kappa)T_{\kappa v_x}\exp_x v_x) & \text{for} \quad v_x \in T_xQ\cap O \\ 0 & \text{else} \end{cases} \tag{10.2}$$

is smooth and itself a cut-off function. The volume densities $\mu_x$ and $\mu$ are now related by a smooth function $\rho : Q\times Q \supseteq \widetilde{O} \to \mathbb{R}^+$ satisfying

$$\left(\exp_x^{-1}\right)^*(\mu_x) = \rho(x,\cdot)\mu. \tag{10.3}$$

Next assume to be given vector bundles $E$ and $F$ over $Q$, a Hermitian metric $\langle\cdot,\cdot\rangle^F$ on $F$ and covariant derivatives $\nabla^E$ and $\nabla^F$, where the latter one is supposed to preserve the Hermitian metric. Then, if $x,y \in Q$ are sufficiently close to each other, or in other words if $(x,y) \in \widetilde{O}$, we will denote by $\tau^E_{(x,y)} : E_y = \pi^{-1}(y) \to E_x$ the parallel translation in $E$ along the unique geodesic with respect to $\nabla$ connecting $y$ with $x$ in $\exp_y(T_yQ\cap O)$. Moreover, $\gamma_\kappa(x,y) = \exp_x(\kappa\exp_x^{-1}(y))$ will denote the $\kappa$-*midpoint* of this geodesic. Now define for every section $e \in \Gamma^\infty(E)$ the lift $e^\chi \in \Gamma^\infty(\pi^*_{TQ}(E))$ by

$$e^\chi(v_x) = \begin{cases} \chi(v_x)\tau^E_{(x,\exp_x v_x)}(e(\exp_x v_x)) & \text{for all} \quad v_x \in T_xQ\cap O, \\ 0 & \text{else,} \end{cases} \tag{10.4}$$

where $x$ runs through $Q$. So finally we have all the ingredients to formulate, how pseudodifferential operators are associated to symbols.

**Theorem 10.1** *Let $\kappa \in [0,1]$ be an ordering parameter and $\hbar \in \mathbb{R}^+$ a deformation parameter. Define for every symbol $a \in \mathrm{S}^m(Q;\mathsf{Hom}(E,F))$ and all sections $f \in \Gamma^\infty(F)$ and $e \in \Gamma^\infty(E)$ the kernel function $W_\kappa(a)(f,e) \in \mathcal{C}^\infty(Q)$ by*

$$\big(W_\kappa(a)(f,e)\big)(x) = \frac{1}{(2\pi\hbar)^n}\int_{T_x^*Q}\int_{T_xQ} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle\zeta_x,v_x\rangle}\,\langle f^\chi(-\kappa v_x), a(\zeta_x)(e^\chi((1-\kappa)v_x))\rangle^F\,\mu_x(v_x)\mu_x^*(\zeta_x),$$

*where $\chi$ is a cut-off function as introduced above. Then there exists a unique properly supported pseudodifferential operator $\mathsf{Op}_{\hbar,\kappa}(a) \in \Psi^\infty(Q;E,F)$ of order $m$ fulfilling the relation*

$$\langle f, \mathsf{Op}_{\hbar,\kappa}(a)\,e\rangle = \int_Q \langle f(x), (\mathsf{Op}_{\hbar,\kappa}(a)\,e)(x)\rangle^F\,\mu(x) = \int_Q \big(W_\kappa(a)(f,e)\big)(x)\,\mu(x) \qquad (10.5)$$

*for all $f$ and $e$ with compact support. One calls $\mathsf{Op}_{\hbar,\kappa}(a)$ the $\kappa$-ordered quantization of $a$.*

*If now $\chi'$ is another cut-off function the difference $\mathsf{Op}^{[\chi]}_{\hbar,\kappa}(a) - \mathsf{Op}^{[\chi']}_{\hbar,\kappa}(a)$ is a smoothing operator, hence lies in $\Psi^{-\infty}(Q;E,F)$. In case $a$ is polynomial in the momenta, i.e. if $a \in \mathcal{P}(Q;\mathsf{Hom}(E,F))$, $\mathsf{Op}_{\hbar,\kappa}(a)$ is even a differential operator between $E$ and $F$ and independent of the choice of $\chi$.*

**Remark 10.2** In a few cases we need to specify the dependence of the operator calculus on the choice of connections $\nabla$ on the considered vector bundles or the cut-off function $\chi$. When necessary, we will denote this dependence by $\mathsf{Op}^{[\nabla,\chi]}_{\hbar,\kappa}$.

Proof: For the case, where $E$ and $F$ are trivial line bundles over $Q$, the claim has been shown in Pflaum [21, Thm. 2.1]. By local triviality of $E$ and $F$ the general statement then follows easily. $\square$

It turns out that the quantization map $\mathsf{Op}_{\hbar,\kappa} : \mathrm{S}^\infty(Q;\mathsf{Hom}(E,F)) \to \Psi^\infty(Q;E,F)$ possesses a filtration preserving pseudo-inverse $\sigma_\kappa : \Psi^\infty(Q;E,F) \to \mathrm{S}^\infty(Q;\mathsf{Hom}(E,F))$ called a *symbol map*. More precisely this means that the induced maps $\overline{\mathsf{Op}_{\hbar,\kappa}} : \mathrm{S}^\infty/\mathrm{S}^{-\infty}(Q;\mathsf{Hom}(E,F)) \to \Psi^\infty/\Psi^{-\infty}(Q;E,F)$ and $\overline{\sigma}_\kappa : \Psi^\infty/\Psi^{-\infty}(Q;E,F) \to \mathrm{S}^\infty/\mathrm{S}^{-\infty}(Q;\mathsf{Hom}(E,F))$ are inverse to each other. Let us give an explicit representation for the symbol map $\sigma_\kappa$. So consider a pseudodifferential operator $A \in \Psi^m(Q;E,F)$ and denote by $K^A_\kappa$ the $\pi^*_{TQ}(\mathsf{Hom}(E,F))$-valued distribution defined by

