NON-ABELIAN COHOMOLOGY AND SUPERMANIFOLDS

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Introduction

In this paper, we apply the machinery of non-abelian cohomology to the problem of classification of complex analytic supermanifolds with a given retract. Namely, we associate to any holomorphic vector bundle $\mathbf{E}$ over a complex manifold M a non-abelian cochain complex whose 1-cohomology describes the set of all complex analytic supermanifolds such that the corresponding split supermanifold is $(M, \bigwedge \mathcal{E})$, where $\mathcal{E}$ is the sheaf of holomorphic sections of $\mathbf{E}$. This complex is actually of a type considered by Nijenhuis and Richardson [7] in connection with the deformation theory of algebras, i.e., it is related to a differential graded Lie superalgebra. The corresponding differential Lie superalgebra was introduced in [9]; it gives the cohomology of M with values in the tangent sheaf of the split supermanifold $(M, \bigwedge \mathcal{E})$. For a compact manifold M, our complex gives rise to a finite-dimensional complex analytic variety which can serve as a moduli variety for our classification problem; it is analogous to the Kuranishi family of complex structures on a compact manifold (see [6]). The characteristic feature of the theory is its global character: it deals with classification of structures rather than with their deformations.

The construction mentioned above is exposed in Section 3 of this paper. For completeness, we included proofs of the well-known results of Batchelor [1], Green [3] and Rothstein [12]. The necessary facts of non-abelian homological algebra can be found in Section 1. Section 2 is devoted to the de Rham and Dolbeault cohomologies with values in an arbitrary Lie group, and its results are not used in Section 3. The first two sections are actually an extension of the corresponding sections of our paper [7]. Here we add an exposition of the exact cohomology sequence associated to a short exact sequence of cochain complexes, and especially of the case when this latter exact sequence splits. This machinery is used in Section 2 for studying the de Rham and Dolbeault cohomology with values in the semi-direct product of Lie groups.

All manifolds are assumed to be paracompact. A *smooth*, or *differentiable* manifold, bundle etc. is a manifold, bundle etc. of class C^∞.

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1. Non-abelian homological algebra

1.1. Preliminaries. We give here some definitions and fix our notation.

A set A with a fixed point $e \in A$ will be called a *set with distinguished point*, e being called the *unit point*. We often will denote this object as (A, e). E.g., a group G will be usually regarded as the set with distinguished point (G, e), where $e \in G$ is the unit element. If H is a subgroup of G, then the set G/H of left cosets will be usually regarded as the set with distinguished point $(G/H, H)$. If (A, e) is a set with distinguished point, then any subset B of A containing e determines the *subset with distinguished point* $(B, e) \subset (A, e)$.

If (A, e) and (B, e) are two sets with distinguished points, then a *homomorphism* $\varphi : (A, e) \to (B, e)$ is a mapping $\varphi : A \to B$ such that $\varphi(e) = e$. An *isomorphism* is a bijective homomorphism.

The *kernel* and the *image* of a homomorphism $\varphi : (A, e) \to (B, e)$ are the subsets with distinguished points $\operatorname{Ker}\varphi = (\varphi^{-1}(e), e) \subset (A, e)$ and $\operatorname{Im}\varphi = (\varphi(A), e) \subset (B, e)$. One defines in a usual way an *exact sequence* of sets with distinguished points.

Let τ be an action of a group G on a set with distinguished point (A, e). Then the quotient G/τ is usually endowed with the distinguished point $\tau(G)(e)$. The orbit mapping $\tau^e : G \to A$ given by $\tau^e(g) = \tau(g)(e)$ is a homomorphism of sets with distinguished points. It induced the injective homomorphism $\overline{\tau^e} : G/G_e \to A$, where G_e is the stabilizer of e in G. This homomorphism is an isomorphism of sets with distinguished points if and only if τ is transitive.

Let G and Γ be certain groups. We say that G is a Γ-*group* if an action σ of Γ on G by automorphisms is given. A mapping $\varphi : \Gamma \to G$ is called a *crossed homomorphism* if

$$\varphi(ab) = \varphi(a)(\sigma(a)(\varphi(b))), \quad a, b \in \Gamma.$$

Clearly, φ is a homomorphism of of sets with distinguished points, and $\operatorname{Ker}\varphi$ is a subgroup of Γ.

A crossed homomorphism may be interpreted in other terms. Let us consider the group of bijective transformations of the group G normalizing the group $l(G)$ of left translations $l_g,\ g \in G$. This group $\operatorname{Hol} G$ is called the *holomorph* of G. One verifies easily that

$$\operatorname{Hol} G = l(G) \rtimes \operatorname{Aut} G. \tag{1.1}$$

Any crossed homomorphism $\varphi : \Gamma \to G$ allows to define an action ρ of Γ on G given by the following homomorphism of the group Γ into $\operatorname{Hol} G$:

$$\rho(a) = l_{\varphi(a)} \circ \sigma(a), \quad a \in \Gamma. \tag{1.2}$$

Clearly, ρ coincides with σ on $\operatorname{Ker}\varphi$, and $\operatorname{Im}\varphi$ is the orbit $\rho(\Gamma)(e)$. Conversely, suppose two groups Γ, G and a homomorphism $\Gamma \to \operatorname{Hol} G$ be given. By (1.1), we have the decomposition (1.2), where $\varphi : \Gamma \to G$ and $\sigma : \Gamma \to \operatorname{Aut} G$ are mappings uniquely determined by ρ. One checks easily that σ is a homomorphism, giving a structure of Γ-group on G, and that φ is a crossed homomorphism of Γ into the Γ-group G.

We also denote by r_g the right translation $x \mapsto xg$ and by α_g the inner automorphism $x \mapsto gxg^{-1}$ of G. Clearly, the set $r(G)$ of all right translations is a normal subgroup of $\operatorname{Hol} G$.

Mutatis mutandi, one extends the above definitions onto sheaves of sets and of groups on a topological space X. A *sheaf* will be always understood as an 'espace étalé' over X. The *constant sheaf* of sets associated with a set S is, by definition, the direct product $X \times S$.

1.2. Cochain complex. We recall here some definitions and simple facts from [8], Section 1.

A *cochain complex* is a triple $K = \{K^0, K^1, K^2\}$, where K^0, K^1 are groups, while K^2 is a set with distinguished point, and two actions σ_p of K^0 on K^p, $p = 1, 2$, by automorphism of a group and of a set with distinguished point, respectively, are given. Further, we should have two *coboundary operators* $\delta_0 : K^0 \to K^1$ and $\delta_1 : K^1 \to K^2$ such that δ_0 is a crossed homomorphism into the K^0-group K^1, δ_1 is a homomorphism of sets with distinguished points, and

$$\delta_1 \circ \rho(a) = \sigma_2(a) \circ \delta_1, \quad a \in K^0, \tag{1.3}$$

where ρ is the action of K^0 on K^1 associated with δ_0.

Clearly, $\delta_1(\delta_0(a)) = e$ for any $a \in K^0$. Elements of K^p are called p-cochains. Define the sets of *p-cocycles* $Z^p(K) = \operatorname{Ker} \delta_p$, $p = 0, 1$, the *0-cohomology group*

$$H^0(K) = Z^0(K)$$

and the *1-cohomology set*

$$H^1(K) = Z^1(K)/\rho.$$

The latter is usually endowed with the distinguished point $\varepsilon = \rho(K^0)(e) = \operatorname{Im} \delta_0$. Clearly, $H^0(K)$ coincides with the stabilizer of $e \in K^1$ under the action ρ.

Since σ_1 and δ_0 are completely determined by ρ, they can be omitted in the definition of the cochain complex. The simplified version of the definition is as follows: instead of $\sigma_1, \sigma_2, \delta_0, \delta_1$ we should have homomorphisms $\rho : K^0 \to \operatorname{Hol} K^1$, $\sigma_2 : K^0 \to \operatorname{Aut} K^2$ and $\delta_1 : K^1 \to K^2$ satisfying (1.3). Then δ_0 and σ_1 can be refound from the decomposition

$$\rho(a) = l_{\delta_0(a)} \circ \sigma_1(a), \ a \in K^0,$$

(see (1.2)).

Let $K = \{K^0, K^1, K^2\}$ and $L = \{L^0, L^1, L^2\}$ be two cochain complexes. A *homomorphism* $f : L \to K$ is a triple of homomorphisms $f_p : L_p \to K_p$, $p = 0, 1, 2$, commuting with the actions σ_p and with the coboundary operators (or, which is the same, with ρ, σ_2 and δ_1). Any homomorphism of complexes $f : L \to K$ induces the cohomology homomorphisms $f_p^* : H^p(L) \to H^p(K)$, $p = 0, 1$.

A cochain complex $L = \{L^0, L^1, L^2\}$ is called a *subcomplex* of K if $L_p \subset K_p$, $p = 0, 1, 2$, and the operations in L are induced by those of K. Then the identical mapping $i : L \to K$ is a homomorphism of complexes.

Let $f : L \to K$ is a homomorphism of complexes. One checks easily that $\operatorname{Ker} f = \{\operatorname{Ker} f_0, \operatorname{Ker} f_1, \operatorname{Ker} f_2\}$ is a subcomplex of L (the *kernel* of f), while $\operatorname{Im} f = \{\operatorname{Im} f_0, \operatorname{Im} f_1, \operatorname{Im} f_2\}$ is a subcomplex of K (the *image* of f).

An *exact sequence* of cochain complexes can be defined in an obvious way. In what follows, we will consider short exact sequences of the form

$$e \to L \xrightarrow{f} K \xrightarrow{g} Q \to e.$$

Here we mean that f_p are injective, g_p are surjective, while $\operatorname{Im} f = \operatorname{Ker} g$. In this case, f identifies L with the subcomplex $\operatorname{Ker} g$ of K.

The absense of a natural group operation in the 1-cohomology set is compensated in part by the following 'twisting' construction (see [8], Section 1.2). Retaining the above notation, suppose that a cochain complex K is given. Fixing a cocycle $z \in Z^1(K)$, we define the *twisted complex* K^z by

$$\begin{aligned}
&K^z = \{K^0, K^1, K^2\},\\
&\sigma_1^z(a) = \alpha_z \circ \sigma_1(a) \circ \alpha_{z^{-1}}, \quad a \in K^0,\\
&\sigma_2^z(a) = \sigma_2(a), \quad a \in K^0,\\
&\delta_0^z(a) = (\rho(a)(z))z^{-1} = \delta_0(a)(\sigma_1(a)(z))z^{-1}, \quad a \in K^0,\\
&\delta_1^z = \delta_1 \circ r_z.
\end{aligned}$$

The corresponding action ρ^z has the form

$$\rho^z(a) = r_z^{-1} \circ \rho(a) \circ r_z, \quad a \in K^0.$$

The group $H^0(K^z) = Z^0(K^z)$ is the stabilizer of z under the action ρ. The translation r_z maps bijectively $Z^1(K^z)$ onto $Z^1(K)$, takes 0 to z and induces a bijection $r_z^* : H^1(K^z) \to H^1(K)$. Clearly, $r_z^*(\varepsilon) = \zeta$ is the cohomology class of z in $H^1(K)$. Thus, the twisting operation replaces the distinguished point $\varepsilon \in H^1(K)$ by ζ.

The following property of this operation will be frequently used (see [8], Section 1.3).

Proposition 1.1. *Suppose that a homomorphism of complexes $g : K \to Q$ is given. Fix a certain $z \in Z^1(K)$ and denote $z' = g_1(z) \in Z^1(Q)$. Then g will be a homomorphism of complexes $g : K^z \to Q^{z'}$ as well. Denoting it by g^z, we have the following commutative diagram:*

$$\begin{array}{ccc}
H^1(K^z) & \xrightarrow{(g^z)_1^*} & H^1(Q^{z'}) \\
{\scriptstyle r_z^*}\downarrow & & \downarrow{\scriptstyle r_{z'}^*} \\
H^1(K) & \xrightarrow{g_1^*} & H^1(Q).
\end{array} \tag{1.4}$$

In the situation of Proposition 1.1, we can define the cochain subcomplexes $\operatorname{Ker} g^z \subset K$ and $\operatorname{Im} g^z \subset Q$. Clearly, $\operatorname{Im} g^z = (\operatorname{Im} g)^{z'}$. Also $\operatorname{Ker} g^z = (\operatorname{Ker} g)^z$ whenever $z \in \operatorname{Ker} g_1$. Otherwise, $\operatorname{Ker} g^z$ is, in general, not a twisted complex in our sense. But we often will denote it by L^z, where $L = \operatorname{Ker} g$.

One easily verifies the following properties of the cohomology of L^z.

Proposition 1.2. *In the situation of Proposition 1.1, denote* $L^z = \operatorname{Ker} g^z$. *Then*

$$H^0(L^z) = H^0(K^z) \cap L.$$

Further, the bijection $r_z : L^1 \to L^1 z \subset K^1$ *maps* $Z^1(L^z)$ *onto* $Z^1(K) \cap L^1 z$ *and induces a bijection*

$$r_z^* : H^1(L^z) = Z^1(L^z)/\rho^z(L^0) \to (Z^1(K) \cap L^1 z)/\rho(L^0).$$

Remark. The twisting construction can be generalized as follows. Retaining the above notation, suppose a cochain complex K be given. We fix an element $\omega \in \operatorname{Hol} K^1$ such that $\omega^{-1}(e) \in Z^1(K)$ and define the complex K^ω by

$$\begin{aligned}
&K^\omega = \{K^0, K^1, K^2\},\\
&\rho^\omega(a) = \omega \circ \rho(a) \circ \omega^{-1}, \quad a \in K^0,\\
&\sigma_2^\omega = \sigma_2,\\
&\delta_1^\omega = \delta_1 \circ \omega^{-1}.
\end{aligned}$$

Write $\omega = l_z \circ \alpha$, where $z \in K^1$, $\alpha \in \operatorname{Aut} K^1$. Then the corresponding σ_1 and δ_0 have the form

$$\begin{aligned}
&\sigma_1^\omega(a) = \alpha \circ \sigma_1(a) \circ \alpha^{-1}, \quad a \in K^0,\\
&\delta_0^\omega(a) = z\alpha(\delta_0(a))\sigma_1^\omega(a)(z^{-1}), \quad a \in K^0.
\end{aligned} \tag{1.5}$$

The group $H^0(K^\omega) = Z^0(K^\omega)$ is the stabilizer of $\omega^{-1}(e)$ under ρ. Now, $\omega : K^1 \to K^1$ maps bijectively $Z^1(K)$ onto $Z^1(K^\omega)$ transforming ρ into ρ^ω and hence inducing a bijection $\omega^* : H^1(K) \to H^1(K^\omega)$.

One verifies easily that $(K^{\omega_1})^{\omega_2} = K^{\omega_1 \circ \omega_2}$ for any two $\omega_1, \omega_2 \in \operatorname{Hol} K^1$ such that $\omega_1^{-1}(e) \in Z^1(K)$ and $\omega_2^{-1}(\omega_1^{-1}(e)) \in Z^1(K^{\omega_1})$.

Consider three important special cases.

1) $\omega = l_z$, where $z^{-1} \in Z^1(K)$. By (1.5), we have

$$\begin{aligned}
\sigma_1^\omega &= \sigma_1,\\
\delta_0^\omega(a) &= z\delta_0(a)\sigma_1(a)(z^{-1}) = z\rho(a)(z^{-1}), \quad a \in K^0.
\end{aligned}$$

In this case, we get a bijection

$$l_z^* : H^1(K) \to H^1(K^{l_z}).$$

2) $\omega = r_{z^{-1}} = l_{z^{-1}} \circ \alpha_z$, where $z \in Z^1(K)$. Clearly, K^ω coincides with the twisted complex K^z, and $\omega^* = (r_z^*)^{-1}$.

3) $\omega = \alpha \in \operatorname{Aut} K^1$. By (1.5), $\delta_0^\alpha = \alpha \circ \delta_0$. It follows that the triple $(\mathrm{id}, \alpha, \mathrm{id})$ is an isomorphism of complexes $K \to K^\alpha$.

Note that, up to isomorphism, this construction gives no other complexes than K^z, $z \in Z^1(K)$.

1.3. Exact cohomology sequences. First suppose that a subcomplex L of a cochain complex K is given. Consider the right coset space $L^0\backslash K^0$ and define its subset $H^0(L\backslash K)$ by

$$H^0(L\backslash K) = \{L^0c \mid \delta_0(c) \in L^1\}.$$

It is invariant under the natural right action of $H^0(K)$ on $L^0\backslash K^0$. Endow this set with the distinguished point $\varepsilon = L^0$. We can define the connecting homomorphism $\delta_0^* : H^0(L\backslash K) \to H^1(L)$ assigning to any $L^0c \in H^0(L\backslash K)$ the cohomology class of $\delta_0(c)$. Then we have the following result ([8], Theorem 1.1).

Theorem 1.1. *Let L be a subcomplex of a cochain complex K and $i : L \to K$ the inclusion homomorphism. Then the following sequence of sets with distinguished points is exact:*

$$e \to H^0(L) \xrightarrow{i_0^*} H^0(K) \xrightarrow{p} H^0(L\backslash K) \xrightarrow{\delta_0^*} H^1(L) \xrightarrow{i_1^*} H^1(K).$$

Here $p : c \mapsto L^0c$ is the natural projection. Moreover, $\delta_0^*(\alpha) = \delta_0^*(\beta)$ if and only if $\alpha = \beta c$, where $c \in H^0(K)$.

Now suppose that we are given a short exact sequence of complexes

$$e \to L \xrightarrow{i} K \xrightarrow{g} Q \to e. \tag{1.6}$$

By Theorem 1.1, we get the exact cohomology sequence corresponding to the subcomplex $L = i(L)$ of K. We are going to complete this sequence by the term $H^1(Q)$. We often identify L with $i(L)$ via i.

First, we can interprete $H^0(L\backslash K)$ as $H^0(Q)$. In fact, g induces the bijection $\bar{g} : L^0\backslash K^0 = K^0/L^0 \to Q^0$ mapping $H^0(L\backslash K)$ onto $H^0(Q)$. Under this identification, p turns into g_0^*. Now, we formulate the result.

Theorem 1.2. *Let the exact sequence of cochain complexes (1.6) be given. Then the following sequence of sets with distinguished points is exact:*

$$e \to H^0(L) \xrightarrow{i_0^*} H^0(K) \xrightarrow{g_0^*} H^0(Q) \xrightarrow{\delta_0^*} H^1(L) \xrightarrow{i_1^*} H^1(K) \xrightarrow{g_1^*} H^1(Q).$$

Moreover, there is a natural action ρ^* of the group $H^0(Q)$ on $H^1(L)$ such that the orbit mapping $(\rho^*)^\varepsilon$ coincides with δ_0^*. For any two $\gamma, \gamma' \in H^1(L)$, we have $i_1^*(\gamma) = i_1^*(\gamma')$ if and only if γ, γ' lie on the same orbit of ρ^*.

Proof. Any $q \in Z^0(Q)$ can be written as $q = g_0(k)$ with $k \in K^0$ and $\delta_0(k) \in L^1$. For any $c \in Z^1(L)$, consider $\rho(k)(c) \in Z^1(K)$. Clearly, $g_1(\rho(k)(c)) = \rho(g_0(k))(g_1(c)) = \rho(q)(e) = \delta_0(q) = e$. Hence, $\rho(k)(c) \in Z^1(L)$. One checks easily that the cohomology class of $\rho(k)(c)$ in $H^1(L)$ does not depend of the choice of k. Denoting this class by $\rho^*(q)(\gamma)$, where $\gamma \in H^1(L)$ is the class of c, we get the desired action ρ^*.

Clearly, $i_1^*(\rho^*(q)(\gamma)) = i_1^*(\gamma)$. Conversely, suppose that $i_1^*(\gamma') = i_1^*(\gamma)$. Then a cocycle $c' \in \gamma'$ satisfies $c' = \rho(k)(c)$ for a certain $k \in K^0$. Applying g_1, we get $e = \rho(g_0(k))(e) = \delta_0(g_0(k))$, i.e., $q = g_0(k) \in Z^0(Q)$, and $\gamma' = \rho^*(q)(\gamma)$.

Check the exactness at the term $H^1(K)$. Take $\alpha = i_1^*(\beta)$, where $\beta \in H^1(L)$. Then α is determined by the cocycle $a = i_1(b)$, where $b \in \beta$. Clearly, $g_1(a) = e$, and hence $g_1^*(\alpha) = \varepsilon$. Conversely, suppose that $g_1^*(\alpha) = \varepsilon$ and take $a \in \alpha$. Then $g_1(a) = \delta_0(q)$, where $q \in Q^0$. Write this element as $q = g_0(k)$, where $k \in K^0$. Denote $b = \rho(k^{-1})(a)$ and show that $b \in Z^1(L)$. From $a = \rho(k)(b) = (\delta_0(k))(\sigma_1(k)(b))$ we deduce that $\delta_0(q) = g_1(a) = \delta_0(q)(\sigma_1(q)(g_1(b)))$. Therefore, $g_1(b) = e$. It follows that $\alpha \in \operatorname{Im} i_1^*$.

