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Modular Constructions of Quantum Field Theories with Interactions

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Dedicated to the memory of Harry Lehmann

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Abstract

We extend the previously introduced constructive modular method to nonperturbative QFT. In particular the relevance of the concept of "quantum localization" (via intersection of algebras) versus classical locality (via support properties of test functions) is explained in detail, the wedge algebras are constructed rigorously and the formal aspects of double cone algebras for d=1+1 factorizing theories are determined. The well-known on-shell crossing symmetry of the S-Matrix and of formfactors (cyclicity relation) in such theories is intimately related to the KMS properties of new quantum-local PFG (one-particle *polarization-free) generators* of these wedge algebras. These generators are "on-shell" and their Fourier transforms turn out to fulfill the Zamolodchikov-Faddeev algebra. As the wedge algebras contain the crossing symmetry informations, the double cone algebras reveal the particle content of fields. Modular theory associates with this double cone algebra two very useful chiral conformal quantum field theories which are the algebraic versions of the light ray algebras.

1 Introductory Remarks

It is known that the local field algebras of interaction-free Fermions and Bosons can be directly obtained [1][3] from the modular wedge-localized one-particle spaces, as a kind of inverse of the Bisognano-Wichmann [6] construction, without any reference to the standard (semi)classical parallelism called quantization. In fact the present paper should be viewed as one in a series of paper [4] which attempt a new access to QFT by avoiding quantization and references to (semi)classical approaches in the presence of interactions and using instead modular methods. The common new finding in all these papers is an enlargement of the symmetry concept which previously we have described in terms of a

new kind of "hidden" (i.e. a non-Lagrangian non-Noetherian) symmetry structure [4]. This new concept also turns out to be useful in the present context (section 4) which aims at an intrinsic modular understanding of nonperturbative interactions.

In the following we will continue the investigations, which were started by one of the authors (B.S.), on the possibility to incorporate interacting theories into a constructive modular approach [1][2]. One important step in this approach was the recognition of the close relation between the thermal Hawking-Unruh aspects of the wedge horizon on the one hand and the subject of crossing symmetry in particle physics on the other hand. This was subsequently (but independently) also noticed in [5]. We have however some problems with the arguments given by the latter author of such a relation; in particular we do not believe that it is possible to give a proof of the relation without the construction of the "polarization free generators" of wedge algebras [2]. Although these new objects were already explicitly introduced in the setting of factorizable models in the cited previous work of one of the authors (B. S.), we will take some pain in explaining their principle features since their use constitutes our main new tool in the present nonperturbative constructions.

Our first testing ground will therefore be the d=1+1 factorizing models which are the simplest and (apart from some less interesting low-dimensional superrenormalizable Lagrangian models[1]) presently the only interacting properly renormalizable field theories which have a well-defined particle (and hence a scattering) content and allow for an explicit constructive analytic understanding. We will show that the use of PFG's goes far beyond the understanding of the thermal KMS $\leftrightarrow$on-shell crossing relation.

For our present purpose it is very helpful to first picture the special position of these factorizable theories within the set of general d=1+1 Wightman field theories as representing massive superselection classes in the following way. Imagine that we remove all particles by spatially separating their (centre of) wave packets "on shell" i.e. in the scattering matrix. Using as a hindsight the analytic singularity structure of the multiparticle creation (annihilation) threshold, as well as the relation between momentum space singularities and fall-off behavior in its Fourier transform, we are led to believe that these inelastic multiparticle matrix elements possess decreasing properties in this extreme cluster limit relative to the elastic ones. Even more, the direct multiparticle elastic processes in this limit is expected to become asymptotically negligible as compared with the two particle elastic contribution [2]. In this way we obtain factorizing S-matrices which fulfill the Yang-Baxter consistency equations without reference to quantization and infinitely many conserved Noether currents;

[1]In the quantization approach to interactions it is the operator short distance dimension of the interaction density in Fockspace which separates the models into those small superrenormalizable family (for which the mathematical existence can be controlled), and the more interesting rest which is not accessible by quantization methods (canonical or functional integration). The constructive modular approach within the limit of present knowledge does not depend in any direct way on the short distance behaviour of particular field generators of local nets.

with other words the Y-B structure is not imposed but rather derived from the general principles of QFT in the above explained extreme cluster limit. Although, as a result of lack of detailed knowledge about the analytic structure of admissable S-matrices of QFT this picture is mathematically not rigorous, it is very useful for the motivation behind our investigation and for an intrinsic local quantum physical understanding of the position of factorizable models within general QFT which removes some of their "freak appearance".

In fact it is one of the folklore theorems (but in this case rigorously provable under suitable analytic assumptions) that for d=1+3 theories the limiting S-matrix is S=1. The limiting field theories are then expected to be free fields which only record the superselection charges (internal symmetries) of the original theory. However in d=1+1 the simplification does not go all the way to free theories since the cluster property is not capable to separate the elastic two-particle T-matrix from the identity in the decomposition S=1+T (intuitively because two particles always meet in x-space or mathematically because the two particle energy momentum delta function in d=1+1 happens to be identical to the two particle inner product delta function). All the multiparticle elastic scattering then takes place through two-particle scattering and the Yang-Baxter relation follows as a physical consistency requirement (and is not a mathematical imposition) and allows together with the unitarity and crossing symmetry (closely linked to the TCP symmetry) to separate and solve the construction of the S-matrix from the field theory. This is a peculiar feature of the so constructed simple (but nontrivial) "factorizing representative" of d=1+1; in any other field theory including nonfactorizing d=1+1 models, there is no possibility to extract a pure S-matrix bootstrap setup from the off-shell field theory; rather the S-matrix must be determined together with the fields or (in our setting) the net of local algebras.

The principle purpose of this paper, which generalizes the previous findings of one of the present authors [2] concerning the modular understanding of crossing symmetry, is a more detailed explanation of the useful constructive role of so called "PFG" (one-particle **p***olarization-***f***ree* wedge **g***enerators*) i.e. on-shell but nevertheless wedge localized operators $F(x)$, which create one-particle states without additional pair contributions[2] known from the famous vacuum polarization in QFT. In fact they constitute a system of generators for the wedge algebra. These "fields", unlike pointlike local fields (either associated with Lagrangians or with the Wightman framework), have a *quantum* (but no classical) *localization* i.e. they remain wedge localized even if smeared with test functions which have sharper localized supports inside the wedge region. With quantum localization, this sharper localization has to be obtained from

[2]We have called such operators on previous occasions "vacuum-polarization free" and used the abbreviation FWG in [2], but a better terminology used in the following is to call them "one-particle-*polarization free generators*" (PFG). Wheras the notion of vacuum-polarization which originated with conserved currents in free field theory in the work of Heisenberg and Weisskopf, the present concept of "one-particle polarization", i.e. the impossibility of creating pure one particle states (without pair admixtures) from local operators is totally characteristic for interacting theories (and may be used as a good intrinsic definition of the latter).

the *intersection of wedge algebras*; the intersected algebras in turn have their own new generators. The PFG's are more noncommutative than the pointlike localized fields and they are in fact not derivable from any known quantization approach. But on the other hand they are essential in our nonperturbative construction which uses (quantum) modular wedge localization; although they are themselves nonlocal, they have a very precise relation to local fields. In some vague sense one may say that the semilocal intermediary PFW's have a milder short distance behavior than pointlike local fields, but their use is in agreement with physical principles, whereas the use of cutoff fields is not and, therefore the explanation in terms cutoffs is somewhat misleading. In any case the existence of the theory in the modular approach becomes detached from the "threatening" short distance behavior of pointlike field coordinates. The borderline between renormalizabe/nonrenormalizable theories looses its meaning and instead the existence of local theories becomes linked with the *nontriviality of intersections of certain algebras*.