$$K^A_\kappa(v_x) = \chi(v_x)\tau^F_{(x,\exp_x(-\kappa v_x))}(K^A\rho^{-1})(\exp_x(-\kappa v_x), \exp_x((1-\kappa)v_x))\tau^E_{(\exp_x((1-\kappa)v_x),x)} \qquad (10.6)$$

for all $v_x \in T_xQ \cap O$ and set to zero outside $O$. Then the symbol $\sigma_\kappa(A)$ is given by the oscillatory integral

$$\sigma_\kappa(A)(\zeta_x) = \int_{T_xQ} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\langle\zeta_x,v_x\rangle} K^A_\kappa(v_x)\,\mu_x(v_x). \qquad (10.7)$$

In the following we will provide some useful integral representations for the operator $\mathsf{Op}_{\hbar,\kappa}(a)$. The proof is done by performing straightforward but somewhat lenghty transformations of the involved integrals and applying Gauß Lemma, i.e. $T_{v_x}\exp_x v_x = \tau_{(\exp_x(v_x),x)}v_x$, several times.

**Proposition 10.3** *For every symbol $a \in \mathrm{S}^m(Q;\mathsf{Hom}(E,F))$ and every section $e \in \Gamma^\infty_{\mathrm{cpt}}(E)$, $\mathsf{Op}_{\hbar,\kappa}(a)e$ can be written in the form*

$$\begin{aligned}(\mathsf{Op}_{\hbar,\kappa}(a)e)(x) &= \frac{1}{(2\pi\hbar)^n}\int_{T_x^*Q}\int_{T_xQ} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle\zeta_x,v_x\rangle}\,\tau^F_{(x,\exp_x(\kappa v_x))}a\left((T_{\exp_x(\kappa v_x)}\exp_x^{-1})^*\zeta_x\right)\tau^E_{(\exp_x(\kappa v_x),\exp_x v_x)}e(\exp_x(v_x)) \\ &\qquad \rho(\exp_x(\kappa v_x),x)\,\chi_\kappa(v_x)\,\mu_x(v_x)\,\mu_x^*(\zeta_x).\end{aligned} \qquad (10.8)$$

*Moreover, the Schwartz kernel $K^A$ of $A = \mathsf{Op}_{\hbar,\kappa}(a)$ is given by*

$$K^A(x,y) = \frac{1}{(2\pi\hbar)^n} \int_{T_x^*Q} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle \zeta_x, \exp_x^{-1}(y)\rangle} \, \tau^F_{(x,\gamma_\kappa(x,y))} a\left((T_{\gamma_\kappa(x,y)} \exp_x^{-1})^* \zeta_x\right) \tau^E_{(\gamma_\kappa(x,y),y)} \\ \rho(\gamma_\kappa(x,y),x)\, \rho(x,y)\, \widetilde{\chi_\kappa}(x,y)\, \mu_x^*(\zeta_x), \tag{10.9}$$

*where $\widetilde{\chi_\kappa}$ is the function equal to $\chi_\kappa \circ (\pi_{TQ}, \exp)$ over $\widetilde{O}$ and which is extended by zero on the complement of $\widetilde{O}$ in $Q \times Q$.*

Next we will calculate more explicitly the differential operator $\mathsf{Op}_{\hbar,\kappa}(\mathsf{p})$ respectively its kernel function $W_\kappa(\mathsf{p})(f,e)$ for a vector valued symbol $\mathsf{p} \in \mathcal{P}^k(Q; \mathsf{Hom}(E,F))$ polynomial in the momenta of degree $k$. In local coordinates one can write $\mathsf{p}$ in the form

$$\mathsf{p} = \frac{1}{k!} \sum_{j_1,\cdots,j_k} p_{j_1} \cdots p_{j_k} T^{j_1 \cdots j_k} \circ \pi,$$

where $T^{j_1\cdots j_k} \in \Gamma^\infty(\mathsf{Hom}(E,F))$. Now denote by $v^j$ the local coordinates on $T_xQ$ induced by $q$, and by $V_x : \mathbb{R}^n \to T_xQ$ the function $(v^1,\ldots,v^n) \mapsto v^1 \partial_{q^1}\big|_x + \cdots + v^n \partial_{q^n}\big|_x$, where $x \in Q$ lies in the domain of $q$. Then we compute by integration by parts

$$\begin{aligned}
&(W_\kappa(\mathsf{p})(f,e))\,(x) = \\
&= \frac{(-\mathrm{i}\hbar)^k}{k!} \sum_{j_1,\cdots,j_k} \partial_{v^{j_1}} \cdots \partial_{v^{j_k}}\Big|_{v=0} \left\langle f^\chi(-\kappa V_x(v)), T^{j_1\cdots j_k}(x)\,(e^\chi((1-\kappa)V_x(v)))\right\rangle_x^F \\
&= \frac{(-\mathrm{i}\hbar)^k}{k!} \sum_{j_1,\cdots,j_k} \sum_{l=0}^k \binom{k}{l} (-\kappa)^l (1-\kappa)^{k-l} \\
&\quad \left\langle \partial_{v^{j_1}} \cdots \partial_{v^{j_l}}\Big|_{v=0} \tau^F_{(x,\exp_x V_x)}(f(\exp_x V_x)), T^{j_1\cdots j_k}(x)\left(\partial_{v^{j_{l+1}}} \cdots \partial_{v^{j_k}}\Big|_{v=0} \tau^E_{(x,\exp_x V_x)}(e(\exp_x V_x))\right)\right\rangle_x^F.
\end{aligned}$$

