On nonformal simply connected symplectic manifolds

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Abstract

For any $N \geq 5$ nonformal simply connected symplectic manifolds of dimension $2N$ are constructed. This disproves the formality conjecture for simply connected symplectic manifolds which was introduced by Lupton and Oprea.

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1. Introduction and main results

A smooth manifold M is called symplectic if it carries a nondegenerate closed 2-form ω which is called a symplectic form. In this event a symplectic manifold means a pair (M, ω). Since the skew-symmetric form ω is nondegenerate, M is even-dimensional and moreover such manifold always carries an almost complex structure.

Topology of symplectic manifolds is being intensively developed now ([18]) but still there are no many examples of compact symplectic manifolds. The simplest of them are Kähler manifolds and there are three constructions of new manifolds from old ones. These are a symplectic fibration ([25]), a blow up ([17]) and a fiber connect sum ([8]). The latter two constructions were first outlined in [11].

By results of Gromov ([10]) and Tischler ([26]), any compact symplectic manifold is diffeomorphic to a symplectic submanifold of a complex projective space. The problem posed by Weinstein was to find compact symplectic manifolds not carrying Kähler structure. First example of such manifold was found by Thurston ([25]) and the first simply connected example of such manifold was constructed by McDuff ([17]). Later Gompf constructed simply connected examples of the minimal possible dimension which equals four ([8]).

An important property of Kähler manifolds is their formality established by Deligne, Griffiths, Morgan, and Sullivan in [5], which means that the rational homotopy type is completely determined by the rational cohomology ring. Formality of Kähler manifolds was used for distinguishing non simply connected symplectic manifolds with no Kähler structure in [2, 14]. For non simply connected spaces the notion of formality is only defined for nilpotent spaces, whereas for simply connected spaces it is always defined. Meanwhile the problem of existence of nonformal simply connected symplectic manifolds was open until now and moreover there was a conjecture that such manifolds do not exist (the Lupton–Oprea conjecture, [27]). We disprove it as follows

Main theorem. *For any $N \geq 5$ there exist infinitely many pairwise non-homotopy-equivalent nonformal simply connected symplectic manifolds of dimension $2N$.*

This theorem is proved as follows. Take a symplectic embedding of the Kodaira–Thurston manifold $\widetilde{M}$ which is a symplectic four-dimensional nilmanifold ([25]) into $\mathbf{C}P^N$ for $N \geq 5$ and symplectically blow up $\mathbf{C}P^N$ along $\widetilde{M}$ obtaining a symplectic manifold $\widetilde{\mathbf{C}P}^N$. It was McDuff who established that $\widetilde{\mathbf{C}P}^5$ is a simply connected compact symplectic manifold with no Kähler structure ([17]). We prove that $\widetilde{\mathbf{C}P}^N$ is nonformal for $N \geq 5$ (see Theorem 6.6).

Moreover we construct an infinite series of symplectic nilmanifolds $M(2m)$ of dimension $2m \geq 4$ and prove that they are nonformal (Theorem 6.5) itself and the blow ups of $\mathbf{C}P^N$ along these manifolds embedded into $\mathbf{C}P^N$ are simply connected and nonformal (Theorem 6.6).

Since the blow up at a point of nonformal simply connected manifold is again nonformal, successively applying blows up at points we obtain from one nonformal symplectic manifold infinitely many pairwise non-homotopy-equivalent nonformal symplectic manifolds of the same dimension.

In §7 we formulate and discuss some conjectures on nonformal symplectic manifolds. We also would like to mention in the introduction that, since all compact simply connected manifolds of dimension ≤ 6 are formal ([20]), the following problem stays open:

Problem. *Are there nonformal simply connected symplectic manifolds of dimension* 8 *?*

The results, of the present paper, in slightly weaker form implying the existence of such manifolds of dimension $2N$ for any $N \geq 6$ were announced in [1].

2. Some facts on symplectic manifolds

An existence of an almost complex structure on symplectic manifolds is demonstrated as follows. Take a Riemannian metric $(\cdot,\cdot)$ on M and define an operator A by the condition

$$(Au, v) = \omega(u, v).$$

The operator A is skew-symmetric and hence the operator $A^*A = -A^2$ is symmetric and positive. Take a positive symmetric square root of it, $Q = \sqrt{-A^2} > 0$, and put $J = AQ^{-1}$. It is clear that

$$J^2 = -1,$$

which means that J defines an almost complex structure, and

$$\langle u, v\rangle = \omega(u, Jv)$$

is a Hermitian metric on M, i.e., a Riemannian metric with respect to which J is skew-symmetric, which means that an almost complex structure J is compatible with a symplectic structure ω.

This procedure enables us to introduce a smooth compatible almost complex structure on any smooth family of symplectic vector spaces, i.e. vector spaces with symplectic forms.

Gromov proved that an open almost complex manifold M always carries a compatible symplectic structure ([10]).

For compact manifolds an existence of an almost complex structure does not imply an existence of a symplectic structure and the simplest additional necessary condition is an existence of a 2-form ω such that its powers ω^j are cohomologically nontrivial for $j = 1,\dots,N$: $[\omega]^j \neq 0$ in $H^{2j}(M)$.

A complex manifold M is called a Kähler manifold if it carries a Hermitian metric $h_{ij}dz^i d\bar{z}^j$ such that the form $\omega = h_{ij}dz^i d\bar{z}^j$ is closed. This form is symplectic and therefore any Kähler manifold carries a natural symplectic structure.

The simplest examples of Kähler manifolds are algebraic manifolds which are complex submanifolds of the complex projective spaces. For such manifold a Kähler structure is given by a metric induced from the Fubiny–Study metric by the embedding. Denote by $(\mathbf{C}P^n, \omega_{FS})$ the complex projective space $\mathbf{C}P^n$ with a Kähler form ω_{FS} induced by the Fubiny–Study metric. These symplectic manifolds serve as universal symplectic manifolds in the following sense.

Proposition 2.1. ([10, 26]) *Given a compact symplectic manifold* (M,ω) *of dimension* $2n$ *with the form* ω *integer, i.e.* $[\omega] \in H^2(M;\mathbf{Z}) \subset H^2(M;\mathbf{R})$, *there exists an embedding*

$$f : M \to \mathbf{C}P^{2n+1}$$

such that $f^*\omega_{FS} = \omega$.

Rational forms, which are η with $[\eta] \in H^2(M;\mathbf{Q})$, are dense in $H^2(M;\mathbf{R})$ and hence for any symplectic form ω on M there exists its small closed and nondegenerate perturbation $\omega + \widetilde{\omega}$ such that the form $K(\omega + \widetilde{\omega})$ is integer for some $K \in \mathbf{R}$

and therefore $(M, K(\omega+\widetilde{\omega}))$ is a symplectic submanifold of $(\mathbf{C}P^{2n+1}, \omega_{FS})$. Hence symplectic submanifolds of the complex projective spaces give us all topological types of compact symplectic manifolds.

The first example of a compact symplectic manifold with no Kähler structure is the Kodaira–Thurston manifold $\widetilde{M}$ which is as follows.

Denote by $\mathcal{H}$ the three-dimensional Heisenberg group formed by the upper-triangular matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

with $x, y, z \in \mathbf{R}$ and the multiplication operation. The matrices with x, y, z integers form a uniform lattice $\mathcal{H}_{\mathbf{Z}}$ in $\mathcal{H}$.

On the circle S^1, take a coordinate u defined modulo 1 and put

$$\widetilde{M} = (\mathcal{H}/\mathcal{H}_{\mathbf{Z}}) \times S^1$$

with the form

$$\omega = dx \wedge du + dy \wedge dz. \tag{1}$$

This is a four-dimensional nilmanifold whose one-dimensional real cohomologies are generated by the forms dx, dy, and du, and therefore the first Betti number equals three. Since the odd-dimensional Betti numbers of Kähler manifolds are even, $\widetilde{M}$ does not carry a Kähler structure but admits a complex structure.

Later examples of four-dimensional symplectic nilmanifolds with no complex structure were found ([6], see also §6).

3. On the blow up construction and simply connected symplectic manifolds with no Kähler structure

The blow up construction is defined for any pair $Y \subset X$ where the structure group of the normal bundle to Y reduces to $U(k)$ with $2k = \dim X - \dim Y$.

