SPINOR EQUATIONS IN WEYL GEOMETRY[1]

VOLKER BUCHHOLZ[2]

Humboldt Universität zu Berlin, Institut für Reine Mathematik,
Ziegelstraße 13a , D-10099 Berlin.

ABSTRACT: In this paper, the Dirac, twistor and Killing equations are investigated on CSpin manifolds. These are Weyl manifolds together with a CSpin structure. A conformal Schrödinger-Lichnerowicz formula is presented. It is used in order to show integrability conditions for the twistor and Killing equation.

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0 Introduction

In this paper, we give an introduction to conformal spin geometry, i.e., spin geometry on conformal manifolds, that are additionally equipped with a Weyl structure.
In the first section, we state some basic facts about Weyl manifolds. This includes the definition of density bundles, Weyl structures, curvature terms and Einstein-Weyl structures. The latter arise as necessary conditions for the existence of non-trivial real Killing spinor fields. For a more detailed approach to Weyl geometry or Einstein Weyl geometry we refer to [3] and [8].
The second section is dedicated to conformal spin geometry. In [1], [4], [5] and [6] the properties and integrability conditions of the twistor and Killing equation are studied. Here we want to generalize some of these results to arbitrary Weyl structures. The first result in the area of conformal spin geometry is due to Andrei Moroianu [7]. He studied the integrability conditions for the existence of non-trivial parallel spinors of weight 0. This is naturally generalized by considering Killing spinor fields of arbitrary weight, i.e., spinor fields that satisfy $\nabla^S_X\psi = \beta X\cdot\psi$, where $\beta \in \Gamma(\mathbb{C}\otimes\mathcal{L}^{-1})$ denotes a complex density of weight -1. In more detail, we first establish a basic setting of $CSpin(n) := Spin(n)\times\mathbb{R}_+$ structures on Weyl manifolds and proceed to the properties of the spinor derivative corresponding to the Weyl structure. Investigating the relationship between curvature data of the underlying Weyl manifold and spinor data leads to the central equation (12):

$$\mu^{234}\mathcal{R}'_c\psi = -2\mu^2 Ric'\psi - 2\mu^2 F\psi - \nu\mu F\psi,$$

where $F \in \Omega^2(M^n)$ denotes the induced curvature form of a Weyl structure W on $\mathcal{L}^1$, the density bundle of weight 1. F is usually called the Faraday curvature. The operator $\mu^{ab} : T^{r,s}\otimes S^w \longrightarrow S^w$ is the conformal Clifford product of a spinor field of weight w with the b^{th} and then with the a^{th}

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[2]e-mail: `bv@@mathematik.hu-berlin.de`

component of a tensor field. Furthermore, $\nu : S^w \longrightarrow T^* \otimes S^w$ is defined by $X \rfloor \nu\psi = X \cdot \psi$. The latter equation essentially describes the action of a triple Clifford product of the curvature tensor on a spinor of weight 0. Next, the Dirac operator $\mathcal{D}_W$ is introduced as follows: $\mathcal{D}_W \psi := \mu \nabla^S \psi$. It is important to notice that $\mathcal{D}_W$ reduces the weight of the spinor field, which it is applied to. The investigation of the Dirac operator is completed by the proof of a conformal version of the Schrödinger-Lichnerowicz Formula, i.e.,

$$\mathcal{D}^2\psi = \Delta^S\psi + \frac{1}{4}R\psi + \left(\frac{n-2+2w}{4}\right) F \cdot \psi,$$

where R denotes the scalar curvature of the Weyl structure. After defining the twistor operator $\mathcal{T}_W$ in this setting by $\mathcal{T}_W := \nabla^S + \frac{1}{n}\nu\mathcal{D}$, we compute

$$\nabla^S \mathcal{D}\psi = \frac{n}{n-2}\left[-\frac{1}{2}\mu^2 Ric' + \frac{1}{4(n-1)}R\nu + \left(w - \frac{1}{2}\right)\left(\mu^2 F + \frac{2}{4(n-1)}\nu\mu F\right)\right]\psi$$

on its kernel, where Ric' is the Ricci curvature of the $\mathfrak{o}(n)$-component W' of the Weyl structure W. This equation corresponds to the equation $\nabla^S_X \mathcal{D}\psi = \frac{n}{2(n-2)}\left(\frac{R}{2(n-1)}X - Ric(X)\right) \cdot \psi$ in [1]. We use this result to prove, that the two well known first integrals $C(\psi)$ and $Q(\psi)$ are parallel densities if the weight of ψ is $\frac{1}{2}$ or $d\theta \cdot \psi = 0$. Furthermore, we use it in order show that the zeros of a twistor spinor field form a discrete set. This is followed by a paragraph on Killing spinor fields. We prove that the existence of real Killing spinor fields forces W to be Einstein-Weyl and $R = 4n(n-1)\beta^2$. Finally, we give some two-dimensional examples.

1 Weyl Geometry

Let M^n be a smooth manifold and $(\mathbf{R}, M^n, \pi, GL(n,\mathbb{R})$ its frame bundle. A conformal class c of Riemannian metrics, i.e., $c := [g] := \{\vartheta g | \vartheta \in C^\infty(M, \mathbb{R}_+)\}$, will be called a *conformal structure*. A smooth manifold (M^n, c) together with a conformal structure is said to be a *conformal* manifold. Let $CO(n) := O(n) \times \mathbb{R}_+ = \{\vartheta A : A \in O(n), \vartheta \in \mathbb{R}_+\}$ be the *conformal group*. There is a bijection between the set of all conformal structures and $CO(n)$-reductions $\mathbf{P}$ of $\mathbf{R}$. On (M^n, c) we have the following vector bundles

1. $\mathcal{L}^w := \mathbf{P} \times_{|det|^{\frac{w}{n}}} \mathbb{R}$ is called a density bundle with weight w.
2. $T^w := \mathbf{P} \times_{\rho^w} \mathbb{R}^n$ is the tangent bundle of weight w.
3. $(T^{r,s})^w := \mathbf{P} \times_{(\rho^{r,s})^w} (\bigotimes^r(\mathbb{R}^n)^* \bigotimes^s \mathbb{R}^n)$
4. For any representation (γ^w, V) we obtain $E^w := \mathbf{P} \times_{\gamma^w} V$,

where ρ and $\rho^{r,s}$ are the corresponding standard representations. $w \in \mathbb{R}$ indicates the power of the action of $\mathbb{R}_+$, i.e., $\gamma^w(\vartheta A) = \vartheta^w \gamma^w(A)$. If there is no ambiguity to be expected, the tensor product sign will be omitted in tensor products with densities. Let $l_g \in \mathcal{L}^1$ denote the density corresponding to a metric $g \in c$ to the power of $\frac{1}{n}$. Then we are given the following conformally invariant objects

1. $c := l_g^2 g : T^w \otimes T^{w_1} \longrightarrow \mathcal{L}^{w+w_1}$, $|X|^2 := c(X,X) \in \mathcal{L}^{2w}$ for $X \in T^w$.
2. $(.)_c := l_g^2(.)_g : T^w \longrightarrow (T^*)^w$, $(.)^c := ((.)_c)^{-1}$, $\omega \wedge X := \omega \otimes X - X_c \otimes w^c$ for $X \in \Gamma(T^w)$ and $\omega \in \Gamma((T^*)^{w_1})$, $\omega \wedge X$ is a section of the antisymmetric endomorphisms of T with respect to any gauge $g \in c$,
3. $tr := l_g^{-2} tr_g : (T^{r,s})^w \longrightarrow (T^{r-2,s})^w$, $r \geq 2$.