Corollary. *There is the following exact sequence of sets with distinguished points:*

$$e \to H^1(L)/\rho^* \xrightarrow{\overline{i_1^*}} H^1(K) \xrightarrow{g_1^*} H^1(Q),$$

where $\overline{i_1^*}$ is induced by i_1^*.

Notice that in the situation described above we also have a natural action σ_1^* of $H^0(K)$ on $H^1(L)$. Indeed, if $k \in H^0(K)$ and $c \in Z^1(L)$, then $\sigma_1(k)(c) = \rho(k)(c) \in Z^1(L)$. If $c = \rho(l)(c')$, where $l \in L^0$, then $\sigma_1(k)(\rho(l)(c')) = \rho(kl)(c') = \rho(klk^{-1})(\sigma_1(k)(c'))$, and hence σ_1 induces the desired action.

We use this remark in the following special situation. We say that the exact sequence (1.6) *splits* if there exists a homomorphism of complexes $j : Q \to K$ such that $g \circ j = \mathrm{id}$. Then we have $g_p^* \circ j_p^* = \mathrm{id}$, $p = 0, 1$. It follows that both g_p^* are surjective, while both i_p^* are injective. Moreover, the action ρ^* of the group $H^0(Q)$ can be written as $\rho^* = \sigma_1^* \circ j$. Thus, we get the following result.

Theorem 1.3. *If the sequence (1.6) splits, then there is the following exact sequence of sets with dustinguished points:*

$$e \to H^1(L)/(\sigma_1^* \circ j) \xrightarrow{\overline{i_1^*}} H^1(K) \xrightarrow{g_1^*} H^1(Q) \to e,$$

where $\overline{i_1^*}$ is induced by i_1^*.

All these exact sequence will give us more information if we pass to twisted complexes making use of Proposition 1.1. Consider, e.g., the situation of Theorem 1.3, i.e., suppose that (1.6) splits by a homomorphism $j : Q \to K$. Using j, we may regard Q as a subcomplex of K. Fix a certain $z \in Z^1(Q) \subset Z^1(K)$ and denote by $\zeta \in H^1(Q)$ the cohomology classes of z. Denoting $L^z = \operatorname{Ker} g^z$, we get from (1.6) the short exact sequence

$$e \to L^z \xrightarrow{i^z} K^z \xrightarrow{g^z} Q^z \to e$$

which splits by $j^z : Q^z \to K^z$. Applying to it Theorem 1.3, we get a description of $(g_1^*)^{-1}(\zeta) = r_z^*(\operatorname{Ker}(g^z)_1^*)$ as a quotient of $H^1(L^z)$ by $H^0(Q^z)$.

We apply this remark for proving the following theorem.

Theorem 1.4. *Suppose that we have the following commutative diagram of cochain complexes and their homomorphisms:*

$$\begin{tikzcd}
e \arrow[r] & L \arrow[r, "i"] \arrow[d, "\alpha"'] & K \arrow[r, "g"] \arrow[d, "\beta"] & Q \arrow[r] \arrow[d, "\gamma"] & e \\
e \arrow[r] & A \arrow[r, "k"'] & B \arrow[r, "h"'] & C \arrow[r] & e,
\end{tikzcd}$$

assuming that both lines are short exact sequences and that there exist splitting homomorphisms $j : Q \to K$ and $l : C \to B$. For any fixed $z \in Z^1(Q)$, denote $w = \gamma(z)$, $L^z = \operatorname{Ker} g^z$ and $A^w = \operatorname{Ker} h^w$. A class $\sigma \in H^1(B)$ belongs to $\operatorname{Im} \beta_1^*$ whenever there exists a cocycle $z \in Z^1(Q)$ such that

(a) $\gamma_1^*(\zeta) = h_1^*(\sigma)$;

(b) $(\alpha^z)_1^* : H^1(L^z) \to H^1(A^w)$ *is surjective.*

Proof. Let $z \in Z^1(Q)$ denote a cocycle satisfying (a) and (b). Then we have the commutative diagram

$$\begin{array}{ccccc} H^1(L^z) & \xrightarrow{(i^z)_1^*} & H^1(K^z) & \xrightarrow{r_z^*} & H^1(K) \\ {\scriptstyle (\alpha^z)_1^*}\downarrow & & \downarrow{\scriptstyle (\beta^z)_1^*} & & \downarrow{\scriptstyle \beta_1^*} \\ H^1(A^w) & \xrightarrow[(k^w)_1^*]{} & H^1(B^w) & \xrightarrow[r_w^*]{} & H^1(B). \end{array}$$

Clearly, $(r_w^*)^{-1}(\sigma) \in \operatorname{Im}(k^w)_1^*$. Since $(\alpha^z)_1^*$ is onto, we deduce that $\sigma \in \operatorname{Im}\beta_1^*$.

1.4. Cohomology with values in a sheaf of groups. Let $\mathcal{F}$ be a sheaf of groups on a topological space X and $\mathfrak{U} = (U_i)$ an open cover of X. We are going to construct the *Čech cochain complex* associated with $\mathcal{F}$ and $\mathfrak{U}$, following [8], Section 2.

Let $C^p = C^p(\mathfrak{U}, \mathcal{F})$, $p \geq 0$, be the group of p-cochains of $\mathfrak{U}$ with values in $\mathcal{F}$, i.e.,

$$C^p(\mathfrak{U}, \mathcal{F}) = \prod_{U_{i_0} \cap \ldots \cap U_{i_p} \neq \emptyset} \Gamma(U_{i_0} \cap \ldots \cap U_{i_p}, \mathcal{F}).$$

Consider the triple $C = C(\mathfrak{U}, \mathcal{F}) = (C^0, C^1, C^2)$. To make it a cochain complex, we set

$$\begin{aligned} (\sigma_1(a)(b))_{ij} &= a_j b_{ij} a_j^{-1}, \quad a \in C^0,\ b \in C^1, \\ (\sigma_2(a)(c))_{ijk} &= a_i c_{ijk} a_i^{-1}, \quad a \in C^0,\ c \in C^2, \\ (\delta_0(a))_{ij} &= a_i a_j^{-1}, \\ (\delta_1(b))_{ijk} &= a_{ij} a_{jk} a_{ik}^{-1}. \end{aligned}$$

The action ρ is given by

$$(\rho(a)(c))_{ij} = a_i c_{ij} a_j^{-1}.$$

The sets of cocycles are denoted by $Z^p(\mathfrak{U}, \mathcal{F})$, and the cohomology sets by $H^p(\mathfrak{U}, \mathcal{F})$, $p = 0, 1$. The group $H^0(\mathfrak{U}, \mathcal{F})$ coincides with the group of global sections $\Gamma(X, \mathcal{F})$ and does not depend of $\mathfrak{U}$. This is not true for the 1-cohomology. Passing to the limit over all open covers $\mathfrak{U}$ of X, we get the 1-*cohomology set of* X with values in $\mathcal{F}$ denoted by $H^1(X, \mathcal{F})$. Note that the natural mapping $H^1(\mathfrak{U}, \mathcal{F}) \to H^1(X, \mathcal{F})$ is injective for any open cover $\mathfrak{U}$.

Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves of groups on X and $f : \mathcal{G} \to \mathcal{F}$ a homomorphism of sheaves of groups. Then, for any open cover $\mathfrak{U}$ of X, the homomorphism of complexes $f : C(\mathfrak{U}, \mathcal{G}) \to C(\mathfrak{U}, \mathcal{F})$ is defined in an obvious way. The latter induces cohomology homomorphisms $f^* : H^p(\mathfrak{U}, \mathcal{G}) \to H^p(\mathfrak{U}, \mathcal{F})$, and, after passing to the limit, homomorphisms $f^* : H^p(X, \mathcal{G}) \to H^p(X, \mathcal{F})$, $p = 0, 1$.

In particular, suppose that $f = i$ is the inclusion homomorphism of a subsheaf of groups $\mathcal{G} \subset \mathcal{F}$. Then $L = C(\mathfrak{U}, \mathcal{G})$ is a subcomplex of $K = C(\mathfrak{U}, \mathcal{F})$, and we can apply Theorem 1.1. Note that $H^0(L\backslash K)$ can be identified with the set of global sections $H^0(X, \mathcal{G}\backslash\mathcal{F})$ of the sheaf of sets $\mathcal{G}\backslash\mathcal{F}$. Indeed, if $a \in C^0(\mathfrak{U}, \mathcal{F})$, then the condition $\delta_0(a) \in C^1(\mathfrak{U}, \mathcal{G})$ means that $a_i = b_{ij} a_j$ in any $U_i \cap U_j$, where b_{ij} is a section of $\mathcal{G}$. Thus, the natural projection $p : \mathcal{F} \to \mathcal{G}\backslash\mathcal{F}$ gives $p(a_i) = p(a_j)$ in $U_i \cap U_j$, determining a global section of $\mathcal{G}\backslash\mathcal{F}$. By this isomorphism, the action of $H^0(K)$ on $H^0(L\backslash K)$ goes over to the natural right action of $H^0(X, \mathcal{F})$ on $H^0(X, \mathcal{G}\backslash\mathcal{F})$. Passing to the limit, we get the following result ([8], Theorem 2.1).

Theorem 1.5. *Let $\mathcal{G}$ be a subsheaf of groups of a sheaf of groups $\mathcal{F}$ on X, and let $i : \mathcal{G} \to \mathcal{F}$ denote the natural inclusion. Then the following sequence of sets with distinguished points is exact:*

$$e \to H^0(X, \mathcal{G}) \xrightarrow{i_0^*} H^0(X, \mathcal{F}) \xrightarrow{p} H^0(X, \mathcal{G}\backslash\mathcal{F}) \xrightarrow{\delta_0^*} H^1(X, \mathcal{G}) \xrightarrow{i_1^*} H^1(X, \mathcal{F}).$$

Here p is the natural projection. Moreover, $\delta_0^*(\alpha) = \delta_0^*(\beta)$ if and only if $\alpha = \beta c$, where $c \in H^0(X, \mathcal{F})$.

Suppose now that we have a short exact sequence of sheaves of groups on X:

$$e \to \mathcal{G} \xrightarrow{i} \mathcal{F} \xrightarrow{g} \mathcal{Q} \to e. \tag{1.7}$$

Then any open cover of X admits a refinement $\mathfrak{U}$ such that the corresponding sequence of cochain complexes

$$e \to C(\mathfrak{U}, \mathcal{G}) \xrightarrow{i} C(\mathfrak{U}, \mathcal{F}) \xrightarrow{g} C(\mathfrak{U}, \mathcal{Q}) \to e \tag{1.8}$$

is exact. Thus, the kernel $\operatorname{Ker} g$ of the homomorphism of complexes g identifies with $C(\mathfrak{U}, \mathcal{G})$. Then Theorem 1.2 applies. Passing to the limit over open covers, we get the following result.

Theorem 1.6. *Let the exact sequence of sheaves of groups (1.7) be given. Then the following sequence of sets with distinguished points is exact:*

$$e \to H^0(X, \mathcal{G}) \xrightarrow{i_0^*} H^0(X, \mathcal{F}) \xrightarrow{g_0^*} H^0(X, \mathcal{Q}) \xrightarrow{\delta_0^*} H^1(X, \mathcal{G}) \xrightarrow{i_1^*} H^1(X, \mathcal{F}) \xrightarrow{g_1^*} H^1(X, \mathcal{Q}).$$

Moreover, there is a natural action ρ^* of the group $H^0(X, \mathcal{Q})$ on $H^1(X, \mathcal{G})$ such that the orbit mapping $(\rho^*)^\varepsilon$ coincides with δ_0^*. For any two $\gamma, \gamma' \in H^1(X, \mathcal{G})$, we have $i_1^*(\gamma) = i_1^*(\gamma')$ if and only if γ, γ' lie on the same orbit of ρ^*.

Consider the special case when (1.7) splits, i.e., there exists a homomorphism of sheaves of groups $j : \mathcal{Q} \to \mathcal{F}$ such that $g \circ j = \mathrm{id}$. Then j determines the splitting of the sequence of complexes (1.4) denoted by j, too. Using Theorem 1.3, we easily get the following result.

Corollary. *If the sequence (1.7) splits by a homomorphism $j : \mathcal{Q} \to \mathcal{F}$, then there is the following exact sequence of sets with dustinguished points:*

$$e \to H^1(X, \mathcal{G})/(\sigma_1^* \circ j) \xrightarrow{\overline{i_1^*}} H^1(X, \mathcal{F}) \xrightarrow{g_1^*} H^1(X, \mathcal{Q}) \to e,$$

where $\overline{i_1^*}$ is induced by i_1^*.

Let us describe the action σ_1^* of $H^0(X, \mathcal{F})$ on $H^1(X, \mathcal{G})$ that exists without any splitting assumption. If $a \in H^0(X, \mathcal{F})$ and $c \in Z^1(\mathfrak{U}, \mathcal{G})$, then

$$(\sigma_1(a)(c))_{ij} = a c_{ij} a^{-1}.$$

Therefore, $\sigma_1^*(a)$ is induced by the action α of the group $H^0(X, \mathcal{F})$ on the sheaf $\mathcal{G}$ by conjugations: $\alpha_a(b) = aba^{-1}$.

Now we are going to discuss certain twisting constructions. Let $\mathcal{S}$ be a sheaf of sets in X and let us denote by $\mathcal{A}ut\,\mathcal{S}$ the sheaf of germs of automorphisms of this sheaf. Its sections over an open set $U \subset X$ are, by definition, bijective continuous mappings $\mathcal{S}|U \to \mathcal{S}|U$ leaving each stalk invariant. Clearly, $\mathcal{A}ut\,\mathcal{S}$ is a sheaf of groups. In what follows, we will often use this notation in a more restrictive sense, having in mind that automorhisms should preserve certain algebraic structure given in the sheaf S (it may be a sheaf of groups, a sheaf of superalgebras etc.).

Let $\mathfrak{U} = (U_i)$ be an open cover of X, and fix a cocycle $z \in Z^1(\mathfrak{U}, \mathcal{A}ut\,\mathcal{S})$. Then we can twist $\mathcal{S}$ with the help of z getting a new sheaf $\mathcal{S}^z$ on X. This means that we glue together any two sheaves $\mathcal{S}|U_i$ and $\mathcal{S}|U_j$ over $U_i \cap U_j \neq \emptyset$ identifying $a^i \in (\mathcal{S}|U_i)_x$ and $a^j \in (\mathcal{S}|U_j)_x$, $x \in U_i \cap U_j$, under the condition $a^i = z_{ij}(a^j)$. If S is a sheaf of groups and we use automorphisms of a sheaf of groups, then, clearly, $\mathcal{S}^z$ is a sheaf of groups, too, etc.

The *twisted sheaf* $\mathcal{S}^z$ depends, up to isomorphism, only on the cohomology class $\zeta \in H^1(X, \mathcal{A}ut\,\mathcal{S})$ determined by z. E.g., if z is cohomologous to e, i.e., $z_{ij} = c_i c_j^{-1}$ for a 0-cochain $c \in C^0(\mathfrak{U}, \mathcal{A}ut\,\mathcal{S})$, then $a^i = z_{ij}(a^j)$ implies $c_i^{-1}(a^i) = c_j^{-1}(a^j)$. Thus, the correspondence $(a^i) \mapsto (c_i^{-1}(a^i))$ is an isomorphism of $\mathcal{S}^z$ onto $\mathcal{S}$.

Note that any section $s \in \Gamma(U, \mathcal{S}^z)$ is given by a 0-cochain (s^i), where $s^i \in \Gamma(U \cap U_i, \mathcal{S})$ satisfying $s^i = z_{ij}(s^j)$ over $U \cap U_i \cap U_j$.

Suppose that we have a homomorphism of sheaves $f : \mathcal{S} \to \mathcal{T}$ and fix certain cocycles $z \in Z^1(\mathfrak{U}, \mathcal{A}ut\,\mathcal{S})$ and $w \in Z^1(\mathfrak{U}, \mathcal{A}ut\,\mathcal{T})$ such that $f(z_{ij}(s)) = w_{ij}(f(s))$, $s \in \mathcal{S}$. Then f gives rise to a homomorphism $f^z : \mathcal{S}^z \to \mathcal{T}^w$. Clearly, $\operatorname{Im} f^z = (\operatorname{Im} f)^w$, and $\operatorname{Ker} f^z = (\operatorname{Ker} f)^z$ whenever the kernel is defined.

Consider some special cases.

As above, fix a cocycle $z \in Z^1(\mathfrak{U}, \mathcal{A}ut\,\mathcal{S})$. Since $\mathcal{A}ut\,\mathcal{S}$ is a sheaf of groups, it admits the homomorphism $g \mapsto \alpha_g$ into the sheaf of its automorphisms. Hence, we have the sheaf of groups $(\mathcal{A}ut\,\mathcal{S})^{\alpha_z}$. Note that there is a natural isomorphism of sheaves of groups $(\mathcal{A}ut\,\mathcal{S})^{\alpha_z} \to \mathcal{A}ut\,\mathcal{S}^z$. In fact, any $a \in (\mathcal{A}ut\,\mathcal{S})^{\alpha_z}$ is given by elements $a^i \in (\mathcal{A}ut\,\mathcal{S})|U_i$ such that $a^i = z_{ij} a^j z_{ij}^{-1}$. If $c \in \mathcal{S}^z$ is given by $c^i \in \mathcal{S}|U_i$ such that $c^i = z_{ij}(c^j)$, then $a^i(c^i) = (z_{ij} a^j z_{ij}^{-1})(z_{ij}(c^j) = z_{ij}(a^j(c^j))$. Thus, a determines a germ of an automorphism of $\mathcal{S}^z$.

Proposition 1.3. *Suppose that we have two cocycles $z \in Z^1(\mathfrak{U}, \mathcal{A}ut\,\mathcal{S})$ and $w \in Z^1(\mathfrak{U}, \mathcal{A}ut\,\mathcal{S}^z)$. Then $(\mathcal{S}^z)^w \simeq \mathcal{S}^u$, where $u \in Z^1(\mathfrak{U}, \mathcal{A}ut\,\mathcal{S})$ is given by*

$$u_{ij} = w_{ij}^i \circ z_{ij} = z_{ij} \circ w_{ij}^j.$$

Proof. Any element of $\mathcal{S}^z$ can be written as (t^i), where $t^i \in \mathcal{S}^z|U_i$ and $t^i = w_{ij}(t^j)$. If we identify w_{ij} with (w_{ij}^k), where $w_{ij}^k \in (\mathcal{A}ut\,\mathcal{S})|U_i$ and $w_{ij}^k = z_{kl} w_{ij}^l z_{kl}^{-1}$, then

$$(t^i)^i = w_{ij}^i((t^j)^i) = w_{ij}^i(z_{ij}((t^j)^j)).$$

We see that the desired isomorphism is given by $((t^i)^i) \mapsto (t^i)$.

Let $\mathcal{F}$ be a sheaf of groups, and fix a cocycle $z \in Z^1(\mathfrak{U}, \mathcal{F})$. Using, as above, inner automorphisms, we get the cocycle $\alpha_z \in Z^1(\mathfrak{U}, \mathcal{A}ut\,\mathcal{F})$. Thus, we obtain the sheaf of groups $\mathcal{F}^{\alpha_z}$. The following is proved in [8], Section 2.1.

Proposition 1.4. *There exists an isomorphism h between the cochain complexes $C(\mathfrak{U},\mathcal{F}^{\alpha_z})$ and $C(\mathfrak{U},\mathcal{F})^z$, such that $h_p : C^p(\mathfrak{U},\mathcal{F}^{\alpha_z}) \to C^p(\mathfrak{U},\mathcal{F})^z = C^p(\mathfrak{U},\mathcal{F})$ is defined by*

$$(h_p(c))_{i_0 i_1 \dots i_p} = c^{i_0}_{i_0 i_1 \dots i_p}.$$

Thus, the cohomology of the twisted sheaf $\mathcal{F}^{\alpha_z}$ can be computed with the help of the twisted complex $C(\mathfrak{U},\mathcal{F})^z$. In particular, we get the bijection $r_z^* : H^1(X,\mathcal{F}^{\alpha_z}) \to H^1(X,\mathcal{F})$ mapping ε onto the cohomology class ζ of z.

This connection between twisted sheaves and twisted complexes allows to get more information from our cohomology exact sequences (see Section 3).

1.5. Resolution of a sheaf of groups. The definitions and the results of Sections 2 and 3 extend readily to sheaves on a fixed topological space X. In particular, one can define a *sheaf of cochain complexes* $\mathcal{K}$ on X and get its *cohomology sheaves* $\mathcal{H}^p(\mathcal{K})$, $p = 0, 1$, that are sheaves on X as well. Clearly, with any cochain sheaf $\mathcal{K} = \{\mathcal{K}^0, \mathcal{K}^1, \mathcal{K}^2\}$ one associates the cochain complex of its global sections $\Gamma(X,\mathcal{K}) = \{\Gamma(X,\mathcal{K}^0), \Gamma(X,\mathcal{K}^1), \Gamma(X,\mathcal{K}^2)\}$.