For the aforementioned factorizing models one can argue that their mass shell Fourier transforms obey the Zamolodchikov-Faddeev algebra [2]. In fact the construction of PFG's in factorizing models may be done in a spirit similar to the perturbative construction of local fields in the retarded (Yang-Feldman) formalism. The PFG's viewed in this way are on-shell analogues of the off-shell interacting fields, but unlike the incoming free fields they carry information about the interaction. Whereas the Z-P algebraic structure is essential for the present analytic control of the bootstrap-formfactor- approach, the concept of *quantum wedge localization* relates to a vastly more general algebraic structure of the commutator of F with its modular reflected JFJ which is controlled by a crossing symmetric (a symmetry around the imaginary rapidity $\frac{1}{2}i\pi$) operator $M(\theta)$ as will be explained in the third section. The understanding of the concept of quantum wedge localization for these objects is the main objective in section 3.. Different from the usual locality of local fields or nets of algebras which exists independently of the particular representation, the new quantum locality depends crucially on the presence of the vacuum representation and one-particle states (more precisely on the standard assumptions which led to the validity of rigorous time-dependent scattering theory). It is only after the modular construction of a local net has been accomplished, that one may change the vacuum representation with another locally normal one.

After the role of the PFG's for quantum wedge localization has been presented, we describe the formal aspects of their intersections which leads to double cone algebras in section 4. Via suitably defined relative commutants we construct two light ray theories whose conformal rotation is a "hidden" symmetry. The problem of nontriviality as well as the the explicit construction of the double cone algebra is greatly facilitated in terms of these "smaller" light ray algebras.

2 An Intuitive Argument for Existence of PFG's

In this section we will present some intuitive (mathematically nonrigorous) arguments why we expect one-particle polarization free wedge generators (PFG's) to exist, although we are presently only able to rigorously show this for d=1+1 factorizing systems.

Let us briefly recollect how the perturbative approach deals with interactions. The formulation which is most appropriate for the present purpose is that of Stueckelberg, Bogoliubov and Shirkov [7] and partially also that of Weinberg [8]; here we do not need the more sophisticated version of Epstein and Glaser [9]. The first step from Wigner particles to Fockspace and free fields and the second step of implementing interactions by forming invariant Wick-ordered local polynomials of composite free fields and defining transition operators

$$S(g,h) = T \exp i \int \{g(x)W(x) + h(x)A(x)\}\, dx \tag{1}$$

where for simplicity we used a symbolic notation and wrote the polynomial interaction density as a monomial W and denotes by h the source function(s) of the basic free field coordinate(s) A in terms of which we specify the Fock space and the W. As usual T denotes time-ordering. These operator functionals of the testfunctions in Fock space fulfill the so called Bogoliubov axiomatic and it is of no relevance to us whether this axiomatics has nonperturbative solutions or not; the reader is entitled to take the most pessimistic view concerning their existence. Formally these time-ordered exponentials would represent the scattering operator S_{sc} if the test functions approach the constant function on Minkowski space. If only this "on-shell" value of this time-ordered operator would be known, there would be no possibility of computing off-shell local fields. Therefore Bogoliubov et al. [7] assume (either by functional dependence or in some other way) that there exists an off-shell extrapolation of S_{sc} which can be related (by their functional derivative formalism which is not part of scattering theory) to local fields and is given by (1). This formalism then leads to an expression for the outgoing free field and to the Yang-Feldman equation

$$\begin{aligned} A_{out}(x) &= A_{in}(x) + \int \Delta(x-x')j(x')dx' \equiv S_{sc}^{*} A_{in}(x) S_{sc} \qquad (2) \\ A(x) &= A_{in}(x) + \int \Delta_{ret}(x-x')j(x')dx' \\ j(x) &= K_x S^{*}(h) \frac{\delta}{\delta h(x)} S(h) \mid_{h=0} \end{aligned}$$

Here Δ is the mass-shell Pauli-Jordan commutator function, whereas Δ_{ret} is the off-shell retarded function which is formally the on-shell commutator function multiplied with the step function in time. The alternative right hand way of writing in the first line indicates that the on shell restriction of the $\int \Delta(x-x')j(x')$ contribution is determined by the on-shell object $S_{sc}^{*}[A_{in}(x)S_{sc}]$, i.e. the definition via the off-shell $S(g,h)$ is not needed. The crucial question is now

if these formulas can be used in a suggestive manner for the construction of a semilocal wedge localized $F(x)$ which like $A_{in}(x)$ is still on-shell, but also like $A(x)$ it carries informations about interactions. Assume for simplicity that we are in the situation of a factorizable model with a diagonal S-matrix. Such an elastic S-matrix in terms of the analytic phase shift δ, with $\delta(\theta)\varepsilon(\theta) = \delta_{phys}(|\theta|)$, would be represented by an exponential of a quadrilinear term in the incoming field

$$S_{sc} = \exp i\frac{1}{2}\int\int \delta(\theta-\theta')\varepsilon(\theta-\theta') : \rho(\theta)\rho(\theta') : d\theta d\theta' \tag{3}$$

where the $\rho(\theta)$ is the rapidity space charge density (resp. particle number density in our case of selfconjugate particles). From this we read off the relation between $S_{sc}$and a_{out}

$$\begin{aligned} a_{out}(\theta) = S_{sc}^{*} a_{in}(\theta) S_{sc} = \qquad (4) \\ = \quad a_{in}(\theta)\exp i\int_{-\infty}^{\infty}\delta(\theta-\theta')\rho(\theta')d\theta' \end{aligned}$$

It is now suggestive to try the following nonlocal field as a candidate for a wedge localized PFG field [2]

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{2\pi}}\int\left\{Z(\theta)e^{-ipx} + Z^{*}(\theta)e^{ipx}\right\}d\theta \qquad (5) \\ Z(\theta) &= a_{in}(\theta)\exp i\int_{-\infty}^{\theta}\delta(\theta-\theta')\rho(\theta')d\theta' \end{aligned}$$

The analogue of the off shell split of Δ into Δ_{ret} and Δ_{av} is the split of the on shell rapidity space integrals of the S-matrix. This analogy gains more credibility from the remark that the exponent is indeed some generalization of an integral over the current $j(x)$ a fact which is particularily evident [2] in the case of the Federbush- and Thirring- model There are also indications that this analogy between "off-shell space time and on-shell velocity" or rather its rapidity logarithm may be helpful in understanding certain "rapidity space cluster properties" [10] which are expected to select a subclass of local pointlike operators which recently were observed in formfactors of certain models. And last not least, we expect this analogy to be helpful in a future unraveling of the structure of PFG's in the nonperturbative analysis of non-factorizing QFT's.

Clearly the positive and negative frequency parts of the above $F(x)$ obey a Zamolodchikov-Faddeev algebra [15] (see (15) of the next section). Now the generalization to factorizing systems with more general nondiagonal S-matrices is obvious: generalize the structure on the algebraic side and prove that the so obtained PFG fields are indeed wedge localized in the sense of quantum localization. The motivation for our notation should be obvious; whereas the symbol F denotes the more general PFG operators which are expected to exist in each QFT, its positive and negative frequency components are denoted by the symbol $Z^{\#}$ since for d=1+1 factorizing theories these generalized "creation and annihilation" operators fulfil the Zamolodchikov-Faddeev algebra. Our previous

intuitive argument restricted to the context of factorizing models relates the new concept of PFG with the Z-F algebra in a nontrivial and useful way.

In order to achieve a more detailed understanding of this connection, we need to develop some more formalism. We first return briefly to the wedge localization of free fields.