Using the symmetrized covariant derivate $\mathsf{D}$ one checks that

$$\partial_{v^{j_1}} \cdots \partial_{v^{j_l}}\Big|_{v=0} \tau^F_{(x,\exp_x V_x)}(s(\exp_x V_x)) = \frac{1}{l!}\left\langle \partial_{q^{j_1}} \otimes \cdots \otimes \partial_{q^{j_l}}, \mathsf{D}^l s\right\rangle(x). \tag{10.10}$$

Inserting this relation one finally obtains

$$(W_\kappa(\mathsf{p})(f,e)) = \frac{(-\mathrm{i}\hbar)^k}{k!} \sum_{j_1,\cdots,j_k} \sum_{l=0}^k \binom{k}{l} (-\kappa)^l (1-\kappa)^{k-l} \frac{1}{l!} \frac{1}{(k-l)!} \\ \left\langle \left\langle \partial_{q^{j_1}} \otimes \cdots \otimes \partial_{q^{j_l}}, \mathsf{D}^l f\right\rangle, T^{j_1\cdots j_l}\left(\left\langle \partial_{q^{j_{l+1}}} \otimes \cdots \otimes \partial_{q^{j_k}}, \mathsf{D}^{k-l} e\right\rangle\right)\right\rangle^F. \tag{10.11}$$

Now, let us abbreviate for every $T \in \Gamma^\infty(\bigvee^k TQ \otimes \mathsf{Hom}(E,F))$ and every $l \leq k$ by $T_l : \Gamma^\infty(E) \to \Gamma^\infty(\bigvee^l TQ \otimes F)$ the differential operator which locally is given by

$$T_l(e) = \frac{1}{(k-l)!} \sum_{j_{l+1},\cdots,j_k} \left\langle dq^{j_{l+1}} \otimes \cdots \otimes dq^{j_k}, T\left(\left\langle \partial_{q^{j_{l+1}}} \otimes \cdots \otimes \partial_{q^{j_k}}, \mathsf{D}^{k-l} e\right\rangle\right)\right\rangle.$$

Then we have the following result.

**Theorem 10.4** *Let* $\mathsf{p} \in \mathcal{P}^k(Q; \mathsf{Hom}(E,F))$ *and* $e \in \Gamma^\infty_{\mathsf{cpt}}(E)$*. Then*

$$\mathsf{Op}_{\hbar,\kappa}(\mathsf{p})e = \sum_{l=0}^{k} \frac{(-\mathrm{i}\hbar)^k}{l!(k-l)!}\kappa^l(1-\kappa)^l \left(\mathsf{div}^F_\mu\right)^l (T_l(e)), \tag{10.12}$$

*where* $T \in \Gamma^\infty(\bigvee^k TQ \otimes \mathsf{Hom}(E,F))$ *is the unique tensor field fulfilling* $\mathsf{p} = J(T)$*.*

Proof: The claim follows from Eq. (10.11) by Lemma 1.1 and the definition of the operator $J$. $\square$

**Corollary 10.5** *For every* $\mathsf{p} \in \mathcal{P}(Q; \mathsf{Hom}(E,F))$ *one has*

$$\mathsf{Op}_{\hbar,\kappa}(\mathsf{p}) = \mathsf{Op}_0\left(J \exp\left(-\mathrm{i}\kappa\hbar\, \mathsf{div}^{\mathsf{Hom}(E,F)}_\mu\right) J^{-1}(\mathsf{p})\right) = \mathsf{Op}_0\left(\exp\left(-\mathrm{i}\kappa\hbar\Delta^{\mathsf{Hom}(E,F)}_\mu\right)\mathsf{p}\right). \tag{10.13}$$

Now let us come back to our original goal, namely to give an operator theoretical interpretation of the representations $\varrho^A_\kappa$ on the formal sections of a line bundle $L$ over $Q$. So let $E = F = L$ be a Hermitian line bundle with metric connection $\nabla^L$ associated to a closed formal real two form $B = \lambda B_1$ satisfying (IC). Comparing formula (10.13) with Eq. (8.4) we obtain the following second corollary.

**Corollary 10.6** *For every integral real formal two-form* $B = \lambda B_1$ *and every corresponding line bundle* $L$ *with Chern class* $[\frac{1}{2\pi}B_1]$ *one has the relation*

$$\mathsf{Op}_{\hbar,\kappa}(\mathsf{p}) = \eta^L_\kappa(\mathsf{p})\big|_{\lambda=\hbar}, \qquad \mathsf{p} \in \mathcal{P}(Q; \mathsf{Hom}(L,L)) = \mathcal{P}(Q), \tag{10.14}$$

*where we have substituted the formal variable* $\lambda$ *in* $\eta^L_\kappa(\mathsf{p})$ *by the real value* $\hbar \in \mathbb{R}^+$*.*

**Remark 10.7** By formula (8.4), $\eta^L_\kappa(\mathsf{p})$ is a polynomial in $\lambda$, hence insertion $\hbar$ for $\lambda$ is well-defined indeed.