A sheaf of cochain complexes $\mathcal{R}$ on X is called a *resolution* of a sheaf of groups $\mathcal{F}$ if $\mathcal{H}^0(\mathcal{R}) = \mathcal{F}$ and $\mathcal{H}^1(\mathcal{R}) = \varepsilon$ (the trivial sheaf of sets). This means that the following sequence of sheaves is exact:

$$e \to \mathcal{F} \xrightarrow{i} \mathcal{R}^0 \xrightarrow{\delta_0} \mathcal{R}^1 \xrightarrow{\delta_1} \mathcal{R}^2,$$

where i is the natural inclusion.

The cochain complex of global sections $\Gamma(X,\mathcal{R})$ has, in general, non-trivial 1-cohomology. Applying Theorem 1.3 to the subsheaf $\mathcal{F}$ of $\mathcal{R}^0$, we get the following result establishing a relation between this cohomology set and $H^1(X,\mathcal{F})$ ([8], Theorem 2.2).

Theorem 1.7. *Suppose that a resolution $\mathcal{R}$ of a sheaf of groups $\mathcal{F}$ on X is given. Then the following sequence of sets with distinguished points is exact:*

$$e \to H^1(\Gamma(X,\mathcal{R})) \xrightarrow{\mu} H^1(X,\mathcal{F}) \xrightarrow{i_1^*} H^1(X,\mathcal{R}^0),$$

where μ is injective and is induced by a connecting homomorphism.

It is useful to describe μ explicitly. Take $z \in Z^1(\Gamma(X,\mathcal{R})) = \{c \in \Gamma(X,\mathcal{R}) \mid \delta_1(c) = e\}$. There exists an open cover $\mathfrak{U} = (U_i)$ of X such that $z = \delta_0(a_i)$, where $a_i \in \Gamma(U_i,\mathcal{R}^0)$, for any i. Define $b \in Z^1(\mathfrak{U},\mathcal{R}^0)$ by $b_{ij} = a_i^{-1}a_j$. Then $\delta_0(b_{ij}) = e$ and hence $b \in Z^1(\mathfrak{U},\mathcal{F})$. Then μ maps the cohomology class of z onto that of b.

Let $\mathcal{G}$ be another sheaf of groups on X and $\mathcal{S}$ its resolution. Suppose that we have a homomorphism of sheaves of complexes $\varphi = \{\varphi_0, \varphi_1, \varphi_2\}$ of $\mathcal{S}$ into $\mathcal{R}$. This means that we have the commutative diagram

$$\begin{tikzcd}
e \arrow[r] & \mathcal{G} \arrow[r, "i"] \arrow[d, "\varphi"'] & \mathcal{S}^0 \arrow[r, "\delta_0"] \arrow[d, "\varphi_0"] & \mathcal{S}^1 \arrow[r, "\delta_1"] \arrow[d, "\varphi_1"] & \mathcal{S}^2 \arrow[d, "\varphi_2"] \\
e \arrow[r] & \mathcal{F} \arrow[r, "i"'] & \mathcal{R}^0 \arrow[r, "\delta_0"'] & \mathcal{R}^1 \arrow[r, "\delta_1"'] & \mathcal{R}^2.
\end{tikzcd}$$

Then we have the corresponding homomorphism $\varphi : \Gamma(X,\mathcal{S}) \to \Gamma(X,\mathcal{R})$ of the complexes of global sections. It is useful to mention the following result ([7], Theorem 2.3):

Theorem 1.8. *Let $\varphi : \mathcal{S} \to \mathcal{R}$ be a homomorphism of resolutions of sheaves of groups $\mathcal{G}$ and $\mathcal{F}$, respectively, inducing the homomorphism of sheaves $\varphi : \mathcal{G} \to \mathcal{F}$. Then the following diagram is commutative:*

$$\begin{array}{ccccc} H^1(\Gamma(X,\mathcal{S})) & \xrightarrow{\mu} & H^1(X,\mathcal{G}) & \xrightarrow{i_1^*} & H^1(X,\mathcal{S}^0) \\ \varphi^* \downarrow & & \downarrow \varphi^* & & \downarrow \varphi_0^* \\ H^1(\Gamma(X,\mathcal{R})) & \xrightarrow[\mu]{} & H^1(X,\mathcal{F}) & \xrightarrow[i_1^*]{} & H^1(X,\mathcal{R}^0). \end{array}$$

Consider now the following problem: given a resolution of a sheaf of groups $\mathcal{F}$, how to construct a resolution of a twisted sheaf? Here we will discuss the simplest case of the sheaf $\mathcal{F}^{\alpha_z}$, where $z \in Z^1(\mathfrak{U}, \mathcal{F})$ for an open cover $\mathfrak{U} = (U_i)$ of X.

Suppose that a resolution $\mathcal{R}$ of $\mathcal{F}$ is given. Consider the triple

$$\mathcal{R}(z) = \{(\mathcal{R}^0)^{\alpha_z}, (\mathcal{R}^1)^{\sigma_1(z)}, (\mathcal{R}^2)^{\sigma_2(z)}\}.$$

One checks (see [7], Section 2.3) that this will be a resolution of $\mathcal{F}^{\alpha_z}$ if we define the actions of $(\mathcal{R}^0)^{\alpha_z}$ and the coboundary operators over any U_i as in the complex $\mathcal{R}$. Using Theorem 1.7 and Proposition 1.1, we get a bijection $\mu^{\alpha_z} : H^1(\Gamma(X, \mathcal{R}(z)) \to (i_1^*)^{-1}(i_1^*(\zeta)) \subset H^1(X, \mathcal{F})$, where $\zeta \in H^1(X, \mathcal{F})$ is the cohomology class of z.

There is another way to twist a resolution, and namely that discussed in Section 1.2. Fix a cocycle $c \in Z^1(\Gamma(X, \mathcal{R})) = \Gamma(X, \mathcal{Z}^1(\mathcal{R}))$, where $\mathcal{Z}^1(\mathcal{R}) = \operatorname{Ker}\delta_1 \subset \mathcal{R}^1$. Using the construction of Section 1.2, we get the sheaf of twisted cochain complexes $\mathcal{R}^c = \{\mathcal{R}^0, \mathcal{R}^1, \mathcal{R}^2\}$ with new actions σ_p^c and new coboundary operators δ_p^c. The sheaf $\mathcal{H}^0(\mathcal{R}^c)$ is the stabilizer of c under the action ρ, while we have the bijection $r_c^* : \mathcal{H}^1(\mathcal{R}^c) \to \mathcal{H}^1(\mathcal{R})$. Thus, $\mathcal{H}^1(\mathcal{R}^c) = e$, and $\mathcal{R}^c$ is a resolution of $\mathcal{H}^0(\mathcal{R}^c)$.

Choose an open cover $\mathfrak{U} = (U_i)$ of X in such a way that $c = \delta_0(b_i)$, where $b_i \in \Gamma(U_i, \mathcal{R}^0)$, in any U_i. Then the cohomology class $\gamma \in H^1(\Gamma(X, \mathcal{R})$ of c determines the class $\zeta = \mu(\gamma) \in H^1(X, \mathcal{F})$ given by the Čech cocycle $z = (z_{ij})$ with $z_{ij} = b_i^{-1}b_j$. Since $c = \delta_0(b_i) = \rho(b_i)(0)$, the stabilizer of c in $\mathcal{R}^c$ is given over U_i by $\mathcal{H}^0(\mathcal{R}^c)|U_i = b_i(\mathcal{F}|U_i)b_i^{-1}$. Assigning to any $f \in \mathcal{H}^0(\mathcal{R}^c)$ the family $(f_i) = (b_i^{-1}fb_i)$, we see that $f_i \in \mathcal{F}$ and $f_i = z_{ij}f_jz_{ij}^{-1}$ on $U_i \cap U_j$. Hence, we get an isomorphism $\mathcal{H}^0(\mathcal{R}^c) \to \mathcal{F}^{\alpha_z}$. Thus, $\mathcal{R}^c$ can be regarded as a resolution of $\mathcal{F}^{\alpha_z}$.

Clearly, the same correspondence $h_0 : f \mapsto (b_i^{-1}fb_i)$ is an isomorphism of $\mathcal{R}^0$ onto $(\mathcal{R}^0)^{\alpha_z}$. Similarly, the correspondence $h_p : u \mapsto (\sigma_p(b_i^{-1})(u))$ is an isomorphism of $\mathcal{R}^p$ onto $(\mathcal{R}^p)^{\sigma_p(z)}$, $p = 1, 2$.

Proposition 1.5. *The correspondence $h = \{h_0, h_1 \circ \alpha_c^{-1}, h_2\}$ is an isomorphism of the resolution $\mathcal{R}^c$ onto $\mathcal{R}(z)$.*

Proof. Denote

$$\begin{aligned} f_i &= b_i^{-1}fb_i, \quad f \in \mathcal{R}^0, \\ u_i &= \sigma_p(b_i^{-1})(u), \quad f \in \mathcal{R}^p, \ p = 1, 2. \end{aligned}$$

Then

$$\begin{aligned}
\sigma_p(f_i)(u_i) &= \sigma_p(b_i^{-1} f b_i)\sigma_p(b_i^{-1})(u) = \sigma_p(b_i^{-1})(\sigma_p(f)(u)),\\
\rho(f_i)(u_i) &= \rho(b_i^{-1})\rho(f)\rho(b_i)(u_i) = \sigma_1(b_i^{-1})(c^{-1}\rho(f)(c\sigma_1(b_i)(u_i))\\
&= \sigma_1(b_i^{-1})((l_c^{-1}\circ\rho(f)\circ l_c)(u),\\
\delta_1(u_i) &= \sigma_2(b_i^{-1})\sigma_2(b_i)\delta_1(u_i) = (\sigma_2(b_i^{-1}))\delta_1(\rho(b_i)(\sigma_1(b_i^{-1}))(u))\\
&= \sigma_2(b_i^{-1})(\delta_1(cu)).
\end{aligned}$$

Thus, $\{h_0, h_1, h_2\}$ is an isomorphism of the complex $\mathcal{R}^{l_c^{-1}}$ (see Section 1.1, Remark) onto $\mathcal{R}(z)$. Clearly, this implies our assertion.

2. De Rham and Dolbeault cohomologies

2.1. De Rham complex associated with a Lie group. Here we discuss a cochain complex that coincides, in the classical abelian case, with the initial part of the usual de Rham complex of a smooth manifold. We follow [8], Sections 4 and 5.

Let G be a Lie group. For any smooth manifold M, denote by $\mathcal{F}_G$ the sheaf of germs of smooth G-valued functions on M. In particular, $\mathcal{F} = \mathcal{F}_{\mathbb{R}}$ is the structure sheaf of M. Clearly, $\mathcal{F}_G$ is a sheaf of groups. The 0-cohomology $H^0(M, \mathcal{F}_G)$ is the group F_G of global smooth functions $M \to G$, while the 1-cohomology set $H^1(M, \mathcal{F}_G)$ is usually identified with the set of all smooth principal bundles with the base M and the structure group G (regarded up to isomorphy). The unit element $\varepsilon \in H^1(M, \mathcal{F}_G)$ corresponds to the trivial bundle $M \times G$. If $\mathfrak{U} = (U_i)$ is an open cover of M, then any $z \in Z^1(\mathfrak{U}, \mathcal{F}_G)$ determines the principal bundle with the transition functions z_{ij} got from $M \times G$ by twisting with the help of z; this is the bundle corresponding to the cohomology class $\zeta \in H^1(M, \mathcal{F}_G)$ of the cocycle z.

Clearly, the constant sheaf G on M identifies with the subsheaf of $\mathcal{F}_G$ consisting of locally constant functions. We have $H^0(M, G) = G$ whenever M is connected. The set $H^1(M, G)$ will be interpreted as the set of *flat* principal bundles with the base M and the structure group G, i.e., the bundles with locally constant transition functions regarded up to corresponding isomorphy.

We are going to construct a resolution of the constant sheaf G using differential forms. Let $\mathfrak{g}$ be the tangent Lie algebra of G and $\Phi_{\mathfrak{g}} = \bigoplus_{p\geq 0} \Phi^p_{\mathfrak{g}}$ the graded sheaf of $\mathfrak{g}$-valued smooth forms on M. This is a sheaf of graded Lie superalgebras, the bracket $[\,,\,]$ being induced by that of $\mathfrak{g}$. Denote by $\varpi \in \Gamma(G, \Phi^1_{\mathfrak{g}})$ the canonical 1-form on G assigning to any tangent vector $v \in T_g(G)$ the vector $d_g r_g^{-1}(v) \in \mathfrak{g}$, where $r_g : x \mapsto xg$ is the right translation determined by $g \in G$.

Consider the triple of sheaves of groups $\mathcal{R}_G = \{\mathcal{F}_G, \Phi^1_{\mathfrak{g}}, \Phi^2_{\mathfrak{g}}\}$ and define the actions of $\mathcal{R}^0_G = \mathcal{F}_G$ and the coboundary operators by

$$\begin{aligned}
\sigma_p(g)(\alpha) &= (\operatorname{Ad} g)\circ\alpha, \quad g \in \mathcal{F}_G,\ \alpha \in \Phi^p_{\mathfrak{g}},\ p = 1, 2,\\
\delta_0(g) &= g^*(\varpi), \quad g \in \mathcal{F}_G,\\
\delta_1(\alpha) &= d\alpha - \frac{1}{2}[\alpha, \alpha], \quad \alpha \in \Phi^1_{\mathfrak{g}}.
\end{aligned}$$

Then we have the following result.

Theorem 2.1. *The triple $\mathcal{R}_G$ is a resolution of the constant sheaf G on M. In particular, the following sequence of sheaves is exact:*

$$e \to G \xrightarrow{i} \mathcal{F}_G \xrightarrow{\delta_0} \Phi^1_{\mathfrak{g}} \xrightarrow{\delta_1} \Phi^2_{\mathfrak{g}}.$$

We would like to make some remarks about the definition of δ_0. Regarding $\mathfrak{g}$ as the tangent space $T_e(G)$, we have, for a $g \in \mathcal{F}_G$:

$$\delta_0(g)(v) = (d_{g(x)}r_{g(x)})^{-1}((d_x g)(v)), \quad x \in M, \ v \in T_x(M).$$

In particular,

$$\delta_0 = d \text{ for } G = \mathbb{R},$$
$$\delta_0(g) = (dg)g^{-1} \text{ for } G = \mathrm{GL}_m(\mathbb{R}).$$

Also

$$\rho(g)(\alpha) = (\mathrm{Ad}\, g) \circ \alpha + g^*(\varpi).$$

In particular, for $G = \mathrm{GL}_m(\mathbb{R})$ we have

$$\rho(g)(\alpha) = g\alpha g^{-1} + (dg)g^{-1}.$$

These are the so-called *gauge transformations.*

Now consider the corresponding complex of global sections $R_G = \Gamma(M, \mathcal{R}_G)$. Clearly, $R_G = \{R^0_G, R^1_G, R^2_G\}$, where $R^0_G = F_G$, $R^p_G = \Gamma(M, \Phi^p_{\mathfrak{g}})$, $p = 1, 2$. By Theorem 1.7, we have the exact sequence of sets with distinguished points

$$e \to H^1(R_G) \xrightarrow{\mu} H^1(M, G) \xrightarrow{i_1^*} H^1(M, \mathcal{F}_G).$$

This means that μ maps bijectively the set $H^1(R_G)$ onto the subset of those flat bundles from $H^1(M, G)$ which are trivial as smooth bundles.

The resolution $\mathcal{R}_G$ of the constant sheaf G will be called the *de Rham resolution*, and the complex R_G the *de Rham complex* of M associated with the Lie group G.

It is useful to describe a twisted version of the above construction. Suppose that a cocycle $\mathfrak{z} \in Z^1(\mathfrak{U}, \mathrm{Aut}\, G)$ is given, $\mathrm{Aut}\, G$ being the group of automorphisms of the Lie group G. Then we may consider the twisted sheaf of groups $G^{\mathfrak{z}}$. We can realize it as the sheaf $\mathcal{C}_E$ of locally constant sections of the flat group bundle E which is got from $M \times G$ by twisting with the help of $\mathfrak{z}$. The set $H^1(M, G^{\mathfrak{z}})$ can be interpreted as the set of principal bundles with the structure sheaf $G^{\mathfrak{z}}$ in the sense of Grothendieck (see [5]). Any automorphism of G induces an automorphism of $\mathcal{F}_G$, and hence we get the twisted sheaf $\mathcal{F}^{\mathfrak{z}}_G$ which is the sheaf $\mathcal{E}$ of smooth sections of E. We also have the cocycle $d\mathfrak{z}$ of automorphisms of the Lie algebra $\mathfrak{g}$ giving rise to the twisted sheaves $(\Phi^p_{\mathfrak{g}})^{d\mathfrak{z}}$, $p = 1, 2$; these are the sheaves $\Phi^p_{\mathfrak{e}}$ of p-forms with values in the Lie algebra bundle $\mathfrak{e}$ which is got from $M \times \mathfrak{g}$ by twisting with the help of $d\mathfrak{z}$. Theorem 2.1 implies that $\mathcal{R}_E = \{\mathcal{E}, \Phi^1_{\mathfrak{e}}, \Phi^2_{\mathfrak{e}}\}$ is a resolution of $\mathcal{C}_E = G^{\mathfrak{z}}$ if we define σ_p, $p = 1, 2$, and δ_p, $p = 0, 1$, over any U_i as in the complex $\mathcal{R}_G$. We also get the complex of global sections $R_E = \Gamma(M, \mathcal{R}_E)$. Theorem 1.7 gives a bijection of $H^1(R_E)$ onto the kernel of the natural mapping $H^1(M, \mathcal{C}_E) \to H^1(M, \mathcal{E})$.

Note an important abelian case when $G = \mathbb{R}^n$ and hence $\operatorname{Aut} G = \mathrm{GL}_n(\mathbb{R})$. Then E is a flat vector bundle, and $\mathcal{R}_E$ is a part of the de Rham resolution of $\mathcal{C}_E$. We get the de Rham isomorphism $H^1(R_E) \to H^1(M, \mathcal{C}_E)$.

Returning to the general case, suppose that $\mathfrak{z} = \alpha_z$, i.e., $\mathfrak{z}_{ij} = \alpha_{z_{ij}}$, where $z = (z_{ij}) \in Z^1(\mathfrak{U}, G)$. Denote by P the principal flat bundle over M corresponding to z. Then $E = \operatorname{Int} P$ is the group bundle associated to P by the action α of G onto itself by inner automorphisms, and $\mathfrak{e} = \operatorname{Ad} P$ is the Lie algebra bundle associated by the adjoint representation Ad of G on $\mathfrak{g}$. The resolution $\mathcal{R}_{\operatorname{Int} P}$ of $\mathcal{C}_{\operatorname{Int} P} = G^{\alpha_z}$ constructed above coincides with the twisted resolution $\mathcal{R}_G(z)$ described in Section 1.5. Thus, we get a bijection of $H^1(R_{\operatorname{Int} P})$ onto the subset $(i_1^*)^{-1}(i_1^*(\zeta)) \subset H^1(M, G)$ of all flat principal bundles isomorphic to P as smooth bundles (or, which is the same, onto the set of all flat structures in the bundle P regarded as a smooth principal bundle, up to automorphisms of this smooth bundle).

By definition, $H^1(R_{\operatorname{Int} P})$ is the quotient of $Z^1(R_{\operatorname{Int} P})$ by the action ρ of the group $R^0_{\operatorname{Int} P} = \Gamma(M, \mathcal{I}nt P)$ of smooth global sections of $\operatorname{Int} P$. The vector space $R^1_{\operatorname{Int} P} = \Gamma(M, \Phi^1_{\operatorname{Ad} P})$ can be interpreted as the space of connections in P (regarded as a smooth principal bundle), while $\delta_1(c) \in R^2_{\operatorname{Int} P} = \Gamma(M, \Phi^2_{\operatorname{Ad} P})$ is interpreted as the curvature of a connection c, so that $Z^1(R_{\operatorname{Int} P})$ is the set of connections with zero curvature. Now, $R^0_{\operatorname{Int} P}$ identifies with the group of smooth automorphisms of the principal bundle P (inducing the identical mapping of the base M), and ρ is the action of this group on $\Gamma(M, \Phi^1_{\operatorname{Ad} P})$ by gauge transformations. Also $H^0(R_{\operatorname{Int} P})$ is the group of flat automorphisms of P.