3 Generators of Quantum Wedge Localization

In order to verify the modular wedge localization for the Z_W operators, we first look back at the free n- particle wedge localization [2] and introduce some additional notation. Using free fields the wedge localization spaces may be written:

$$\begin{aligned} \left|\psi^{(n)}\right\rangle &= \int \hat{\psi}^{(n)}(x_n, x_{n-1}, ...x_1) : A(x_1)...A(x_n) : \Omega \qquad (6) \\ supp\hat{\psi}^{(n)} &\subseteq W^{\otimes n},\ A(x) = \frac{1}{\sqrt{2\pi}} \int e^{-ipx} a(\theta) + e^{ipx} a^*(\theta) \end{aligned}$$

where W denotes the right hand wedge and the operator involves a Wick product of free fields A (for the sake of notation chosen scalar and selfconjugate). Instead of the vectors p and x we use their rapidity parametrization:

$$p = m \begin{pmatrix} \cosh\theta \\ \sinh\theta \end{pmatrix},\ x = \rho \begin{pmatrix} \sinh\chi \\ \cosh\chi \end{pmatrix} for\ x \in W \qquad (7)$$

We also prefer the more intrinsic way of writing which avoids the use of field coordinates (which are not unique) and uses the momentum space creation and annihilation operators which are directly linked to the Wigner representation theory of irreducible particle representations:

$$\begin{aligned} \left|\psi^{(n)}\right\rangle &= A_{\psi^{(n)}}\Omega \qquad (8) \\ A_{\psi^{(n)}} &= \int_C ... \int_C \psi^{(n)}(\theta_n, \theta_{n-1}, ...\theta_1) : a(\theta_1)...a(\theta_n) : d\theta_1...d\theta_n \end{aligned}$$

The used path notation C is a self-explanatory notation which generalizes the rapidity representation of the wedge localized fields [16]:

$$\begin{aligned} A(\hat{f}) &= \int f(\theta)a(\theta) + \int f(\theta - i\pi)a(\theta - i\pi) \equiv \int_C f(\theta)a(\theta) \qquad (9) \\ & a(\theta - i\pi) \equiv a^*(\theta),\ f(\theta - i\pi) = \bar{f}(\theta) \end{aligned}$$

C consists of the real $\theta-$axis and the parallel path shifted down by $-i\pi$ and it is only the function f which is analytic in the strip $-\pi < Im\theta < 0$ and conjugate-symmetric (i.e. fulfilling the Schwarz reflection principle) around the $\mathrm{Im}\,\theta = \frac{1}{2}i\pi$ line. For the operators this is only a notational convention and implies no

analyticity[3]. The f analyticity is equivalent to the localization property of $\hat{f}$, and the analytic properties of the state vectors ψ and the path notation in (8) is a n-variable generalization of (9) In the application to the vacuum in (8) of course only the creation contribution from the lower rim of the strip survives. It is easy to see that these vectors fulfill the modular localization equation [2]:

$$S\left|\psi^{(n)}\right\rangle = \left|\psi^{(n)}\right\rangle \tag{10}$$
$$S = J\Delta^{\frac{1}{2}},\ J = TCP, \Delta^{it} = U(\Lambda(2\pi t))$$

The antilinear unbounded Tomita involution S (be aware to avoid confusions with the notation for the S-matrix!) consists (for Bosons) of the TCP reflection J and the analytically continued (by functional calculus) wedge affiliated Lorentz boost $U(\Lambda(\chi))$ and the analytical strip properties guarantee that the localized vectors $\left|\psi^{(n)}\right\rangle$ are in the domain of $\Delta^{\frac{1}{2}}$ and hence of S. The action of $\Delta^{\frac{1}{2}}$ corresponds to the continuation to the lower rim and the action of J is just complex conjugation in momentum space (for selfconjugate situations):

$$\psi^{(n)}(\theta_n, \theta_{n-1}, ...\theta_1) \overset{S}{\to} \overline{\psi^{(n)}(\theta_n - i\pi, \theta_{n-1} - i\pi, ...\theta_1 - i\pi)} \tag{11}$$

so that modular localization equation (10) states that the value of the wave function on the $-i\pi$-shifted boundary equals the complex conjugate of the upper boundary. Note that the hermiticity of $A_{\psi^{(n)}}$ implies the reality condition $\overline{\psi^{(n)}(\theta_n, \theta_{n-1}, ...\theta_1)} = \psi^{(n)}(\theta_1, ...\theta_n)$ without analytic properties. For Bosons the wave functions are of course symmetric. In case of Fermions it is well known that the Tomita reflection J has a Klein twist in addition to the TCP.

The closure $H_R^{(n)}$of the real subspace of solutions of (10) contains the spatial part of modular wedge localization. By applying the generators of the wedge algebra (9) to the vacuum n-times, we generate a dense set of localized state vectors in the complex space which turns out to be the n-particle component of the well known Reeh-Schlieder set of vectors. This dense space becomes the Hilbert space $H_R^{(n)} + iH_R^{(n)}$ if one forms the closure in the graph norm:

$$\langle\psi_2, \psi_1\rangle \equiv (\psi_2, \psi_1) + (S\psi_1, S\psi_2) \tag{12}$$

Let us now pass to the case with interaction. For simplicity of notation we assume that the S-matrix of the factorizing interacting model describes the interaction of only one kind of particle (neutral, without bound states. Different from the free case, it follows from scattering theory [2] that J carries all the interaction whereas the Δ^{it} remains unchanged. Whenever necessary we will add a suffix 0 for the free (incoming) objects. With this notation we have:

$$J = S_{sc}J_0,\ \Delta^{it} = \Delta_0^{it} \tag{13}$$

[3]Operators in QFT, either in x-space or in momentum space, are never analytic, although some unfortunate notation and terminology especially in chiral conformal QFT suggests this. For more remarks see [2].

where S_{sc} denotes the S-matrix. The realization that the scattering connection between asymptotic (e.g. incoming) free fields and interacting fields (resp. algebras) keeps the unitary representations of the connected part of the Poincaré group unmodified and only changes those disconnected components which contain the anti-unitary time reversal is well-known [17]. What is new is the realization that the interpretation within the modular framework attributes to the S-matrix the property of a relative modular invariant for wedges. To see this, one only has to remember that thanks to the the Bisognano-Wichmann connection of the Lorentz boost and the TCP operation with modular theory, the relation $J = S_{sc}J_0$ is just the TCP transformation law of the S-matrix with the TCP operator being expressed in terms of the Tomita conjugation. In d=1+1 the J is identical with the TCP operator whereas in higher dimensions it is different by a π-rotation which commutes with the S_{sc}.

In analogy with the free n-particle Hilbert spaces $H_R^{(0)}$ we make the Ansatz:

$$\begin{aligned} \left|\psi^{(n)}\right\rangle &= A_{\psi^{(n)}}\Omega \qquad (14)\\ A_{\psi^{(n)}} &= \int_C \psi^{(n)}(\theta_n,\theta_{n-1},...\theta_1) : Z(\theta_1)...Z(\theta_n) : \end{aligned}$$

Here the $Z's$ are also mass shell annihilation and creation operators but with more complicated nonlocal commutation relations:

$$\begin{aligned} Z^{\#}(\theta_1)Z^{\#}(\theta_2) &= S_{sc}(\theta_1-\theta_2)Z^{\#}(\theta_2)Z^{\#}(\theta_1),\ Z^{\#}\equiv Z\ or\ Z^* \qquad (15)\\ Z(\theta_1)Z^*(\theta_2) &= S_{sc}^{-1}(\theta_1-\theta_2)Z^*(\theta_2)Z(\theta_1)+\delta(\theta_1-\theta_2)\\ crossing &: S_{sc}(\theta)=S_{sc}(i\pi-\theta)=S_{sc}^*(\theta-i\pi) \end{aligned}$$