Moreover, we define some operators on $T^{r,s}$:

1. Let (ab) denote the transposition of the components a and b,
2. $Sym := Id + (12)$, $Alt = Id - (12)$, $Zyk := Id + (23)(12) + (12)(23)$, $Zyk^{1234} := Id + (12)(23)(34) + (34)(23)(12) + (13)(24)$.

A non-vanishing section of $\mathcal{L}^w$ is called gauge. l_g is a gauge and because of the bijective relationship to $g \in c$, metrics are also said to be gauges.

Definition 1.1 (Weyl structure) *A torsion-free connection $W : T\mathbf{P} \longrightarrow \mathfrak{co}(n)$ on a conformal manifold (M^n, c) is called Weyl structure. ∇ shall denote the induced covariant derivatives on associated vector bundles. The term Weyl manifold describes the triple (M^n, c, W).*

The operators c, tr and $(.)_c$ are parallel with respect to any Weyl structure. In general, Weyl structures are not given uniquely on (M^n, c). Let $\Gamma \in \Omega^1_{Ad}(P, \mathfrak{co}(n))$ be the difference of two Weyl structures W, W_1 on (M^n, c), i.e., $\Gamma := W_1 - W$. On vector bundles E, we will denote the section of linear operators corresponding to Γ by Γ^E. $\Gamma^{\mathcal{L}^1}$ is a globally defined 1-form, which we will call $\gamma \in \Omega(M^n)$. Subsequently, we obtain $\Gamma^{\mathcal{L}^w}_X l = w\gamma(X) l$. For Γ^{T^w} we obtain

$$\Gamma^{T^w}_X = \gamma \wedge X + w\gamma(X), \quad X \in \Gamma(T), \tag{1}$$

which can be written locally as

$$\rho^w_* \Gamma = \overline{\gamma} \wedge Id + w\overline{\gamma} \otimes Id. \tag{2}$$

$\overline{\gamma}$ is characterized by $\gamma = [p, \overline{\gamma}], r \in \Gamma(P)$. On $\mathcal{L}^1$ the curvature of a Weyl structure is given by $Alt \nabla^{T^* \otimes \mathcal{L}^1} \nabla^{\mathcal{L}^1} =: F \in \Omega^2(M)$. This globally defined 2-form is also called Faraday curvature. Choosing a gauge g on M^n provides a 1-form $\theta \in \Omega^1(M^n)$ in the following way: $\nabla l_g = \theta \otimes l_g$. θ depends on the gauge g as follows: Let $g_1 = e^{-2f} g$ be the result of a conformal transformation of g, where $f \in C^\infty(M^n)$. Then

$$\theta_1 = \theta + df. \tag{3}$$

For any gauge, we obtain $F = d\theta$. Since the Lie algebra of the conformal group splits into two components, there is also a splitting of a Weyl structure into a metric part W' and a scalar part $\theta' \otimes Id$.

$$W = W' + \theta' \otimes Id, \quad W' : T\mathbf{P} \longrightarrow \mathfrak{o}(n), \quad \theta' \in \Omega^1(P)$$

It is straight forward that W' is a metric connection on $\mathbf{P}$ itself, but it is not necessarily torsion-free. The action of W' coincides with W on associated vector bundles of weight 0.

Definition 1.2 *A Weyl structure $W = W' + \theta' \otimes Id$ is exact (closed) if and only if θ' is exact (closed).*

Theorem 1.3 *The following statements are equivalent:*

(i) W is exact.

(ii) θ is exact with respect to any gauge.

(iii) θ vanishes for at least one gauge of the conformal class.

(iv) W is the Levi-Civita connection belonging to a gauge of the conformal class.

(v) There is a W-parallel gauge, i.e., there exists a gauge $l \in \Gamma(\mathcal{L}^1)$ such that $\nabla l = 0$.

(vi) $\nabla^{\mathcal{L}^1}$ is trivial.

A Weyl structure is closed if and only if it is locally exact if and only if $F = d\theta = 0$.

PROOF: The first equivalence is due to the fact that $\pi^*\theta = \theta'|_{\mathbf{Q_g}}$, where $\mathbf{Q_g}$ denotes the $O(n)$-reduction of $\mathbf{P}$ corresponding to g. The other equivalences are straight forward. They can be checked in [8]. □

For a given Weyl manifold (M^n, c, W) the curvature tensor $\mathcal{R} \in \Gamma(T^{3,1})$ is defined by

$$\mathcal{R}(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

for any vector fields $X, Y, Z \in \Gamma(T)$.
Using the isomorphism $(.)_c$, which is imposed by the conformal structure, we can *lower* the last index, i.e., we can identify $\mathcal{R}$ with the 4-form $\mathcal{R}_c \in \Gamma(\mathcal{L}^2 \otimes T^{4,0})$, which takes values in $\mathcal{L}^2$. This leads us to the following definition of the Ricci curvature:

$$Ric := tr^{14}\mathcal{R}_c \text{ and } Ric' := tr^{14}\mathcal{R}'_c = Ric + F,$$

where the primed objects belong to the connection W'. Ric is *not* symmetric. In fact, we obtain:

$$\frac{1}{2}AltRic = -\frac{n}{2}F. \tag{4}$$

Finally, we define the scalar curvature

$$R := tr(Ric) = trtr^{14}\mathcal{R}_c \in \Gamma(\mathcal{L}^{-2}),$$

which is not a function, but a density of weight -2. As in Riemannian spin geometry the symmetry properties of the curvature tensor play an important role in further calculations. The next theorem shows that, in some symmetry aspects, $\mathcal{R}$ differs strongly from its Riemannian opposite.