Let (X, ω) be a compact symplectic manifold of dimension $2N$ and let Y be a symplectic submanifold, of X, of dimension $2(N-k)$. For any $p \in Y$ consider a subspace $E_p \subset T_pX$ which is an orthogonal complement to T_pY with respect to ω. These subspaces form a bundle $E = E_Y$, over Y, isomorphic to the normal bundle to Y. Since Y is a symplectic submanifold, we have

1) the restriction of ω onto E_pY is nondegenerate for any $p \in Y$;

2) the normal bundle to Y in X is naturally identified with E.

By the method exposed in §2 construct a fiberwise almost complex structure on E compatible with the restrictions of ω onto fibres. This implies that the structure group of E reduces to $U(k) = SO(2k) \cap Sp(k)$.

Now let Y be a submanifold of X and the structure group of the normal bundle E to Y be $U(k)$. Identify fibres of E with $\mathbf{C}^k$ and consider another bundle

$$\widetilde{E} \to Y$$

with fibres isomorphic to the canonical line bundle over $\mathbf{C}P^{k-1}$. This canonical line bundle is

$$L \to \mathbf{C}P^{k-1}$$

with

$$L = \{(z, l) \in \mathbf{C}^k \times \mathbf{C}P^{k-1} | z \in l\}.$$

The condition $z \in l$ is written as

$$z_i l_j = z_j l_i \quad \text{for } i, j = 1, \ldots, k$$

with $(l_1, \ldots, l_k)$ the homogeneous coordinates on $\mathbf{C}P^{k-1}$. This bundle is associated with E which means that the action of the structure group on $\widetilde{E}$ is as follows

$$A \cdot (z, l) = (Az, Al)$$

where $A \cdot z = Az$ is the corresponding action of the structure group of E.

The fibres of E are endowed with a Hermitian metric and we denote by E_r and $\widetilde{E}_r$ the submanifolds of E and $\widetilde{E}$ defined by the condition on z: $|z| \leq r$.

The fibre spaces $\left(\widetilde{E}_1 \setminus \widetilde{E}_1\right) \to Y$ and $(E_1 \setminus E_0) \to Y$ are canonically isomorphic and their fibres are diffeomorphic to a punctured disc $\{z \in \mathbf{C}^k | 0 < |z| \leq 1\}$. The fibre space $E_0 \to Y$ is just the zero section of $E \to Y$ and the fibre space

$$\widetilde{Y} = \widetilde{E}_0 \to Y$$

is called the projectivization of $E \to Y$ and its fibres are diffeomorphic to $\mathbf{C}P^{k-1}$.

Now we are able to construct the blow up of X along Y. For that take a closed tubular neighborhood V of Y in X and naturally identify it with the fibre space E_1. Now take the manifold

$$\widetilde{X} = \overline{(X \setminus V)} \cup_{\partial V} = \widetilde{E}_1$$

with $\widetilde{E}_1$ glued to the boundary of $\overline{(X \setminus V)}$ by the natural isomorphism $\partial E_1 = \partial \widetilde{E}_1$.

The manifold $\widetilde{X}$ is called the blow up of X along Y and roughly speaking is obtained by replacing a closed neighborhood V, of Y, which is a disc bundle over Y by a manifold with boundary $\widetilde{V}$ which is a disc bundle over $\widetilde{Y}$.

There is a natural projection

$$\pi : \widetilde{X} \to X$$

which is a diffeomorphism onto outside Y and whose restriction onto $\pi^{-1}(Y)$ is the fibering map $\widetilde{Y} \to Y$.

Notice that $\partial V = \partial E_1 = \partial \widetilde{E}_1$ fibers over Y with the fibre S^{2k-1} and $\widetilde{E}_1$ fibers over Y with the fibre $\overline{\mathbf{C}P^k \setminus D^{2k}}$ where an overline stands for the closure. The fibre of the latter bundle is homotopy equivalent to $\mathbf{C}P^{k-1}$ and the embedding

$$\widetilde{i} : \partial V \to \widetilde{E}_1 = \widetilde{V}$$

extends to a commutative diagram of the fibre bundles

$$\begin{tikzcd} \partial V = \partial \widetilde{E}_1 \arrow[rr] \arrow[dr, "S^{2k-1}"'] & & \widetilde{E}_1 \arrow[dl, "\sim \mathbf{C}P^{k-1}"] \\ & Y & \end{tikzcd} \tag{2}$$

and the horizontal mapping $\widetilde{i}$ preserves fiberings.

The Leray–Hirsch theorem implies

Proposition 3.1. *1) The cohomology ring of the projectivization* $\pi : \widetilde{Y} \to Y$ *of a vector bundle* $E \to Y$ *is isomorphic to*

$$H^*(\widetilde{Y}) = H^*(Y)[a] / \langle a^k + c_1 a^{k-1} + \ldots + c_{k-1} a + c_k \rangle$$

with $c_1, \dots, c_k$ the Chern classes of the vector bundle $E \to Y$. The induced homomorphism $\pi^* : H^*(Y) \to H^*(\widetilde{Y})$ is an monomorphism.

2) The cohomology class $a \in H^2(\widetilde{Y})$ maybe taken such that $\widetilde{i}^*(a) = 0$ in $H^2(\partial V)$.

The proof of the first statement of this proposition is exposed, for instance, in [12]. It may be explained also in terms of spectral sequences of fiberings from (2) and from the functoriality of these sequences follows that a, a generator of $E_2^{0,2} = H^0(Y; H^2(\mathbf{C}P^{k-1}))$ is mapped by $\widetilde{i}$ into $H^0(Y; H^2(S^{2k-1})) = 0$. This finishes the proof of the proposition.

Proposition 3.2. ([17]) *If $k \geq 2$, then $\pi_1(\widetilde{X}) = \pi_1(X)$.*

The proof of this proposition is as follows. By the Van Kampen theorem,

$$\pi_1(\widetilde{X}) = \pi_1(\overline{X \setminus V}) *_{\pi_1(\partial V)} \pi_1(\widetilde{E}_1).$$

Notice that V retracts onto Y. Since $\dim X - \dim Y \geq 4$, any 2-disc immersed into $X \setminus Y$ is deformed into a disc with no intersections with Y and this implies that $\pi_1(\overline{X \setminus V}) = \pi_1(X)$.

Since both fibres in (2) are simply connected, the projections induce isomorphisms of the fundamental groups. The diagram is commutative and therefore the embedding $\partial \widetilde{E}_1 \to \widetilde{E}_1$ also induces an isomorphism of the fundamental groups which together with the Van Kampen proves the proposition.

The simplest example is the blow up along a point $p = Y$ in $2n$-dimensional manifold X which results in adding the complex projective space:

$$\widetilde{X} = X \# \overline{\mathbf{C}P^n}$$

where an overline stands for the opposite orientation.

In [11] Gromov outlined the blow up for symplectic manifolds and the detailed exposition of it was done by McDuff ([17]):

Proposition 3.3. ([11, 17]) *If Y is a compact symplectic submanifold of a symplectic manifold (X, ω) then $\widetilde{X}$ carries a symplectic form $\widetilde{\omega}$ which equals $\pi^*\omega$ outside a neighborhood of $\pi^{-1}(Y)$.*

Before explaining the McDuff example recall two fundamental properties of compact Kähler manifolds M:

1) for any k the cohomology group $H^k(M; \mathbf{C})$ has a decomposition

$$H^k(M; \mathbf{C}) = \oplus_{p+q=k} H^{p,q}(M; \mathbf{C})$$

where the elements of $H^{p,q}(M; \mathbf{C})$ are realized by forms of the type $\alpha dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q}$ and the complex conjugation establishes isomorphisms $H^{p,q}(M; \mathbf{C}) \to H^{q,p}(M; \mathbf{C})$ (the Hodge decomposition). This implies, in particular, that if k is odd then $\dim H^k(M; \mathbf{C})$ is even;

2) for a manifold M of complex dimension n the mapping

$$H^{n-k}(M; \mathbf{C}) \xrightarrow{\times [\omega]^k} H^{n+k}(M; \mathbf{C})$$

is an isomorphism for any $k = 0, \dots, n$ (the Hard Lefschetz condition).

Now we are able to describe

The McDuff manifold.

The form (1) is integer and therefore, by the Gromov–Tischler theorem (see Proposition 2.1), there exists a symplectic embedding of the Kodaira–Thurston manifold $\widetilde{M}$ into $\mathbf{C}P^5$. Define $\widetilde{\mathbf{C}P}^5$ as the blow up of $\mathbf{C}P^5$ along $\widetilde{M}$.