In a different and more formal context this algebraic structure (the "Z-F" algebra) was introduced by Zamolodchikov and later completed (by adding the $\delta-$function term) by Faddeev. The states they generate are connected with the in-states by

$$\begin{aligned} &Z^*(\theta_1)...Z^*(\theta_n)\Omega = a^*(\theta_1)...a^*(\theta_n)\Omega \qquad (16)\\ &for\ \theta_1>\theta_2>...>\theta_n \end{aligned}$$

and the identification for permuted orders following from the Z-F algebra. It should be stressed that these consistent identifications are only tentative, pending on their reproduction by (LSZ, Haag-Ruelle) time-dependent scattering theory applied to the local operators which are still under construction. In the case of nondiagonal S-matrices it has not been possible to guess an operator formula on the basis of the above identification of states; more comments on this problem can be found in the next section. If we treat the $Z^{\#}(\theta)$ in analogy to mass shell creation and annihilation operators, we (following our intuitive discussion in the previous section) should form the "field":

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{2\pi}}\int \left(e^{-ipx}Z(\theta)+e^{ipx}Z^*(\theta)\right)d\theta \qquad (17)\\ Z^*(\theta) &= Z(\theta-i\pi) \qquad (18) \end{aligned}$$

It is of course no surprise that this field is noncausal. It is however a bit better than that; it is "wedge local" i.e. the smeared operator $F(f) = \int F(x)f(x)d^2x$ with supp$f \in W$ generates a *-algebra of the interacting theory localized in the wedge. In formula

$$[JF(f)J, F(g)] = 0, \quad \text{supp}f, g \in W \tag{19}$$

To prove this one first notices that the $Z(\theta)^\#$ commutes with the $JZ(\theta)^\#J$ underneath the Wick-ordering. The reason is that the exponentials involve integrals over number density which extend over complementary rapidity regions. The numerical phase factors which originate from the commutation of these exponential factors with the $a^{\#\prime}s$ mutually compensate. There remains the contraction between the pre-exponential $a^{\#\prime}s$ which leads to

$$\int_C \bar{f}(\theta)g(\theta - i\pi) \exp i \int \delta_{sc}(\theta - \theta')n(\theta')d\theta' \tag{20}$$

Shifting the integration by -$i\pi$ as required by the lower boundary of C, and using the crossing symmetry (particle-antiparticle Schwartz reflection symmetry around -$i\frac{\pi}{2}$) in the form: $\delta_{sc}(\theta - i\pi) = \delta_{sc}(-\theta) + 2\pi ni$, we see that this contraction is equal to that in the opposite order (which has the negative exponential). Again one easily verifies that this argument goes through if one only uses the commutation and vacuum annihilation properties of the $Z^{\#\prime}s$.

The mathematical control over the $Z^\#(\theta)$ and $F(f)$-operators is not more difficult than that of the standard creation and annihilation operators $a^\#(\theta)$ and the smeared free fields $A(f)$. Since the vacuum is annihilated by the $Z^\prime s$, the n-particle vectors are generated by n-fold application of $\int Z^*(\theta)f(\theta)d\theta$ onto Ω. With $S^{(2)}$ being a crossing symmetric solution of the bootstrp program, the n-particle component of the action of this operator on a state vector ψ is then

$$\begin{aligned} & (\int Z^*(\theta)f(\theta)d\theta\psi)^{(n)}(\theta_1, ...\theta_n) \qquad (21) \\ = \quad & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f(\theta_i) \prod_{k=1}^{i-1} S^{(2)}(\theta_k - \theta_i)\psi^{(n-1)}(\theta_1, ..\check{\theta}_i, ..\theta_n) \end{aligned}$$

where we used $\check{\theta}$ for the omission of a variable and where we have suppressed the indices on the wave functions on which the two-body S-matrix acts. As in the bosonic case, the norm of this wave function obeys the standard inequality involving the number operator $\mathbf{N}$

$$\left|\left| \int Z^*(\theta)f(\theta)d\theta\psi \right|\right| \leqslant ||f||\, ||\mathbf{N}\psi|| \tag{22}$$

and hence the closability of the $Z^{\#\prime}s$ and the selfadjointness of $F(f)$ for real test functions follows in a well-known manner [18], despite the fact that the total particle space

$$\mathcal{H} = \oplus_n \mathcal{H}_n \tag{23}$$

is not manifestly identical to a Boson/Fermion Fock space. In fact for none of the calculations we need a formula for $Z^{\#}$ in terms of free incoming fields as in (5), the general algebraic relations (15) together with the vacuum annihilation property is all we use for the construction of wedge localized generators in the space (23). The check of the Tomita relation

$$J\Delta^{\frac{1}{2}}A\Omega = A^*\Omega \tag{24}$$

can be done directly on products of the generators which are associated to the von Neumann algebras or alternatively by checking the KMS relation (see (27)) together with the transformation into the commutant by J.

Contrary to the previous free case, the sharpening of the support of the test function does not improve the localization within the wedge. This is equivalent to the statement that the reflection with J does not create an operator which is localized at the geometrically mirrored region in the opposite wedge W'. It only fulfills the commutation relation with respect to the full W'. In fact the breakdown of parity covariance is important for the existence of such nonlocal but wedge-localized fields since fields which are covariant under all transformations are expected to be either point local or completely delocalized (i.e. not even in a wedge). This breakdown of parity covariance for the Z-fields does not mean that the theory violates parity symmetry but only that these auxiliary fields only fulfill that symmetry and localization requirement which are expressible in terms of wedge algebras without reference to possible pointlike local field generators. We will later see that any sharper localization requires the operator to be an *infinite power series* in the $Z's$:

$$A = \sum \frac{1}{n!} \int_C ... \int_C a_n(\theta_n, \theta_{n-1}, ...\theta_1) : Z(\theta_1)....Z(\theta_n) : \tag{25}$$

where we again used the previously explained path notation. The sharper localization leads in fact to relations between the $a_n's$ (later) which, with one coefficient being different from zero forces higher ones to be nonvanishing as well. Therefore the PFG's only serve as a natural basis for smaller than wedge algebras, they themselves are not generators of these algebras. These statements comply with the physical idea that whereas the noncompact wedge region is big enough to allow the identification of particle states (it contains Lorentz boosts as automorphisms), it is on the other hand small enough to contain no annihilators as required by having a unique relation between vector states and operators (the Reeh-Schlieder relation) which is the prerequisite for the modular theory. This delicale balance is broken if one passes to compact localized algebras which favor the field side with the sharp particle number (and iven the one-particle aspect) loosing its observable meaning.

The proof of (19) used the crossing symmetry of the S-matrix which appears in the Z-F algebra (15). Note that since the commutations of $Z's$ produce unitary factors with the same two-particle S-matrix, the a_n must compensate this unitary factors upon commuting $\theta's$.

$$a_n(\theta_1, ...\theta_i, \theta_{i+1}, ..\theta_n) = S_{sc}^{(2)}(\theta_i - \theta_{i+1})a_n(\theta_1, ...\theta_{i+1}, \theta_i, ..\theta_n) \tag{26}$$

In this respect the phase factors are like statistics terms. However since the a_n have analytic properties in the multi-θ strip (actually for compact localization the analytic region is much bigger), these phase factors must be consistent with the univaluedness in the analytic domain.