Now we consider the special case when P is trivial (as a smooth bundle). This means that the cohomology class ζ of z lies in $\operatorname{Im} \mu$. More precisely, we may suppose that $z_{ij} = b_i^{-1} b_j$, where $b_i \in \Gamma(U_i, \mathcal{F}_G)$. Then $c = \delta_0(b_i)$ is a well-defined global form from $Z^1(R_G)$, and its cohomology class γ satisfies $\mu(\gamma) = \zeta$. By Proposition 1.5, there is the isomorphism h of the sheaf of twisted complexes $(\mathcal{R}_G)^c$ onto $\mathcal{R}_{\operatorname{Int} P}$ given by

$$\begin{aligned} h_0(f) &= (b_i^{-1} f b_i), \quad f \in \mathcal{F}_G, \\ h_p(u) &= (\operatorname{Ad} b_i^{-1})(u), \quad u \in \Phi^p_{\mathfrak{g}},\ p = 1, 2. \end{aligned} \tag{2.1}$$

Clearly, $h_0(\mathcal{H}^0((\mathcal{R}_G)^c)) = \mathcal{C}_{\operatorname{Int} P} = G^{\alpha_z}$. As a corollary, we get an isomorphism of complexes $h : (R_G)^c \to R_{\operatorname{Int} P}$.

The construction of the de Rham complex gives a covariant functor $G \mapsto R_G$ from the category of Lie groups into that of cochain complexes. In particular, to any smooth homomorphism of Lie groups $f : G \to Q$ there corresponds the homomorphism $f = \{f_0, f_1, f_2\} : R_G \to R_Q$, where $f_0(g) = f \circ g$, $g \in R^0_G = F_G$, and $f_p(c) = df \circ c$, $c \in R^p_G$, $p = 1, 2$. In particular, to a Lie subgroup of G there corresponds a subcomplex of R_G, to a short exact sequence of Lie groups a short exact sequence of cochain complexes, etc.

Now we will apply these remarks to studying the de Rham complex of the semi-direct product of Lie groups. Suppose that $G = N \rtimes S$, where N and S are Lie subgroups of a Lie group G, the subgroup N being normal. Denote by $i : N \to G$ and $j : S \to G$ the inclusion mappings and by $f : G \to S$ the canonical projection. Then we get the exact sequence

$$e \to N \xrightarrow{i} G \xrightarrow{f} S \to e \tag{2.2}$$

which splits by $j : S \to G$ satisfying $f \circ j = \mathrm{id}$. Write down the corresponding short exact sequence of de Rham complexes:

$$e \to R_N \xrightarrow{i} R_G \xrightarrow{f} R_S \to e. \tag{2.3}$$

It splits by $j : R_S \to R_G$. By Theorem 1.3, it implies the following exact sequence of sets with distinguished points:

$$e \to H^1(R_N)/H^0(R_S) \xrightarrow{\overline{i_1^*}} H^1(R_G) \xrightarrow{\overline{f_1^*}} H^1(R_S) \to e,$$

where $\overline{i_1^*}$ is induced by i_1^* and the action of $H^0(R_S)$ on $H^1(R_N)$ is determined by the adjoint action of S on the ideal $\mathfrak{n} \subset \mathfrak{g}$. Note that $H^0(R_S) = S$ whenever M is connected. Thus, $f_1^* : H^1(R_G) \to H^1(R_S)$ is surjective, and i_1^* induces a bijection between $H^1(R_N)/H^0(R_S)$ and $\operatorname{Ker} f_1^*$. Moreover, $f_1^* \circ j_1^* = \mathrm{id}$, and hence $j_1^* : H^1(R_S) \to H^1(R_G)$ is injective. Since S is a subgroup of G, we may regard the complex R_S as a subcomplex of R_G. It follows that $R_G^0 = R_N^0 \rtimes R_S^0$, $R_G^p = R_N^p \oplus R_S^p$, $p = 1, 2$.

Fix a cocycle $c \in Z^1(R_S) \subset Z^1(R_G)$ and denote by γ the corresponding cohomology class from $H^1(R_S)$. We want to describe the subset $(f_1^*)^{-1}(\gamma)$. Consider the homomorphism of twisted complexes $f^c = f : (R_G)^c \to (R_S)^c$ and denote $(R_N)^c = \operatorname{Ker} f$. Then we obtain from (2.3) the split short exact sequence of complexes

$$e \to (R_N)^c \xrightarrow{i} (R_G)^c \xrightarrow{f} (R_S)^c \to e. \tag{2.4}$$

Applying Theorem 1.3 and Proposition 1.1, we get the bijection

$$r_c^* \circ \overline{(i_1^c)^*} : H^1((R_N)^c)/H^0((R_S)^c) \to (f_1^*)^{-1}(\gamma) \subset H^1(R_G).$$

Clearly (see Proposition 1.2), it is determined by the translation $r_c : a \mapsto a + c$, $a \in Z^1((R_N)^c)$.

We are going now to give a description of the complex $(R_N)^c$. Let $\zeta = \mu(\gamma) \in H^1(M, S) \subset H^1(M, G)$ be the cohomology class corresponding to γ by Theorem 1.7. It is given by the Čech 1-cocycle $z = (z_{ij})$ of a cover $\mathfrak{U} = (U_i)$ of M, where $z_{ij} = b_i^{-1} b_j$ and $b_i \in \Gamma(U_i, \mathcal{F}_S)$ satisfy $c = \delta_0(b_i)$ in any U_i. Denote by P and $\tilde{P}$ the principal flat bundles with structure groups S and G, respectively, corresponding to z. As we have seen above, Proposition 1.5 allows to define the isomorphisms of complexes $h : (R_S)^c \to R_{\operatorname{Int} P}$ and $\tilde{h} : (R_G)^c \to R_{\operatorname{Int} \tilde{P}}$ given on the sheaf level by (2.1). Denote by E the flat group bundle with the fibre N determined by the cocycle $(\alpha_{z_{ij}}|N) \in Z^1(\mathfrak{U}, \operatorname{Aut} N)$. One easily checks that $\tilde{h}$ restricted to $(R_N)^c$ induces an isomorphism of this complex onto R_E. This gives us the following commutative diagram, whose vertical arrows are isomorphisms of cochain complexes:

$$\begin{tikzcd}
e \arrow[r] & (R_N)^c \arrow[r, "i"] \arrow[d, "\tilde{h}"'] & (R_G)^c \arrow[r, "f"] \arrow[d, "\tilde{h}"] & (R_S)^c \arrow[r] \arrow[d, "h"] & e \\
e \arrow[r] & R_E \arrow[r, "i"'] & R_{\operatorname{Int} \tilde{P}} \arrow[r, "f"'] & R_{\operatorname{Int} P} \arrow[r] & e.
\end{tikzcd} \tag{2.5}$$

Thus, we have the isomorphisms $\tilde{h}_1^* : H^1((R_N)^c) \to H^1(R_E)$ and $h_0 : H^0((R_S)^c) \to H^0(R_{\operatorname{Int} P})$, the latter group being the group of flat automorphisms of P. The action of this group on $H^1(R_E)$ corresponding to the bottom line of (2.5) is, clearly, induced by the adjoint action of S on $\mathfrak{n}$. As a result, we have the following theorem.

Theorem 2.2. *Suppose that we have a short exact sequence of Lie groups (2.2) which is split by a homomorphism $j : S \to G$. Then $j_1^* : H^1(R_S) \to H^1(R_G)$ is injective, while $f_1^* : H^1(R_G) \to H^1(R_S)$ is surjective. For any $\gamma \in H^1(R_S)$, we have a natural bijection between $(f_1^*)^{-1}(\gamma) \subset H^1(R_G)$ and $H^1(R_E)/H^0(R_{\mathrm{Int}\,P})$, the class $j_1^*(\gamma)$ corresponding to the orbit of the unit in $H^1(R_E)$. Here E is the flat group bundle with the fibre N determined by the cocycle $(\alpha_{z_{ij}}|N)$, where $z = (z_{ij})$ is a cocycle giving the cohomology class $\mu(\gamma) \in H^1(M, S)$, and P is the principal flat bundle with structure group S corresponding to z.*

2.2. Dolbeault complex associated with a complex Lie group.

Here we consider a theory, analogous to that exposed in Section 2.1, in which the flat bundles and locally constant sections or functions are replaced by holomorphic ones. The abelian model is the classical Dolbeault complex of a complex analytic manifold. We follow [8], Section 6.

Let G be a complex Lie group. For any complex manifold M, denote by $\mathcal{F}_G$ the sheaf of germs of smooth G-valued functions on M. In particular, $\mathcal{F} = \mathcal{F}_{\mathbb{C}}$ is the structure sheaf of M. Clearly, $F_G = H^0(M, \mathcal{F}_G)$ is the group of all G-valued holomorphic functions on M, while $H^1(M, \mathcal{F}_G)$ can be interpreted as the set of all principal holomorphic bundles with the base M and the structure group G, considered up to holomorphic isomorphisms leaving any point of M fixed. The unit cohomology class ε corresponds to the trivial bundle $M \times G$. We also will consider the sheaf of smooth G-valued functions on M denoting it by $\mathcal{F}_G^\infty$.

We are going to construct a resolution of the sheaf F_G using differential forms. As in Section 2.1, consider the tangent Lie algebra $\mathfrak{g}$ of G and the sheaf $\Phi_\mathfrak{g}$ of $\mathfrak{g}$-valued smooth forms on M. This is a sheaf of bigraded Lie superalgebras, the bigrading being defined by $\Phi_\mathfrak{g} = \bigoplus_{p,q\geq 0} \Phi_\mathfrak{g}^{p,q}$, where $\Phi_\mathfrak{g}^{p,q}$ is the sheaf of forms of type (p,q). Let $\Pi_{p,q} : \Phi^{p+q} \to \Phi^{p,q}$ be the natural projection. Denote

$$\partial c = \Pi_{p+1,q} dc, \ \bar\partial c = \Pi_{p,q+1} dc, \quad c \in \Phi_{p,q},$$

Then

$$d = \partial + \bar\partial, \quad \partial^2 = \bar\partial^2 = [\partial, \bar\partial] = 0.$$

Consider the triple of sheaves of groups $\bar{\mathcal{R}}_G = \{\mathcal{F}_G^\infty, \Phi_\mathfrak{g}^{0,1}, \Phi_\mathfrak{g}^{0,2}\}$ and define the actions of $\bar{\mathcal{R}}_G^0 = \mathcal{F}_G^\infty$ and the coboundary operators by

$$\begin{aligned} \sigma_p(g)(\alpha) &= (\mathrm{Ad}\, g) \circ \alpha, \quad g \in \mathcal{F}_G^\infty, \ \alpha \in \Phi_\mathfrak{g}^{0,p}, \ p = 1, 2, \\ \delta_0(g) &= \Pi_{0,1} g^*(\varpi), \quad g \in \mathcal{F}_G^\infty, \\ \delta_1(\alpha) &= \Pi_{0,2}(d\alpha - \frac{1}{2}[\alpha, \alpha]) = \bar\partial\alpha - \frac{1}{2}[\alpha,\alpha], \quad \alpha \in \Phi_\mathfrak{g}^{0,1}. \end{aligned} \tag{2.6}$$

Then we have the following result (see [8], Theorem 6.1).

Theorem 2.3. *The triple $\bar{\mathcal{R}}_G$ is a resolution of the sheaf F_G on M. In particular, the following sequence of sheaves is exact:*

$$e \to F_G \xrightarrow{i} \mathcal{F}_G^\infty \xrightarrow{\delta_0} \Phi_\mathfrak{g}^{0,1} \xrightarrow{\delta_1} \Phi_\mathfrak{g}^{0,2}.$$

Note that

$$\delta_0 = \bar{\partial} \text{ for } G = \mathbb{C},$$
$$\delta_0(g) = (\bar{\partial} g)g^{-1} \text{ for } G = \mathrm{GL}_m(\mathbb{C}).$$

Also

$$\rho(g)(\alpha) = (\mathrm{Ad}\, g) \circ \alpha + \Pi_{0,1} g^*(\varpi).$$

In particular, for $G = \mathrm{GL}_m(\mathbb{C})$ we have

$$\rho(g)(\alpha) = g\alpha g^{-1} + (\bar{\partial} g)g^{-1}.$$

Now consider the corresponding complex of global sections $\bar{R}_G = \Gamma(M, \bar{\mathcal{R}}_G)$. Clearly, $\bar{R}_G = \{\bar{R}^0_G, \bar{R}^1_G, \bar{R}^1_G\}$, where $\bar{R}^0_G = F_G$, $\bar{R}^p_G = \Gamma(M, \Phi^{0,p}_{\mathfrak{g}})$, $p = 1, 2$. By Theorem 1.7, we have the exact sequence of sets with distinguished points

$$e \to H^1(\bar{R}_G) \xrightarrow{\mu} H^1(M, \mathcal{F}_G) \xrightarrow{i_1^*} H^1(M, \mathcal{F}^\infty_G).$$

This means that μ maps bijectively the set $H^1(\bar{R}_G)$ onto the subset of those flat bundles from $H^1(M, \mathcal{F}_G)$ which are trivial as smooth bundles.

The resolution $\bar{\mathcal{R}}_G$ of the sheaf $\mathcal{F}_G$ will be called the *Dolbeault resolution*, and the complex $\bar{R}_G$ the *Dolbeault complex* of M associated with the complex Lie group G.

As in Section 2.1, there exists a twisted version of the Dolbeault resolution. Denote by $\mathrm{Aut}\, G$ the group of (holomorphic) automorphisms of G and choose a cocycle $\mathfrak{z} \in Z^1(\mathfrak{U}, \mathcal{F}_{\mathrm{Aut}\, G})$. Clearly, $\mathcal{F}_{\mathrm{Aut}\, G}$ acts on $\mathcal{F}_G$ and $\mathcal{F}^\infty_G$. We get the holomorphic group bundle E obtained by twisting $M \times G$ with the help of $\mathfrak{z}$ and the sheaves $\mathcal{E} = \mathcal{F}^{\mathfrak{z}}_G$ and $\mathcal{E}^\infty = (\mathcal{F}^\infty_G)^{\mathfrak{z}}$ of holomorphic and smooth sections of E, respectively. The cocycle $d\mathfrak{z}$ gives rise to the twisted sheaves $(\Phi^{0,p}_{\mathfrak{g}})^{d\mathfrak{z}}$, $p = 1, 2$; these are the sheaves $\Phi^{0,p}_{\mathfrak{e}}$ of p-forms with values in the holomorphic Lie algebra bundle $\mathfrak{e}$ which is got from $M \times \mathfrak{g}$ by twisting with the help of $d\mathfrak{z}$. Then $\bar{\mathcal{R}}_E = \{\mathcal{E}, \Phi^{0,1}_{\mathfrak{e}}, \Phi^{0,2}_{\mathfrak{e}}\}$ is a resolution of $\mathcal{E}$ if we define σ_p, $p = 1, 2$, and δ_p, $p = 0, 1$, over any U_i as in the complex $\bar{\mathcal{R}}_G$. We also get the complex of global sections $\bar{R}_E = \Gamma(M, \bar{\mathcal{R}}_E)$.

If $G = \mathbb{C}^n$, $\mathrm{Aut}\, G = \mathrm{GL}_n(\mathbb{C})$, then E will be a holomorphic vector bundle. Clearly, $\bar{\mathcal{R}}_E$ is a part of the Dolbeault–Serre resolution of E, and we get the classical isomorphism $H^1(\bar{\mathcal{R}}_E) \to H^1(M, \mathcal{E})$.

Consider the case when $\mathfrak{z} = \alpha_z$, where $z = (z_{ij}) \in Z^1(\mathfrak{U}, \mathcal{F}_G)$. Denote by P the principal holomorphic bundle over M corresponding to z. Then $E = \mathrm{Int}\, P$ is the group bundle associated to P by the action α of G onto itself by inner automorphisms, and $\mathfrak{e} = \mathrm{Ad}\, P$ is the Lie algebra bundle associated by the adjoint representation Ad of G on $\mathfrak{g}$. The resolution $\bar{\mathcal{R}}_{\mathrm{Int}\, P}$ of $\mathcal{E}$ constructed above coincides with the twisted resolution $\bar{\mathcal{R}}_G(z)$ described in Section 1.5. Thus, we get a bijection $\mu^{\alpha_z} : H^1(\bar{R}_{\mathrm{Int}\, P})$ onto the subset $(i_1^*)^{-1}(i_1^*(\zeta)) \subset H^1(M, \mathcal{F})$ of all holomorphic principal bundles isomorphic to P as smooth bundles (or, which is the same, onto the set of all holomorphic structures in the bundle P regarded as a smooth principal bundle, up to automorphisms of this smooth bundle).

Now we consider the special case when P is trivial (as a smooth bundle). We may suppose that $z_{ij} = b_i^{-1} b_j$, where $b_i \in \Gamma(U_i, \mathcal{F}^\infty_G)$. Then $c = \delta_0(b_i)$ is a well-defined global form from $Z^1(\bar{R}_G)$, and its cohomology class γ satisfies $\mu(\gamma) = \zeta$, where ζ is the cohomology class of z. By Proposition 1.5, there is the isomorphism h of the sheaf of twisted complexes $(\bar{\mathcal{R}}_G)^c$ onto $\bar{\mathcal{R}}_{\mathrm{Int}\,P}$ given by

$$
\begin{aligned}
&h_0(f) = (b_i^{-1} f b_i), \quad f \in \mathcal{F}_G^\infty, \\
&h_p(u) = (\mathrm{Ad}\, b_i^{-1})(u), \quad u \in \Phi_{\mathfrak{g}}^{0,p}, \ p = 1, 2.
\end{aligned}
\tag{2.7}
$$

Clearly, $h_0(\mathcal{H}^0((\bar{\mathcal{R}}_G)^c)) = \mathcal{E} = (\mathcal{F}_G)^{\alpha_z}$. As a corollary, we get an isomorphism of complexes $h : (\bar{R}_G)^c \to \bar{R}_{\mathrm{Int}\,P}$.

As in the case of the de Rham complex, the correspondence $G \mapsto \bar{R}_G$ is a covariant functor from the category of complex Lie groups into that of cochain complexes. In particular, to any holomorphic homomorphism of Lie groups $f : G \to Q$ there corresponds the homomorphism $f = \{f_0, f_1, f_2\} : \bar{R}_G \to \bar{R}_Q$. In particular, to a complex Lie subgroup of G there corresponds a subcomplex of $\bar{R}_G$, to a short exact sequence of complex Lie groups a short exact sequence of cochain complexes, etc.

Now we will study the Dolbeault complex of the semi-direct product of complex Lie groups. Suppose that $G = N \rtimes S$, where N and S are complex Lie subgroups of a complex Lie group G, the subgroup N being normal. Denoting by $i : N \to G$ and $j : S \to G$ the inclusion mappings and by $f : G \to S$ the canonical projection, we get the exact sequence

$$
e \to N \xrightarrow{i} G \xrightarrow{f} S \to e
\tag{2.8}
$$

which splits by $j : S \to G$. The argument proving Theorem 2.2 gives us the following theorem.

Theorem 2.4. *Suppose that we have a short exact sequence (12) of complex Lie groups split by a homomorphism $j : S \to G$. Then $j_1^* : H^1(\bar{R}_S) \to H^1(\bar{R}_G)$ is injective, while $f_1^* : H^1(\bar{R}_G) \to H^1(\bar{R}_S)$ is surjective. For any $\gamma \in H^1(\bar{R}_S)$, we have a natural bijection between $(f_1^*)^{-1}(\gamma) \subset H^1(\bar{R}_G)$ and $H^1(\bar{R}_E)/H^0(\bar{R}_{\mathrm{Int}\,P})$, the class $j_1^*(\gamma)$ corresponding to the orbit of the unit in $H^1(\bar{R}_E)$. Here E is the holomorphic group bundle with the fibre N determined by the cocycle $(\alpha_{z_{ij}}|N)$, where $z = (z_{ij})$ is a cocycle giving the cohomology class $\mu(\gamma) \in H^1(M, \mathcal{F}_S)$, and P is the holomorphic principal bundle with structure group S corresponding to z.*

There is an important relationship between the de Rham and the Dolbeault resolutions associated with the same complex Lie group G. It corresponds to the natural inclusion $\iota_G : G \to \mathcal{F}_G$. Clearly, we have the commutative diagram, where the coboundary operators (2.6) of the Dolbeault resolution are denoted by $\bar{\delta}_p$, $p = 0, 1$:

$$
\begin{array}{ccccccccc}
e & \longrightarrow & G & \xrightarrow{i} & \mathcal{F}_G^\infty & \xrightarrow{\delta_0} & \Phi_{\mathfrak{g}}^1 & \xrightarrow{\delta_1} & \Phi_{\mathfrak{g}}^2 \\
 & & \Big\downarrow{\scriptstyle \iota_G} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \Pi_{0,1}} & & \Big\downarrow{\scriptstyle \Pi_{0,2}} \\
e & \longrightarrow & F_G & \xrightarrow[i]{} & \mathcal{F}_G^\infty & \xrightarrow[\bar{\delta}_0]{} & \Phi_{\mathfrak{g}}^{0,1} & \xrightarrow[\bar{\delta}_1]{} & \Phi_{\mathfrak{g}}^{0,2}.
\end{array}
$$

Thus, we get the homomorphism of resolutions $\Pi = \{\mathrm{id}, \Pi_{0,1}, \Pi_{0,2}\}$ inducing $\iota_G : G \to \mathcal{F}_G$. By Theorem 1.8, it implies the commutative diagram

$$\begin{array}{ccccc} H^1(R_G) & \xrightarrow{\mu} & H^1(M,G) & \xrightarrow{i^*} & H^1(M,\mathcal{F}_G^\infty) \\ {\scriptstyle \Pi_1^*}\big\downarrow & & \big\downarrow{\scriptstyle \iota_G^*} & & \big\| \\ H^1(\bar{R}_G) & \xrightarrow[\bar{\mu}]{} & H^1(M,\mathcal{F}_G) & \xrightarrow[i^*]{} & H^1(M,\mathcal{F}_G^\infty), \end{array}$$

where $\bar{\mu}$ denotes the mapping μ for the Dolbeault resolution.