Theorem 1.4 (Symmetry properties)

$$\begin{aligned} \mathcal{R}_c &= -(12)\mathcal{R}_c \\ \mathcal{R}'_c &= -(12)\mathcal{R}'_c \\ \mathcal{R}'_c &= -(34)\mathcal{R}'_c \\ \mathcal{R}'_c &= (13)(24)\mathcal{R}'_c + [(13)+(23)-(14)-(24)]\,F\otimes c. \end{aligned} \tag{5}$$

PROOF: Only the last equation has to be proved explicitly. We have $Zyk\mathcal{R}'_c = -ZykF\otimes c$, which is just a version of the first Bianchi identity for $\mathcal{R}$. This yields:

$$Zyk\mathcal{R}'_c Zyk^{1234} = -ZykF\otimes cZyk^{1234}.$$

We choose vector fields X, Y, Z, T. Then:

$$\begin{aligned} Zyk\mathcal{R}'_c Zyk^{1234}(X,Y,Z,T) &= \mathcal{R}'_c(X,Y,Z,T) + \mathcal{R}'_c(Y,Z,X,T) + \mathcal{R}'_c(Z,X,Y,T) \\ &+ \mathcal{R}'_c(Y,Z,T,X) + \mathcal{R}'_c(Z,T,Y,X) + \mathcal{R}'_c(T,Y,Z,X) + \mathcal{R}'_c(Z,T,X,Y) + \mathcal{R}'_c(T,X,Z,Y) \\ &+ \mathcal{R}'_c(X,Z,T,Y) + \mathcal{R}'_c(T,X,Y,Z) + \mathcal{R}'_c(X,Y,T,Z) + \mathcal{R}'_c(Y,T,X,Z) \\ &= 2\,(\mathcal{R}'_c(Z,X,Y,T) + \mathcal{R}'_c(T,Y,Z,X)) = 2(12)(23)(\mathcal{R}'_c - (13)(24)\mathcal{R}'_c)(X,Y,Z,T). \end{aligned}$$

Similarly, we get for $F\otimes c$: $ZykF\otimes cZyk^{1234} = 2Zyk^{1234}F\otimes c$. Putting all this together implies:

$$\mathcal{R}'_c = (13)(24)\mathcal{R}'_c - (12)(23)Zyk^{1234}F\otimes c = (13)(24)\mathcal{R}'_c + [(13)+(23)-(14)-(24)]F\otimes c,$$

since

$$\begin{aligned} (12)(23)Zyk^{1234}F\otimes c &= (12)(23)[Id + (12)(23)(34) + (34)(23)(12) + (13)(24)]F\otimes c \\ &= [-(13) - (13)(23)(12) + (14) + (23)(24)(34)]F\otimes c \\ &= [-(13) - (23) + (14) + (24)]F\otimes c. \end{aligned}$$

□

1.1 Einstein-Weyl structures

Definition 1.5 *For $n \geq 3$ a Weyl structure W on (M^n, c, W) is said to be Einstein-Weyl if an only if*

$$Ric = \frac{R}{n} \cdot c - \frac{n}{2}F \quad or \quad Ric' = \frac{R}{n} \cdot c - \frac{n-2}{2}F. \tag{6}$$

The symmetric part of Ric reduces to its trace if and only if W is Einstein-Weyl. (M^n, c, W) is called an Einstein-Weyl manifold.

Theorem 1.6 ([8]) *Let (M^n, c, W) be an Einstein-Weyl manifold. The following conditions are equivalent.*

(i) *W is closed or (M^n, c, W) is non-compact, 4-dimensional, and F is harmonic,*

(ii) $\delta F = 0$,

(iii) $\nabla R = 0$,

(iv) *W is either exact or $R = 0$,*

where $\delta := -tr^{12}\nabla$ is the co-differential of a Weyl structure.

2 Conformal Spin geometry

Let $Cl(\mathbb{R}^n, g_0)$ be the Clifford algebra over $\mathbb{R}^n$ with respect to the standard scalar product g_0. Then the following equation shall be satisfied $x \cdot x = -|x|^2$. Within $Cl(\mathbb{R}^n, g_0)$ we find the group $Spin(n)$, which is the universal two-fold covering space of $SO(n)$ for $n \geq 3$. Let $\lambda : Spin(n) \longrightarrow SO(n)$ denote the covering map. Elements of the group $CSpin(n) := Spin(n) \times \mathbb{R}_+$ are taken as products of $Spin(n)$ and $\mathbb{R}_+$ in $Cl(\mathbb{R}^n, g_0)$. We define a two-fold covering $\lambda^c : CSpin(n) \longrightarrow CO(n)_+ = \{A \in CO(n) | det(A) > 0\}$ by

$$\lambda^c(\vartheta a) := \vartheta \lambda(a).$$

The complexified Clifford algebra $Cl^{\mathbb{C}}(\mathbb{R}^n, g_0)$ can be identified with $End(\Delta_n)$ if n is even and with $End(\Delta_n) \oplus End(\Delta_n)$ if n is odd, where Δ_n denotes $\mathbb{C}^{2^{[\frac{n}{2}]}}$). We denote this identification by κ_{g_0} and its restriction to $CSpin(n)$ by κ. The spinor representation κ^w with weight w is defined as follows:

$$\kappa^w(\vartheta a) = \vartheta^w \kappa(a).$$

Let (M^n, c) be an oriented, conformal manifold and $(\mathbf{P}_{CO_+}, M^n, \pi_{CO_+}, CO(n)_+)$ the corresponding $CO(n)_+$-principal fibre bundle.

Definition 2.1 *A CSpin structure on (M^n, c) is a pair $(\mathbf{P}_{CSpin}, \Lambda^c)$, where $(\mathbf{P}_{CSpin}, \pi_{CSpin}, M^n, CSpin(n))$ is a $CSpin(n)$-principal fibre bundle on M^n and $\Lambda^c : \mathbf{P}_{CSpin} \longrightarrow \mathbf{P}_{CO_+}$ is a two-fold covering that makes the following diagramme commute.*

$$\begin{tikzcd}
\mathbf{P}_{CSpin} \times CSpin(n) \arrow[r] \arrow[d, "\Lambda^c \times \lambda^c"'] & \mathbf{P}_{CSpin} \arrow[r, "\pi_{CSpin}"] \arrow[d, "\Lambda^c"'] & M^n \\
\mathbf{P}_{CO_+} \times CO(n)_+ \arrow[r] & \mathbf{P}_{CO_+} \arrow[ur, "\pi_{CO_+}"] &
\end{tikzcd}$$

A Weyl manifold (M^n, c, W) carrying a CSpin structure is called a CSpin manifold.