We now show that the KMS property relative to the boost is equivalent to the crossing symmetry of the $S^{(2)}$-coefficients in the Z-F algebra. The first nontrivial correlation function which deviates from that of free fields is the 4-point function. The KMS property reads

$$\langle F(f_{1'})F(f_{2'})F(f_2)F(f_1)\rangle_{therm} = \left\langle F(f_{2'})F(f_2)F(f_1)F(f_{1'}^{2\pi i})\right\rangle_{therm} \tag{27}$$

where the superscript $2\pi i$ is the imaginary KMS shift in the boost parameter. Each side is the sum of two terms, the direct term associated with

$$\begin{aligned} F(f_2)F(f_1)\Omega &= \int f_2(\theta_2 - i\pi)f_1(\theta_1 - i\pi)Z^*(\theta_1)Z^*(\theta_2)\Omega + c-number\cdot\Omega \\ &= \int f_2(\theta_2 - i\pi)f_1(\theta_1 - i\pi)S^{(2)}(\theta_2-\theta_1)a^*(\theta_1)a^*(\theta_2)\Omega + (28) \end{aligned}$$

and the analogous formula for the bra-vector. For the inner product there are two contraction terms consisting of direct and crossed contraction (in indices 1 or 2) of the $a^{\#}s$. Only the second one gives an S-matrix factor in the integrand. The c-number term an the left hand side cancels the direct term on the right hand side. The equality of the crossed terms on both sides gives (using the denseness of the analytic wave functions)

$$S^{(2)}(\theta_2-\theta_1) = S^{(2)}(\theta_1 - \theta_{1'} + i\pi)\mid_{\theta_{1'}=\theta_2} \tag{29}$$

i.e. one obtains the above crossing relation for the two particle S-matrix. Higher inner products involve products of S-matrices and it is easy to see that the KMS condition for the Z-F algebra is equivalent to the crossing property of the S-matrix. The presence of operators A which are localized in a e.g. double cone (without loss of generality within a wedge) does not influence the validity of the KMS condition.

$$\begin{aligned} &\langle F(f_{1'})F(f_{2'})...F(f_{m'})AF(f_n)...F(f_2)F(f_1)\rangle_{therm} = \qquad (30) \\ = \quad &\left\langle F(f_{2'})..F(f_m)AF(f_n)..F(f_2)F(f_1)F(f_{1'}^{2\pi i})\right\rangle_{therm} \end{aligned}$$

The rapidity space formulation of this KMS condition will turn out (see next section) to be the cyclicity relation for the formfactor of that local operator which hitherto [13] was a special consequence for factorizing systems of the formally derived crossing symmetry which in turn follows from the LSZ scattering theory together with analytic assumptions.

It is not difficult to find an axiomatic generalization of PFW operators from this illustrative model theory to the general situation of d=1+1 wedge localized generators. Let us assume that we have a PFG operator $F(f)$ and its J-transform $JF(g)J$ such that the commutator in rapidity space after splitting

off the f,g factors which we denote by M commutes with the Poincaré-group. This assumption is certainly satisfied in the factorizing case (see below). Then the fluctuation terms do not contribute to the commutator and we obtain for the structure of the matrix elements of the $F-JFJ$ commutator between multiparticle states (the $F-JFJ$ formfactors)

$$\begin{aligned} & {}^{out}\langle p'_1...p'_n|\left[JF(f)J,F(g)\right]|p_1...p_m\rangle^{in} \\ = & \int \bar{f}(\theta)g(\theta-i\pi)M(\theta'_1,...,\theta'_n;\theta-i\pi;\theta_1...,\theta_m)d\theta \\ & -\int \bar{g}(\theta)\bar{f}(\theta+i\pi)M^J(\theta'_1,...,\theta'_n,\theta;\theta_1...,\theta_m)d\theta \end{aligned} \tag{31}$$

Here we assumed the validity of the following commutation relations

$$\begin{aligned} \left[Z^J(p),Z(p')\right] &= 0=\left[Z^{J*}(p),Z^*(p')\right] \\ \left[Z^J(p),Z^*(p')\right] &= 2\omega\delta(p-p)M(p) \end{aligned} \tag{32}$$

i.e. we assume that the relative commutators between $Z's$ and $Z^{J}'s$ generalize those between free field annihilation/creation operators in momentum space in that they are Poincaré-invariant operators (but not necessarily multiples of the identity as for free fields). In terms of the F's we used the following notation

$$\begin{aligned} \left[JF^{(-)}(f)J,F^{(+)}(g)\right] &= \int\int \bar{f}(\theta')g(\theta-i\pi)\delta(\theta'-\theta)M(\theta-i\pi)d\theta'd\theta \\ M(\theta'_1,...,\theta'_n;\theta-i\pi,\theta_1...,\theta_m) &= {}^{out}\left\langle\theta'_1,...,\theta'_n\right|M(\theta-i\pi)\left|\theta_1...,\theta_m\right\rangle^{in} \end{aligned} \tag{33}$$

Note that the second contribution on the right hand side of (31) results from the identity

$$\begin{aligned} & {}^{out}\left\langle\theta'_1,...,\theta'_n\right|\left[F^{(-)}(g),JF^{(+)}(f)J\right]\left|\theta_1...,\theta_m\right\rangle^{in} \\ = & {}^{out}\left\langle\theta'_1,...,\theta'_n\right|JJ\left[F^{(-)}(g),JF^{(+)}(f)J\right]\left|\theta_1...,\theta_m\right\rangle^{in} \\ = & \overline{{}^{out}\left\langle\theta'_1,...,\theta'_n\right|J\left[JF^{(-)}(g)J,F^{(+)}(f)\right]J\left|\theta_1...,\theta_m\right\rangle^{in}} \end{aligned} \tag{34}$$

If now M is "crossing symmetric" around $\mathrm{Im}\,\theta=-\frac{1}{2}i\pi$ as the left hand side in (32) suggests, i.e. M and M^J are strip-analytic and

$$\begin{aligned} & M(\theta'_1,...,\theta'_n;-\frac{1}{2}i\pi-\vartheta;\theta_1...,\theta_m) \\ = & M^J(\theta'_1,...,\theta'_n;-\frac{1}{2}i\pi+i\vartheta;\theta_1,...,\theta_m) \end{aligned} \tag{35}$$

then we have established wedge locality i.e. the F are really wedge generators and hence the validity of the following theorem

Theorem 1 *The quantum wedge localization of the PFG F is equivalent to the existence and crossing symmetry of their Z_W-JZ_WJ formfactors.*

There remains the task to *prove this crossing symmetry for the factorizing models*. From the structure of the Z-F algebra and the definition one concludes that $M(\theta)$ has the form

$$M(\theta) = T(\theta) \tag{36}$$

where $T(\theta)$ is Lorentz (θ-translation) covariant, leaves the vacuum unchanged and fulfills the following commutation relation with the Z's

$$T(\theta)Z(\theta') = S^{(2)}(\theta - \theta')Z(\theta')T(\theta) \tag{37}$$

which together with $T(\theta)\Omega = \Omega$ fix the operator. In the case of a diagonal S-matrix the T has the following expression in terms of the incoming fields

$$T(\theta) = e^{i\int_{-\infty}^{+\infty} \delta(\theta-\theta')n(\theta')d\theta'} \tag{38}$$

Here as before we left it to the reader to convince himself that one can do all the computations without knowing the formulas for Z's or T's in terms of incoming fields, which covers the general factorizable setting with many particles which may form multiplets. We will later on comment on how to obtain such an operator relation for this case.