By the last corollary one can interpret the pseudodifferential calculus $\mathsf{Op}_{\hbar,\kappa}$ as an operator realization for the formal representation $\eta^L_\kappa$. Let us now briefly recall that for two symbols $a \in \mathrm{S}^m(Q; \mathsf{Hom}(L,L))$ and $b \in \mathrm{S}^{\tilde{m}}(Q; \mathsf{Hom}(L,L))$ the pseudodifferential operator $\mathsf{Op}_{\hbar,\kappa}(a)\mathsf{Op}_{\hbar,\kappa}(b)$ lies in $\Psi^{m+\tilde{m}}(Q; L, L)$. Additionally, over every coordinate patch $U$ of $Q$ trivializing $L$ one even has an asymptotic expansion of the symbol $\sigma_0(\mathsf{Op}_0(a)\mathsf{Op}_0(b))$ of the form

$$\sigma_0(\mathsf{Op}_0(a)\mathsf{Op}_0(b))|_{T^*U} \sim \sum_{k=0}^{\infty} \hbar^k \sum_{\substack{|\alpha|,|\beta|+|\gamma|\leq k \\ |\alpha|+|\beta|-|\delta|=k}} \pi^*(r^\delta_{k,\alpha\beta\gamma}) p_\delta \partial^\alpha_{p.} a \partial^\beta_{p.} \partial^\gamma_{q.} b, \tag{10.15}$$

where $r^\delta_{k,\alpha\beta\gamma} \in \mathcal{C}^\infty(U)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{N}^n$ denote multi-indices. (cf. [27, Sect. 5] and [20, Thm. 6.4]). By the corollary and relation (10.15) we can now conclude that for all symbols $a \in \mathrm{S}^m(Q)$ and $b \in \mathrm{S}^{\tilde{m}}(Q)$ there is an asymptotic expansion of the operator product $\mathsf{Op}_{\hbar,\kappa}(a)\mathsf{Op}_{\hbar,\kappa}(b)$ of the form

$$\mathsf{Op}_{\hbar,\kappa}(a)\mathsf{Op}_{\hbar,\kappa}(b) \sim \mathsf{Op}_{\hbar,\kappa}(ab) + \sum_{k=1}^{\infty} \hbar^k \mathsf{Op}_{\hbar,\kappa}(C_k(a,b)), \tag{10.16}$$

where the $C_k$ denote the $k$-th bidifferential operator of the star product $\star^B_\kappa$. Moreover Eq. (10.14) shows that $\eta^L_\kappa$ is induced by $\mathsf{Op}_{\hbar,\kappa}$.

In the following we want to generalize this result to the case of the star product $\star^B_\kappa$ when $B = \lambda B_1 + \lambda^2 B_2 + O(\lambda^3)$ is real formal two-form fulfilling (IC). Now, choose a formal vector potential $A = \lambda^2 A_2 + O(\lambda^3)$ like in Section 8 such that $dA = B - \lambda B_1$. Next we choose a smooth map $\mathbb{R}^+ \to \Omega^1(Q)$, $\hbar \mapsto A_\hbar$ having Taylor expansion $A$ around $\hbar = 0$. Then let $B_\hbar = \hbar B_1 + dA_\hbar$, and $\nabla^L_\hbar = \nabla^L + \frac{\mathrm{i}}{\hbar} A_\hbar$ and check that the curvature of $\nabla^L_\hbar$ is just $\mathrm{i} B_\hbar$. Finally we associate to every symbol $a \in \mathrm{S}^\infty(Q)$ and all $\hbar \in \mathbb{R}^+$, $\kappa \in [0,1]$ the pseudodifferential operator $\mathsf{Op}^A_{\hbar,\kappa}(a) = \mathsf{Op}^{[\nabla^L_\hbar,\chi]}_{\hbar,\kappa}(a)$. Then we arrive at the following main result.

**Theorem 10.8** *For all real formal series $B = \lambda B_1 + \lambda^2 B_2 + O(\lambda^3)$ and $A = \lambda^2 A_2 + O(\lambda^3)$ of two- resp. one-forms such that $B$ is integral and $B = \lambda B_1 + dA$ the operator map $\mathsf{Op}^A_{\hbar,\kappa}$ comprises an operator representation of $\star^B_\kappa$ on the Hermitian line bundle $L$ with Chern class $\frac{1}{2\pi}[B_1]$ and metric connection $\nabla^L$. Moreover the formal representation $\eta^A_\kappa$ of $\star^B_\kappa$ is induced by the operator calculus $\mathsf{Op}^A_{\hbar,\kappa}$. In case $A = 0$ one has $\mathsf{Op}^A_{\hbar,\kappa} = \mathsf{Op}_{\hbar,\kappa}$.*

Proof: The relation $\mathsf{Op}^A_{\hbar,\kappa} = \mathsf{Op}_{\hbar,\kappa}$ is obvious in case $A = 0$, hence the claim follows for $A = 0$. Now, fix $\hbar$ and let $\hat{\hbar} \in \mathbb{R}^+$ be arbitrary. Then again by the above considerations the formal representation $\tilde{\eta}_\kappa$ of $\star^{\frac{\lambda}{\hbar} B_\hbar}_\kappa$ defined by the Hermitian connection $\nabla^L_\hbar$ and the formal one-form $\tilde{A} = 0$ is induced by the operator calculus $\mathsf{Op}^{[\nabla^L_\hbar]}_{\hat{\hbar},\kappa}$, or formally

$$\mathsf{Op}^{[\nabla^L_\hbar]}_{\hat{\hbar},\kappa}(a) = \sum_{l=0}^{\infty} \hat{\hbar}^l \tilde{\eta}_{\kappa,l}(a) \bmod \hat{\hbar}^\infty,$$

where $\tilde{\eta}_{\kappa,l}$ depends on $\hbar$. From Eq. (5.6) it follows that one has $\sum_{l=0}^\infty \hbar^l \tilde{\eta}_{\kappa,l}(a) = \sum_{l=0}^\infty \hbar^l \eta^A_{\kappa,l}(a)$ for $\hbar = \hat{\hbar}$. Consequently

$$\mathsf{Op}^A_{\hbar,\kappa}(a) = \sum_{l=0}^{\infty} \hbar^l \eta^A_{\kappa,l}(a) \bmod \hbar^\infty$$

which entails the claim. □

# 11 GNS construction for pseudodifferential operators

In this section we show how the symbol and pseudodifferential operator calculus for the Weyl ordered type, i.e. for $\kappa = 1/2$ can be understood as a particular GNS construction analogously to the formal case as considered in Theorem 9.3.