The existence of $CSpin$ structures is equivalent to the existence of $Spin$ structures, since $Spin(n)$ is maximally compact in $CSpin(n)$. For the remainder of this paper, $M^n := (M^n, c, W)$ shall be a $CSpin$ manifold. Coming from the $CSpin$ structure we are given the following vector bundles, operators and identities:

1. $S^w := \mathbf{P}_{CSpin(n)} \times_{\kappa^w} \Delta_n$ is the spinor bundle with weight w. $S := S^1$ denotes the ordinary spin bundle,
2. the conformal, hermitian product $(.,.) := l_g^2(.,.)_g : \Gamma(S^w \otimes S^{w_1}) \longrightarrow \Gamma(\mathcal{L}^{w+w_1})$ and
3. the conformal Clifford product $\mu := l_g \mu_g : \Gamma(T^w \otimes S^{w_1}) \longrightarrow \Gamma(S^{w+w_1})$,

where $(.,.)_g$ and μ_g denote the hermitian product and the Clifford product given on $(M^n, g \in c)$. One can easily check from the definitions (cp. [2]) that they are conformally invariant. Via λ^c we realise all occurring bundles of the first section as associated bundles of $\mathbf{P}_{CSpin(n)}$. Therefore, we can use $(.)_c$ in order to expand the μ on arbitrary (r,s)-tensor fields. We will use indices in order to indicate which components should be multiplied with a spinor in which order. The operator $\mu^{ab} : T^{r,s} \otimes S^w \longrightarrow S^w$ is the conformal Clifford product of a spinor field of weight w with the b^{th} and then with the a^{th} component of a tensor field. Example:

$$\mu^{21}\gamma \otimes X \otimes \omega \otimes \psi = \omega \otimes X \cdot \gamma \cdot \psi,$$

where $\gamma \otimes X \in (T^{1,1})^w$ and $\psi \in \Delta_n^{w_1}$. Whenever there are no indices, the Clifford product ranges over all components of the corresponding tensor. The operator $\nu : \Delta_n^w \longrightarrow T^{1,0} \otimes \Delta_n^w$ is defined as follows:

$$X \rfloor \nu\psi = \mu X \otimes \psi = X \cdot \psi.$$

Then $\mu\nu = -n$ holds. Both, μ and ν have been introduced in order to abbreviate notation. Some well known identities have the following appearance:

$$tr\nu\omega\psi = w \cdot \psi \tag{7}$$

$$Re(\nu\psi, \nu\psi) = (\psi, \psi)c := |\psi|^2 c \tag{8}$$

2.1 The spinor derivative of a Weyl structure

Let (M^n, c, W) be a $CSpin$ manifold. Whenever local considerations are made, we assume that a local section $r \in \Gamma(\mathbf{P})$ has already been chosen. In a gauge $g \in c$, r shall automatically denote a section of $\mathbf{Q_g}$. By r^S we denote a lift of r into the $CSpin$ structure. In addition, a section of a vector bundle X shall be locally described by $X = [r, \overline{X}]$. Let W^S be the lift of W into the $CSpin$ structure. Then locally, the following diagramme commutes:

$$\begin{tikzcd}
 & T\mathbf{P}_{CSpin} \arrow[r, "W^S"] \arrow[d, "\lambda^c_*"'] & \mathfrak{cspin}(\mathfrak{n}) \arrow[d, "\lambda^c_*"'] \\
T \arrow[ur, "dr^S"] \arrow[r, "dr"] & T\mathbf{P}_{CO_+} \arrow[r, "W"] & \mathfrak{co}(n)
\end{tikzcd}$$

where $\mathfrak{co}(n) = \mathfrak{o}(n) + \mathbb{R} \cdot Id$ and $\mathfrak{cspin}(\mathfrak{n}) := m_2 + \mathbb{R} \cdot 1 = e_i \cdot e_j : 1 \leq i < j \leq n + \mathbb{R} \cdot 1$ are the Lie algebras of $CO(n)$ and $CSpin(n)$, respectively. For m_2 see [2]. We denote the corresponding covariant derivative on the spinor bundles with weight S^w by $\nabla^{S,w}$. However, we will often use the simply ∇^S or even only ∇ if the context permits. For $\Gamma = W_1 - W$ we now compute Γ^{S^w}.

Theorem 2.2 *Let $X \in \Gamma(T)$ be a vector field, $\psi \in \Gamma(S^w)$ a spinor field of weight w. Then*

$$\Gamma^{S^w}\psi = -\frac{1}{2}\nu\gamma \cdot \psi + \left(w - \frac{1}{2}\right)\gamma \otimes \psi \tag{9}$$

is satisfied.

PROOF: For the basis $\{E_{ij}\} \subset \mathfrak{o}(\mathfrak{n})$, where $E_{ij}(e_i) = e_j$, we obtain $(\lambda_*^c)^{-1}E_{ij} = \frac{1}{2}e_ie_j$ and, hence,

$$\kappa_*^w(\lambda_*^c)^{-1}E_{ij} = \frac{1}{4}\mu E_{ij}$$

is satisfied. Therefore, by using (2) in the second step, we obtain locally

$$\begin{aligned}
[r^S, \kappa_*^w\Gamma^S(dr^S(X))\overline{\psi}] &= [r^S, \kappa_*^w(\lambda_*^c)^{-1}\Gamma(dr(X))\overline{\psi}] = [r^S, \kappa_*^w(\lambda_*^c)^{-1}\left(\overline{\gamma\wedge X} + \overline{\gamma}(\overline{X})\right)\overline{\psi}] \\
&= \frac{1}{4}(\gamma\cdot X - X\cdot\gamma)\cdot\psi + w\gamma(X)\psi = -\frac{1}{2}X\cdot\gamma\cdot\psi + \left(w - \frac{1}{2}\right)\gamma(X)\psi.
\end{aligned}$$

□

Corollary 2.3 *Fix a gauge $g \in c$. Then the difference between the spinor derivatives of W and W^g acts on S^w as follows:*

$$\nabla_X^{S,w}\psi - \nabla^{S,g}\psi = -\frac{1}{2}\nu^g\theta\cdot\psi - \left(w + \frac{1}{2}\right)\theta\otimes\psi.$$

Both, the conformal hermitian product $(.,.)$ and the conformal Clifford product μ are W-parallel, since $\overline{(.,.)}$ and $\overline{\mu}$ are $CSpin$-invariant and constant on M^n. This yields, in particular,

$$\nabla^{T^*\otimes S}\nu\phi = (12)\nu\nabla^S\phi. \tag{10}$$

We define the spinorial curvature by

$$\mathcal{R}^{S,w} := Alt\nabla^{T^*\otimes S}\circ\nabla^S,$$

which is locally equivalent to $\pi^*\overline{\mathcal{R}}^{S,w} = \kappa_*^w\Omega^S$, where Ω^S is the curvature form of W^S. Especially the second equation of the following lemma turns out to be a very important tool, when relating *Spin* data to *Weyl* data.