4 The Construction of the Double-Cone Algebra

Finally we to indicate (again for the simplest one particle model) how one proceeds from the quantum localized Z's to the double cone algebras. Formally we have to look for solutions of

$$JBJ = A, \quad A, B \in \mathcal{A}(\mathcal{O}_a) \tag{39}$$

the equation for the elements of the algebraic intersection

$$\mathcal{A}(\mathcal{O}_a) = U(-\frac{a}{2})\mathcal{A}(W)U(\frac{a}{2}) \cap U(\frac{a}{2})\mathcal{A}(W')U(-\frac{a}{2}) \tag{40}$$

The wedge algebra is the von Neumann algebra associated with the *-algebra

$$\mathcal{A}_{formal}(W) = \left\{ A \mid A = \sum_n \frac{1}{n!} \int_C d\theta_1 ... \int_C d\theta_n a_n(\theta_n, \theta_{n-1} .... \theta_1) : Z(\theta_1) .... Z(\theta_n) : \right\} \tag{41}$$

where the a_n are meromorphic in the lower strip. Here we call a would-be von Neumann operator algebra $\mathcal{A}_{formal}$ if we only study aspects of sesquilinear forms i.e. modulo convergence properties required by bona fide operators. These coefficient functions are related to the KMS property of the previously introduced mixed correlation functions

$$\begin{array}{rcl} & \langle AF(f_1)...F(f_n)\rangle_{therm} & \\ A & \in & \mathcal{A}(O_a), \quad F(f_i) \in \mathcal{A}(W_a) \end{array} \tag{42}$$

Originally the KMS property gives the analyticity in the regions of ordered imaginary parts for monomial vectors $\prod_i Z(f_i)\Omega$, but as a result of the denseness of the analytic $f_i's$, one retains meromorphy (analyticity modulo poles) for the coefficient functions of general nth components a_n of wedge localized operators. In fact the expansion of A in terms of $F's$ may be directly written in x-space

$$\begin{aligned} A &= \sum \frac{1}{n!} \int ... \int \hat{a}(x_1, ..., x_n) : F(x_n)...F(x_1) : \qquad (43) \\ &= \sum \frac{1}{n!} \int_C ... \int_C a(\theta_1, ...\theta_n) : Z(\theta_n)...Z(\theta_1) : \\ & \langle AF(f_1)...F(f_n)\rangle_{therm} = \int ... \int \hat{a}_n(x_1, ..., x_n) f(x_1)...f(x_n) + lower\ terms \end{aligned}$$

The compact localization allows to extend the meromorphy to the product of the θ-planes. As before in the case of the KMS property of the pure F correlation functions was equivalent to the crossing symmetry of the S-matrix, the KMS property of the mixed functions involving one A with compact localization (or a field $A(x)$ with $x \in W$) to a crossing symmetry relation, the so-called cyclicity relation in the rapidity variables θ.

Theorem 2 *The KMS property of the mixed wedge correlation functions is equivalent to the cyclicity relation of the coefficient functions in the rapidity representation*

$$a_n(\theta_2, ..., \theta_n, \theta_1 - 2\pi i) = a_n(\theta_1, ..., \theta_n) \qquad (44)$$

The proof uses the rapidity representation (43). For the highest coefficient in the KMS relation one obtains the cyclicity relation by straightforward computation again (as in the previous S-matrix case) using the density of the boundary values of the strip analytic test functions $f_i(\theta_i)$

For many considerations it is safer to argue directly in terms of the thermal correlators. in x-space than with the momentum rapidity $a_n's$. Even the following derivation of the pole structure should be transcribed into the thermal correlators, but we will present it in the more familiar momentum space rapidity form. In order to avoid questions about the necessity to introduce "fused" $Z(\theta)'s$ and $F's$ for fused (bound states[4]) particles into the expansions (41) and other questions related to the completeness of $Z's$, we will simply assume the absence of such particles and defer a more general discussion to a future publication.

Now $B \in \mathcal{A}_{formal}(W)'$ has the analogue representation to (41) in terms of Z^J and coefficient functions b_n which are upper strip-meromorphic. Shifting the apex of the opposite wedge by a translation a into W means that we multiply the b_n-coefficients with $\exp i\sum p_i(\theta_i)a$. Let us call the shifted b's $b_n^{(a)}$. The formal double cone algebra $\mathcal{A}_{formal}(\mathcal{O}_a)$ is defined as $\mathcal{A}_{formal} \cap$

[4]The notion of bound states carries connotations of Q.M. which are somewhat misleading in relativistic field theories with (apart from free fields) a virtual particle polarization structure. The correct hierarchy is that of basic versus fused charges whose low-lying carries are (infra)particles.

$U(a)\mathcal{A}_{formal}(W)'U^*(a)$. In order to compute matrix elements of Z^J within vector states created by $Z's$ we use the cumulant formula

$$\exp i\int\sum_i \delta(\theta_i-\theta')\dot{a}^*(\theta')a(\theta')d\theta' =: \exp\int(1-\Pi_i S(\theta_i-\theta'))a^*(\theta')a(\theta')d\theta' : \tag{45}$$

The computation is facilitated by using the following equivalent modular characterization of the intersection algebra in terms of either $A\in\mathcal{A}(W_{\pm a})$ and

$$\begin{aligned} A &= A_s + iA_s \\ JA_sJ &= A_s \end{aligned} \tag{46}$$

where we decomposed a general operator of the double cone algebra into a J-selfconjugate and antiselfconjugate part. Here A belongs to the shifted algebra, but J is the Tomita involution of the original wedge algebra, i.e. is our previous J. Since the J does not mix even and odd terms, we will assume that A has only even terms in its power series.

Matching first the diagonal matrixelements for the $A_s's$ and $JAJ's$, we obtain the following recursion for the boundary values of the meromorphic functions:

$$\begin{aligned} &a_0 = \bar{a}_0 \\ &a_2(\theta_1,\theta_2-i\pi) = \bar{a}_2(\theta_1,\theta_2+i\pi)+\delta(\theta_1-\theta_2)(1-S(\theta_1-\theta_2-i\pi))a_0 \\ &a_4(\theta_1,\theta_2,\theta_3-i\pi,\theta_4-i\pi) = \bar{a}_4(\theta_1,\theta_2,\theta_3+i\pi,\theta_4+i\pi) \\ &+\delta(\theta_1-\theta_3)(1-S(\theta_1-\theta_4-i\pi)S(\theta_2-\theta_3+i\pi)S(\theta_1-\theta_2))a_2(\theta_2,\theta_4-i\pi) \\ &+3\ more\ such\ terms \\ &a_6(\theta_1,.....) \quad = \quad etc. \end{aligned} \tag{47}$$

with an analogous recursion for the odd coefficient functions. Taking into account the meromorphic properties of the coefficient functions, these δ-function terms in the boundary value mean that the meromorphic functions have poles if two rapidities coalesce modulo $i\pi$. The matching conditions also contain the possibility of continued extension into the tensor product of complex planes. In this way one obtains meromorphic functions in the multi-θ plane which have a Paley-Wiener type of increase in the imaginary direction which is related to the size a of the double cone. Again one easily notices that these consideration can be done solely on the basis of the algebraic properties of the $Z^\#$ operators together with their annihilation property of the vacuum vector. Possible poles in the coefficient functions from other fused particle states appear in crossing symmetric pairs and compensate; only the above kinematical poles enter the formal determination of the double cone algebras $\mathcal{A}(\mathcal{O}_a)$. Hence our method applies to the general case of factorizable theories with admissable nondiagonal bootstrap S-matrices [19].

In addition to the above kinematical poles there are poles inside the physical strip associated with "bound states" [5]. As in the case of wedge localization they

[5]Strictly speaking the hierarchy of elementary versus bound states is limited to quantum

do not contribute to the structure of $\mathcal{A}_{formal}(\mathcal{O}_a)$ as a result of their pairwise occurrence due to crossing symmetry.

As a result, we obtain the well-known "kinematical pole structure" of formfactors (the coefficients in the series (25) known from the work of Smirnov [14][16] (based on recipes) and later abstracted from LSZ scattering formalism in [13]. In our approach this structure (as the crossing symmetry) is equivalent to the J-invariance of the symmetrically placed double cone algebra.