To achieve this we will first introduce an integral representation of the formal operator $N_\kappa$. Hereby we can assume without restriction that the cut-off function $\chi$ is the square of another cut-off function $\sqrt{\chi}$, and likewise for $\chi_\kappa$. Moreover, we denote by $\mathrm{S}^m_{\mathrm{cpt}}(Q; \mathsf{Hom}(E,F))$ those symbols $a$ in $\mathrm{S}^m(Q; \mathsf{Hom}(E,F))$ such that $\overline{\pi(\mathrm{supp} a)} \subset Q$ is compact. Then the following proposition determines the relation between the different orderings in the various operator calculi.

**Proposition 11.1** *i.) The operator $N^{\mathrm{op}}_\kappa : \mathrm{S}^\infty(Q; \mathsf{Hom}(E,F)) \to \mathrm{S}^\infty(Q; \mathsf{Hom}(E,F))$ defined by*

$$\begin{aligned}(N^{\mathrm{op}}_\kappa a)(\zeta'_x) = &\frac{1}{(2\pi\hbar)^n} \int_{T^*_x Q} \int_{T_x Q} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\langle \zeta_x, v_x\rangle} \sqrt{\chi_\kappa}(v_x) \rho(\exp_x(\kappa v_x), x) \\ &\tau^F_{(x,\exp_x(\kappa v_x))} a\left( (T_{\exp_x(\kappa v_x)} \exp_x^{-1})^*(\zeta'_x + \zeta_x) \right) \tau^E_{(\exp_x(\kappa v_x),x)} \mu_x(v_x) \mu^*_x(\zeta_x)\end{aligned} \tag{11.1}$$

*for $a \in \mathrm{S}^\infty(Q; \mathsf{Hom}(E,F))$ maps $\mathrm{S}^m_{\mathrm{cpt}}(Q; \mathsf{Hom}(E,F))$ to $\mathrm{S}^m_{\mathrm{cpt}}(Q; \mathsf{Hom}(E,F))$ and fulfills the equation*

$$\mathsf{Op}^{[\chi]}_\kappa(a) = \mathsf{Op}^{[\sqrt{\chi_\kappa}]}_0(N^{\mathrm{op}}_\kappa a). \tag{11.2}$$

*ii.)* *There exists a cut-off function* $\psi_\kappa : TQ \to [0,1]$ *such that*

$$\mathsf{Op}_0^{[\psi_\kappa]}(a) = \mathsf{Op}_0^{[\sqrt{\chi_\kappa}]}(N_\kappa^{\mathrm{op}}\overline{N_\kappa^{\mathrm{op}}}a) = \mathsf{Op}_\kappa^{[\chi]}(\overline{N_\kappa^{\mathrm{op}}}a) \tag{11.3}$$

*for every* $a \in \mathrm{S}^m(Q; \mathsf{Hom}(E,F))$. *Hereby* $\overline{N_\kappa^{\mathrm{op}}}$ *is defined as* $N_\kappa^{\mathrm{op}}$ *by Eq. (11.1) only with complex conjugated phase factor* $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\langle\zeta_x, v_x\rangle}$. *In particular* $N_\kappa^{\mathrm{op}}$ *and* $\overline{N_\kappa^{\mathrm{op}}}$ *are quasi-inverse to each other, i.e.* $N_\kappa^{\mathrm{op}}\overline{N_\kappa^{\mathrm{op}}} - \mathsf{id}$ *and* $\overline{N_\kappa^{\mathrm{op}}}N_\kappa^{\mathrm{op}} - \mathsf{id}$ *are smoothing.*

*iii.)* *For every* $\mathsf{p} \in \mathcal{P}(Q; \mathsf{Hom}(E,F))$ *and all* $\kappa \in [0,1]$ *one has*

$$N_\kappa^{\mathrm{op}}\mathsf{p} = \exp\left(-\mathrm{i}\kappa\hbar\Delta_\mu^{\mathsf{Hom}(E,F)}\right)\mathsf{p} \quad \text{and} \quad \overline{N_\kappa^{\mathrm{op}}}\mathsf{p} = \exp\left(\mathrm{i}\kappa\hbar\Delta_\mu^{\mathsf{Hom}(E,F)}\right)\mathsf{p}. \tag{11.4}$$

*In general* $N_\kappa^{\mathrm{op}}$ *and* $\overline{N_\kappa^{\mathrm{op}}}$ *possess Taylor expansions which are given explicitly by*

$$N_\kappa^{\mathrm{op}}a = \exp\left(-\mathrm{i}\kappa\hbar\Delta_\mu^{\mathsf{Hom}(E,F)}\right)a \bmod \hbar^\infty \quad \text{and} \quad \overline{N_\kappa^{\mathrm{op}}}a = \exp\left(\mathrm{i}\kappa\hbar\Delta_\mu^{\mathsf{Hom}(E,F)}\right)a \bmod \hbar^\infty. \tag{11.5}$$