Proposition 3.4. ([17]) *1) The dimension of $H^3(\widetilde{\mathbf{C}P}^5;\mathbf{C})$ equals 3. Therefore $H^*(\widetilde{\mathbf{C}P}^5;\mathbf{C})$ has no Hodge decomposition and $\widetilde{\mathbf{C}P}^5$ has no Kähler structure.*

2) The symplectic manifold $\widetilde{\mathbf{C}P}^5$ does not satisfy the Hard Lefschetz condition.

4. Minimal models and formality

A) Differential graded algebras and their minimal models.

Recall that a differential graded algebra is a graded algebra

$$\mathcal{A} = \oplus_{k\geq 0}\mathcal{A}^k$$

with a differential $d : \mathcal{A} \to \mathcal{A}$ of degree 1 , i.e. $d(\mathcal{A}^k)) \subset \mathcal{A}^{k+1}$, such that

1) $x \wedge y = (-1)^{kl} y \wedge x$ for $x \in \mathcal{A}^k, y \in \mathcal{A}^l$;

2) $d(x \wedge y) = dx \wedge y + (-1)^k x \wedge y$ for $x \in \mathcal{A}^k$ (the Leibnitz rule);

3) $d^2 = 0$.

This implies that the cohomology ring $H^*(\mathcal{A})$ of $\mathcal{A}$ is defined and this graded ring, supplied with the zero differential $d = 0$, is also a differential graded algebra.

In the sequel we consider algebras over the ground field F which may be $\mathbf{Q}$ (the rational numbers), $\mathbf{R}$ (the real numbers), or $\mathbf{C}$ (the complex numbers) and in addition assume that $\dim_F \mathcal{A}^k$ is finite for any k.

A homomorphism f of two differential graded algebras $(\mathcal{A}, d_\mathcal{A})$ and $(\mathcal{B}, d_\mathcal{B})$ is a homomorphism $f : \mathcal{A} \to \mathcal{B}$ of algebras such that

$$f(\mathcal{A}^k) \subset \mathcal{B}^k \quad \text{and} \quad f(da) = df(a),$$

i.e. it preserves the grading and the differential. This implies that any such homomorphism induces a homomorphism

$$f^* : H^*(\mathcal{A}) \to H^*(\mathcal{B})$$

of the cohomology rings.

An algebra is called connected if $H^0(\mathcal{A}) = F$ with F the ground field and it is called one-connected if in addition $H^1(\mathcal{A}) = 0$.

Let $x_1, x_2, \dots$ a sequence of weighted variables with $\deg x_i \geq 1$ for any i. Denote by $\Lambda(x_1, \dots)$ a free graded commutative algebra generated by $x_1, \dots$.

A differential graded algebra $\mathcal{M}$ is called minimal if

1) $\mathcal{M} = \Lambda(x_1, \dots)$ for some family of free generators and there is only finitely many generators of any fixed degree;

2) $dx_i \in \Lambda(x_1, \dots, x_{i-1})$.

Notice that for one-connected algebra the condition 2) maybe replaced by

2') $d(\mathcal{M}) \subset \mathcal{M}^+ \wedge \mathcal{M}^+$ where $\mathcal{M}^+ = \oplus_{k>0}\mathcal{M}^k$.

We say that $\mathcal{M}(\mathcal{A})$ is a minimal model for $\mathcal{A}$ if

1) $\mathcal{M}(\mathcal{A})$ is a minimal algebra;

2) there is a homomorphism $h : \mathcal{M} \to \mathcal{A}$ inducing an isomorphism of the cohomology rings.

The fundamental theorem of Sullivan says

Proposition 4.1. ([23]) *Every one-connected differential graded algebra has a minimal model unique up to isomorphism.*

Now, for any simply connected polyhedron X such that ranks of its homotopy groups are finite, construct the minimal algebra $\mathcal{M}_X$ over $\mathbf{Q}$ as follows.

Take the Postnikov tower of X which is a family of spaces X_k and mappings $f_k : X \to X_k$ such that

1) $\pi_j(X_k) = 0$ for $j > k$;

2) f_k induces isomorphisms $f_{k*} : \pi_j(X) \to \pi_j(X_k)$ for $j \leq k$;

3) there are commutative up to homotopy diagrams

$$\begin{array}{ccc} X & \stackrel{f_{k+1}}{\to} & X_{k+1} \\ & f_k \searrow & \downarrow \\ & & X_k \end{array}$$

where $p_{k+1} : X_{k+1} \to X_k$ is a fibration with the fibre $K(\pi_{k+1}(X), k+1)$.

By the Cartan–Serre theorem,

1) $H^*(K(\mathbf{Z}, n; \mathbf{Q})$ is a free graded commutative algebra $\Lambda(x)_n$, over $\mathbf{Q}$, generated by x with $\deg x = n$ which means that

$$\Lambda(x)_n = \mathbf{Q}[x] \text{ for } n \text{ even and } \Lambda(x)_n = F[x]/x^2 \text{ for } n \text{ odd;}$$

2) $H^j(K(G,n); \mathbf{Q}) = 0$ for any finite commutative group G and $j \geq 1$.
This remains true after replacing $\mathbf{Q}$ by $\mathbf{R}$ or $\mathbf{C}$.

Using the Cartan–Serre theorem together with the Postnikov tower, construct $\mathcal{M}_X$ inductively.

For $X = K(\pi, n)$ with $n \geq 2$ put $M_X = H^*(K(\pi, n); \mathbf{Q})$.

Assume that for X_k the algebra $\mathcal{M}_k = \mathcal{M}_{X_k}$ is defined and $H^*(M_k; \mathbf{Q}) = H^*(X_k; \mathbf{Q})$. Consider the cohomology spectral sequence for the fibration $X_{k+1} \to X_k$ and take the transgression homomorphism

$$d_{k+1} : E^{k+1,0} = H^{k+1}(K(\pi_{k+1}(X), k+1); \mathbf{Q}) \to E^{0,k+2} = H^{k+2}(X_k; \mathbf{Q}).$$

Let $y_1, \dots, y_i$ be linear generators of $H^{k+1}(K(\pi_{k+1}(X), k+1); \mathbf{Q})$ and for any y_j take an element w_j in $\mathcal{M}_k$ representing $d_{k+1}y_j$. Now put $dy_j = w_j$ and define

$$\mathcal{M}_{k+1} = \mathcal{M}_k \otimes_d \Lambda(\mathrm{Hom}\,(\pi_{k+1}(X), \mathbf{Q}))_{k+1}$$

where

$$\Lambda(\mathrm{Hom}\,(\pi_{k+1}(X), \mathbf{Q}))_{k+1} = H^{k+1}(K(\pi_{k+1}(X), k+1); \mathbf{Q}),$$
$$d : \mathrm{Hom}\,(\pi_{k+1}(X), \mathbf{Q}) \to \mathcal{M}_k^{k+1},$$

and the differential on $\mathcal{M}_{k+1}$ is determined by its restrictions onto $\mathcal{M}_k$ and Λ. Taking another representatives of dy_j we obtain isomorphic algebras.

Now consider the series of such extensions

$$\ldots \subset \mathcal{M}_k \subset \mathcal{M}_{k+1} \subset \ldots \subset \mathcal{M}_\infty,$$

take the limit algebra $\mathcal{M}_\infty = \cup_{k \geq 0} \mathcal{M}_k$ and put $\mathcal{M}_X = \mathcal{M}_\infty$.

It follows from the construction that there is a natural isomorphism

$$\mathrm{Hom}\,(\pi_*(X), \mathbf{Q}) = \mathcal{M}_X / \mathcal{M}_X \wedge \mathcal{M}_X. \tag{3}$$

Another important properties were also established by Sullivan who had invented minimal models.

Proposition 4.2. ([23]) *1) For a simply connected compact polyhedron X there is a differential algebra $\mathcal{E}(X)$ of $\mathbf{Q}$-polynomial forms on X and $\mathcal{M}_X$ is the minimal model for $\mathcal{E}(X)$;*

2) The algebra $\mathcal{M}_M \otimes \mathbf{R}$ is the minimal model for the algebra $\mathcal{E}^\infty(M)$ of smooth differential forms on a compact manifold M.