More generally, suppose that we have a flat group bundle E corresponding to a cocycle $\mathfrak{z} \in Z^1(\mathfrak{U}, \mathrm{Aut}\, G)$. Regarding E as a holomorphic Lie group bundle, we can define the homomorphism $\Pi = \{\mathrm{id}, \Pi_{0,1}, \Pi_{0,2}\}$ of the resolution $\mathcal{R}_E$ into the resolution $\bar{\mathcal{R}}_E$ inducing the natural inclusion $\iota_E : \mathcal{C}_E \to \mathcal{E}$. By Theorem 1.8, the following diagram is commutative:

$$\begin{array}{ccccc} H^1(R_E) & \xrightarrow{\mu} & H^1(M,\mathcal{C}_E) & \xrightarrow{i^*} & H^1(M,\mathcal{E}^\infty) \\ {\scriptstyle \Pi_1^*}\big\downarrow & & \big\downarrow{\scriptstyle \iota_E^*} & & \big\| \\ H^1(\bar{R}_E) & \xrightarrow[\mu]{} & H^1(M,\mathcal{E}) & \xrightarrow[i^*]{} & H^1(M,\mathcal{E}^\infty). \end{array}$$

In conclusion, we want to formulate an application of Theorem 1.4. Suppose that we have the semi-direct product $G = N \rtimes S$ of complex Lie groups. For any 1-cocycle $z = (z_{ij}) \in Z^1(\mathfrak{U}, S)$ of a cover $\mathfrak{U} = (U_i)$ of M, such that $z_{ij} = b_i^{-1} b_j$, $b_i \in \Gamma(U_i, \mathcal{F}_S^\infty)$, we denote by $E(z)$ the flat group bundle with the fibre N determined by the cocycle $(\alpha_{z_{ij}}|N) \in Z^1(\mathfrak{U}, \mathrm{Aut}\, N)$.

Theorem 2.5. *Suppose that we have the semi-direct product $G = N \rtimes S$ of complex Lie groups. A cohomology class $\sigma \in H^1(\bar{R}_G)$ belongs to* $\mathrm{Im}(\Pi_1^*)$ *whenever there exists a form $\omega \in Z^1(R_S)$ satisfying the following conditions:*

(a) $\Pi_1^*(\zeta) = h_1^*(\sigma)$, where $\zeta \in H^1(R_S)$ is the cohomology class of ω;

(b) The mapping $\Pi_1^* : H^1(R_{E(z)}) \to H^1(\bar{R}_{E(z)})$ is surjective, where $z = (z_{ij}) \in Z^1(\mathfrak{U}, S)$ is defined by $z_{ij} = b_i^{-1} b_j$, $\delta_0(b_i) = \omega$ in U_i.

Proof. Apply Theorem 1.4 to the commutative diagram

$$\begin{array}{ccccccccc} e & \longrightarrow & R_N & \xrightarrow{i} & R_G & \xrightarrow{g} & R_S & \longrightarrow & e \\ & & {\scriptstyle \Pi}\big\downarrow & & \big\downarrow{\scriptstyle \Pi} & & \big\downarrow{\scriptstyle \Pi} & & \\ e & \longrightarrow & \bar{R}_N & \xrightarrow[k]{} & \bar{R}_G & \xrightarrow[h]{} & \bar{R}_S & \longrightarrow & e \end{array}$$

and use the proof of Theorem 2.2.

3. Classification of supermanifolds

3.1. Supermanifolds and vector bundles. Let $(M, \mathcal{F})$ be a fixed ringed space that can be either a real differentiable (or analytic) manifold of real dimension n, or a complex analytic manifold of complex dimension n. We want to classify, up to isomorphisms leaving any point of M fixed, all real differentiable (respectively, real or complex analytic) supermanifolds of dimension $n|m$ with reduction $(M,\mathcal{F})$. This can be easily made in terms of non-abelian cohomology.

Let us denote by $\xi_1,\dots,\xi_m$ the standard basis of k^m, where $k=\mathbb{R}$ or $\mathbb{C}$, and form the Grassmann algebra $A_m=\bigwedge_k(\xi_1,\dots,\xi_m)=\bigwedge_k(k^m)$. We may regard it either as an algebra graded by the exterior powers $A_m^p=\bigwedge_k^p(k^m)$, or as a superalgebra ($\mathbb{Z}_2$-graded algebra). The same is true for the sheaf of algebras $\mathcal{F}_{A_m}=\mathcal{F}\otimes A_m$. Consider the sheaf of groups $\mathcal{A}ut\,\mathcal{F}_{A_m}$ formed by germs of automorphisms of this sheaf of superalgebras. Take again an open cover $\mathfrak{U}=(U_i)$ of M and consider a cocycle $z\in Z^1(\mathfrak{U},\mathcal{A}ut\,\mathcal{F}_{A_m})$. The twisted sheaf $\mathcal{O}=\mathcal{F}^z_{A_m}$ is a sheaf of k-superalgebras on M that can be regarded as the structure sheaf of a supermanifold $(M,\mathcal{O})$ of dimension $n|m$. Denote by $\mathcal{J}$ the subsheaf of ideals of $\mathcal{O}$ generated by the subsheaf $\mathcal{O}_{\bar 1}$ of odd elements. One verifies easily that $\mathcal{O}/\mathcal{J}\simeq\mathcal{F}$, i.e., the reduction (the underlying manifold) of $(M,\mathcal{O})$ is $(M,\mathcal{F})$. Up to isomorphisms leaving any point of M fixed, this supermanifold depends on the cohomology class $\zeta\in H^1(M,\mathcal{A}ut\,\mathcal{F}_{A_m})$ only. Clearly, each supermanifold of dimension $n|m$ with reduction $(M,\mathcal{F})$ can be got in this way, using its transition functions. Thus, $H^1(M,\mathcal{A}ut\,\mathcal{F}_{A_m})$ can be interpreted as the set of all supermanifolds of dimension $n|m$ with reduction $(M,\mathcal{F})$, regarded up to isomorphisms leaving any point of M fixed.

Note that we have the injection of the sheaf of groups $j:\mathcal{F}_{\mathrm{GL}_m(k)}\to\mathcal{A}ut\,\mathcal{F}_{A_m}$ which extends, in a natural way, any linear transformation of k^n to an automorphisms of the algebra A_m. The image of j will be the subsheaf of automorphisms preserving the $\mathbb{Z}$-grading. If $z=j_1(z')$, where $z'\in Z^1(\mathfrak{U},\mathcal{F}_{\mathrm{GL}_m(k)})$, then the twisting by z preserves the $\mathbb{Z}$-grading, and hence the corresponding structure sheaf $\mathcal{O}$ will be a sheaf of $\mathbb{Z}$-graded algebras. We shall interprete $H^1(M,\mathcal{F}_{\mathrm{GL}_m(k)})$ as the set of all differentiable, respectively analytic, k-vector bundles of rank m over M, up to isomorphisms leaving any point of M fixed. One sees easily that $\mathcal{O}=\bigwedge_{\mathcal{F}}\mathcal{E}$, where $\mathcal{E}=\mathcal{F}^{z'}_{k^m}$ is the sheaf of differentiable, respectively analytic, sections of the vector bundle $\mathbf{E}$ corresponding to the cocycle z'. The supermanifolds $(M,\mathcal{O})$ of this special form (and those isomorphic to these supermanifolds) are called *split*. Thus, the set of split supermanifolds is the image of the mapping $j_1^*:H^1(M,\mathcal{F}_{\mathrm{GL}_m(k)})\to H^1(M,\mathcal{A}ut\,\mathcal{F}_{A_m})$.

Let $(M,\mathcal{O})$ be a supermanifold of dimension $n|m$ with reduction $(M,\mathcal{F})$. Then we have the folowing filtration of $\mathcal{O}$ by powers of its ideal subsheaf $\mathcal{J}$ generated by $\mathcal{O}_{\bar 1}$:

$$\mathcal{O}=\mathcal{J}^0\supset\mathcal{J}\supset\mathcal{J}^2\supset\dots \tag{3.1}$$

It is finite, since $\mathcal{J}^{m+1}=0$. Clearly, $\mathcal{O}/\mathcal{J}\simeq\mathcal{F}$, and $\mathcal{E}=\mathcal{J}/\mathcal{J}^2$ is a locally free sheaf of $\mathcal{F}$-modules of rank m over $(M,\mathcal{F})$.

In the case when $(M,\mathcal{O})$ is split, i.e., $\mathcal{O}=\bigwedge_{\mathcal{F}}\mathcal{E}$, where $\mathcal{E}$ is the sheaf of differentiable, respectively analytic sections of a vector bundle, the filtration (3.1) is accociated with the grading, i.e.,

$$\mathcal{J}^p=\bigoplus_{q\geq p}\bigwedge_{\mathcal{F}}^{q}\mathcal{E},\quad p\geq 0.$$

Now consider the sheaf $\mathcal{A}ut\,\mathcal{O}$ of automorphisms of the sheaf of superalgebras $\mathcal{O}$, and, for any $p\geq 0$, its subsheaf

$$\mathcal{A}ut_{(2p)}\mathcal{O}=\{a\in\mathcal{A}ut\,\mathcal{O}\mid a(f)-f\in\mathcal{J}^{2p}\text{ for any }f\in\mathcal{O}\}. \tag{3.2}$$

Then we get the following finite filtration:

$$\mathcal{A}ut\,\mathcal{O} = \mathcal{A}ut_{(0)}\mathcal{O} \supset \mathcal{A}ut_{(2)}\mathcal{O} \supset \dots . \tag{3.3}$$

Any automorphism preserves the filtration (3.1), and hence any $\mathcal{A}ut_{(2p)}\mathcal{O}$ is a subsheaf of normal subgroups.

Consider again the simplest case when $\mathcal{O} = \mathcal{F}_{A_m}$, A_m being the fixed Grassmann algebra $\bigwedge_k(k^m)$. Then $\mathcal{J}/\mathcal{J}^2 = \mathcal{F}_{k^m}$. Any automorphism of $\mathcal{O}$ leaves $\mathcal{J}$ and $\mathcal{J}^2$ invariant and hence induces an automorphism of $\mathcal{F}_{k^m}$, i.e., an element of $\mathcal{F}_{\mathrm{GL}_m(k)}$. We got a homomorphism of sheaves of groups $g : \mathcal{A}ut\,\mathcal{O} \to \mathcal{F}_{\mathrm{GL}_m(k)}$. Clearly, $g \circ j = \mathrm{id}$. Also $\operatorname{Ker} g = \mathcal{A}ut_{(2)}\mathcal{O}$. Thus, we have the exact sequence of sheaves of groups

$$e \to \mathcal{A}ut_{(2)}\mathcal{F}_{A_m} \xrightarrow{i} \mathcal{A}ut\,\mathcal{F}_{A_m} \xrightarrow{g} \mathcal{F}_{\mathrm{GL}_m(k)} \to e \tag{3.4}$$

that splits by $j : \mathcal{F}_{\mathrm{GL}_m(k)} \to \mathcal{A}ut\,\mathcal{F}_{A_m}$.

The mappings

$$g_1^* : H^1(M, \mathcal{A}ut\,\mathcal{F}_{A_m}) \to H^1(M, \mathcal{F}_{\mathrm{GL}_m(k)}),$$
$$j_1^* : H^1(M, \mathcal{F}_k) \to H^1(M, \mathcal{A}ut\,\mathcal{F}_{A_m})$$

have a nice interpretation in terms of supermanifolds and vector bundles. Namely, g_1^* assigns to any supermanifold $(M, \mathcal{O})$ with reduction $(M, \mathcal{F})$ the vector bundle $\mathbf{E}$ over M, whose sheaf of differentiable, respectively analytical, sections is isomorphic to $\mathcal{E} = \mathcal{J}/\mathcal{J}^2$, $\mathcal{J}$ being the ideal subsheaf of $\mathcal{O}$ described above. Now, j_1^* assigns to any vector bundle $\mathbf{E}$ the corresponding split supermanifold with the structure sheaf $\mathcal{O} = \bigwedge_{\mathcal{F}}\mathcal{E}$. Since $g_1^* \circ j_1^* = \mathrm{id}$, g_1^* is surjective. Now, $j_1^* \circ g_1^*$ assignes to any supermanifold $(M, \mathcal{O})$ the split supermanifold $(M, \bigwedge_{\mathcal{F}}\mathcal{E})$, where $\mathcal{E} = \mathcal{J}/\mathcal{J}^2$. The latter one is called the *retract* of $(M, \mathcal{O})$. One checks easily that its structure sheaf is isomorphic to the sheaf of graded algebras associated with the filtration (3.1) of $\mathcal{O}$.

We see that the classification problem for supermanifolds of dimension $n|m$ and reduction $(M, \mathcal{F})$ reduces to the folowing two problems:

(1) To classify vector bundles of rank m over $(M, \mathcal{F})$, i.e., to compute $H^1(M, \mathcal{F}_{\mathrm{GL}_m(k)})$.
(2) For a fixed vector bundle $\mathbf{E}$ of rank m over $(M, \mathcal{F})$, to classify the supermanifolds whose retracts are isomorphic to $(M, \bigwedge_{\mathcal{F}}\mathcal{E})$, i.e., to describe the inverse image $(g_1^*)^{-1}(\zeta)$ of the corresponding class $\zeta \in H^1(M, \mathcal{F}_{\mathrm{GL}_m(k)})$.

In what follows, we will study the second problem. In cohomological terms, it is solved by the following result (see [3]).

Theorem 3.1. *Let $\mathbf{E}$ be a vector bundle over $(M, \mathcal{F})$ and $(M, \mathcal{O})$ the corresponding split supermanifold. Then there is a bijection between the set of all supermanifolds having $(M, \mathcal{O})$ as their retract (considered up to isomorphisms leaving fixed any point of M) and the quotient $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O})/\operatorname{Aut}\mathbf{E}$, where the action is induced by the action of* $\operatorname{Aut}\mathbf{E}$ *on the sheaf $\mathcal{A}ut_{(2)}\mathcal{O}$ by conjugations.*

If $\omega \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O})$ is determined by a cocycle $w \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\mathcal{O})$, then the orbit of ω corresponds to the sheaf $\mathcal{O}^w$, where w is regarded as a 1-cocycle with values in $\mathcal{A}ut\,\mathcal{O}$.

Proof. Let $z \in Z^1(\mathfrak{U}, \mathcal{F}_{\mathrm{GL}_m(k)})$ be a cocycle representing the vector bundle $\mathbf{E}$. Then we have the cocycle $j_1(z) \in Z^1(\mathfrak{U}, \mathcal{A}ut\,\mathcal{F}_{A_m})$ which we will denote again by z. Let $(M, \mathcal{O})$ be the split supermanifold corresponding to $\mathbf{E}$. Then we may identify $\mathcal{A}ut\,\mathcal{O} = \mathcal{A}ut(\mathcal{F}_{A_m})^z$ with $(\mathcal{A}ut\,\mathcal{F}_{A_m})^{\alpha_z}$, and $\mathcal{A}ut\,\mathcal{E} = \mathcal{A}ut(\mathcal{F}_{k^m})^z$ with $(\mathcal{F}_{\mathrm{GL}_m(k)})^{\alpha_z}$. Therefore $g : \mathcal{A}ut\,\mathcal{F}_{A_m} \to \mathcal{F}_{\mathrm{GL}_m(k)}$ induces a homomorphism $g(z) : \mathcal{A}ut\,\mathcal{O} \to \mathcal{E}$. Clearly, $\operatorname{Ker} g(z) = \mathcal{A}ut_{(2)}\mathcal{O}$. Hence we get the following exact sequence that is a generalization of (3.4) to $(M, \mathcal{O})$:

$$e \to \mathcal{A}ut_{(2)}\mathcal{O} \stackrel{i(z)}{\to} \mathcal{A}ut\,\mathcal{O} \stackrel{g(z)}{\to} \mathcal{A}ut\,\mathcal{E} \to e. \tag{3.5}$$

We also have an extension homomorphism $j(z) : \mathcal{A}ut\,\mathcal{E} \to \mathcal{A}ut\,\mathcal{O}$ splitting the sequence (3.5).

Applying to (3.5) Corollary of Theorem 1.4, we get the following exact sequence of sets with distinguished points:

$$e \to H^1(M, \mathcal{A}ut_{(2)}\mathcal{O})/\operatorname{Aut}\mathcal{E} \stackrel{\overline{i(z)^*_1}}{\to} H^1(M, \mathcal{A}ut\,\mathcal{O}) \stackrel{g(z)^*_1}{\to} H^1(M, \mathcal{A}ut\,\mathcal{E}) \to e,$$

where the action of the group $\operatorname{Aut}\mathcal{E} = \operatorname{Aut}\mathbf{E}$ on $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O})$ is induced by the action of this group on $\mathcal{A}ut_{(2)}\mathcal{O}$ by conjugations $\alpha_{j(a)}$, $a \in \operatorname{Aut}\mathbf{E}$.

Since the cochain complexes for the sheaves $\mathcal{A}ut\,\mathcal{O}$, $\mathcal{A}ut\,\mathcal{E}$ identify with the twisted complexes for $\mathcal{A}ut\,\mathcal{F}_{A_m}$, $\mathcal{F}_{\mathrm{GL}_m(\mathbb{C})}$, we may apply Proposition 1.1. It identifies $\operatorname{Ker} g(z)^*_1 = \operatorname{Im} i(z)^*_1$ with $(g^*_1)^{-1}(\zeta)$, where $\zeta \in H^1(M, \mathcal{F}_{\mathrm{GL}_m(k)})$ is the cohomology class of z. Since $\overline{i(z)^*_1}$ is a bijection of $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O})/\operatorname{Aut}\mathbf{E}$ onto $\operatorname{Im} i^*_1$, the proof of the first assertion is finished.

Now, let $\omega \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O})$ be determined by $w \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\mathcal{O})$. We want to calculate a cocycle representing the class $r^*_z(i(z)^*_1(\omega))$. Identifying $\mathcal{A}ut\,\mathcal{O} = \mathcal{A}ut\,(\mathcal{F}_{A_m})^z$ with $(\mathcal{A}ut\,\mathcal{F}_{A_m})^{\alpha_z}$, we write w as a cochain (w^k_{ij}) with values in $\mathcal{A}ut\,\mathcal{F}_{A_m}$, where $w^k_{ij} = z_{kl} w^l_{ij} (z_{kl})^{-1}$. Then the desired cocycle is $\tilde{z}$, where $\tilde{z}_{ij} = w^i_{ij} z_{ij}$. By Proposition 1.3, it follows that $(\mathcal{F}_{A_m})^{\tilde{z}} = ((\mathcal{F}_{A_m})^z)^w = \mathcal{O}^w$. Thus, the second assertion is proved.

3.2. Tangent sheaf and exponential mapping. We retain the notation of Section 3.1. For an arbitrary supermanifold $(M, \mathcal{O})$, denote by $\mathcal{T} = \mathcal{D}er\,\mathcal{O}$ its tangent sheaf. This is a sheaf of Lie superalgebras with respect to the Lie bracket $[\ ,\ ]$.

Here we suppose that $(M, \mathcal{O})$ is the split supermanifold corresponding to a vector bundle $\mathbf{E}$ over M, i.e., that $\mathcal{O} = \bigwedge_{\mathcal{F}} \mathcal{E}$. Then $\mathcal{O}$ and $\mathcal{T}$ are sheaves of graded algebras, the $\mathbb{Z}$-grading in $\mathcal{T}$ being given by

$$\mathcal{T} = \bigoplus_{p \geq -1} \mathcal{T}_p, \text{ where } \mathcal{T}_p = \{u \in \mathcal{T} \mid u(\mathcal{O}_q) \subset \mathcal{O}_{q+p} \text{ for all } q \in \mathbb{Z}\}.$$

We also note that $\mathcal{T}$ is a locally free graded sheaf of $\mathcal{F}$-modules on the manifold $(M, \mathcal{F})$. This means that it can be regarded as the sheaf of differentiable, respectively analytic, sections of a certain graded vector bundle over M (see [2]). We call it the *supertangent bundle* and denote by $\mathbf{ST}(M, \mathcal{O})$. Note that for any $p \geq -1$ there exists a natural exact sequence of vector bundles

$$0 \to \mathbf{E}^* \otimes \bigwedge^{p+1} \mathbf{E} \to \mathbf{ST}_p(M, \mathcal{O}) \to \mathbf{T}(M) \otimes \bigwedge^{p} \mathbf{E} \to 0, \tag{3.6}$$

where $\mathbf{T}(M)$ is the tangent bundle of M.