Lemma 2.4

$$\begin{aligned}
\mathcal{R}^{S,w} &= \frac{1}{4}\mu^{34}\mathcal{R}'_c + wF & (11)\\
\mu^{234}\mathcal{R}'_c &= -2\mu^2Ric' - 2\mu^2F - \nu\mu F & (12)\\
\mu\mathcal{R}'_c &= 2R + 2(n-2)\mu F & (13)
\end{aligned}$$

PROOF: (1) Let $(e_1, ..., e_n)$ be a local conformal, weightless frame field i.e., $c(e_i, e_j) = \delta_{ij}$. E_{ij} shall be defined as in the proof of Theorem 2.2. According to the splitting $W = W' + \theta'\otimes Id$ we get for the corresponding curvature form: $\Omega = \Omega' + F'$. π shall denote the projection of $\mathbf{P}_{CSpin}$ on M^n.

$$\begin{aligned}
\pi^*\overline{\mathcal{R}}^S &= \kappa_*^w\Omega^S = \kappa_*^w(\lambda_*^c)^{-1}\Omega = \kappa_*^w(\lambda_*^c)^{-1}\Omega' + wF' = \kappa_*^w(\lambda_*^c)^{-1}\left(\frac{1}{2}\sum_{ij}\overline{c}(\pi^*\overline{\mathcal{R}'}\overline{e_i}, \overline{e_j})E_{ij}\right) + wF' \\
&= \frac{1}{4}\sum_{ij}\overline{e}_i\overline{e}_j\rfloor^{34}\pi^*\overline{\mathcal{R}'}_c\overline{e}_i\cdot\overline{e}_j\cdot + wF' = \frac{1}{4}\overline{\mu}^{34}\pi^*\overline{\mathcal{R}}'_c + wF'.
\end{aligned}$$

This yields the assertion, since $F' = \pi^*F$.

(2) First we use the symmetry properties of $F\otimes c$ to calculate:

$$\begin{aligned}
&-\mu^{124}[(13) + (23) - (14) - (24)]F\otimes c \\
&= \left[-\mu^{324} - \mu^{134} + \mu^{124}(14) - \mu^{124} - 2\mu^1tr^{24}\right]F\otimes c \\
&= \left[2\mu^{234} + 2\mu^2tr^{23} - \mu^{214}(14) - 2tr^{12}\mu^4(14) - \mu^{124} + 2\mu^2tr^{23}\right]F\otimes c \\
&= \left[2\mu^{234} + 4\mu^2tr^{23} + \mu^{214} + 2\mu^1tr^{14} - 2\mu^1tr^{23} - \mu^{124}\right]F\otimes c \\
&= \left[2\mu^{234} + 4\mu^2tr^{23} - \mu^{124} + 4\mu^2tr^{23} - \mu^{124}\right]F\otimes c \\
&= \left[2\mu^{234} + 8\mu^2tr^{23} - 2\mu^{124}\right]F\otimes c = (-2n+8)\mu^2F - 2F\cdot\nu
\end{aligned}$$

and

$$\begin{aligned} -\mu^{234} ZykF \otimes c &= -\mu^{234}\left[Id + (23)(12) + (12)(23)\right] F \otimes c \\ &= -\left[\mu^{234} - \mu^{234}(23) + \mu^{124}\right] F \otimes c = -\left[2\mu^{234} + 2\mu^2 tr^{23} + \mu^{124}\right] F \otimes c \\ &= (2n-2)\mu^2 F - F \cdot \nu. \end{aligned}$$

This implies, by using $F \cdot \nu = \nu\mu F + 4\mu^2 F$ and equation (5):

$$\begin{aligned} &\mu^{234}\mathcal{R}'_c = -\mu^{234}\left[\left[(12)(23) + (23)(12)\right]\mathcal{R}'_c - ZykF \otimes c\right] \\ &= \mu^{234}(23)\mathcal{R}'_c - \mu^{124}\mathcal{R}'_c - \mu^{234} ZykF \otimes c \\ &= \mu^{234}(23)\mathcal{R}'_c - \mu^{124}(13)(24)\mathcal{R}'_c - \mu^{124}[(13) + (23) - (14) - (24)]F \otimes c - \mu^{234} ZykF \otimes c \\ &= \mu^{234}(23)\mathcal{R}'_c + \mu^{324}\mathcal{R}'_c + 2\mu^2 tr^{24}\mathcal{R}'_c + (-2n+8)\mu^2 F - 2F\cdot\nu + (2n-2)\mu^2 F - F \cdot \nu \\ &= -2\mu^{234}\mathcal{R}'_c - 6\mu^2 Ric' + 6\mu^2 F - 3F\cdot\nu = -2\mu^{234}\mathcal{R}'_c - 6\mu^2 Ric' - 6\mu^2 F - 3\nu\mu F. \end{aligned}$$

(3) From (12), we obtain

$$\begin{aligned} \mu\mathcal{R}'_c &= \mu\mu^{234}\mathcal{R}'_c - 2\mu Ric' - \mu F + n\mu F = -\mu(Alt(Ric') + Sym(Ric')) + (n-2)\mu F \\ &= (n-2)\mu F + 2R + (n-2)\mu F = 2(n-2)\mu F + 2R. \end{aligned}$$

□

2.2 The Dirac equation

The definition of the Dirac operator in conformal geometry does not differ from the procedure known from the Riemannian context. It arises as a composition of the conformal Clifford product with the conformal spinor derivative. Since the Clifford product is weightless, the Dirac operator has weight -1.

Definition 2.5 *In conformal Spin geometry, the Dirac operator $\mathcal{D}_W : \Gamma(S^w) \longrightarrow \Gamma(S^{w-1})$ is defined by:*

$$\mathcal{D}_W := \mu\nabla^{S,w}.$$

If there is no ambiguity to be expected, we will omit the index W. The following theorem is an immediate implication of (9).

Theorem 2.6 (Dependence of $\mathcal{D}_W$ on W) *Let Γ be the difference of two given Weyl structures W, W_1 on M^n. If we set $\mathcal{D}_\Gamma := \mathcal{D}_1 - \mathcal{D}$ the change of Dirac operator on S^w due to the change of the Weyl structure is given by:*

$$\mathcal{D}_\Gamma = \left(\frac{n-1}{2} + w\right)\mu\gamma.$$

Corollary 2.7 *The Dirac operator $\mathcal{D}_W$ is independent of W, if it is applied to spinor fields of weight $-\frac{n-1}{2}$. Furthermore, in a gauge $g \in c$, $\mathcal{D}_W$ is related to the Dirac operator $\mathcal{D}_g$ defined by the Levi-Civita connection as follows:*

$$\mathcal{D}_W = l_g^{-1}\left(\mathcal{D}_g + \left(\frac{n-1}{2} + w\right)\mu_g\gamma\right). \tag{14}$$

2.2.1 The conformal Laplacian and the conformal Schrödinger-Lichnerowicz formula

Definition 2.8 *The conformal Laplacian is given by $\Delta^{S,w} : \Gamma(S^w) \longrightarrow \Gamma(S^{w-2})$*

$$\Delta^{S,w} := -tr\nabla^{T^*\otimes S} \circ \nabla^S.$$

Theorem 2.9 *Let* $\Gamma := W_1 - W$ *and* $\gamma := \Gamma^{\mathcal{L}^1}$. *Then*

$$\Delta_1^{S,w} - \Delta^{S,w} = -\gamma \cdot \mathcal{D} + w\delta\gamma + \frac{n-1-4(n-2)w-4w^2}{4}|\gamma|^2 + (-2w-n+1)\nabla^S_{\gamma^c} + \frac{1}{4}\mu d\gamma,$$

where $\nabla^S_{\gamma^c} := l_g^{-2}\nabla^S_{\gamma^g}$.