Proof: For the proof of Eq. (11.2) first check the claim on $\mathrm{S}^{-\infty}(Q; \mathsf{Hom}(E,F))$ by straightforward computation and extend it to $\mathrm{S}^\infty(Q; \mathsf{Hom}(E,F))$ by a continuity argument. Likewise one shows part ii.). The fact that $\mathrm{S}^m_{\mathrm{cpt}}(Q; \mathsf{Hom}(E,F))$ is mapped into itself under $N_\kappa^{\mathrm{op}}$ follows from the fact that $\overline{\pi(\mathsf{supp}a)}$ and $\mathsf{supp}(\chi_\kappa|_{T_xQ})$ are compact and the integral representation (11.1). Corollary 10.5 and part i.) entail part iii.) by first checking the claim on polynomial symbols $\mathsf{p} \in \mathcal{P}(Q; \mathsf{Hom}(E,F))$. □

From now on we specialize to the case of Weyl ordered calculus and $E = F = L$, where $L$ is a Hermitian line bundle with Hermitian connection. The following two lemmas are crucial for the GNS construction we have in mind.

**Lemma 11.2** *Let* $s_0 \in \Gamma^\infty(L)$ *be a section such that* $s|_U \neq 0$ *for some open set* $U \subset Q$. *Then for every* $s \in \Gamma^\infty_{\mathrm{cpt}}(L)$ *with* $\mathsf{supp}s \subset U$ *there is a symbol* $a \in \mathrm{S}^{-\infty}_{\mathrm{cpt}}(Q)$ *such that* $\mathsf{Op}_0(a)s_0 = s$.

Proof: Let $u \in \mathcal{C}^\infty_{\mathrm{cpt}}(Q)$ be the unique function with $\mathsf{supp}u \subset U$ such that $us_0 = s$ and let $\sigma : TU \to \mathbb{C}$ be the unique smooth function satisfying $s_0^\chi(v_x) = \sigma(v_x)s_0(x)$. Then $\sigma|_{T_xU}$ has compact support due to the choice of the cut-off function $\chi$ and fulfills $\sigma(0_x) = 1$ for all $x \in U$. Hence

$$S(x) := \frac{1}{(2\pi\hbar)^{n/2}}\int_{T_xQ} |\sigma(v_x)|^2 \, \mu_x(v_x) > 0$$

is a strictly positive smooth function on $U$, so $\widetilde{b}(v_x) := \frac{1}{S(x)}\overline{\sigma(v_x)}$ is a well-defined smooth function on $TU$ such that $\widetilde{b}|_{T_xQ}$ has compact support for every $x \in U$. Its fiberwise Fourier transform

$$b(\zeta_x) := \frac{1}{(2\pi\hbar)^{n/2}}\int_{T_xQ} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\langle\zeta_x, v_x\rangle}\,\widetilde{b}(v_x)\,\mu_x(v_x)$$

is an element of $\mathrm{S}^{-\infty}(U)$. Finally let $a = b(\pi^*u) \in \mathrm{S}^{-\infty}_{\mathrm{cpt}}(Q)$ and check by a straightforward computation using the inverse fiberwise Fourier transform that this $a$ fulfills the claim. □

**Lemma 11.3** *Let* $s_0 \in \Gamma^\infty(L)$ *be a section such that* $s_0(x_0) = 0$ *and* $s_0$ *is transversal at* $x_0$ *for some* $x_0 \in Q$. *Then there exists an open neighborhood* $U$ *of* $x_0$ *such that for every* $s \in \Gamma^\infty_{\mathrm{cpt}}(L)$ *with* $\mathsf{supp}s \subset U$ *there exists a symbol* $a \in \mathrm{S}^{-\infty}_{\mathrm{cpt}}(Q)$ *satisfying* $\mathsf{Op}_0(a)s_0 = s$.

PROOF: First choose a contractible chart $V$ around $x_0$ with coordinates $q^1, \ldots, q^n$ and induced coordinates $v^1, \ldots, v^n$ resp. $p_1, \ldots, p_n$ for the tangent resp. cotangent vectors. Moreover, let $e \in \Gamma^\infty(L|_V)$ be a local non-vanishing section and let $\sigma_k : TV \to \mathbb{C}$ be the unique smooth functions with $(\frac{\partial}{\partial v^k} s_0^\chi)(v_x) = \sigma_k(v_x) e(x)$ for $k = 1, \ldots, n$. Then $\sigma_k|_{T_xV}$ has compact support for every $x \in V$. Due to the transversality of $s_0$ at $x_0$ we notice that not all of the $\sigma_k(0_{x_0})$ vanish whence the function

$$S(x) := \frac{1}{(2\pi\hbar)^{n/2}} \sum_{k=1}^{n} \int_{T_xQ} |\sigma_k(v_x)|^2 \, \mu_x(v_x) \geq 0$$

is smooth and non-negative over $V$ and satisfies $S(x_0) > 0$ due to the transversality of $s_0$ at $x_0$. Hence we can find an open neighborhood $U \subset V$ of $x_0$ such that over $U$ the function $S$ is strictly positive. Now, let $s \in \Gamma^\infty_{\mathrm{cpt}}(L|_U)$ and $u \in \mathcal{C}^\infty_{\mathrm{cpt}}(U)$ be the unique function with $s = ue$. Furthermore define $\widetilde{b}_k(v_x) = \frac{1}{S(x)} \overline{\sigma_k(v_x)}$ for $v_x \in T_xU$ and $x \in U$. Then $\widetilde{b}_k$ is smooth and $\widetilde{b}_k|_{T_xU}$ has compact support. Thus its fiberwise Fourier transform $b_k$ is an element of $\mathrm{S}^{-\infty}(U)$. Next define the global symbol $a \in \mathrm{S}^{-\infty}_{\mathrm{cpt}}(Q)$ by

$$a(\zeta_x) = \frac{\mathrm{i}}{\hbar} u(x) \sum_{k=1}^{n} b_k(\zeta_x) p_k(\zeta_x).$$