For these reasons, $\mathcal{M}_X$ is called the minimal model for X, up to isomorphism it describes the rational homotopy type of X.

Important property of minimal models which is based on their construction via the Postnikov towers and is also partially justified by (3) and Proposition 4.2 is the following.

Proposition 4.3. *Every simplicial mapping of polyhedra or smooth mapping of manifolds $f : X \to Y$ induces a homomorphism of the minimal models*

$$\hat{f} : \mathcal{M}_Y \to \mathcal{M}_X$$

such that the induced homomorphisms $f^* : H^*(Y; \mathbf{Q}) \to H^*(X; \mathbf{Q})$ and $\hat{f}^* : H^*(\mathcal{M}_Y; \mathbf{Q}) \to H^*(\mathcal{M}_X; \mathbf{Q})$ coincide as

$$f^* h_Y^* = h_X^* \hat{f}^*,$$

where h_X and h_Y are maps of minimal models in $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ correspondingly .

For a detailed exposition of minimal models see [5, 9] where the minimal model is also defined for compact nilpotent polyhedra. The nilpotence of a polyhedron means that its fundamental group is nilpotent and its actions on higher homotopy groups are also nilpotent.

B) Minimal models for nilmanifolds.

Recall that a nilmanifold $N = G/\Gamma$ is the compact quotient space where G is a simply connected nilpotent group and Γ a lattice in G.

We only formulate few facts about the minimal models for nilmanifolds. Given a nilmanifold $X = G/\Gamma$, there is a uniquely defined (up to isomorphism) graded differential algebra $\mathcal{M} = \mathcal{M}_X$ over $\mathbf{Q}$ satisfying, in particular, the following conditions

1) $\mathcal{M}$ is freely generated by elements $x_1, \dots, x_k$ of degree 1;
2) $\mathcal{M}$ is a minimal algebra;
3) $H^*(\mathcal{M}_X) = H^*(X; \mathbf{Q})$.

This algebra is called the minimal model for a nilmanifold X and has a nice and clear algebraic origin.

Let $\mathcal{G}$ be the Lie algebra of a nilpotent group G and $\mathcal{G}^*$ be the dual algebra to $\mathcal{G}$. The Lie brackets define a mapping

$$\mathcal{G} \times \mathcal{G} \stackrel{[\cdot,\cdot]}{\to} \mathcal{G},$$

and it is easily checked that the dual mapping

$$d : \mathcal{G}^* \to \mathcal{G}^* \times \mathcal{G}^*$$

is a differential and in this event the equality $d^2 = 0$ is equivalent to the Jacoby identity. The nilpotence of $\mathcal{G}$ implies minimality of the algebra $\Lambda(\mathcal{G}^*, d)$ which will be the minimal model for G/Γ.

The differential of this minimal model is written in terms of the structure constants as follows. Let $\{e^1, \dots, e^k\}$ be a basis for $\mathcal{G}$ and $\{\omega_1, \dots, \omega_k\}$ be the dual basis for $\mathcal{G}^*$. Let in this basis the Lie brackets are given by

$$[e^i, e^j] = \sum_k c_k^{ij} e^k.$$

Then the differential d is as follows

$$d\omega_k = \sum_{i,j} c_k^{ij} \omega_i \wedge \omega_j. \qquad (4)$$

Elements of the Lie algebra are naturally associated with leftinvariant vector fields on G and elements of $\mathcal{G}^*$ are represented in this event by leftinvariant 1-forms on G.

Assume that G possesses uniform lattices. By the results of Mal'tsev ([15]) this happens if, and only if, the structure constants c_k^{ij} are rational. Let $\Gamma \subset G$ be a uniform lattice and $X = G/\Gamma$ be the corresponding nilmanifold. By the Nomizu theorem ([21]), there is a natural isomorphism

$$H^*(X; F) = H^*(\widehat{\Omega}(1), F)$$

where $\widehat{\Omega}(1)$ is a subcomplex generated as a graded differential algebra by leftinvariant 1-forms on G and the isomorphism is induced by an embedding of the complexes of forms. For $F = \mathbf{Q}$ it needs to consider forms with rational periods and in this events the rationality of the structure constants guarantees the "rationality" of the differential (4).

We have

Proposition 4.4. *Given $X = G/\Gamma$ an n-dimensional nilmanifold, its minimal model $\mathcal{M}_X$ is correctly defined and is as follows:*

1) $\mathcal{M}_X = (\Lambda(x_1, \dots, x_n); d)$ with $\deg x_i = 1$ for $i = 1, \dots, n$;

2) $x_k = \sum_{i,j} c_k^{ij} x_i \wedge x_j$.

Example. The minimal model of the Kodaira–Thurston manifold $\widetilde{N}$ is freely generated by elements $\eta_1, \dots, \eta_4$ of degree 1 such that

$$d\eta_1 = d\eta_2 = d\eta_4 = 0, \; d\eta_3 = \eta_1 \wedge \eta_2. \qquad (5)$$

These elements are realized via Proposition 4.2 by the following left invariant forms on $\widetilde{N}$:

$$\eta_1 = dx, \; \eta_2 = dy, \; \eta_3 = xdy + dz, \; \eta_4 = du.$$

By the Gompf theorem ([8]), any finitely presented group is the fundamental group of a four-dimensional symplectic compact manifold and therefore the minimal model is not well defined for all symplectic manifolds.

C) Formality of differential algebras and spaces.

An algebra $\mathcal{A}$ is called formal if there is a homomorphism of differential graded algebras

$$(\mathcal{A}, d) \to (H^*(\mathcal{A}), 0)$$

inducing an identity of cohomologies.

A polyhedron or a smooth manifold X is called formal if its minimal model $\mathcal{M}_X$ is formal. For such space it is said that its rational homotopy type is a formal consequence of its cohomology.

Among the examples of formal spaces there are Riemannian locally symmetric spaces ([9]), classifying spaces ([23]), compact Kähler manifolds ([5]), and simply connected compact manifolds of dimension ≤ 6 ([20]).

The simplest example of a nonformal manifold is a three-dimensional non simply connected but nilpotent space $\mathcal{H}/\mathcal{H}_{\mathbf{Z}}$.

Proposition 4.5. *The nilmanifold $\mathcal{H}/\mathcal{H}_Z$ is nonformal.*

By Proposition 4.4, the minimal model $\mathcal{M}$ for $\mathcal{H}/\mathcal{H}_Z$ is as follows: $\mathcal{M} = \Lambda(x_1, x_2, x_3), \deg x_i = 0$, and $dx_1 = dx_2 = 0, dx_3 = x_1 \wedge x_2$. Assuming that this manifold is formal, there exists a homomorphism $\psi : \mathcal{M} \to H^*(\mathcal{M})$ inducing an identity of cohomologies. Notice that $\psi(x_1) \cup \psi(x_2) \in \psi(x_1) \cup H^1(\mathcal{M}) = 0$ which implies that $\psi(x_1 \wedge x_3) = \psi(x_1) \cup \psi(x_3) = 0$. The element $x_1 \wedge x_3$ is closed but not exact in $\mathcal{M}$ and hence $\psi(x_1 \wedge x_3) \neq 0$. We arrives at a contradiction which proves the proposition.

By (5), the same arguments show that the Kodaira–Thurston manifold $\widetilde{M}$ is also nonformal.

An analysis of this situation leads to the following criterion. The minimal model for X is isomorphic as a graded commutative algebra to

$$\mathcal{M} = \otimes_{k \geq 0} \Lambda(V_k)_k$$

where $V_k = \mathrm{Hom}(\pi_k(X), k), \mathbf{Q})$ for a simply connected space X. In each V_k take the subspace C_k of closed elements.