The proof is a simple, but tedious calculation and shall be omitted. One has to use a weightless conformal frame field, which is also W-parallel in $m \in M^n$.

Theorem 2.10 (Conformal Schrödinger-Lichnerowicz formula) *Let* $\psi \in \Gamma(S^w)$. *Then*

$$\mathcal{D}^2\psi = \Delta^S\psi + \frac{1}{4}R\psi + \left(\frac{n-2+2w}{4}\right) F \cdot \psi. \tag{15}$$

Proof: This can be obtained directly. The final reduction of the curvature terms is due to the equations (11) and (13).

$$\begin{aligned}
\mathcal{D}^2\psi &= \mu\nabla^S\mu\nabla^S\psi = \mu\nabla^{T^*\otimes S}\nabla^S\psi = \mu\frac{1}{2}(Alt\nabla^{T^*\otimes S}\nabla^S + Sym\nabla^{T^*\otimes S}\nabla^S)\psi \\
&= \mu\frac{1}{2}\mathcal{R}^S\psi - tr\nabla^{T^*\otimes S}\nabla^S\psi = \Delta^S\psi + \frac{1}{8}\mu\mathcal{R}'{}_c\psi + \frac{1}{2}wF\cdot\psi \\
&= \Delta^S\psi + \frac{1}{4}R\psi + \left(\frac{n-2}{4} + \frac{w}{2}\right) F \cdot \psi.
\end{aligned}$$

$\square$

2.3 The twistor equation

Definition 2.11 (twistor operator) *We define the twistor operator* $\mathcal{T}_W : \Gamma(S^w) \longrightarrow \Gamma(T^* \otimes S^w)$ *of a CSpin manifold* (M^n, c, W) *by* $\mathcal{T}_W^w := \nabla^{S,w} + \frac{1}{n}\nu\mathcal{D}$.

Theorem 2.12 (Dependence of $\mathcal{T}_W$ on W) *Let* $\Gamma := W_1 - W$ *be the difference between two Weyl structures and let* $\mathcal{T}_\Gamma^w := \mathcal{T}_1^w - \mathcal{T}^w$. *Then*

$$\mathcal{T}_\Gamma^w = \left(w - \frac{1}{2}\right)\left(\frac{1}{n}\nu\mu\gamma + \gamma\right). \tag{16}$$

holds.

Proof:

$$\mathcal{T}_\Gamma^w = \Gamma^{S,w} + \frac{1}{n}\nu\mathcal{D}_\Gamma = -\frac{1}{2}\nu\mu\gamma + \left(w - \frac{1}{2}\right)\gamma + \frac{n-1+2w}{2n}\nu\mu\gamma = \left(w - \frac{1}{2}\right)\left(\frac{1}{n}\nu\mu\gamma + \gamma\right).$$

$\square$

Corollary 2.13 *The twistor operator is independent of* W *if it is applied to spinor fields of weight* $\frac{1}{2}$. *Furthermore, in a gauge* $g \in c$, *the relationship between the Riemannian twistor operator* $\mathcal{T}_g$ *and* $\mathcal{T}_W$ *is as follows:*

$$\mathcal{T}_W = \mathcal{T}_g + \left(w - \frac{1}{2}\right)\left(\theta + \frac{1}{n}\nu\mu\theta\right). \tag{17}$$

Let (M^n, c, W) be a $CSpin$ manifold and $\psi \in \Gamma(S^w)$ a twistor spinor field. Then $\nabla^S\psi = -\frac{1}{n}\nu\mathcal{D}\psi$ is true and, therefore, (10) and (7) applied in the last two steps imply:

$$\Delta^S\psi = -tr\nabla^{T^*\otimes S}\nabla^S\psi = \frac{1}{n}tr\nabla^{T^*\otimes S}\nu\mathcal{D}\psi = \frac{1}{n}tr\nu\nabla^S\mathcal{D}\psi = \frac{1}{n}\mathcal{D}^2\psi \tag{18}$$

is satisfied. From the conformal Schrödinger-Lichnerowicz formula we obtain:

$$\mathcal{D}^2 = \frac{n}{4(n-1)}R + \frac{(n-2+2w)n}{4(n-1)}\mu F. \tag{19}$$

This leads to

$$\nabla^S\mathcal{D}\psi = \frac{n}{n-2}\left[-\frac{1}{2}\mu^2 Ric' + \frac{1}{4(n-1)}R\nu + \left(w-\frac{1}{2}\right)\left(\mu^2 F + \frac{2}{4(n-1)}\nu\mu F\right)\right]\psi. \tag{20}$$

Proof: We use (12) and (11) in the first and second step. Then finally, after some direct calculations, we use (19).

$$\begin{aligned}
&-\frac{1}{2}\mu^2 Ric'\psi - \frac{1}{2}\mu^2 F\psi - \frac{1}{4}\nu\mu F\psi = \frac{1}{4}\mu^{234}\mathcal{R}'_c \\
=\;& \mu^2\mathcal{R}^S\psi - w\mu^2 F\psi = \mu^2 Alt\nabla^{T^*\otimes S}\nabla^S\psi - w\mu^2 F\psi \\
=\;& -\frac{1}{n}\mu^2 Alt\nabla^{T^*\otimes S}\nu\mathcal{D}\psi - w\mu^2 F\psi = \frac{1}{n}\mu^2 Alt\nu\nabla^S\mathcal{D}\psi - w\mu^2 F\psi \\
=\;& \frac{1}{n}\mu^2\nu\nabla^S\mathcal{D}\psi - \frac{1}{n}\mu^1\nu\nabla^S\mathcal{D}\psi - w\mu^2 F\psi = -\frac{1}{n}\nabla^S\mathcal{D}^2\psi - \frac{2}{n}\nabla^S\mathcal{D}\psi + \nabla^S\mathcal{D} - w\mu^2 F\psi \\
=\;& -\frac{n}{4n(n-1)}R\nu\psi - \frac{(n-2-2w)}{4(n-1)}\nu\mu F\psi + \frac{n-2}{n}\nabla^S\mathcal{D}\psi - w\mu^2 F\psi.
\end{aligned}$$

□

Theorem 2.14 *If the term* $\left(w-\frac{1}{2}\right)\left(\mu^2 F + \frac{2}{4(n-1)}\nu\mu F\right)\psi$ *reduces to a* single *Clifford product or even vanishes, e.g. if* $w = \frac{1}{2}$ *or* $F\cdot\phi = 0$ *is satisfied, the sections*

$$C(\psi) := Re(\psi, \mathcal{D}\psi) \in \Gamma(\mathcal{L}^{2w-1})$$

and

$$Q(\psi) := |\psi|^2|\mathcal{D}\psi|^2 - trRe(\mathcal{D}\psi, \nu\psi)^2 \in \Gamma(\mathcal{L}^{4w-2})$$

are W-parallel.