Following [12], we shall apply the tangent sheaf for studying the sheaves of automorphisms $\mathcal{A}ut_{(2p)}\mathcal{O}$, $p \geq 1$, defined by (3.2). Consider the even part of $\mathcal{T}$; this is the graded sheaf of Lie algebras

$$\mathcal{T}_{\bar{0}} = \bigoplus_{p \geq 0} \mathcal{T}_{2p}.$$

This sheaf admits the natural finite filtration

$$\mathcal{T}_{\bar{0}} = \mathcal{T}_{\bar{0}}^{(0)} \supset \mathcal{T}_{\bar{0}}^{(2)} \supset \dots, \text{ where } \mathcal{T}_{\bar{0}}^{(2p)} = \bigoplus_{q \geq p} \mathcal{T}_{2q}. \tag{3.7}$$

Define the *exponential mapping* $\exp : \mathcal{T}_{\bar{0}}^{(2)} \to \mathcal{A}ut_{(2)}\mathcal{O}$ expressed by the usual exponential series. Since our filtrations are finite, this series will be actually a polynomial. One also defines the inverse mapping $\log = \exp^{-1}$. Thus, exp is an isomorphism of sheaves of sets (but, in general, not of groups). Clearly, exp maps each subsheaf $\mathcal{T}_{\bar{0}}^{(2p)}$, $p \geq 1$, onto $\mathcal{A}ut_{(2p)}\mathcal{O}$.

To get a homomorphism of sheaves of groups, we define the mapping $\lambda_{2p} : \mathcal{A}ut_{(2p)}\mathcal{O} \to \mathcal{T}_{2p}$ assigning to any $a \in \mathcal{A}ut_{(2p)}\mathcal{O}$ the $(2p)$-component $(\log a)_{2p}$ of $\log a$. By the Campbell–Hausdorff formula, we have

$$\log(ab) = \log a + \log b + \frac{1}{2}[\log a, \log b] + \dots, \tag{3.8}$$

and hence

$$(\log(ab))_{2p} = (\log a)_{2p} + (\log b)_{2p}.$$

Thus, for any $p \geq 1$, λ_{2p} is a homomorphism of sheaves of groups. Clearly, $\operatorname{Ker} \lambda_{2p} = \mathcal{A}ut_{(2p+2)}\mathcal{O}$. Therefore we have the exact sequence of sheaves of groups

$$e \to \mathcal{A}ut_{(2p+2)}\mathcal{O} \to \mathcal{A}ut_{(2p)}\mathcal{O} \xrightarrow{\lambda_{2p}} \mathcal{T}_{2p} \to 0. \tag{3.9}$$

Note that $\mathcal{A}ut\,\mathcal{O}$ acts on $\mathcal{T}$ by the automorphisms $\alpha_g : v \mapsto gvg^{-1}$, leaving invariant $\mathcal{T}_{\bar{0}}$ and the filtration (3.7). We have

$$\log \alpha_g(a) = \alpha_g(\log a), \quad a \in \mathcal{A}ut_{(2)}\mathcal{O},$$
$$\alpha_{\exp u}(v) = \exp(\operatorname{ad} u)(v) = v + [u, v] + \frac{1}{2!}[u, [u, v]] + \dots, \quad u, v \in \mathcal{T}_{\bar{0}}^{(2)}.$$

We will use (3.9) for proving the following vanishing theorem (see also [10], Lemma 3.1 and Proposition 3.2).

Theorem 3.2. *Fix an integer $p \geq 1$.*

(1) *If $H^1(M, \mathcal{T}_{2q}) = 0$ for all $q \geq p$, then $H^1(M, \mathcal{A}ut_{(2q)}\mathcal{O}) = \varepsilon$ for all $q \geq p$.*
(2) *If $H^1(M, \mathcal{T}_{2q}) = 0$ for all $q \geq p+1$, then $\lambda_{2p}^* : H^1(M, \mathcal{A}ut_{(2p)}\mathcal{O}) \to H^1(M, \mathcal{T}_{2p})$ is injective.*

Proof. To prove (1), we use the induction by $-p$. For $p \geq m+1$ the assertion is true. Suppose it to be true for $p+1$. To prove it for p, we should assume that $H^1(M, \mathcal{T}_{2q}) = 0$ for all $q \geq p$. Then by the induction hypothesis we have $H^1(M, \mathcal{A}ut_{(2q)}\mathcal{O}) = \varepsilon$ for all

$q \geq p+1$. We have to prove that $H^1(M, \mathcal{A}ut_{(2p)}\mathcal{O}) = \varepsilon$. By Theorem 1.6, (3.9) gives the exact sequence

$$H^1(M, \mathcal{A}ut_{(2p+2)}\mathcal{O}) \longrightarrow H^1(M, \mathcal{A}ut_{(2p)}\mathcal{O}) \stackrel{(\lambda_{2p})_1^*}{\longrightarrow} H^1(M, \mathcal{T}_{2p}). \tag{3.10}$$

It easily implies our assertion, since both extreme terms are trivial.

To prove (2), we assume that $H^1(M, \mathcal{T}_{2q}) = 0$ for all $q \geq p+1$. Then, by (1), $H^1(M, \mathcal{A}ut_{(2q)}\mathcal{O}) = \varepsilon$ for all $q \geq p+1$. In particular, the left-hand term in (3.10) is trivial, whence $\operatorname{Ker}(\lambda_{2p})_1^* = \varepsilon$. But we have to prove that $(\lambda_{2p})_1^*)^{-1}((\lambda_{2p})_1^*)(\zeta)) = \{\zeta\}$ for any $\zeta \in H^1(M, \mathcal{A}ut_{(2p)}\mathcal{O})$.

Let $z \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2p)}\mathcal{O})$ denote a cocycle representing ζ. We can regard z as a cocycle with values in $\mathcal{A}ut_{(2)}\mathcal{O}$. Since $\mathcal{A}ut_{(2q)}\mathcal{O}$, $q \geq p$, are subsheaves of normal subgroups, we may consider the twisted sheaves $(\mathcal{A}ut_{(2q)}\mathcal{O})^{\alpha_z}$. Now we construct exact sequences similar to (3.9). Any element of $(\mathcal{A}ut_{(2q)}\mathcal{O})^{\alpha_z}$ can be expressed as a system (a_i), where $a_i \in \mathcal{A}ut_{(2q)}\mathcal{O}|U_i$, satisfying $a_i = z_{ij}a_j z_{ij}^{-1}$. One sees easily that $(\log a_i)_{2q} = (\log a_j)_{2q}$. Therefore λ_{2q} determines a homomorphism of sheaves $\lambda_{2q}^z : (\mathcal{A}ut_{(2q)}\mathcal{O})^{\alpha_z} \to \mathcal{T}_{2q}$. Clearly, $\operatorname{Ker}\lambda_{2q}^z = (\mathcal{A}ut_{(2q+2)}\mathcal{O})^{\alpha_z}$, and hence we get the exact sequence

$$e \to (\mathcal{A}ut_{(2q+2)}\mathcal{O})^{\alpha_z} \to (\mathcal{A}ut_{(2q)}\mathcal{O})^{\alpha_z} \stackrel{\lambda_{2q}^z}{\to} \mathcal{T}_{2q} \to 0. \tag{3.11}$$

As in the assertion (1), we can deduce from (3.11) that

$$H^1(M, (\mathcal{A}ut_{(2q)}\mathcal{O})^{\alpha_z}) = \varepsilon \text{ for all } q \geq p+1.$$

Now consider (3.11) for $q = p$. The corresponding exact cohomology sequence shows that $\operatorname{Ker}(\lambda_{2p}^z)_1^* = \varepsilon$. But α_z is a cocycle of inner automorphisms of $\mathcal{A}ut_{(2p)}\mathcal{O}$, and hence we have the bijection $r_{\alpha_z}^* : H^1(M, (\mathcal{A}ut_{(2p)}\mathcal{O})^{\alpha_z}) \to H^1(M, \mathcal{A}ut_{(2p)}\mathcal{O})$ that maps $\operatorname{Ker}(\lambda_{2p}^z)_1^*$ onto $((\lambda_{2p})_1^*)^{-1}((\lambda_{2p})_1^*(\zeta))$. Thus, $(\lambda_{2p})_1^*$ is injective.

Using Theorems 3.1 and 3.2, we can solve our problem in some special cases. The following two theorems are well-known (see [1]).

Theorem 3.3. *Any real differentiable or analytic supermanifold is split.*

Proof. Let $\mathcal{T}$ be the tangent sheaf of a split real differentiable supermanifold $(M, \mathcal{O})$. The sheaves $\mathcal{T}_{2q}$ are fine, and hence $H^1(M, \mathcal{T}_{2q}) = 0$ for all q. Hence, by Theorem 3.2 (1), $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}) = \varepsilon$. By Theorem 3.1, this implies that any supermanifold with retract $(M, \mathcal{O})$ is isomorphic to $(M, \mathcal{O})$. The same argument can be applied in the real analytic case, since $H^1(M, \mathcal{T}_{2q}) = 0$ for all q in this case, too.

The same argument gives the following result.

Theorem 3.4. *If M is a Stein manifold, then any complex analytic supermanifold with reduction M is split.*

We will nead a similar result for a more complicated situation. Let $(M, \mathcal{F})$ be again a complex analytic manifold and $\mathbf{E}$ a holomorphic vector bundle over M. Denote by $\mathcal{F}^\infty$ the sheaf of differentiable complex-valued functions on M and by $\mathcal{E}^\infty$ the sheaf of differentiable sections of $\mathbf{E}$. Then the sheaf of algebras

$$\mathcal{O}^\infty = \bigwedge_{\mathcal{F}^\infty} \mathcal{E}^\infty$$

is the sheaf of differentiable sections of $\bigwedge \mathbf{E}$. The ringed space $(M, \mathcal{O}^\infty)$ is the *differentiable complex supermanifold* corresponding to $(M, \mathcal{O})$. We have

$$\mathcal{O}^\infty = \mathcal{F}^\infty \otimes \mathcal{O}.$$

Let $\mathcal{T}^\infty = \mathcal{D}er\, \mathcal{O}^\infty$ denote the tangent sheaf of $(M, \mathcal{O}^\infty)$.

A derivation $v \in \mathcal{T}^\infty$ will be called *pseudoholomorphic* if $v(\bar{f}) = 0$ for all $f \in F$. Clearly, the subset $\mathcal{PT}$ of all pseudoholomorphic derivations is a subsheaf of graded Lie superalgebras and, moreover, of $\mathcal{O}^\infty$-submodules in $\mathcal{T}^\infty$. The sheaves of even derivations $\mathcal{T}^\infty_{\bar{0}} \supset \mathcal{PT}^\infty_{\bar{0}}$ are sheaves of graded Lie algebras and admit filtrations by ideal subsheaves $\mathcal{T}^{\infty(2p)}_{\bar{0}}$ and $\mathcal{PT}^{(2p)}_{\bar{0}}$ similar to (3.7).

As in the real differentiable case, we have the filtration of $\mathcal{O}^\infty$ by the powers of the ideal subsheaf $\mathcal{J}^\infty$ generated by odd elements, giving rise to the fitration of the automorphism sheaf $\mathcal{A}ut\, \mathcal{O}^\infty$ by subsheaves of normal subgroups

$$\mathcal{A}ut\, \mathcal{O}^\infty = \mathcal{A}ut_{(0)}\mathcal{O}^\infty \supset \mathcal{A}ut_{(2)}\mathcal{O}^\infty \supset \mathcal{A}ut_{(4)}\mathcal{O}^\infty \supset \dots,$$
$$\text{where } \mathcal{A}ut_{(2p)}\mathcal{O}^\infty = \{a \in \mathcal{A}ut\, \mathcal{O}^\infty \mid a(f) - f \in (\mathcal{J}^\infty)^{2p} \text{ for all } f \in \mathcal{O}^\infty\}.$$

Now, define the subsheaves

$$\mathcal{PA}ut\, \mathcal{O}^\infty = \{a \in \mathcal{A}ut\, \mathcal{O}^\infty \mid a(\bar{f}) = \bar{f} \text{ for all } f \in \mathcal{F}\},$$
$$\mathcal{PA}ut_{(2p)}\mathcal{O}^\infty = \mathcal{A}ut_{(2p)}\mathcal{O}^\infty \cap \mathcal{PA}ut\, \mathcal{O}^\infty.$$

Then

$$\mathcal{PA}ut\, \mathcal{O}^\infty = \mathcal{PA}ut_{(0)}\mathcal{O}^\infty \supset \mathcal{PA}ut_{(2)}\mathcal{O}^\infty \supset \mathcal{PA}ut_{(4)}\mathcal{O}^\infty \supset \dots.$$

We also have the exponential mapping

$$\exp : \mathcal{T}^{\infty(2)}_{\bar{0}} \to \mathcal{A}ut_{(2)}\mathcal{O}^\infty.$$

It is an isomorphism of sheaves of sets and maps $\mathcal{T}^{\infty(2p)}_{\bar{0}}$ onto $\mathcal{A}ut_{(2p)}\mathcal{O}^\infty$ and $\mathcal{PT}^{(2p)}_{\bar{0}}$ onto $\mathcal{PA}ut_{(2p)}\mathcal{O}^\infty$, $p \geq 1$. Using $\log = \exp^{-1}$, we get the following exact sequences similar to (3.9):

$$e \to \mathcal{PA}ut_{(2p+2)}\mathcal{O}^\infty \to \mathcal{PA}ut_{(2p)}\mathcal{O}^\infty \to \mathcal{PT}_{2p} \to 0. \tag{3.12}$$

Since $\mathcal{PT}_{2p}$ are fine sheaves, their 1-cohomology is 0. Then, as in Theorem 3.3, we deduce from (3.12) the following result.

Theorem 3.5. *For any holomorphic vector bundle* $\mathbf{E}$ *over a complex manifold* M, *we have* $H^1(M, \mathcal{PA}ut_{(2p)}\mathcal{O}^\infty) = \varepsilon$ *for all* $p \geq 1$.

3.3. Resolutions of automorphism sheaves. Let $M = (M, \mathcal{F})$ be a complex manifold, $\mathbf{E}$ a holomorphic vector bundle over $(M, \mathcal{F})$, and $\mathcal{E}$ the sheaf of holomorphic sections of $\mathbf{E}$. We will study again the split supermanifold $(M, \mathcal{O})$, where $\mathcal{O} = \bigwedge_{\mathcal{F}} \mathcal{E}$. In what follows, we will omit the subscript $\mathcal{F}$ in the notation of exterior algebras or tensor products over this sheaf of algebras. The goal of this section is to construct resolutions of the non-abelian sheaves $\mathcal{A}ut_{(2p)}\mathcal{O}$, $p \geq 1$.

First, we construct a fine resolution of the tangent sheaf $\mathcal{T} = \mathcal{D}er\,\mathcal{O}$. Consider the classical bigraded sheaf $\Phi = \bigoplus_{p,q} \Phi^{p,q}$ of smooth complex-valued differential forms on M. Since $\mathcal{T}$ is a locally free analytic sheaf, we can form the standard Dolbeault–Serre resolution of $\mathcal{T}$ that is the bigraded sheaf $\mathcal{R}$ of $\mathbf{ST}$-valued smooth forms of type $(0,*)$. More precisely, we set

$$\begin{aligned} \mathcal{R}_{p,q} &= \Phi^{0,q} \otimes \mathcal{T}_p, \\ \mathcal{R} &= \bigoplus_{p \geq -1, q \geq 0} \mathcal{R}_{p,q}, \\ \bar{\partial}(\varphi \otimes u) &= (\bar{\partial}\varphi) \otimes u, \quad \varphi \in \mathcal{R}_{0,q}, u \in \mathcal{T}_p. \end{aligned}$$

Then the sequence

$$0 \to \mathcal{T} \xrightarrow{i} \mathcal{R}_{*,0} \xrightarrow{\bar{\partial}} \mathcal{R}_{*,1} \xrightarrow{\bar{\partial}} \dots . \tag{3.13}$$

is the desired resolution. However, it is convenient to write this resolution in a more complicated form, expressing $\mathbf{ST}$-valued forms as derivations of the sheaf $\hat{\Phi}$ of smooth $\bigwedge \mathbf{E}$-valued forms.

Thus, we first form the standard Dolbeault–Serre resolution of $\mathcal{O}$, setting

$$\begin{aligned} \hat{\Phi}^{p,q} &= \Phi^{0,q} \otimes \mathcal{O}_p, \\ \hat{\Phi} &= \bigoplus_{p,q \geq 0} \hat{\Phi}^{p,q}, \\ \bar{\partial}(\varphi \otimes u) &= (\bar{\partial}\varphi) \otimes u, \quad \varphi \in \Phi^{0,q}, u \in \mathcal{O}_p. \end{aligned}$$

Then, regarding $\hat{\Phi}$ as a sheaf of graded algebras with respect to the total degree, consider the sheaf of graded Lie algebras $\hat{\mathcal{T}} = \mathcal{D}er\,\hat{\Phi}$. Denote

$$\bar{D} = \operatorname{ad} \bar{\partial}.$$

Clearly, $\bar{D}$ is a derivation of degree 1 (and of bidegree $(0,1)$) of $\hat{\mathcal{T}}$, and

$$\bar{D}^2 = \frac{1}{2}[\bar{D}, \bar{D}] = \frac{1}{2} \operatorname{ad}[\bar{\partial}, \bar{\partial}] = 0.$$

By definition, we have

$$\bar{D}u = \bar{\partial} \circ u - (-1)^{p(u)} u \circ \bar{\partial}, \quad u \in \mathcal{S}.$$

Set

$$\mathcal{S} = \{u \in \hat{\mathcal{T}} \mid u(\bar{f}) = u(d\bar{f}) = 0 \text{ for any } f \in \mathcal{F}\}.$$

One sees readily that $\mathcal{S}$ is a subsheaf of bigraded subalgebras and of $\mathcal{F}^\infty$-submodules of $\hat{\mathcal{T}}$. Further, for any $u \in \mathcal{S}$ and any $f \in \mathcal{O}$ we have

$$(\bar{D}u)(\bar{f}) = (\bar{D}u)(d\bar{f}) = 0,$$

and so $\bar{D}(\mathcal{S}) \subset \mathcal{S}$.

As in Section 3.2, consider the sheaves

$$\mathcal{O}^\infty = \mathcal{F}^\infty \otimes \mathcal{O} = \hat{\Phi}^{*,0},$$

$\mathcal{T}^\infty = \mathcal{D}er\,\mathcal{O}^\infty$ and the subsheaf $\mathcal{PT}$ of pseudoholomorphic derivations. Then we can identify $\mathcal{PT}$ with $\mathcal{S}_{*,0}$. To do it, note that $\hat{\Phi}$ is a locally free sheaf of $\mathcal{O}^\infty$-modules. Suppose that $x_1, \dots, x_n$ and $\xi_1, \dots, \xi_m$ are respectively even and odd local coordinates of $(M, \mathcal{O})$. Then a local basis of this sheaf is formed by the sections $d\bar{x}_{i_1} \dots d\bar{x}_{i_q} \otimes 1 = d\bar{x}_{i_1} \dots d\bar{x}_{i_q}$, $i_1 < \dots < i_q$. For any $u \in \mathcal{PT}$, we put

$$u(\sum_{i_1<\dots<i_q} f_{i_1\dots i_q} d\bar{x}_{i_1} \dots d\bar{x}_{i_q}) = \sum_{i_1<\dots<i_q} u(f_{i_1\dots i_q}) d\bar{x}_{i_1} \dots d\bar{x}_{i_q},$$

where $f_{i_1\dots i_q} \in \mathcal{O}^\infty$.

On the other hand, we have a natural injection $\mathcal{T} \to \mathcal{PT}$ that extends any derivation germ u of $\mathcal{O}$ to a derivation germ $\tilde{u}$ of $\mathcal{O}^\infty$ satisfying $\tilde{u}(\bar{f}) = 0$ for any $f \in \mathcal{F}$. This injection is easily described in terms of the local coordinates introduced above. Namely, suppose that $u = \sum_i a_i \frac{\partial}{\partial x_i} + \sum_j b_j \frac{\partial}{\partial \xi_j}$. Then $\tilde{u}$ is expressed by the same formula in the local coordinates $x_1, \dots, x_n, \bar{x}_1, \dots, \bar{x}_n, \xi_1, \dots, \xi_m$ of $(M, \mathcal{O}^\infty)$.

This allows to define a natural injection $i : \mathcal{T} \to \mathcal{S}_{*,0}$. Thus, we get the sequence

$$0 \to \mathcal{T} \xrightarrow{i} \mathcal{S}_{*,0} \xrightarrow{\bar{D}} \mathcal{S}_{*,1} \xrightarrow{\bar{D}} \dots. \tag{3.14}$$

By [9], Theorem 1, this is a fine resolution of $\mathcal{T}$ isomorphic to (3.13). Moreover, i is a homomorphism of graded Lie algebra sheaves, and hence the natural bracket in $\mathcal{S}$ may be used to calculate the bracket in $H^*(M, \mathcal{T})$ induced by the Lie bracket defined in $\mathcal{T}$.