Proposition 4.6. ([5]) *$\mathcal{M}$ is formal if, and only if, there is in each V_k the complement N_k to C_k, $V_k = C_k \oplus N_k$, such that any closed form in the ideal I_N generated by the elements of N_k, $I_N = I(\oplus N_k)$, is exact.*

This criterion has the following topological interpretation. By (3), roughly speaking the elements of C_k come from the Hurewicz homomorphism $\pi_*(X) \to H_*(X)$, i.e., there are such decompositions (not unique)

$$H_k(X; \mathbf{Q}) = S_k \oplus R_k \quad \text{and} \quad H^k(X; \mathbf{Q}) = S^k \oplus R^k \tag{6}$$

such that

1) S_k is the image of the Hurewicz homomorphism, i.e., the set generated over $\mathbf{Q}$ by spherical cycles;

2) S^k and R^k are dual to S_k and R_k, with respect to the Kronecker product

$$\langle u, v \rangle = \int_v u \quad \text{for } u \in H^k, v \in H_k,$$

which means that

$$\langle u, v \rangle = 0 \text{ for } u \in S^k, v \in R_k \text{ or } u \in R^k, v \in S_k.$$

Given such decompositions, we call the subspace

$$S = \oplus_k S^k \subset H^*(X)$$

the subspace of spherical cocycles. They satisfy the following condition: for any spherical cocycle $u \in S$ there exists a mapping $f : S^l \to X$ such that $f^*(u) \neq 0$. Moreover if for $u \in H^*(X)$ we have $f^*(u) \neq 0$ for some mapping $f : S^l \to X$ then $u \neq 0$ in $H^*(X; \mathbf{Q})/(\oplus R^k)$.

Denote $\oplus_{k \geq 1} H^k(X; \mathbf{Q})$ by $H^+(X; \mathbf{Q})$ and notice that the restriction of the natural projection

$$H^+(X; \mathbf{Q}) \to H^+(X; \mathbf{Q}) / \left(H^+(X; \mathbf{Q}) \cdot H^+(X; \mathbf{Q}) \right)$$

onto S is a monomorphism.

The space X is formal if, and only if, there is a decomposition (6) such that the whole cohomology ring $H^*(X; \mathbf{Q})$ is multiplicatively generated by the elements of $\oplus_k S^k$, i.e., a graded algebra $H^*(X; \mathbf{Q})$ is generated by spherical cocycles.

D) Massey products.

We define only the triple Massey product.

Let $a \in \mathcal{M}^p, b \in \mathcal{M}^q$, and $c \in \mathcal{M}^r$ represent nontrivial cohomology classes such that $[a] \cup [b] = 0$ and $[b] \cup [c] = 0$. Therefore there exist $g \in \mathcal{M}^{p+q-1}$ and $h \in \mathcal{M}^{q+r-1}$ such that $a \wedge b = dg$ and $b \wedge c = dh$.

Define a cycle

$$k = g \wedge c + (-1)^{p-1} a \wedge h.$$

Its cohomology class is defined modulo $([a]H^{q+r-1}(\mathcal{M}) + [c]H^{p+q-1}(\mathcal{M}))$ and is called the triple Massey product

$$\langle [a], [b], [c] \rangle \in H^{p+q+r-1}(\mathcal{M})/([a]H^{q+r-1}(\mathcal{M}) + [c]H^{p+q-1}(\mathcal{M})).$$

It follows from Proposition 4.6 that

Proposition 4.7. *If there is a nontrivial Massey product of classes from $H^*(\mathcal{M})$, then the algebra $\mathcal{M}$ is not formal.*

5. Cohomology of a blow up of a symplectic manifold

In this paragraph we expose some computations, of cohomologies of a symplectic blow up, part of which was done in [17].

Let (X, ω) be a compact symplectic manifold of dimension $2N$ and let Y be a symplectic submanifold, of X, of dimension $2(N-k)$. Denote by $\widetilde{X}$ the blow up of X along Y, by $\pi : \widetilde{X} \to X$ the projection which is a degree one mapping, by V the closure of sufficiently small tubular neighborhood of Y in X and by $\widetilde{V} = \pi^{-1}(V)$ the pullback of V by π. The boundaries of $\widetilde{V}$ and V are diffeomorphic to a S^{2k-1}-bundle over Y and are diffeomorphic to each other by π. By j and $\widetilde{j}$ denote the natural embeddings

$$j : V \to X \quad \text{and} \quad \widetilde{j} : \widetilde{V} \to \widetilde{X}.$$

and by f and $\widetilde{f}$ denote the natural embeddings of pairs

$$f : (X, \emptyset) \to (X, V) \quad \text{and} \quad \widetilde{f} : (\widetilde{X}, \emptyset) \to (\widetilde{X}, \widetilde{V}).$$

There are the exact cohomology sequences, for pairs $(\widetilde{X}, \widetilde{V})$ and (X, V), related by the induced homomorphisms π^*:

$$\begin{array}{cccccccccc}
\dots & \leftarrow & H^{i+1}(\widetilde{X}) & \xleftarrow{\widetilde{f}^*} & H^{i+1}(\widetilde{X}, \widetilde{V}) & \xleftarrow{\partial} & H^i(\widetilde{V}) & \xleftarrow{\widetilde{j}^*} & H^i(\widetilde{X}) & \xleftarrow{\widetilde{f}^*} \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\dots & \leftarrow & H^{i+1}(X) & \xleftarrow{f^*} & H^{i+1}(X, V) & \xleftarrow{\partial} & H^i(V) & \xleftarrow{j^*} & H^i(X) & \xleftarrow{f^*} \\
 & & & & & & & & & \\
\leftarrow & H^i(\widetilde{X}, \widetilde{V}) & \xleftarrow{\partial} & H^{i-1}(\widetilde{V}) & \xleftarrow{\widetilde{j}^*} & H^{i-1}(\widetilde{X}) & \leftarrow & \dots. & & \\
 & \uparrow & & \uparrow & & \uparrow & & & & \\
\leftarrow & H^i(X, V) & \xleftarrow{\partial} & H^{i-1}(V) & \xleftarrow{j^*} & H^{i-1}(X) & \leftarrow & \dots. & &
\end{array} \tag{7}$$

First, formulate some simple properties of the diagram of sequences (7).

Proposition 5.1. *The vertical homomorphism*

$$\pi^* : H^i(X, V) \to H^i(\widetilde{X}, \widetilde{V})$$

is an isomorphism for any $i \geq 0$.

Proof. Denote by $\operatorname{Int} V$ and $\operatorname{Int} \widetilde{V}$ the interiors of V and $\widetilde{V}$ which are fibre bundles over Y and notice that by the excision lemma ([22]) there are isomorphisms induced by embeddings of pairs

$$H^i(\widetilde{X}, \widetilde{V}) = H^i(\widetilde{X} \setminus \operatorname{Int} \widetilde{V}, \widetilde{V} \setminus \operatorname{Int} \widetilde{V}), \quad H^i(X, V) = H^i(X \setminus \operatorname{Int} V, V \setminus \operatorname{Int} V).$$

Notice that $\pi^* : H^*(\widetilde{X} \setminus \operatorname{Int} \widetilde{V}, \widetilde{V} \setminus \operatorname{Int} \widetilde{V}) \to H^*(X \setminus \operatorname{Int} V, V \setminus \operatorname{Int} V)$ is also an isomorphism and this proves the proposition.

Proposition 5.2. *For the cohomology groups over the field F of rational, real or complex numbers the vertical homomorphism*

$$\pi^* : H^i(X; F) \to H^i(\widetilde{X}; F)$$

is a monomorphism for any $i \geq 0$.

Proof. Without loss of generality we consider the case of real cohomologies. Symplectic manifolds are oriented and therefore for any nontrivial cohomology class $[\tau] \in H^i(X; \mathbf{R})$ there exists a Poincare dual class $[\eta] \in H^{2N-i}(X; \mathbf{R})$ such that

$$[\tau] \cup [\eta] = [\mathrm{vol}_X] \in H^{2N}(X; \mathbf{R}), \quad \langle [\mathrm{vol}_X], [X] \rangle = \int_X \mathrm{vol}_X = 1.$$

The homomorphism

$$\pi^* : H^{2N}(X; \mathbf{R}) \to H^{2N}(\widetilde{X}; \mathbf{R})$$

is a multiplication by $\deg \pi$ and, since $\deg \pi = 1$, we have $\pi^*([\mathrm{vol}_X]) = [\mathrm{vol}_{\widetilde{X}}]$. Since π^* is a homomorphism of the cohomology rings, we have

$$\pi^*([\tau]) \cup \pi^*([\eta]) = \pi^*([\mathrm{vol}_X]) = [\mathrm{vol}_{\widetilde{X}}]$$

which implies that $\pi^*([\tau]) \neq 0$. This proves the proposition.

It is clear from the proof of Proposition 5.2 that

Proposition 5.3. *If the cohomology ring $H^*(X)$ is torsion free then the vertical homomorphism*

$$\pi^* : H^i(X) \to H^i(\widetilde{X})$$

is a monomorphism for any $i \geq 0$.