Proof: In (20) there are only single Clifford products left. Then (8) yields

$$\nabla C(\psi) = Re(\nabla^S\psi, \mathcal{D}\psi) + Re(\psi, \nabla^S\mathcal{D}\psi) = Re(-\frac{1}{n}\nu\mathcal{D}\psi, \mathcal{D}\psi) = 0$$

and

$$\begin{aligned}
\nabla Q(\psi) \;=\;& 2Re(\nabla^S\psi, \psi)Re(\mathcal{D}\psi, \mathcal{D}\psi) + 2Re(\psi, \psi)Re(\nabla^S\mathcal{D}\psi, \mathcal{D}\psi) \\
& -2tr^{23}Re(\nabla^S\mathcal{D}\psi, \nu\psi)Re(\mathcal{D}\psi, \nu\psi) + \frac{2}{n}tr^{13}Re(\mathcal{D}\psi, \nu\nu\mathcal{D}\psi)Re(\mathcal{D}\psi, \nu\psi) \\
=\;& 0.
\end{aligned}$$

□

2.3.1 The zeros of a twistor spinor field

In this section we show that the zeros of a twistor spinor field are a discrete set in M^n. Let M^n be connected. We define $E^w := S^w \oplus S^{w-1}$ and regard the covariant derivative ∇^{E^w}, which is characterized by

$$\nabla^{E^w} = \begin{pmatrix} \nabla^{S,w} & \frac{1}{n}\nu \\ K^w & \nabla^{S,w-1} \end{pmatrix},$$

where $K^w = \nabla^{S,w-1}\mathcal{D} : \Gamma(S^w) \longrightarrow \Gamma(S^{w+1})$.

Theorem 2.15 *For all twistor spinor fields $\phi \in \Gamma(S^w)$*

$$\nabla^{E^w}\begin{pmatrix} \phi \\ \mathcal{D}\phi \end{pmatrix} = 0$$

holds. Conversely, any ∇^{E^w}-parallel section $\begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \Gamma(E^w)$ yields:

$$\mathcal{T}_W\phi = 0 \quad \text{and} \quad \psi = \mathcal{D}\phi.$$

Since parallel sections on vector bundles over connected manifolds are uniquely determined by their value in a single point, we obtain:

Corollary 2.16 *The dimension of the space of all twistor spinor fields of connected CSpin manifold is less than or equal to $2^{[\frac{n}{2}]+1}$. Furthermore, a twistor spinor field ψ on a connected CSpin manifold for that $\psi(m) = 0$ and $\mathcal{D}\psi(m) = 0$ in a point $m \in M^n$ is trivial.*

Theorem 2.17 *The set $N_\psi := \{\psi \in \Gamma(S^w) : \ \psi(m) = 0 \text{ and } \mathcal{T}_W\phi = 0\}$ of a twistor spinor field $0 \neq \psi \in \Gamma(S^w)$ is discrete in M^n.*

Proof: (20) yields $\nabla^S\mathcal{D}\phi(m) = 0$ and for $g \in c$:

$$0 = 2Re(\nabla\psi, \psi)(m) = \nabla|\psi|^2(m) = \nabla^g|\psi|^2(m),$$

since $\nabla l^w = \nabla^g l^w + w\theta \otimes l^w$. $|\psi|^2$ is a density, i.e a section of $\mathcal{L}^{2w}$. Therefore, we get for vector fields X, Y on M^n:

$$\nabla_X\nabla_Y|\psi|^2(m) = \nabla^g_X\nabla_Y|\psi|^2(m) + 2w\theta(X)\nabla_Y|\psi|^2(m) = \nabla^g_X\nabla^g_Y|\psi|^2(m).$$

If we choose X and Y to be W-parallel in m, we finally obtain by applying (8)

$$\begin{aligned} \nabla_X\nabla_Y|\psi|^2(m) &= 2\nabla_X(\nabla_Y\psi, \psi)(m) = -\frac{2}{n}\nabla_X(Y \cdot \mathcal{D}\psi, \psi)(m) \\ &= \frac{2}{n^2}(Y \cdot \mathcal{D}\psi, X \cdot \mathcal{D}\psi)(m) = \frac{2}{n^2}c(X,Y)|\mathcal{D}\psi|^2(m). \end{aligned}$$

The combination of the latter two equations yields that $Hess_m(|\psi|^2)$ is not degenerated if $\mathcal{D}\psi(m)$ is not trivial. Therefore, m is an isolated point of $N(\psi)$. Otherwise, it follows from the last corollary that ψ must be trivial.

□

2.4 The Killing equation

In this section, M^n shall be connected.

Definition 2.18 (Killing spinor fields) *A spinor field $\psi \in \Gamma(S^w)$ is called a Killing spinor field if it satisfies the following differential equation:*

$$\nabla^S \psi = \beta \nu \psi, \quad \beta \in \Gamma(\mathbb{C} \otimes \mathcal{L}^{-1}),$$

where β is the Killing density of ψ.

A non-trivial Killing spinor field vanishes nowhere on a connected manifold, since it is parallel with respect to the covariant derivative $\nabla^S - \beta\nu$. It is obvious, that any Killing spinor field satisfies the twistor equation and can be taken as an eigenspinor of the Dirac operator with the eigenvalue $-n\beta$. We now investigate the integrability conditions for the existence of non-trivial Killing spinor fields. See previous section for examples.

Theorem 2.19 *Let $\psi \in \Gamma(S^w)$ be a Killing spinor field.*

(i) For β purely imaginary and $w \neq -\frac{n-2}{2}$ it follows that: $(F \cdot \psi, \psi) = 0$.

(ii) If $\beta \neq 0$ is real, W is exact and Einstein-Weyl for any $w \in \mathbb{R}$. Furthermore, $R = 4n(n-1)\beta^2$.

(iii) For $\beta = 0$, W is exact for any $w \neq 0$, i.e., Einstein-Weyl. If $n > 2$ and $w = 0$, W is also Einstein-Weyl. W is closed if $n \neq 4$ or M^n is compact. F is harmonic if $n = 4$ and M^n non-compact.

Remark: For the latter case ($n = 4$ and M^n compact) Moroianu gave in [7] an example of a $CSpin$ manifold together with a non-closed Weyl structure that carries non-trivial parallel spinor fields.

PROOF:

1. This is an immediate consequence of

$$R\psi + 2\left(\frac{n-2}{2} + w\right) F \cdot \psi = 4(n-1)n\beta^2\psi - \frac{4(n-1)}{n}\nabla\beta \cdot \psi, \tag{21}$$

which itself follows from (19):

$$\frac{n}{4(n-1)}R\psi + \frac{(n-2+2w)n}{4(n-1)}F \cdot \psi = \mathcal{D}^2\psi = -n\mu\nabla^S(\beta\psi) = -n\nabla\beta \cdot \psi + n^2\beta^2\psi.$$

The imaginary part of the product of (21) with ψ is as follows:

$$\left(w + \frac{n-2}{2}\right)(F \cdot \psi, \psi) = 0,$$

i.e., we have shown the assertion.