Now we go over to the construction of resolutions for certain sheaves of automorphisms of $\mathcal{O}$. Consider the subsheaf $\mathcal{PA}ut\,\mathcal{O}^\infty$ of $\mathcal{A}ut\,\mathcal{O}^\infty$ defined in Section 3.2. Any $a \in \mathcal{PA}ut\,\mathcal{O}^\infty$ can be extended to an automorphism of $\hat{\Phi}$; the extension is denoted by the same symbol a and is given by

$$a(\sum_{i_1<\dots<i_q} f_{i_1\dots i_q} d\bar{x}_{i_1} \dots d\bar{x}_{i_q}) = \sum_{i_1<\dots<i_q} a(f_{i_1\dots i_q}) d\bar{x}_{i_1} \dots d\bar{x}_{i_q},$$

where $f_{i_1\dots i_q} \in \mathcal{O}^\infty$. One checks easily that thus we get an injection $\mathcal{PA}ut\,\mathcal{O}^\infty \to \mathcal{A}ut\,\hat{\Phi}$. Further, $u \mapsto aua^{-1}$ is, clearly, an automorphism of $\hat{\mathcal{T}}$ for any $a \in \mathcal{A}ut\,\hat{\Phi}$. If $a \in \mathcal{PA}ut\,\mathcal{O}^\infty$, then this automorphism maps $\mathcal{S}$ onto itself. Therefore, we get an action $\mathrm{Int} : a \mapsto \mathrm{Int}\,a$ of the sheaf $\mathcal{PA}ut\,\mathcal{O}^\infty$ on $\mathcal{S}$ by the automorphisms $\mathrm{Int}\,a(u) = aua^{-1}$.

Now we define the mapping $\delta_0 : \mathcal{PA}ut\,\mathcal{O}^\infty \to \hat{\mathcal{T}}_{*,1}$ by

$$\delta_0(a) = \bar{\partial} - a\bar{\partial}a^{-1},$$

and the mapping $\delta_1 : \hat{\mathcal{T}}_{*,1} \to \hat{\mathcal{T}}_{*,2}$ by

$$\delta_1(u) = \bar{D}u - \frac{1}{2}[u, u] = -\frac{1}{2}[u - \bar{\partial}, u - \bar{\partial}].$$

Clearly, $\delta_1(0) = 0$.

Proposition 3.1.

(1) *The mapping* δ_0 *is a crossed homomorphism of* $\mathcal{PA}ut\,\mathcal{O}^\infty$ *into* $\mathcal{S}_{*,1}$.

(2) *The corresponding affine action of* $\mathcal{PA}ut\,\mathcal{O}^\infty$ *on* $\mathcal{S}_{*,1}$ *is given by*

$$\rho(a)(u) = a(u - \bar{\partial})a^{-1} + \bar{\partial}.$$

(3) *The mapping* δ_1 *maps* $\mathcal{S}_{*,1}$ *into* $\mathcal{S}_{*,2}$ *and satisfies*

$$\delta_1(\rho(a)(u)) = a\delta_1(u)a^{-1}.$$

Proof. One checks easily that $\delta_0(a) \in \mathcal{S}_{*,1}$ whenever $a \in \mathcal{PA}ut\,\mathcal{O}^\infty$. Also

$$\delta_0(ab) = \bar{\partial} - ab\bar{\partial}b^{-1}a^{-1} = \bar{\partial} - a\bar{\partial}a^{-1} + a(\bar{\partial} - b\bar{\partial}b^{-1})a^{-1} = \delta_0(a) + a\delta_0(b)a^{-1}.$$

Now, by definition,

$$\rho(a)(u) = aua^{-1} + \delta_0(a) = a(u - \bar{\partial})a^{-1} + \bar{\partial}.$$

Finally,

$$\delta_1(\rho(a)(u)) = -\frac{1}{2}[a(u - \bar{\partial})a^{-1}, a(u - \bar{\partial})a^{-1}] = a\delta_1(u)a^{-1}.$$

Let us introduce the formal series

$$E(x) = \frac{\exp x - 1}{x} = 1 + \frac{1}{2!}x + \frac{1}{3!}x^2 + \dots .$$

Proposition 3.2. *For any* $v \in \mathcal{T}_{\bar{0}}^{\infty(2)}$ *we have*

$$\delta_0(\exp v) = E(\operatorname{ad} v)(\bar{D}v).$$

Proof. By definition, we have

$$\begin{aligned}\delta_0(\exp v) &= \bar{\partial} - (\exp v)\bar{\partial}(\exp v)^{-1} = \bar{\partial} - (\exp \operatorname{ad} v)(\bar{\partial})\\ &= E(\operatorname{ad} v)(-[v, \bar{\partial}]) = E(\operatorname{ad} v)(\bar{D}v),\end{aligned}$$

since v is an even derivation.

Proposition 3.3. *Any* $a \in \mathcal{A}ut_{(2)}\mathcal{O}$ *can be uniquely extended to an automorphism* $b \in \mathcal{PA}ut_{(2)}\mathcal{O}^\infty$ *such that* $b\bar{\partial}b^{-1} = \bar{\partial}$. *This gives us, for any* $p \geq 1$, *a natural injective homomorphism* $j : \mathcal{A}ut_{(2p)}\mathcal{O} \to \mathcal{PA}ut_{(2p)}\mathcal{O}^\infty$ *whose image is* $(\operatorname{Ker} \delta_0) \cap \mathcal{PA}ut_{(2p)}\mathcal{O}^\infty$.

Proof. Any $b \in \mathcal{PA}ut_{(2)}\mathcal{O}^\infty$ satisfying $\delta_0(b) = 0$ maps $\mathcal{O}$ onto itself. Thus, we get the restriction homomorphism $(\operatorname{Ker} \delta_0) \cap \mathcal{PA}ut_{(2)}\mathcal{O}^\infty \to \mathcal{A}ut_{(2)}\mathcal{O}$. If $b|\mathcal{O} = \mathrm{id}$, then $\log b(f) = 0$ for any $f \in \mathcal{O}$. Since $\log b(\bar{f}) = 0$ for any $f \in \mathcal{F}$, we have $\log b = 0$, and $b = \mathrm{id}$. Thus, b is uniquely determined by its restriction onto $\mathcal{O}$. On the other hand, if $a \in \mathcal{A}ut_{(2)}\mathcal{O}$, then $\log a$, as was shown above, can be uniquely prolonged to a derivation $v \in \mathcal{PT}_{\bar{0}}^{(2)}$ satisfying $[\bar{\partial}, v] = 0$. Then $(\exp v)\bar{\partial}(\exp v)^{-1} = (\exp \operatorname{ad} v)(\bar{\partial}) = \bar{\partial}$, and thus $\exp v$ is the desired extension of a onto $\mathcal{O}^\infty$.

Define

$$\mathcal{S}_q^{(2p)} = \bigoplus_{k \geq p} \mathcal{S}_{2k,q}.$$

Theorem 3.6. *For any $p \geq 1$, the sequence*

$$\mathcal{PA}ut_{(2p)}\mathcal{O}^\infty \xrightarrow{\delta_0} \mathcal{S}_1^{(2p)} \xrightarrow{\delta_1} \mathcal{S}_2^{(2p)} \tag{3.15}$$

is a resolution of the sheaf $\mathcal{A}ut_{(2p)}\mathcal{O}$.

Proof. We see from Proposition 3.1 that (3.15) gives us a sheaf of cochain complexes, if we define the action σ_q of $\mathcal{K}_p^0 = \mathcal{PA}ut_{(2p)}\mathcal{O}^\infty$ on $\mathcal{K}_p^q = \mathcal{S}_q^{(2p)}$, $q = 1, 2$, as Int. By Proposition 3.3, the 0-cohomology of it is isomorphic to $\mathcal{A}ut_{(2p)}\mathcal{O}$. We have only to check that the 1-cohomology is trivial, i.e., that $\operatorname{Ker}\delta_1 \subset \operatorname{Im}\delta_0$ in (3.16).

Since this is true for large p, we can proceed by induction in $-p$. Suppose that the sequence (3.15) associated with $p+1$ is exact. Take $u \in \mathcal{S}_1^{(2p)}$ such that $\delta_1(u) = 0$. If $u_{2p} \in \mathcal{S}_{2p,1}$ is the $2p$-component of u, then, clearly, the $2p$-component of $\delta_1(u)$ is $\bar{D}u$. Hence $\bar{D}u = 0$. Since (3.14) is exact, there exists $v \in \mathcal{S}_{2p,0}$ such that $\bar{D}v = u$. Using Proposition 3.2, we see that

$$\begin{aligned} u' &= \rho(\exp(-v))(u) = \delta_0(\exp(-v)) + (\exp(-v))u(\exp v) \\ &= -E(-\operatorname{ad} v)(\bar{D}v) + \exp(-\operatorname{ad} v)(u) \end{aligned}$$

lies in $\mathcal{S}_1^{(2p+2)}$. Clearly, $\delta_1(u') = 0$. By the inductive hypothesis, $u' = \delta_0(a)$, where $a \in \mathcal{PA}ut_{(2p+2)}\mathcal{O}^\infty$. Then

$$\begin{aligned} u &= \rho(\exp v)(u') = \rho(\exp v)(\delta_0(a)) = \rho(\exp v)(\rho(a)(0)) \\ &= \rho((\exp v)a)(0) = \delta_0((\exp v)a). \end{aligned}$$

Clearly, $(\exp v)a \in \mathcal{PA}ut_{(2p)}\mathcal{O}^\infty$, and the exactness is proved.

Now we consider the corresponding complexes of global sections. For any $p \geq 1$, denote

$$\begin{aligned} \mathrm{K}_p^0 &= \mathrm{PAut}_{(2p)}\,\mathcal{O}^\infty = \Gamma(M, \mathcal{PA}ut_{(2p)}\mathcal{O}^\infty), \\ \mathrm{K}_p^q &= \mathrm{S}_q^{(2p)} = \Gamma(M, \mathcal{S}_q^{(2p)}),\ q = 1, 2, \end{aligned}$$

and define, as above, the operators $\delta_q : \mathrm{K}_p^q \to \mathrm{K}_p^{q+1}$, $q = 0, 1$, and the actions $\sigma_q = \mathrm{Int}$ of K_p^0 on K_p^q, $q = 1, 2$. Then, using Theorems 3.6, 1.7 and 3.5, we get the following result.

Theorem 3.7. *For any $p \geq 1$, we have an isomorphism of sets with distinguished points*

$$\mu_p : H^1(\mathrm{K}_p) \to H^1(M, \mathcal{A}ut_{(2p)}\mathcal{O}).$$

The mapping μ_p can be expressed quite explicitly. Take $z \in Z^1(\mathrm{K}_p)$. There exists an open cover $\mathfrak{U} = (U_i)$ of M such that $z = \delta_0(a_i)$, where $a_i \in \Gamma(U_i, \mathcal{PA}ut_{(2p)}\mathcal{O}^\infty)$, for any i. Define $b \in Z^1(\mathfrak{U}, \mathcal{PA}ut_{(2p)}\mathcal{O}^\infty)$ by $b_{ij} = a_i^{-1}a_j$. One sees that $b \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2p)}\mathcal{O})$. Then μ_p maps the cohomology class of z onto that of b.

Due to Theorem 3.1, it is important to interprete the natural action of $\operatorname{Aut}\mathbf{E}$ on $H^1(M, \mathcal{A}ut_{(2p)}\mathcal{O})$ in terms of the complex K_p. Clearly, any holomorphic automorphism of $\mathbf{E}$ is smooth, and hence we have the natural injection of the sheaf $\mathcal{A}ut\,\mathcal{E}$ into $\mathcal{A}ut\,\mathcal{E}^\infty$. On the other hand, any element of $\mathcal{A}ut\,\mathcal{E}^\infty$ extends uniquely to an automorphism of $\mathcal{O}^\infty$ leaving fixed any element of $\mathcal{F}^\infty$, and thus we get the inclusions $\mathcal{A}ut\,\mathcal{E} \subset \mathcal{A}ut\,\mathcal{E}^\infty \subset \mathcal{PA}ut\,\mathcal{O}^\infty$. As we saw above, there is an action Int of the group $\operatorname{Aut}\mathbf{E} = \operatorname{Aut}\mathcal{E}$ on the resolution (3.15) of $\mathcal{A}ut_{(2p)}\mathcal{O}$, $p \geq 1$. After verifying that $\operatorname{Int} a$, for any $a \in \operatorname{Aut}\mathbf{E}$, is an automorphism of this resolution, we get from Theorem 1.8 the following assertion.

Proposition 3.4. *The action of* $\operatorname{Aut}\mathbf{E}$ *on the sheaf* $\mathcal{A}ut_{(2p)}\mathcal{O}$ *by conjugations induces an action on the 1-cohomology of this sheaf corresponding, via the isomorphism* μ_p *of Theorem 3.7, to the action of this group on* $H^1(\mathrm{K}_p)$ *induced by the action* Int *on the resolution (3.16).*

Note that $\operatorname{Aut}\mathbf{E}$ is non-trivial whenever $m = \operatorname{rk}\mathbf{E} \geq 1$. In fact, to any $c \in \mathbb{C}^\times$ there corresponds the dilatation d_c of $\mathbf{E}$ by the scalar c.

Proposition 3.5. *We have*

$$(\operatorname{Int} d_c)(u) = c^{2q}u, \quad c \in \mathbb{C}^\times, \ u \in \Gamma(M, \mathcal{S}_{2q,r}).$$

Proof. For any $f \in \mathcal{O}_k$, $\varphi \in \Phi^{0,l}$ we have $u(\varphi \otimes f) \in \Gamma(M, \mathcal{S}_{2q+k,r+l})$. Therefore

$$\begin{aligned}((\operatorname{Int} d_c)(u))(\varphi \otimes f) &= (d_c u d_{-c})(\varphi \otimes f) = d_c(u(c^{-k}\varphi \otimes f)) \\ &= c^{-k} d_c(u(\varphi \otimes f)) = c^{-k}c^{2q+k}u(\varphi \otimes f) = (c^{2q}u)(\varphi \otimes f).\end{aligned}$$

3.4. $(2p)$**-projections.** Here we consider an application of the resolution constructed in Section 3.3.

Suppose that $p \geq 1$. In Section 3.2, we have defined the homomorphism of sheaves of groups $\lambda_{2p} : \mathcal{A}ut_{(2p)}\mathcal{O} \to \mathcal{T}_{2p}$. Actually, (3.8) implies that

$$(\log(ab))_{2q} = (\log a)_{2q} + (\log b)_{2q}$$

for $p \leq q \leq 2p-1$, whenever $a, b \in \mathcal{A}ut_{(2p)}\mathcal{O}$. Therefore we can define a homomorphism of sheaves of groups $\hat{\lambda}_{2p} : \mathcal{A}ut_{(2p)}\mathcal{O}^\infty \to \mathcal{T}_{2p}^\infty$ by

$$\hat{\lambda}_{2p}(a) = \sum_{q=p}^{2p-1} (\log a)_{2q}.$$

Clearly, $\operatorname{Ker}\hat{\lambda}_{2p} = \mathcal{A}ut_{(4p)}\mathcal{O}^\infty$. Also $\hat{\lambda}_2 = \lambda_2$. Further, $\operatorname{Ker}\hat{\lambda}_{2p}$ maps $\mathcal{P}\mathcal{A}ut_{(2p)}\mathcal{O}^\infty$ onto $\bigoplus_{q=p}^{2p-1} \mathcal{P}\mathcal{T}_{2q}$. From the definitions one sees easily that the following diagram is commutative:

$$\begin{array}{ccccccccc}
e & \longrightarrow & \mathcal{A}ut_{(2p)}\mathcal{O} & \xrightarrow{j} & \mathcal{P}\mathcal{A}ut_{(2p)}\mathcal{O}^\infty & \xrightarrow{\delta_0} & \mathcal{S}_1^{(2p)} & \xrightarrow{\delta_1} & \mathcal{S}_2^{(2p)} \\
 & & \Big\downarrow{\scriptstyle\hat{\lambda}_{2p}} & & \Big\downarrow{\scriptstyle\hat{\lambda}_{2p}} & & \Big\downarrow{\scriptstyle\pi_{2p}} & & \Big\downarrow{\scriptstyle\pi_{2p}} \\
0 & \longrightarrow & \bigoplus_{q=p}^{2p-1}\mathcal{T}_{2q} & \xrightarrow[i]{} & \bigoplus_{q=p}^{2p-1}\mathcal{S}_{2q,0} & \xrightarrow[\bar{D}]{} & \bigoplus_{q=p}^{2p-1}\mathcal{S}_{2q,1} & \xrightarrow[\bar{D}]{} & \bigoplus_{q=p}^{2p-1}\mathcal{S}_{2q,2}.
\end{array}$$

Here π_{2p} is the natural projection of $\mathcal{S}_r^{(2p)}$ onto $\bigoplus_{q=p}^{2p-1}\mathcal{S}_{2q,r}$, $r = 1, 2$. Thus, we get a homomorphism π_{2p} of a resolution of $\mathcal{A}ut_{(2p)}\mathcal{O}$ onto a resolution of $\bigoplus_{q=p}^{2p-1}\mathcal{T}_{2q}$ inducing $\hat{\lambda}_{2p}$ on these sheaves. By Theorem 1.5, we may identify $(\hat{\lambda}_{2p})_1^*$ with the mapping $(\pi_{2p})_1^* : H^1(K_p) \to \bigoplus_{q=p}^{2p-1} H^1(\mathrm{S}_{2q,*})$, where $\mathrm{S}_{2q,*}$ is the complex of sections of the resolution (3.14) of $\mathcal{T}_{2q}$. The following theorem was essentially proved in [12] using the Čech cochain complex.

Theorem 3.8. *If $H^2(M, \mathcal{T}_{2q}) = 0$ for all $q \geq 2p$, then $(\hat{\lambda}_{2p})_1^* : H^1(M, \mathcal{A}ut_{(2p)}\mathcal{O}) \to \bigoplus_{q=p}^{2p-1} H^1(M, \mathcal{T}_{2q})$ is surjective.*

Proof. Take $u_{2q} \in \mathrm{S}_{2q,1}$, $q = p, \ldots, 2p-1$, satisfying $\bar{D}u_{2q} = 0$. We want to construct elements $u_{2k} \in \mathrm{S}_{2k,1}$, $k \geq 2p$, such that $u = u_{2p} + \ldots + u_{4p-1} + u_{4p} + \ldots$ satisfies $\bar{D}u - \frac{1}{2}[u, u] = 0$. It suffices to prove the following inductive assertion:

For any $s \geq 2p$, there exist elements $u_{2k} \in \mathrm{S}_{2k,1}$, $2p \leq k \leq s$, such that $v(s) = u_{2p} + \ldots + u_{4p-1} + u_{4p} + \ldots + u_{2s}$ satisfies

$$\delta_1(v(s)) = \bar{D}v(s) - \frac{1}{2}[v(s), v(s)] \in \mathrm{S}_2^{(2s+2)}. \tag{3.16}$$

This is true for $s = 2p$. Suppose that $v(s-1) = u_{2p} + \ldots + u_{4p-1} + u_{4p} + \ldots + u_{2s-2}$ ist constructed. Denote

$$w = \delta_1(v(s-1)) = w_{2s} + w_{2s+2} + \ldots,$$

where $w_{2k} \in \mathrm{S}_{2k,2}$. Then

$$\begin{aligned} \bar{D}w &= -\frac{1}{2}\bar{D}[v(s-1), v(s-1)] = -[\bar{D}v(q-1), v(s-1)] \\ &= -[w + \frac{1}{2}[v(s-1), v(s-1)], v(s-1)] = -[w, v(s-1)] \in \mathrm{S}_2^{(2s+2)}. \end{aligned}$$

In particular, $\bar{D}w_{2s} = 0$. Since $H^2(M, \mathcal{T}_{2s}) = 0$, we have $w_{2s} = -\bar{D}u_{2s}$, where $u_{2s} \in \Gamma(M, \mathcal{S}_{2s,1})$. Then define $v(s) = v(s-1) + u_{2s}$. One checks easily that $v(s)$ satisfies (3.16), and our assertion is proved by induction.

Corollary 1. *If $H^1(M, \mathcal{T}_{2q}) = H^2(M, \mathcal{T}_{2q}) = 0$ for all $q \geq p+1$, then $(\lambda_{2p})_1^* : H^1(M, \mathcal{A}ut_{(2p)}\mathcal{O}) \to H^1(M, \mathcal{T}_{2p})$ is an isomorphism.*

Proof. By Theorem 3.2 (1), $(\lambda_{2p})_1^*$ is injective. Under our conditions, $(\lambda_{2p})_1^* = (\hat{\lambda}_{2p})_1^*$, and hence it is surjective by Theorem 3.8.