The most important case in which Proposition 5.3 is applied is $X = \mathbf{C}P^N$.

Since $\widetilde{V}$ retracts onto $\widetilde{Y}$ and V retracts onto Y with preserving the fibration, we derive from Proposition 3.1 that

Proposition 5.4. *The vertical homomorphism*

$$\pi^* : H^i(V) \to H^i(\widetilde{V})$$

is a monomorphism for any $i \geq 0$.

Now consider consequences of (7) in some special case.

Proposition 5.5. *Given Y a $2(N-k)$-dimensional symplectic submanifold of $X = \mathbf{C}P^N$, there are the following short exact sequences:*

1) for $i = 2l$ with $0 \leq l \leq (N-k)$,

$$\begin{array}{ccccccccc} & & & & H^i(\widetilde{V}) & \stackrel{\widetilde{j}^*}{\leftarrow} & H^i(\widetilde{X}) & \leftarrow & 0 \\ & & & \partial \swarrow & & & & & \\ 0 & \stackrel{\widetilde{f}^*}{\leftarrow} & H^{i+1}(\widetilde{X}, \widetilde{V}) & & \uparrow \pi^* & & \uparrow \pi^* & & ; \\ & & & \pi^* \cdot \partial \nwarrow & & & & & \\ & & & & H^i(V) & \stackrel{j^*}{\leftarrow} & \mathbf{Z} = H^i(X) & \leftarrow & 0 \end{array} \qquad (8)$$

2) for $i = 2l$ with $(N-k+1) \leq l \leq N$,

$$0 \leftarrow H^i(\widetilde{V}) \overset{\widetilde{j}^*}{\leftarrow} H^i(\widetilde{X}) \overset{\pi^* f^*}{\leftarrow} \mathbf{Z} = H^i(X) = H^i(X,V) \overset{\partial}{\leftarrow} 0; \tag{9}$$

3) for $i = 2l+1$ with $i+1 \leq \dim Y = 2(N-k)$,

$$\begin{array}{ccccccccc} 0 & \leftarrow & H^{i+1}(\widetilde{X},\widetilde{V}) & \overset{\partial}{\leftarrow} & H^i(\widetilde{V}) & \overset{\widetilde{j}^*}{\leftarrow} & H^i(\widetilde{X}) & \leftarrow & 0 \\ & & & \pi^*\partial \nwarrow & \uparrow \pi^* & & & & \\ & & & & H^i(V) & \leftarrow & 0 & & \end{array} ; \tag{10}$$

4) for $i = 2l+1$ with $i+1 > \dim Y = 2(N-k)$,

$$0 \leftarrow \pi^*(H^{i+1}(X)) = \mathbf{Z} \leftarrow H^{i+1}(\widetilde{X},\widetilde{V}) \overset{\partial}{\leftarrow} H^i(\widetilde{V}) \overset{\widetilde{j}^*}{\leftarrow} H^i(\widetilde{X}) \leftarrow 0. \tag{11}$$

Proof.

1) Since an embedding $Y \subset V \subset X$ is symplectic, $j^* : H^i(X) \to H^i(V) = H^i(Y)$ is a monomorphism for any $i \leq 2(N-k)+1$. This implies that

$$f^* = 0 \quad \text{for} \quad i \leq 2(N-k)+1. \tag{12}$$

Since the diagram (7) is commutative and $\pi^* : H^i(X,V) \to H^i(\widetilde{X},\widetilde{V})$ is an isomorphism, we have

$$\tilde{f}^* = 0 \quad \text{for} \quad i \leq 2(N-k)+1. \tag{13}$$

Now (7) together with (12) and (13) imply (8).

2) For $i = 2l > 2(N-k)$, $H^i(V) = H^{i-1}(V) = 0$ and $H^{i+1}(X) = 0$. Therefore the lower exact sequence in (7) together with Proposition 5.1 imply

$$H^{i+1}(X,V) = H^{i+1}(\widetilde{X},\widetilde{V}) = 0 \quad \text{for } i = 2l > 2(N-k) \tag{14}$$

and

$$H^i(X) \overset{f^*}{\approx} H^i(X,V) \quad \text{for } i = 2l > 2(N-k). \tag{15}$$

Now (7) together with (14) and (15) imply (9).

3) For $i = 2l+1$, $H^i(X) = 0$. Together with (13) for $i+1 = 2l+2 \leq \dim Y$ this imply (10).

4) For $i = 2l+1$ with $i+1 > \dim Y$, we have $H^{i+1}(V) = 0$ and $H^i(X) = 0$. The first equality implies that the homomorphism $\widetilde{j}^* : H^{i+1}(\widetilde{X}) \to H^{i+1}(\widetilde{V})$ is trivial: $\widetilde{j}^* = 0$. The second equality together with Proposition 5.1 and the commutativity of (7) imply that the homomorphism $\widetilde{f}^* : H^i(\widetilde{X},\widetilde{V}) \to H^i(\widetilde{X})$ is also trivial. In this event the corresponding fragment of (7) reduces to (11).

Proposition 5.5 is proved.

Recall that the minimal model of $\mathbf{C}P^m$ is a differential graded algebra $\mathcal{M}_{\mathbf{C}P^m}$ freely generated by elements x and y, with $\deg x = 2$ and $\deg y = 2k-1$, and the differential acts as follows: $dx = 0, dy = x^k$.

The minimal model of $\widetilde{Y}$ is computed by using Proposition 3.1 as follows.

Proposition 5.6. *Given $\mathcal{M}_Y$ the rational homotopy type of Y and a $U(k)$-vector bundle $E \to Y$, the rational homotopy type $\mathcal{M}_{\widetilde{Y}}$ of the projectivization $\widetilde{Y}$ is isomorphic to*

$$\mathcal{M}_{\widetilde{Y}} = \mathcal{M}_Y \otimes_d \mathcal{M}_{\mathbf{C}P^{k-1}}$$

which is a differential graded algebra freely generated by elements of $\mathcal{M}_Y$ and $\mathcal{M}_{\mathbf{C}P^{k-1}}$ and the differential d acts on $\mathcal{M}_{\widetilde{Y}}$ as follows

1) its restriction onto $\mathcal{M}_Y$ equals the differential of $\mathcal{M}_Y$;

2) $dx = 0$ and

$$dy = x^k + c_1 x^{k-1} + \ldots + c_{k-1}x + c_k$$

where the elements $c_j \in \mathcal{M}_Y$ represent the rational Chern classes $c_j(E)$ via the isomorphism $H^*(\mathcal{M}_Y) = H^*(Y; \mathbf{Q})$.

This Proposition being quite clear may be proved directly but follows from general facts on relations between minimal models and Serre fibrations ([24]).

6. Examples of nonformal simply connected symplectic manifolds

A) A series of nonformal symplectic nilmanifolds.

Consider the algebra $W(1)$ of formal vector fields on the line. This is a topological infinite-dimensional algebra for which a basis is given by linear differential operators

$$e_k = x^{k+1}\frac{d}{dx},\ k = -1, 0, 1, \ldots,$$

the Lie brackets in this basis takes the form

$$[e_i, e_j] = (j - i)e_{i+j}, \quad i, j \geq -1. \tag{16}$$

The algebra $W(1)$ has a natural filtration

$$\ldots \subset \mathcal{L}_1(1) \subset \mathcal{L}_0(1) \subset \mathcal{L}_{-1}(1) \subset W(1)$$

where $\mathcal{L}_k(1)$ is a subalgebra spanned by $e_k, e_{k+1}, \ldots$. here we use the notations from [7] which are widely accepted. We also refer to this book for information of cohomologies of the Lie algebras $\mathcal{L}_k(1)$ and their relations to other subjects.