Theorem 3 *Necessary and sufficient for the series (25) to fulfill (39) is that the coefficient functions are meromorphic in the multi θ-plane with certain fall-off behavior in imaginary direction and following pole structure on the boundary of the old strip*

$$\begin{aligned} & a_{n+2}(\theta + i\pi + i\varepsilon, \theta, \theta_1, ..., \theta_n) \mid_{\varepsilon \to 0} \\ = \quad & \frac{1}{\varepsilon}\left[1 - \prod_{i=1}^{n} S^{(2)}(\theta - \theta_i)\right] a_n(\theta_1, ..., \theta_n) \end{aligned} \tag{48}$$

Since each pair of $\theta's$ can always be transported into this canonical position by the application of (26), there is no loss of generality in this way of writing.

The formfactors of pointlike fields at the origin which have appeared in the literature [14] [13] result by taking the limit a$\to$ 0, prescribing the Lorentz transformation property of the would be field and defining a convention which picks a particular (composite) field in the remaining infinite dimensional field space. Since fields at the origin are sesquilinear forms and not operators, they are improper members of $\mathcal{A}_{formal}(\mathcal{O})$. The operator problems show up in the computation of correlation functions of these fields. In fact there are two formidable tasks in the Smirnov-Karowski-Weisz approach. On the one hand one must construct a basis in field space (the analogue to the Wick-polynomials in the free field case) which even in the simplest nontrivial so-called SinhGordon model led to presently intractable looking problems about symmetric polynomials in the $x_i \equiv \exp\theta_i$ [10]. But this is not enough, according to Wightman one also has to take care of the operator problems of the smeared fields who'sss prerequisite is the existence of the correlation functions. Apart from those special cases with a constant S-matrix as the Ising- and Federbush models, this has not been achieved. In the latter cases one notices immediately that there are quadratic expressions in the $Z's$ for which the iteration yields a vanishing factor from the factor in front of a_2 on the right hand side in (48). Since the operator aspects of such quadratic expressions as the energy-momentum tensor are easily controlled, it is clear that not only $\mathcal{A}_{formal}(\mathcal{O})$ but also $\mathcal{A}(\mathcal{O})$ is nonempty. For rapidity dependent S-matrices the iteration cannot stop and the construction of even a single operator causes serious mathematical problems.

mechanics and looses its meaning in the presence of (virtual) vacuum polarization. The only hierarchy which remains meaningful is that between basic and fused charges; on the level of particles QFT practices "Nuclear Democracy" i.e. interactions couple all channels which are not separated by superselection rules.

Using the algebraic approach, one only faces a "light" version of these problems. The reason is that the modular construction liberates the existence proof of a model from the technical burden of field coordinates and sometimes even insures the existence of local copies of symmetry operators. One only has to show that the double cone algebras are not trivial i.e. contain operators which are not multiples of unity. This step would then allow to omit the subscript "formal" in the above space of sesquilinear forms and write $A \in \mathcal{A}(\mathcal{O}_a) = U(-\frac{a}{2})\mathcal{A}(W)U(\frac{a}{2}) \cap U(\frac{a}{2})\mathcal{A}(W')U(-\frac{a}{2})$. Let us first note that a trivial double cone algebra ($\mathcal{A}(\mathcal{O}_a) = C \cdot \mathbf{1}$) would violate several properties of Local Quantum Physics. On the one hand the additivity property

$$\begin{aligned} W &= \cup_d \mathcal{O}(0,d) \\ \mathcal{A}(W) &= \vee \mathcal{A}(\mathcal{O}(0,d)) \end{aligned} \tag{49}$$

where $\mathcal{O}(0,d)$ is a family of double cones inside W which have a common left corner with W, does not hold. Furthermore the so called split property cannot hold in such a case for it is one of the consequences [11] that a double cone algebra cannot be just a multiple of the identity because it must contain at least a local version of the symmetries of the theory e. g. an algebraic analog of the energy momentum tensor. But in our situation we need an argument which refers only to the commuting wedge algebras $\mathcal{A}(W), \mathcal{A}(W') \equiv J\mathcal{A}(W)J$ and the d=1+1 Poincaré group. If we were to assume the split property for two dimensional wedges [12], we would immediately obtain the unitary equivalence of the double cone algebra to a tensor product of wedge algebras

$$\mathcal{A}(\mathcal{O}_a) \simeq \mathcal{A}(W_{\frac{a}{2}}) \otimes \mathcal{A}(W'_{-\frac{a}{2}}) \tag{50}$$

so that the nontriviality of the double cone algebra would be manifest. This elegant argument has however one drawback; all factorizable models are very different from superrenormalizable models which are locally normal with respect to free theories and therefore inherit the split property of the latter and as a result we have doubts about the validity of the split property for wedges for factorizing models. Therefore we will try yet another argument based on the idea that the double cone algebra possesses a nontrivial "light cone reduction". The mathematical formulation of this idea starts from the relative commutant associated with the following inclusion

$$\mathcal{A}(W_{a_\pm}) \subset \mathcal{A}(W) \tag{51}$$

$$i.e.\ \mathcal{A}(W_{a_\pm})' \cap \mathcal{A}(W) \tag{52}$$

Here W is the d=1+1 standard wedge and $W_{a_\pm}$ are the upper and lower wedges obtained by having W slide into itself along the upper/lower light ray ($\pm$horizon) with the distance $a_\pm$. The geometric position of the associated relative commutant $\mathcal{A}(W_{a_\pm})' \cap \mathcal{A}(W)$ is the 1-dimensional interval on the upper/lower light ray which starts at the origin and ends at $a_\pm$. It may be considered to be the limiting case of the 2-dim. relative commutant which is obtained by shifting the W into itself so that the upper/lower horizons of the shifted wedge do not

yet touch the standard wedge. From these upper/lower relative commutants we may form

$$\mathcal{M}_{\pm} \equiv \bigcup_t \Delta_W^{it} \left(\mathcal{A}(W_{a_{\pm}})' \cap \mathcal{A}(W) \right) \subset \mathcal{M} \equiv \mathcal{A}(W) \tag{53}$$

This strict inclusion reflects the interpretation as being the conformal light ray limit. A manifest way of exposing this structural property of this limiting theory on the light ray is to emphasize that the improvement of symmetry on the light ray[6] [21] to full conformal symmetry goes together with a decrease in the size of the cyclically generated space from the vacuum

$$\begin{aligned} \mathcal{H}_{\pm} &\equiv \mathcal{M}_{\pm}\Omega \subset M\Omega \equiv \mathcal{H} = P_{\pm}\Omega \\ E_{\pm}(\mathcal{M}) &= P_{\pm}\mathcal{M}P_{\pm} = \mathcal{M}_{\pm} \end{aligned} \tag{54}$$

The projector $P_{\pm}$ onto this smaller space is associated with a conditional expectation $E_{\pm}$, in agreement with the fact that the two theories share the same modular group. To recover the full algebra from its light cone pieces, we have to combine $\mathcal{M}_{\pm}$. There are two ways of doing this

$$\begin{aligned} \mathcal{M}_{(+,-)} &= \mathcal{M}_+ \vee \mathcal{M}_- \\ \mathcal{M}_{\otimes} &= \mathcal{M}_+ \otimes \mathcal{M}_- \end{aligned} \tag{55}$$

The second way belongs to the construction of two-dimensional conformal QFT from their chiral components whereas the first one gives the equality of the combined light cones to reproduce the original massive algebra in case that the vacuum is cyclic since $clos(\mathcal{M}_{(+,-)}\Omega) = clos(\mathcal{M}\Omega) \leftrightarrow \mathcal{M}_{(+,-)} = \mathcal{M}$.