Then another straightforward computation using the inverse fiberwise Fourier transform and involving one partial integration shows that $\mathsf{Op}_0(a) s_0 = s$. ☐

**Proposition 11.4** *Let $s_0 \in \Gamma^\infty(L)$ be a transversal section and $s \in \Gamma^\infty_{\mathrm{cpt}}(L)$ arbitrary. Then there exists a symbol $a \in \mathrm{S}^{-\infty}_{\mathrm{cpt}}(Q)$ such that $\mathsf{Op}_0(a) s_0 = s$.*

PROOF: The claim follows by the above two lemmas and a partition of unity argument analogously to the proof of Lemma 9.1. ☐

Now let us come to the GNS construction for pseudodifferential operators. Let $\mathcal{A} \subset \mathsf{End}(\Gamma^\infty_{\mathrm{cpt}}(L))$ be the algebra generated (not topologically but only algebraically) by pseudodifferential operators of the form $\mathsf{Op}_{\hbar,\mathrm{w}}(a)$, where $a \in \mathrm{S}^{-\infty}_{\mathrm{cpt}}(Q)$. Then $\mathcal{A}$ is a *-algebra with the *-structure given by the (formal) operator adjoint on the pre-Hilbert space $\Gamma^\infty_{\mathrm{cpt}}(L)$. The *-involution now fulfills $\mathsf{Op}_{\hbar,\mathrm{w}}(a)^* = \mathsf{Op}_{\hbar,\mathrm{w}}(\overline{a})$ which follows from the definition of $\mathsf{Op}_{\hbar,\mathrm{w}}$ in Theorem 10.1. Note that in general $\mathcal{A}$ has no unit element. The aim is now to define a positive functional on this algebra such that the GNS pre-Hilbert space (over $\mathbb{C}$) is isometric to $\Gamma^\infty_{\mathrm{cpt}}(L)$ and such that the induced GNS representation coincides with the usual action of $\mathcal{A}$ on the sections of $L$. So choose a transversal section $s_0 \in \Gamma^\infty(L)$ and set

$$\omega_{s_0}(A) = \langle s_0, A s_0 \rangle_{L,\mu}, \qquad A \in \mathcal{A}. \tag{11.6}$$

Then $\omega_{s_0} : \mathcal{A} \to \mathbb{C}$ comprises a well-defined functional due to the fact that every operator $\mathsf{Op}_{\hbar,\mathrm{w}}(a)$ maps $s_0$ onto some section with compact support. Then the following lemma is obvious.

**Lemma 11.5** *The so-called expectation value functional $\omega_{s_0}$ is positive with Gel'fand ideal given by $\mathcal{J}_{s_0} = \{A \in \mathcal{A} \mid A s_0 = 0\}$.*

**Remark 11.6** In case $\mathcal{A}$ were the algebra of bounded operators on a Hilbert space and $s_0$ a non-zero vector, the analogue of the above functional would reproduce the Hilbert space as GNS representation space and the usual action of operators as GNS representation. In our case the result is not quite as easy to achieve. The main problem lies in the proof of the surjectivity of the isomorphism in the following theorem.

**Theorem 11.7** *Let $s_0 \in \Gamma^\infty(L)$ be a transversal section and $\omega_{s_0} : \mathcal{A} \to \mathbb{C}$ its expectation value functional. Then the quotient pre-Hilbert space $\mathfrak{H}_{s_0} = \mathcal{A}/\mathcal{J}_{s_0}$ is isometric to $\Gamma^\infty_{\mathrm{cpt}}(L)$ by the unitary map*

$$\mathfrak{H}_{s_0} \ni \psi_A \mapsto As_0 \in \Gamma^\infty_{\mathrm{cpt}}(L), \qquad A \in \mathcal{A}. \tag{11.7}$$

*The GNS representation of $\mathcal{A}$ on $\mathcal{A}/\mathcal{J}_{s_0}$ is unitarily equivalent to the action of $\mathcal{A}$ on $\Gamma^\infty_{\mathrm{cpt}}(L)$ under the unitary map (11.7).*

PROOF: First note that (11.7) is well-defined indeed and isometric, hence injective. The surjectivity of (11.7) is less trivial but follows from Prop. 11.4 and relation (11.3) for $\kappa = 1/2$. Then the unitary equivalence of the GNS representation with the usual action of $\mathcal{A}$ on $\Gamma^\infty_{\mathrm{cpt}}(L)$ follows immediately. □

# A Positive functionals and formal GNS construction

For the reader's convenience let us recall some basic facts on positive functionals and GNS constructions in deformation quantization. For a detailed exposition and proofs we refer the reader to [8] and [5, App. A].

The main observation which leads to the notion of a formal GNS representation is the fact that $\mathbb{R}[[\lambda]]$ is an *ordered ring* by the following simple definition: a nonzero element $a = \sum_{r=0}^\infty \lambda^r a_r \in \mathbb{R}[[\lambda]]$ is called *positive* if $a_{r_0} > 0$, where $r_0 := \min\{r \in \mathbb{N} \mid a_r \neq 0\}$. As easily verified the sum and product of two positive elements is again positive and each element is either negative, null, or positive. Moreover, after extending complex conjugation coefficientwise to the ring $\mathbb{C}[[\lambda]] = \mathbb{R}[[\lambda]] \oplus \mathrm{i}\mathbb{R}[[\lambda]]$ the following relations and inequalities hold: for $z \in \mathbb{C}[[\lambda]]$ one has $\overline{z}z \geq 0$, and $\overline{z}z = 0$ is true if and only if $z = 0$. Note that we regard $\lambda$ as real, i.e. we set $\overline{\lambda} = \lambda$.