In the sequel we consider a series of finite-dimensional nilpotent Lie algebras

$$\mathcal{V}_n = \mathcal{L}_1(1)/\mathcal{L}_{n+1}(1),\ n = 3, 4, \ldots.$$

We denote by V_n the corresponding Lie groups. Hence $\mathcal{V}_n$ is an n-dimensional algebra with the basis $\{e_1, \ldots, e_n\}$ and the Lie brackets

$$[e_i, e_j] = \begin{cases} (j-i)e_{i+j}, & \text{for } i + j \leq n \\ 0, & \text{for } i + j > n. \end{cases} \tag{17}$$

The structure constants of $\mathcal{V}_n$ are rational and therefore $\mathcal{V}_n$ possesses uniform lattices. We take one of them which may be called canonical ([15]). The group V_n is isomorphic to $(\mathcal{V}_n, \times)$ where the multiplication $\times$ is given by the Campbell–Hausdorff formula. The basis $\{e_1, \ldots, e_n\}$ multiplicatively generates a subgroup Γ_n and we obtain as a result an infinite family of finite-dimensional nilmanifolds

$$M(n) = V_n/\Gamma_n,\ n = 3, 4, \ldots.$$

The group V_3 is the Heisenberg group $\mathcal{H}$, $M(3)$ is $\mathcal{H}/\mathcal{H}_Z$, and $M(3) \times S^1$ is the Kodaira–Thurston manifold $\widetilde{N}$.

There are only three 4-dimensional nilpotent groups: 1) a commutative group $\mathbf{R}^4$ which possesses leftinvariant Kähler structure; 2) $V_3 \oplus \mathbf{R}$ which possesses leftinvariant complex and symplectic structures but no leftinvariant Kähler structure; 3)

V_4, a three step nilpotent group, which possesses a leftinvariant symplectic structure but no leftinvariant complex structure ([6]).

Let $\{\omega_1, \dots, \omega_n\}$ be a basis for leftinvariant 1-forms, on V_n, dual to the basis $\{e_1, \dots, e_k\}$. Then (17) implies (see also §4)

$$d\omega_k = (k-2)\omega_1 \wedge \omega_{k-1} + (k-4)\omega_2 \wedge \omega_{k-2} + \dots. \tag{18}$$

By Proposition 4.4, we have

Proposition 6.1. *The minimal model for $M(n) = V_n/\Gamma_n$ is as follows:*

$$\mathcal{M}(n) = (\Lambda(x_1, \dots, x_n), d) \text{ with } \deg x_k = 1 \text{ for } k = 1, \dots, n,$$

$$dx_1 = dx_2 = 0, \; dx_k = (k-2)x_1 \wedge x_{k-1} + (k-4)x_2 \wedge x_{k-2} + \dots \text{ for } k \geq 3.$$

In fact, an algebra $M(n)$ is bigraded with the second grading given by $\deg' x_i = i$. In this event the bidegree of the differential is $(1,0)$.

Proposition 6.2. *The form*

$$\Omega_{2m} = (2m-1)\omega_1 \wedge \omega_{2m} + (2m-3)\omega_2 \wedge \omega_{2m-1} + \dots + \omega_m \wedge \omega_{m+1}$$

is a leftinvariant symplectic form on V_{2m} for $m \geq 2$.

Proof of this proposition is as follows. Take an extension of $\mathcal{V}_{2m}$ by adding a new generator ω_{2m+1} such that $d\omega_{2m+1} = \Omega_{2m}$. It follows from (18) that this would be $\mathcal{V}_{2m+1}$ and hence $d^2\omega_{2m+1} = d\Omega_{2m} = 0$. This proves the proposition.

Since Ω_{2m} is invariant, then its pushforward on the quotient space V_{2m} is well-defined and we preserve for the pushfoward the same notation. It is clear that the integrals of Ω_{2m} over cycles in $H_2(V_{2m}; \mathbf{Z})$ are integer. This implies

Corollary 6.3. *The nilmanifolds $M(2m)$ possess integer symplectic forms.*

We will not compute cohomologies of these nilmanifolds and restrict ourselves by the following fact.

Proposition 6.4. *For any $m \geq 2$,*

1) $H^1(M(2m); \mathbf{Q})$ is spanned by $[x_1]$ and $[x_2]$;

2) $[x_2 \wedge x_3] \neq 0$ in $H^2(M(2m); \mathbf{Q})$.

Since $\mathcal{M}(2m)$ contains only two closed generators, the first statement is clear. Notice now that $x_2 \wedge x_3$ has bidegree $(2,5)$ and remind that d has bidegree $(1,0)$. If $x_2 \wedge x_3 = du$, then u has to be proportional to x_5 but, by (18), $dx_5 = 3x_1 \wedge x_4 + x_2 \wedge x_3$. The proposition is proved.

Theorem 6.5. *The symplectic nilmanifolds $M(2m)$ are nonformal.*

Taking into account that in $dx_3 = x_1 \wedge x_2$ in $\mathcal{M}(2m)$, a proof of the theorem is obtained from Proposition 6.4 by the same conversations as the proof of Proposition 4.5 above.

B) Nonformal simply connected symplectic manifolds.

By Proposition 2.1, there exist symplectic embeddings of $\widetilde{M} = M(3) \times S^1$ into $\mathbf{C}P^N$ with $N \geq 5$ and of $M(2m)$ into $\mathbf{C}P^N$ with $N \geq 2m+1$ such that the symplectic form (1) and Ω_{2m} are the pullbacks of the Fubiny–Study forms ω_{FS} on $\mathbf{C}P^N$ under these embeddings.

Now, realizing these nilmanifolds as symplectic submanifolds of complex projective spaces, denote by $\widetilde{\mathbf{C}P}^N$ the symplectic blow up of $\mathbf{C}P^N$ along $\widetilde{N}$ and denote by $\widetilde{X}_m(N)$ the symplectic blow up of $\mathbf{C}P^N$ along $M(2m)$.

Theorem 6.6. *For $m \geq 2$ and $N \geq 2m+1$ the symplectic manifolds $\widetilde{\mathbf{C}P}^N$ and $\widetilde{X}_m(N)$ are simply connected and nonformal.*

Proof of Theorem 6.6.

By Proposition 3.2, these manifolds are simply connected.

We will use the same notation as in §5 and denote by Z the closure of the complement to $\widetilde{V}$ and V in X and Y. We use the following notation for the embeddings

$$Z \xleftarrow{\widetilde{i}_1} \partial V \xrightarrow{\widetilde{i}} \widetilde{V}, \quad Z \xrightarrow{\widetilde{j}_1} \widetilde{X} \xleftarrow{\widetilde{j}} \widetilde{V}$$

and notice that $\widetilde{j}$ is already introduced in §5 and $\widetilde{i}$ is introduced in §3. We also consider rational cohomologies.

Take the exact Mayer–Vietoris sequence for the pair $(Z, \widetilde{V})$:

$$\ldots \to H^q(Z \cup \widetilde{V}) \xrightarrow{\widetilde{j}_1^* \oplus \widetilde{j}^*} H^q(Z) \oplus H^q(\widetilde{V}) \xrightarrow{\widetilde{i}_1^* - \widetilde{i}^*} H^q(Z \cap \widetilde{V}) \to \ldots.$$

By (8),(10), and (11), $\widetilde{j}^*$ is a monomorphism for $q \leq \dim Y + 1$ and we derive that there are the following splittings:

$$0 \to H^q(\widetilde{X}) \xrightarrow{\widetilde{j}_1^* \oplus \widetilde{j}^*} H^q(Z) \oplus H^q(\widetilde{V}) \xrightarrow{\widetilde{i}_1^* - \widetilde{i}^*} H^q(\partial \widetilde{V}) \to 0 \quad \text{for } q \leq \dim Y + 1. \tag{19}$$

For any Y of the form $M(2m)$ or $M(3) \times S^1$ its minimal model contains closed generators x_1 and x_2 and a generator x_3 such that $dx_3 = x_1 \wedge x_2$ (see (1) and Propositions 4.4 and 6.1).

Take elements $(0, a \cup [x_1])$ and $(0, a \cup [x_2])$ in $H^3(Z) \oplus H^3(\widetilde{V})$ and notice that by Proposition 3.1 $\widetilde{i}^*(a) = 0$ which together with (19) imply that there are $u_1, u_2 \in H^3(\widetilde{X})$ such that $\widetilde{j}^*(u_k) = a \cup [x_k]$ for $k = 1, 2$.

We are left to prove two lemmas.

Lemma 6.7. *For $m \geq 3$ the triple Massey product $\langle u_2, u_1, u_2 \rangle$ is defined and nontrivial in $H^8(\widetilde{X}_m(N))/u_2 \cup H^5(\widetilde{X}_m(N))$.*

Lemma 6.8. *For symplectic manifolds $(\widetilde{\mathbf{C}P}^N)$ and $\widetilde{X}_2(N)$ the triple Massey product $\langle u_2, v, u_2 \rangle$ is defined where $v = \pi^*([\omega])$ and ω is a symplectic form on $Y = \widetilde{N}$ or $M(4)$. These products are also nontrivial in $H^7/(u_2 \cup H^4)$.*

Proof of Lemma 6.7.