2. For $w \neq 0$ we obtain $\nabla_X Re(\psi,\psi) = 2Re(\nabla^S_X\psi,\psi) = 2Re(\beta X \cdot \psi, \psi) = 0$, by assumption. Hence, W is exact. Let now $w = 0$. By using the definition of Killing spinor fields we obtain: $\mathcal{R}^S\psi = Alt\nabla\nu\psi + 2\beta^2((\nu^{21} + c)\psi$, where $X, Y\rfloor\nu^{21}\psi = Y \cdot X \cdot \psi$. This yields:

$$\mu^2\mathcal{R}^S\psi = -n\nabla\beta \otimes \psi - \nabla\beta \cdot \nu\psi - 2\beta^2(n-1)\nu\psi.$$

Together with (12) we obtain:

$$\mu^2 Ric'\psi = 2n\nabla\beta \otimes \psi + 2\nabla\beta \cdot \nu\psi + 4\beta^2(n-1)\nu\psi - \mu^2 F\psi - \frac{1}{2}\nu F \cdot \psi. \tag{22}$$

$R = 4n(n-1)\beta^2$ by (21) and the assumption and thus we obtain again from (21):

$$F \cdot \psi = -\frac{4(n-1)}{n(n-2)}\nu\nabla\beta \cdot \psi. \tag{23}$$

Inserting (23) into (22) yields:

$$\mu^2 Ric'\psi = 2n\nabla\beta \otimes \psi + 2\nabla\beta \cdot \nu\psi + \frac{R}{n}\nu\psi - \mu^2 F\psi - \frac{2(n-1)}{n(n-2)}\nu\nabla\beta \cdot \psi. \tag{24}$$

The operator $\nabla\beta \cdot \nu$ consists of double Clifford products and scalar parts. We now rearrange (24) accordingly.

$$\begin{aligned}\mu^2 Ric'\psi &= 2\left(n - \frac{2(n-1)}{n(n-2)}\right)\nabla\beta\otimes\psi + \left(2 - \frac{2(n-1)}{n(n-2)}\right)\nabla\beta\cdot\nu\psi + \frac{R}{n}\nu\psi - \mu^2 F\psi \\ &= 2\left(n - 1 - \frac{n-1}{n(n-2)}\right)\nabla\beta\otimes\psi + \left(1 - \frac{(n-1)}{n(n-2)}\right)\mu^{12}Alt^{12}\nabla\beta\otimes c\psi + \frac{R}{n}\nu\psi - \mu^2 F\psi.\end{aligned}$$

If we multiply this equation by ψ, we see that $\nabla\beta$ must vanish since all the other terms are purely imaginary. Therfore, W is exact for any $w \in \mathbb{R}$. Hence, ψ is a (non-trivial) Killing spinor field with respect to a gauge g. (M^n, g) is Einstein and (M^n, c, W) is Einstein-Weyl.

3. W has to be exact for $w \neq 0$, since $|\psi|^2$ is a parallel section of $\mathcal{L}^{2w}$. For $w = 0$, the equations (11) and (12) together with the assumption yield

$$\mu^2 Ric'\psi = -\mu^2 F\psi - 2\nu F \cdot \psi.$$

Then (21), $R = 0$ and the assumption impose $F \cdot \psi$ to vanish. Since ψ vanishes nowhere however, $Ric' = -F$. Therefore, the symmetric part of Ric' reduces to its trace (which is 0), i.e., W is Einstein-Weyl. Hence, Theorem 1.6 is applicable, which yields all remaining assertions.

□

2.5 Two dimensional examples

1. *Killing spinor fields of weight $\frac{1}{2}$*:
We can find imaginary Killing spinor fields ψ of weight $\frac{1}{2}$ on $(\mathbb{R}^n, [g], x_1 dx^2)$. They have to be a solution of

$$X(\psi) = \frac{1}{2}X \cdot \theta \cdot \psi + \beta_g X \cdot \psi = X \cdot \left(\frac{1}{2}x_1\partial_2 + \beta_g\right) \cdot \psi, \tag{25}$$

where $X \in \mathbb{R}^n$ and $\Gamma(\mathbb{C} \otimes \mathcal{L}^1) \ni \beta = \beta_g l_g$ hold true. $(\partial_1, ..., \partial_n)$ are said to be the standard basis in $\mathbb{R}^n$. If one uses the following representation of the Clifford algebra

$$\partial_1 \longmapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \partial_2 \longmapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

one obtains

$$X \cdot \begin{pmatrix} \beta_g & \frac{i}{2}m_1 \\ \frac{i}{2}m_1 & \beta_g \end{pmatrix}\psi_0 = 0.$$

We find a non-trivial kernel of the matrix for all X if and only if $\beta_g = \pm\frac{i}{2}m_1$. An element of this kernel must be of the form $\psi_0 = \begin{pmatrix} a \\ -a \end{pmatrix}$ with $a \in \mathbb{C}$. We obtain, just as stated in Theorem 2.19:

$$(F\psi_0, \psi_0) = x_1 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ a \end{pmatrix} \cdot \begin{pmatrix} a \\ -a \end{pmatrix} = x_1 \begin{pmatrix} a \\ a \end{pmatrix} \cdot \begin{pmatrix} a \\ -a \end{pmatrix} = 0.$$

2. *Parallel spinor fields of weight* 0*:*
On $(\mathbb{R}^2, [g], x_1 dx^2)$ we have to solve:

$$X(\psi) = \frac{1}{2}X \cdot \theta \cdot \psi + \frac{1}{2}\theta(X)\psi = \frac{1}{2}X_1 x_1 \partial_1 \cdot \partial_2 \cdot \psi$$

for $\psi \in \Gamma(S^0)$, where $X = \sum_{i=1}^n X_i \partial_i$.
We use the following representation of the Clifford algebra

$$\partial_1 \longmapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \partial_2 \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Therfore

$$\partial_1 \cdot \partial_2 \longmapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

and so we are given

$$X(\psi^+) = \frac{i}{2}X_1 x_1 \psi^+, \quad X(\psi^-) = -\frac{i}{2}X_1 x_1 \psi^-,$$

where ψ^+ and ψ^- correspond to the splitting $\Delta_2 = \Delta_2^+ \oplus \Delta_2^-$. It is, however, not difficult to determine the solution of this system.

$$\psi^+(x) := \exp\left(\frac{i}{4}x_1^2\right)\psi_0^+, \quad \psi^-(x) := \exp\left(-\frac{i}{4}x_1^2\right)\psi_0^-,$$

where $\psi_0^\pm \in \mathbb{C}$.

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