Now let $M$ be a symplectic or Poisson manifold with star product $*$ such that the pointwise complex conjugation $f \mapsto \overline{f}$ is an anti-automorphism of $*$, i.e. we have $\overline{f * g} = \overline{g} * \overline{f}$ for all $f, g \in \mathcal{C}^\infty(M)[[\lambda]]$. Then a $\mathbb{C}[[\lambda]]$-linear functional $\omega : \mathcal{C}^\infty(M)[[\lambda]] \to \mathbb{C}[[\lambda]]$ is called *positive* if and only if $\omega(\overline{f} * f) \geq 0$. Note that one considers $\mathbb{C}[[\lambda]]$-linear and $\mathbb{C}[[\lambda]]$-valued functionals instead of $\mathbb{C}$-linear and $\mathbb{C}$-valued ones. The crucial point is that the notion of positivity extends naturally to this framework. One immediately verifies now the Cauchy-Schwarz inequality for positive functionals

$$\begin{gathered} \omega(\overline{f} * g) = \overline{\omega(\overline{g} * f)} \\ \omega(\overline{f} * g)\overline{\omega(\overline{f} * g)} \leq \omega(\overline{f} * f)\omega(\overline{g} * g), \end{gathered} \tag{A.1}$$

where $f, g \in \mathcal{C}^\infty(M)[[\lambda]]$. This implies by pure algebraic reasoning as in the well-known case of $C^*$-algebras that

$$\mathcal{J}_\omega := \left\{ f \in \mathbb{C}^\infty(M)[[\lambda]] \;\middle|\; \omega(\overline{f} * f) = 0 \right\} \tag{A.2}$$

is a left-ideal, the so-called *Gel'fand ideal* of $\omega$ in $\mathcal{C}^\infty(M)[[\lambda]]$.

Given a positive $\mathbb{C}[[\lambda]]$-linear functional $\omega$ with Gel'fand ideal $\mathcal{J}_\omega$ one considers the quotient $\mathfrak{H}_\omega := C^\infty(M)[[\lambda]]/\mathcal{J}_\omega$ which carries both a $\mathcal{C}^\infty(M)[[\lambda]]$-left module structure and a Hermitian product: one defines the Hermitian product by

$$\langle \psi_f, \psi_g \rangle := \omega(\overline{f} * g) \in \mathbb{C}[[\lambda]], \tag{A.3}$$

where $\psi_f, \psi_g \in \mathfrak{H}_\omega$ denote the equivalence classes of $f, g$. Indeed $\langle \cdot, \cdot \rangle$ is $\mathbb{C}[[\lambda]]$-linear in the second argument and satisfies $\langle \psi_f, \psi_g \rangle = \overline{\langle \psi_g, \psi_f \rangle}$ as well as the positivity requirement $\langle \psi_f, \psi_f \rangle \geq 0$ for all $\psi_f \in \mathfrak{H}_\omega$ and the non-degeneracy $\langle \psi_f, \psi_f \rangle = 0$ if and only if $\psi_f = 0$. Thus $\mathfrak{H}_\omega$ becomes with this $\mathbb{C}[[\lambda]]$-valued Hermitian product a pre-Hilbert space over the ring $\mathbb{C}[[\lambda]]$. Moreover the $\mathcal{C}^\infty(M)[[\lambda]]$-left module structure induces the *GNS representation* $\pi_\omega$ of $\mathcal{C}^\infty(M)[[\lambda]]$ on $\mathfrak{H}_\omega$ given by

$$\pi_\omega(f)\psi_g := \psi_{f*g}, \tag{A.4}$$

which turns out to be a representation of the star product algebra. This representation is even a $^*$-representation, i.e. we have

$$\langle \psi_g, \pi_\omega(f)\psi_h \rangle = \langle \pi_\omega(\overline{f})\psi_g, \psi_h \rangle \tag{A.5}$$

for all $\psi_g, \psi_h \in \mathfrak{H}_\omega$ and $f \in \mathcal{C}^\infty(M)[[\lambda]]$. In this sense we also write $\pi(f)^* = \pi(\overline{f})$.

Finally we would like to mention that the domain of $\omega$ need not be the whole algebra $\mathcal{C}^\infty(M)[[\lambda]]$ but rather a two-sided ideal $\mathcal{B}$ stable under complex conjugation $\overline{\mathcal{B}} = \mathcal{B}$. Then $\mathcal{J}_\omega$ turns out to be even a left-ideal in $\mathcal{C}^\infty(M)[[\lambda]]$ due to the Cauchy-Schwarz inequality. Thus the GNS representation $\pi_\omega$ of $\mathcal{B}$ on $\mathfrak{H}_\omega = \mathcal{B}/\mathcal{J}_\omega$ extends naturally to a $^*$-representation of the whole algebra $\mathcal{C}^\infty(M)[[\lambda]]$, a situation which is present throughout this paper. Here the two-sided ideal is given e.g. by $\mathcal{C}^\infty_Q(T^*Q)[[\lambda]]$.

For further generalizations to arbitrary ordered rings as well as more results in the context of deformation quantization including the field of formal Laurent series $\mathbb{C}((\lambda))$ and its field extension $\mathbb{C}\langle\langle\lambda\rangle\rangle$ we refer to [8]. For a detailed treatment of the implementation of symmetries as unitary operators in GNS representations, the 'pull-back' of positive functionals and the corresponding GNS representations we refer to [5].

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