Since $[x_1 \wedge x_2] = 0$ in $M_{\widetilde{V}}$, we have

$$\widetilde{j}^*(u_1 \cup u_2) = \widetilde{j}^*(u_1) \cup \widetilde{j}^*(u_2) = a^2 \cup ([x_1] \cup [x_2]) = 0.$$

But $\deg(u_1 \cup u_2) = 6 \leq \dim M(2m)$ and, since, by (8), $\widetilde{j}^* : H^6(\widetilde{X}) \to H^6(\widetilde{V})$ is a monomorphism, $u_1 \cup u_2 = 0$ in $H^6(\widetilde{X})$. Therefore the triple product $\langle u_2, u_1, u_2 \rangle$ is defined.

The image of the triple product in $H^8(\widetilde{V})$ equals $a^3 \cup ([x_3] \cup [x_2])$ modulo $(a \cup [x_2]) \cup H^5(\widetilde{V})$. It is easily computed by using Proposition 5.6 that

$$a^3 \cup ([x_3] \cup [x_2]) \neq 0 \ \text{ modulo } (a \cup [x_2]) \cup H^5(\widetilde{V}).$$

We have $\widetilde{j}^*(u_2 \cup H^5(\widetilde{X})) \subset (a \cup [x_2]) \cup H^5(\widetilde{V})$ and this implies that the triple product $\langle u_2, u_1, u_2 \rangle$ is also nontrivial *modulo* $u_2 \cup H^5(\widetilde{X})$ and proves the lemma.

Proof of Lemma 6.8.

By Proposition 5.6, $a = [x] \in H^2(\widetilde{V})$.

Compute

$$\widetilde{j}(u_2 \cup v) = [x \wedge x_2] \cup [Ax_1 \wedge x_4 + x_2 \wedge x_3] =$$

$$A[x \wedge x_2 \wedge x_1 \wedge x_4] = -A[d(x \wedge x_3 \wedge x_4)]$$

where $A = 1$ for $Y = \widetilde{M}$ and $A = 3$ for $Y = M(4)$. Therefore $\widetilde{j}^*(u_2 \wedge v) = 0$.

By (10), $\widetilde{j}^* : H^5(\widetilde{X}) \to H^5(\widetilde{V})$ is a monomorphism and therefore

$$u_2 \cup v = 0.$$

Hence the triple product $\langle u_2, v, u_2 \rangle$ is defined and its image in $H^7(\widetilde{V})$ equals $-Aa^2 \cup [x_2 \wedge x_3 \wedge x_4]$ modulo $(a \cup [x_2]) \cup H^4(\widetilde{V})$. It is easily computed by using Proposition 5.6 that

$$a^2 \cup [x_2 \wedge x_3 \wedge x_4] \neq 0 \ \text{ modulo } (a \cup [x_2]) \cap H^5(\widetilde{V}).$$

Since $\widetilde{j}^*(u_2 \cup H^4(\widetilde{X})) \subset (a \cup [x_2]) \cup H^4(\widetilde{V})$, this implies that the triple product $\langle u_2, v, u_2 \rangle$ is nontrivial *modulo* $u_2 \cup H^4(\widetilde{X})$ and proves the lemma.

Now Theorem 6.6 follows from Lemmas 6.7 and 6.8 and Proposition 4.7.

Proposition 6.9. *Let X_1 and X_2 be simply connected manifolds of dimension N such that there is a nontrivial triple Massey product in $H^q(X_1)$ with $q \leq N-3$. Then there is a nontrivial Massey product in $H^q(X_1 \# X_2)$.*

Note that one can proof a general proposition: *Connected sum of two simply connected manifolds is formal if and only if each summand is formal*, but actually we need the slight version stated above.

Proof. Take the Mayer–Vietoris sequence for the pair $(\overline{X_1 \setminus D}, \overline{X_2 \setminus D})$ where D is an N-disc and an overline stands for the closure. Since $H^q(S^{N-1}) = H^q(\overline{X_1 \setminus D} \cap \overline{X_2 \setminus D}) = 0$ for $1 \leq q \leq N-2$, the embeddings $j_1 : \overline{X_1 \setminus D} \to X_1 \# X_2$ and $j_2 : \overline{X_2 \setminus D} \to X_1 \# X_2$ induce isomorphisms

$$H^q(X_1 \# X_2) \xrightarrow{j_1^* \oplus j_2^*} H^q(\overline{X_1 \setminus D}) \oplus H^q(\overline{X_2 \setminus D})$$

for $1 \leq q \leq N-3$.

Considering the Mayer–Vietoris sequence for the pair $((\overline{X_1 \setminus D}, \overline{D})$ we conclude that there are isomorphisms

$$H^q(X_1) \to H^q(\overline{X_1 \setminus D})$$

for $1 \leq q \leq N-3$.

Therefore, if there is a nontrivial triple Massey product of degree $\leq N-3$ in the cohomologies of X_1 it survives in the cohomologies of $X_1 \# X_2$. This proves the proposition.

Corollary 6.10. *For any $k \geq 1$ symplectic manifolds $\widetilde{\mathbf{C}P}^N \# k\overline{\mathbf{C}P}^N$ and $\widetilde{X}_m(N) \# k\overline{\mathbf{C}P}^N$ are nonformal.*

Proof is clear, because the nontrivial triple products in $\widetilde{\mathbf{C}P}^N$ and $\widetilde{X}_2(N)$, constructed in the proof of Lemma 6.8, are of degree 7 and the nontrivial triple products in $\widetilde{X}_m(N)$, with $m \geq 3$, constructed in the proof of Lemma 6.7, are of degree 8 and therefore all of them survives symplectic blows up at points.

7. Some remarks on formality and the symplectic Hodge theory

The nice relation of formality to the symplectic Hodge theory was recently established by Merkulov ([19]).

First recall some definitions from the symplectic Hodge theory.

Take, for simplicity, local coordinates $\{x^1, \dots, x^{2N}\}$ on a manifold with a symplectic form

$$\omega = \omega_{ij} dx^i \wedge dx^j.$$

and take the inverse tensor, to ω_{ij}, defined by

$$\omega^{ij}\omega_{jk} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}.$$

The symplectic inner product of differential k-forms is

$$\langle \alpha, \beta \rangle = \omega^{i_1 j_1} \dots \omega^{i_k j_k} \alpha_{i_1 \dots i_k} \beta_{j_1 \dots j_k}.$$

Now define the symplectic star operator by the condition

$$\alpha \wedge *\beta = \frac{1}{n!} \langle \alpha, \beta \rangle \, \omega^n.$$

The latter definition enables us to introduce a codifferential δ of degree -1:

$$\delta = * \cdot d \cdot *, \delta^2 = 0.$$

Finally call a differential form α symplectically harmonic if

$$d\alpha = \delta\alpha = 0.$$

These definitions are due to Koszul ([13]) and Brylinski ([3]).

Brylinski proved that on a compact Kähler manifold M each cohomology class in $H^*(M; \mathbf{C})$ is realized by a symplectically harmonic form and he also conjectured that this analog of the Hodge theorem for Riemannian manifolds is valid in general ([3]).

Mathieu proved that a compact symplectic manifold satisfies the Brylinski conjecture if, and only if it satisfies the Hard Lefschetz condition ([16]). This theorem implies that the Kodaira–Thurston manifold $\widetilde{M}$ and the McDuff manifold $\widetilde{\mathbf{C}P}^5$ serve as counterexamples to the Brylinski conjecture.

Recently Merkulov proved that if a simply connected symplectic manifold satisfies the Hard Lefschetz condition then it is formal ([19]).

This lead us to the following

Conjecture. *A simply connected compact symplectic manifold M is formal if and only if it satisfies the Hard Lefschetz condition, or, which is equivalent, each cohomology class in $H^*(M)$ is realized by a symplectically harmonic form.*

Moreover, on nonformal manifolds the cohomology classes not realized by symplectically harmonic forms are exactly nontrivial Massey products.

We also would like to state

Conjecture. *Let $Y \to X$ be a symplectic embedding of a nonformal simply connected manifold into a simply connected compact manifold X. Then the blow up of X along Y is nonformal.*

Final remarks.

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