Corollary 2. *If $m = \mathrm{rk}\,\mathbf{E} \leq 3$, then $(\lambda_2)_1^* : H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}) \to H^1(M, \mathcal{T}_2)$ is an isomorphism.*

For another special case, where Corollary 1 applies, see Theorem 3.9 below.

3.5. Case of the cotangent bundle. Suppose that $\mathbf{E} = \mathbf{T}^*(M)$ is the cotangent bundle of M. Then $\mathcal{O} = \Omega$ is the sheaf of holomorphic forms. Therefore $\hat{\Phi}^{p,q} = \Phi^{0,q} \otimes \Omega^p = \Phi^{p,q}$. Thus, $\hat{\Phi}$ is the sheaf Φ of all smooth complex-valued forms on M. Due to Frölicher–Nijenhuis, the derivations of this sheaf can be easily described in terms of vector-valued forms. We will mention two results concerning this case.

First, there is a situation, where the conditions of Corollary 1 of Theorem 3.8 are fulfilled for $p = 1$. We denote by $\mathrm{Gr}_{n,k}$ the Grassmannian of k-planes in $\mathbb{C}^n$. The following is proved in [10], Theorem 4.4.

Theorem 3.9. *Let M be an irreducible compact simply connected Hermitian symmetric space of dimension ≥ 2 and let $\mathbf{E} = \mathbf{T}^*(M)$. Then the conditions of Corollary 1 of Theorem 3.8 are fulfilled for $p = 1$, and hence $(\lambda_2)_1^* : H^1(M, \mathcal{A}ut_{(2)}\Omega) \to H^1(M, \mathcal{T}_2)$ is an isomorphism. Now,*

$$\dim H^1(M, \mathcal{T}_2) = \begin{cases} 2 & \text{if } M = \mathrm{Gr}_{n,k},\ 2 \leq k \leq n-2, \\ 1 & \text{otherwise.} \end{cases}$$

Using Theorem 3.1 and Proposition 3.5, we get (see [10], Theorem 4.6) the following

Corollary. *Suppose that M be an irreducible compact simply connected Hermitian symmetric space of dimension ≥ 2. If $M = \mathrm{Gr}_{n,k}$, $2 \leq k \leq n-2$, then there exists a 1-parameter analytic family of pairwise non isomorphic non-split supermanifolds with retract (M, Ω). Otherwise, there exists precisely one non-split supermanifold with this property.*

Another result gives, in particular, a construction of non-split supermanifolds with retract (M, Ω) for any compact Kähler manifold M. Let M be an arbitrary complex manifold and $\omega \in \Gamma(M, \Phi^{1,1})$ be a (1,1)-form satisfying $d\omega = 0$. Consider the derivation $u = \omega\partial$ of Φ. Clearly, $u \in S_{2,1}$. Moreover,

$$\begin{aligned} \bar{D}u &= [\bar{\partial}, \omega\partial] = (\bar{\partial}\omega)\partial + \omega\bar{\partial}\partial = \bar{\partial}\omega(\partial) + \omega(\bar{\partial}\partial + \partial\bar{\partial}) = 0, \\ [u,u] &= [\omega\partial, \omega\partial] = 2\omega(\partial\omega)\partial = 0. \end{aligned}$$

Thus, $\delta_1(u) = 0$, and u determines a cohomology class $\tilde{u} \in H^1(M, \mathcal{A}ut_{(2)}\Omega)$. Using $(\lambda_2)_1^*$, the following theorem can be proved (see [11], Theorem 2.1):

Theorem 3.10. *If M is a compact Kähler manifold of dimension ≥ 2, then the correspondence $\omega \mapsto \tilde{u}$ gives rise to an injective mapping $H^{1,1}(M, \mathbb{C}) \to H^1(M, \mathcal{A}ut_{(2)}\Omega)$.*

3.6. Application of Hodge theory. Under assumptions of Section 3.3, endow M and $\mathbf{E}$ with smooth Hermitian metrics. Then we get natural Hermitian metrics in the vector bundles $\mathbf{E}^* \otimes \bigwedge \mathbf{E}$ and $\mathbf{T}(M) \otimes \bigwedge \mathbf{E}$. Using (3.6), one can introduce a Hermitian metric in $\mathbf{ST}(M)$ such that the subbundle of $\mathbf{ST}_p(M)$, $p \geq 0$, orthogonal to $\mathbf{E}^* \otimes \bigwedge^{p+1} \mathbf{E}$, is projected isometrically onto $\mathbf{T}(M) \otimes \bigwedge^p \mathbf{E}$. Suppose that M is compact. Then we can develop the standard Hodge theory in the complex $(\Gamma(M, \mathcal{R}), \bar{\partial})$ or in its subcomplex $\mathrm{S}^{(2)} = (\Gamma(M, \mathcal{S}^{(2)}), \bar{D})$ (see [4]). Let $\bar{D}^*$ denote the operator conjugate to $\bar{D}$ and $\square = [\bar{D}, \bar{D}^*]$ the Beltrami–Laplace operator. Their bidegrees are $(0,-1)$ and $(0,0)$, respectively. Then we have the orthogonal decomposition

$$\mathrm{S}^{(2)} = \mathbf{H} \oplus \bar{D}\,\mathrm{S}^{(2)} \oplus \bar{D}^*\,\mathrm{S}^{(2)}, \tag{3.17}$$

where $\mathbf{H} = \operatorname{Ker} \square$ is the bigraded subspace of harmonic elements. Moreover,

$$\mathrm{id} = H + \square G = H + \bar{D}\bar{D}^* G + \bar{D}^*\bar{D} G, \tag{3.18}$$

where H is the projection onto $\mathbf{H}$ in (3.17) and G is the so-called Green operator commuting with $\bar{D}$ and $\bar{D}^*$. Then

$$\begin{aligned} \operatorname{Ker} \bar{D} &= \mathbf{H} \oplus \bar{D}\,\mathrm{S}^{(2)}, \\ \operatorname{Ker} \bar{D}^* &= \mathbf{H} \oplus \bar{D}^*\,\mathrm{S}^{(2)}. \end{aligned} \tag{3.19}$$

In particular,

$$\begin{aligned} \mathbf{H}_{2p,q} &\simeq H^q(M, \mathcal{T}_{2p}), \quad p \geq 1,\ q \geq 0. \\ \mathbf{H}_{*,q} &\simeq H^q(M, \mathcal{T}^{(2)}), \quad q \geq 0. \end{aligned}$$

Clearly, the subcomplexes $\mathrm{S}^{(2p)} = \Gamma(M, \mathcal{S}^{(2p)})$, $p \geq 1$, are invariant with respect to $\bar{D}^*$, $\square$ and G.

Let us denote $L_0 = \bar{D}^*(\mathrm{S}_1^{(2)})$ and $L_1 = \operatorname{Ker} \bar{D}^* \cap \mathrm{S}_1^{(2)}$. Then (3.17) and (3.19) imply

$$\mathrm{S}_1^{(2)} = \bar{D}(L_0) \oplus L_1. \tag{3.20}$$

We are going now to establish a non-linear analogue of this decomposition.

Define the mapping $Q : L_0 \times L_1 \to \mathrm{S}_1^{(2)}$ by

$$Q(v_2 + v_4 + \ldots + v_{2r}, u) = \rho(\exp v_{2r}) \ldots \rho(\exp v_2)(u).$$

Here $v_{2q} \in (L_0)_{2q}$, $q = 1, \ldots, r$, $u \in L_1$, $r = [m/2]$, where $m = \operatorname{rk} \mathbf{E}$. Note that $\mathrm{S}^{(2p)} = 0$ for $p \geq r+1$.

Theorem 3.11.

(1) *The mapping* $Q : L_0 \times L_1 \to \mathrm{S}_1^{(2)}$ *is bijective.*
(2) Q *defines a bijection* $L_0 \times (Z^1(\mathrm{K}_2) \cap L_1) \to Z^1(\mathrm{K}_2)$.
(3) *The natural mapping of* $Z^1(\mathrm{K}_2) \cap L_1$ *into* $H^1(\mathrm{K}_2) = H^1(M, \mathcal{A}ut_{(2)}\mathcal{O})$ *is surjective.*

Proof. (1) Let us prove, for any $p = 1, \ldots, r$ the following assertion:

(A_p). Assume that

$$w = w_2 + \ldots + w_{2p-2} + w_{2p} + \ldots \in \mathrm{S}_1^{(2)}$$

is given, where $w_{2q} \in \mathrm{S}_{2q,1}$ and $w_{2q} \in L_1$ for $q = 1, \ldots, p-1$. Then there exist $v_{2p} \in (L_0)_{2p}$ such that

$$\rho(\exp v)(w) = w_2 + \ldots + w_{2p-2} + \tilde{w}_{2p} + \tilde{w}_{2p+2} \ldots,$$

where $\tilde{w}_{2q} \in \mathrm{S}_{2q,1}$ and $\tilde{w}_{2p} \in L_1$.

If (A_p) is proved for $p = 1, \ldots, r$, then for any $w \in \mathrm{S}_1^{(2)}$ we can find elements $v_{2p} \in (L_0)_{2p}$, $p = 1, \ldots, r$, such that $\rho(\exp v_{2r}) \ldots \rho(\exp v_2)(w) \in L_1$. This implies that Q is surjective.

To prove (A_p), we note that by (3.20) $w_{2p} = -\bar{D}v_{2p} + \tilde{w}_{2p}$, where $v_{2p} \in (L_0)_{2p}$, $\tilde{w}_{2p} \in (L_1)_{2p}$. Using Proposition 4.2, we see that

$$\begin{aligned} \rho(\exp v_{2p})(w) &= \delta_0(\exp v_{2p}) + (\exp v_{2p}) w (\exp v_{2p})^{-1} \\ &= E(\operatorname{ad} v_{2p})(\bar{D}v_{2p}) + (\exp \operatorname{ad} v_{2p})(w) \\ &= w_2 + \ldots + w_{2p-2} + w_{2p} - \bar{D}v_{2p} + w' \\ &= w_2 + \ldots + w_{2p-2} + \tilde{w}_{2p} + w', \end{aligned}$$

where $w' \in \mathrm{S}_1^{(2p+2)}$.

To prove the injectivity of Q, suppose that

$$\rho(\exp v_{2r}) \ldots \rho(\exp v_2)(u) = \rho(\exp v'_{2r}) \ldots \rho(\exp v'_2)(u'),$$

where $v_{2q}, v'_{2q} \in (L_0)_{2q}$, $u, u' \in L_1$. We have to prove that $v_{2q} = v'_{2q}$ for all q. Assume that it is true for $q = r, \ldots, p+1$. Then

$$\rho(\exp v_{2p})(w) = \rho(\exp v'_{2p})(w'),$$

where

$$w = \rho(\exp v_{2p-2}) \dots \rho(\exp v_2)(u), \quad w' = \rho(\exp v'_{2p-2}) \dots \rho(\exp v'_2)(u').$$

Write

$$w = w_2 + \dots + w_{2p-2} + w_{2p} + \dots , \quad w' = w'_2 + \dots + w'_{2p-2} + w'_{2p} + \dots ,$$

where $w_{2q}, w'_{2q} \in \mathrm{S}_{2q,1}$. Using Proposition 4.2, we see that

$$\begin{aligned} \rho(\exp v_{2p})(w) &= w_2 + \dots + w_{2p-2} + w_{2p} + \bar{D}v_{2p} + \dots , \\ \rho(\exp v'_{2p})(w) &= w'_2 + \dots + w'_{2p-2} + w'_{2p} + \bar{D}v'_{2p} + \dots , \end{aligned}$$

where the dots denote elements lying in $\mathrm{S}_1^{(2p+2)}$. Therefore $w_{2p} + \bar{D}v_{2p} = w'_{2p} + \bar{D}v'_{2p}$, whence $\bar{D}v_{2p} = \bar{D}v'_{2p}$, and $v_{2p} = v'_{2p}$, since $v_{2p}, v'_{2p} \in L_0$. Thus, our assertion is proved by induction.

(2) follows from (1), since $Z^1(K_2)$ is invariant under ρ, and (3) follows from (2).

Theorem 3.11 implies that any element of $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O})$ is represented by a cocycle from the subset $\mathbf{K} = Z^1(\mathrm{K}_2) \cap L_1$. We will study now the structure of this algebraic subset of the Frechet space $\mathrm{S}_1^{(2)}$.

Proposition 3.6. *The subset* $\mathbf{K}$ *is connected.*

Proof. For any $u = \sum_{k\ge 1} u_{2k}$, where $u \in \mathrm{S}_{2k,1}$, consider the complex curve $u(t) = \sum_{k\ge 1} t^k u_{2k}$, $t \in \mathbb{C}$, connecting u with $0 = u(0)$. If $u \in Z^1(\mathrm{K}_2)$, then $u(t) \in Z^1(\mathrm{K}_2)$ for any $t \in \mathbb{C}$. This follows, e.g., from Proposition 3.5, since d_c is an automorphism of the complex K_2 for any $c \in \mathbb{C}^\times$. Therefore, $u(t) \in \mathbf{K}$ for any $t \in \mathbb{C}$ whenever $u \in \mathbf{K}$.

Consider now the algebraic subset of $\mathrm{S}_1^{(2)}$ given by

$$u - \frac{1}{2}\bar{D}^* G[u,u] = Hu. \tag{3.21}$$

Clearly, (3.18) implies that $\mathbf{K} \subset \mathbf{K}_0$. Also, $\mathbf{K}_0 \subset L_1$.

Theorem 3.12.

(1) *There exists a polynomial mapping* $\varphi : \mathbf{H}_{*,1} \to L_1$ *such that* $\varphi(\mathbf{H}_{*,1}) = \mathbf{K}_0$, $H \circ \varphi = \mathrm{id}$, $\varphi \circ H = \mathrm{id}$ *on* $\mathbf{K}_0$, $\varphi(0) = 0$, $d_0\varphi = \mathrm{id}$. *In particular,* $H : \mathbf{K}_0 \to \mathbf{H}_{*,1}$ *is a bijection.*

(2) H *determines a bijection of* $\mathbf{K}$ *onto the connected algebraic subset* $\mathbf{V}$ *of* $\mathbf{H}_{*,1} \simeq H^1(M, \mathcal{T}^{(2)})$ *given by the equation*

$$H[\varphi(h), \varphi(h)] = 0, \quad h \in \mathbf{H}_{*,1}. \tag{3.22}$$

Proof. We are going to solve the equation (3.21), where $h = Hu \in \mathbf{H}_{*,1}$ is a given harmonic element, while u is an unknown element of $\mathrm{S}_1^{(2)}$ (necessarily lying in L_1). Writing $h = \sum_{k\geq 1} h_{2k}$, $u = \sum_{k\geq 1} u_{2k}$, where $h_{2k}, u_{2k} \in \mathrm{S}_{2k,1}$, we obtain the following equations:

$$\begin{aligned}
&u_2 = h_2,\\
&u_4 - \frac{1}{2}\bar{D}^* G[u_2, u_2] = h_4,\\
&\dots\dots\dots\dots\dots\dots\\
&u_{2k} - \frac{1}{2}\bar{D}^* G\sum_{s=1}^{k-1}[u_{2s}, u_{2(k-s)}] = h_{2k},\\
&\dots\dots\dots\dots\dots\dots
\end{aligned}$$

One proves by induction that they admit a unique solution $u = \varphi(h)$, where $\varphi : \mathbf{H}_{*,1} \to L_1$ is a polynomial mapping. Thus,

$$\varphi(h) - \frac{1}{2}\bar{D}^* G[\varphi(h), \varphi(h)] = h \tag{3.23}$$

for any $h \in \mathbf{H}_{*,1}$. Clearly, φ satisfies the conditions of (1).

To prove (2), we have to characterize those $h \in \mathbf{H}_{*,1}$, for which $\bar{D}\varphi(h) = \frac{1}{2}[\varphi(h), \varphi(h)]$. It follows from (3.18) and (3.23) that

$$\begin{aligned}
&\bar{D}\varphi(h) - \frac{1}{2}[\varphi(h), \varphi(h)]\\
=&Hu + \frac{1}{2}(\bar{D}\bar{D}^* G[\varphi(h), \varphi(h)] - [\varphi(h), \varphi(h)])\\
=&-\frac{1}{2}(H[\varphi(h), \varphi(h)] + \bar{D}^*\bar{D}G[\varphi(h), \varphi(h)]).
\end{aligned} \tag{3.24}$$

If $\bar{D}\varphi(h) - \frac{1}{2}[\varphi(h), \varphi(h)] = 0$, then, clearly, $H[\varphi(h), \varphi(h)] = \bar{D}^*\bar{D}G[\varphi(h), \varphi(h)] = 0$. Thus, $h \in \mathbf{V}$. Conversely, suppose that $h \in \mathbf{V}$, i.e., $H[\varphi(h), \varphi(h)] = 0$. We prove that $\bar{D}^*\bar{D}G[\varphi(h), \varphi(h)] = 0$ which implies that $\bar{D}\varphi(h) - \frac{1}{2}[\varphi(h), \varphi(h)] = 0$. In fact, we see from (3.24) that

$$\begin{aligned}
&\bar{D}^*\bar{D}G[\varphi(h), \varphi(h)] = 2\bar{D}^* G[\bar{D}\varphi(h), \varphi(h)]\\
=&\bar{D}^* G[[\varphi(h), \varphi(h)] - \bar{D}^*\bar{D}G[\varphi(h), \varphi(h)], \varphi(h)]\\
=&-\bar{D}^* G[\bar{D}^*\bar{D}G[\varphi(h), \varphi(h)], \varphi(h)].
\end{aligned}$$

Denoting $\psi(h) = \bar{D}^*\bar{D}G[\varphi(h), \varphi(h)]$, we obtain

$$\psi(h) = -\bar{D}^*[\psi(h), \psi(h)].$$

Writing $\psi(h) = \sum_{k\geq 1} \psi_{2k}(h)$, where $\psi_{2k}(h) \in \mathrm{S}_{2k,1}$, we easily prove by induction that $\psi_{2k}(h) = 0$ for any $k \geq 1$. Thus, $\psi(h) = 0$.

The connectedness of $\mathbf{V}$ follows from Proposition 3.6.

Corollary. *Suppose that* $H^2(M, \mathcal{T}^{(4)}) = 0$, *where* $\mathcal{T}^{(4)} = \bigoplus_{p\geq 2} \mathcal{T}_{2p}$. *Then* $\mathbf{K} = \mathbf{K}_0$, *and the mapping* $H : \mathbf{K} \to \mathbf{H}_{*,1}$ *is a bijection.*

Proof. Under our condition, the equation (3.22) is satisfied identically, whence $\mathbf{V} = \mathbf{H}_{*,1}$.

Thus, $\mathbf{K}$ is bijectively projected onto the connected algebraic subvariety $\mathbf{V}$ of the graded vector space $\mathbf{H}_{*,1} \simeq H^1(M, \mathcal{T}^{(2)})$ of finite dimension given by (3.22). This is an analogue of the *Kuranishi family* of complex structures on a compact manifold (see [6,7]). By Theorems 3.11 and 3.12, we can regard $\mathbf{V}$ as a moduli variety for the cohomology set $H^1(M, \mathcal{A}ut_{(2)}\mathcal{O})$. Of course, this cohomology set is in general a non-trivial quotient of $\mathbf{V}$.

As an example, consider the case $n = \dim M = 1$. Then, clearly, $Z^1(\mathrm{K}_2) = \operatorname{Ker} \bar{D} \cap S_1^{(2)}$, $L_1 = \mathbf{H}_{*,1}$, and hence $\mathbf{V} = \mathbf{K}_0 = \mathbf{H}_{*,1}$.

Theorem 3.13. *Suppose that* M *is a compact Riemannian surface and* $m \leq 5$. *Two cocycles* $u, u' \in \mathbf{H}_{*,1}$ *represent the same cohomology class in* $H^1(\mathrm{K}_2)$ *if and only if there exists* $v_2 \in \Gamma(M, \mathcal{T}_2)$ *such that*

$$\begin{aligned} u'_2 &= u_2, \\ u'_4 &= u_4 + H[v_2, u_2]. \end{aligned} \tag{3.25}$$

Proof. Let $v = v_2 + v_4 \in \mathrm{S}_0^{(2)}$ satisfies $u' = \rho(v)(u)$. Using Proposition 3.2, we see that

$$\begin{aligned} u'_2 &= u_2 + \bar{D}v_2, \\ u'_4 &= u_4 + \bar{D}v_4 + \frac{1}{2}[v_2, \bar{D}v_2] + [v_2, u_2]. \end{aligned} \tag{3.26}$$

Clearly, (3.26) implies (3.25), where $\bar{D}v_2 = 0$. Conversely, if (3.25) holds for a certain $v_2 \in \Gamma(M, \mathcal{T}_2)$, then we can find a $v_4 \in \mathrm{S}_{4,0}$ such that $u'_4 = u_4 + \bar{D}v_4 + [v_2, u_2]$. Clearly, (3.26) will be satisfied, whence $u' = \rho(v_2 + v_4)(u)$.

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