In order to see what is going on in the case of factorizing models in more detail, let us write down the commutation relation which characterize those operators $A \in \mathcal{M}$ (25) which commute with (the generators of) $\mathcal{M}_+$

$$\left[A, \int f_{a'_+}(x) Z(x) d^2x \right] = 0, \; a'_+ \geq a_+ \tag{56}$$

Clearly the n^{th} order contribution in the $Z(\theta)$ power series receives two contributions, one with a coefficient function of the form

$$\int_C d\theta (\Pi S(\theta_i - \theta) - 1) a_{n-1}(\theta_1, ..\theta_{n-1}) e^{imu \exp\theta} \tag{57}$$

which results from the product of S-matrices from commuting the $Z(\theta)$ to the left hand side, as well as contraction terms

$$\int_C d\theta \int_C d\theta_n a_{n+1}(\theta_1, ...\theta_{n-1}, \theta_n, \theta) e^{imu \exp\theta} \tag{58}$$

[6]The fact that this net on the light ray which is indexed by intervals is not only covariant under light ray translations and scale transformations but also under conformal rotations follows from [20].

A pause of thought reveals that this relative commutator condition does not tell us anything on the pole structure of the coefficient functions which we do not know already from theorem (48), however due to the presence of only one light cone coordinate the vanishing of the commutator is identical to problems in the algebraic formulation of chiral nets on the line with tranlation and scale covariance. The one dimensional operator version of the Paley-Wiener theorem gives a much better control on the remaining meromorphy properties. The reduction to the light ray fields gives a powerful tool for the transition from the wedge algebras to the compactly localized double cone algebras. We will give a detailed account of its application to factorizable models. Here we will be content to present some more structural properties in the sequel.

The light ray reductions are bona fide conformal nets since the translation-dilation situation together with the modular inclusion $\mathcal{M}'_{a_\pm} \subset \mathcal{M}$ is known to correspond to a SL(2,R)-covariant local conformal net [22]. It is very important not to equate this net with the in a naive way neither with the (divergent) light cone restriction nor with the (Lagrangian) light cone quantization[7] of fields. For factorizing theories whose local generating fields are more singular for short distances than free fields (they are not superrenormalizable as it would be required by local normality), the canonical structure is not defined and the light cone restriction is divergent since the field near to the + light cone does generally not become independent on the - light cone variable. It is only through the associated algebraic net (generated by the local fields) that the light ray restriction is defined. This algebraic reprocessing is even necessary in superrenormalizable theories since the light cone quantization in the presence of interactions is nonlocal (it is a PFG) even when it is well defined. Only by reprocessing the nonlocal data via relative commutants one has a chance to reconstitute a local net on the light ray. With other words the modular method is essential for understanding and physically interpreting light ray physics. Simple examples show that the light ray restriction generates a space which is genuinly smaller than the total Hilbert space.

The simplest illustration of this mechanism is obtained by looking at the free massive Fermion field

$$\begin{aligned} \{\psi(x), \psi^*(y)\} &= \left(\gamma_\mu \partial^\mu - m\right) i\Delta(x-y) \\ \{\psi_1(x), \psi_2^*(y)\} &\to \pm m \end{aligned} \tag{59}$$

where the x, y in the last relation are on the upper/lower horizon so that the distance is $\pm$time-like. In this case no algebraic reprocessing is needed, we obtain two conformal theories with different chirality which applied to the vacuum generate $H_\pm$ subspaces.. The fact that they did not simply originate from a mass zero limit shows up in their nontrivial m-dependent relative commutation

[7]The formal light cone quantization procedure leads to to a kind of quantum mechanics as a result of the surpression of vacuum polarization [23]. In none of the presentations of the light cone formalism an attempt is made to reconstruct the original local fields from the light ray data. We believe that this is not possible without the algebraic reprocessing by modular methods as we used in this paper.

relation which is the memory of the relative time-like difference. In fact the light cone restriction is a peculiar chiral conformal theory, a kind of hidden massive theory, which by the adjunction of the opposite light cone translation can be cropped into a wedge-localized massive theory

$$alg\,\{\mathcal{M}_+, U_-(a \geq 0)\} = \mathcal{M} \tag{60}$$

In fact these properties of simple models are believed to be a general structural features of light ray restrictions, a point to which we plan to return in a separate paper.

As mentioned before, all modular constructions for factorizing theories can be based on the algebraic structure as well as the vacuum annihilation properties of the Z-operators. After having constructed the local net one would like to check however the interpretation of the n-particle Z-states in terms of incoming states by establishing an operator formula (which generalizes the well-known exponential formula for diagonal S-matrices) based on the (Haag-Ruelle, LSZ) time dependent scattering theory. The explicit knowledge of the local double cone net together with the isolated one particle states (the spectral gap property) allows for the construction of the incoming fields in terms of the $Z's$ and by inversion also the (a priori unknown for the nondiagonal models) formulas for the $Z's$ in terms of the incoming free fields. At this point the correctness of the scattering interpretation of the coefficient functions (25) in terms of formfactors between scattering states is confirmed by selfconsistency.

5 Outlook

We have succeeded to reformulate the existence of "quantum localized" PFG's in interacting QFT in terms of crossing symmetric on-shell operators $M(\theta)$ which appear in the commutation relation between the $Z's$ and their Tomita transformed JZJ and we identified these operators with well-defined (through their commutation relations) expressions $T(\theta)$ within the more restricted context of integrable models. As a side result we obtained a relation between the extremely deep crossing symmetry[8]. It would have been somewhat disappointing if the understanding of this most characteristic and profound aspect of QFT could have been achieved by already well-known properties as Haag-Ruelle scattering theory and the thermal KMS condition.without the necessity to introduce a new concept as the role of PFW's and their connection with an associated modular light ray theory.

Two questions immediately arise: can one use this gain of conceptual insight for the improvement of the Bootstrap-Formfactor program, and secondly could the new algebraic structure based on the general possible existence of PFG's be useful for a kind of "modular revival" (this time build on noncommutative

[8]In fact it was the ill-understood (outside some formal perturbative calculations) crossing symmetry, a kind of TCP relation which is not global but holds for individual particles, for which the desire for a more profound understanding led first to Veneziano's dual model and then (after the original motivation was forgotten) to string theory.

Tomita modular theory instead of the commutative measure theory underlying euclidean functional integrals) of the old constructive program to get beyond free fields and perturbation theory?

As far as the first question is concerned, we remark that the outstanding problem is the physical understanding and the mathematical description of the "operator content" of local fields. By this we mean the understanding of their expansion coefficients in terms of an auxiliary basis (25) which leads to formfactors. As it turns out, this is a very difficult task even for the simplest case of (diagonal Toda) factorizing field theories. The identification of the sequences of symmetric polynomials which characterize the formfactors of the list of (composite) fields up to this date has not been accomplished. Our algebraic viewpoint which shifts the emphasis from individual field coordinates to double cone algebras, reveals that the traditional field construction carries a lot of nonintrinsic structure which adds little to the securing of the existence and the understanding of the important dynamical properties of these models. The construction of a useful basis of local fields (i.e. a basis in the local field space), which parallels the construction of Wick polynomials in the free case, is a highly technical problem with a large amount of arbitrariness. This is of course a phenomenon which had to be expected on the analogy with coordinates in differential geometry. In fact in (nonperturbative) LQP the technical difficulties in using field coordinates versus intrinsically defined local nets if anything become even more awkward than the use of coordinates in differential geometry. Perturbation theory which requires the selection of a set of field coordinates is no good guide for these aspects, but even there one realizes the clumsiness of field coordinates if one tries to understand the insensitivity of the S-matrix (in the present setting its role as a relative modular invariant for wedge algebras) against local changes of field coordinates. Even formulations which are usually considered to be the quintessence of the nonperturbative approach as e.g. the Wightman approach are not completely intrinsic. In the present work we have used modular ideas which force us to understand the existence and physical content of QFT in terms of net of von Neumann algebras. The crucial step, the construction of the double cone algebras, is facilitated by the use of (far off-shell) light ray algebras which turned out to be "hidden" chiral conformal theories. This approach, whose complexity increases with the distance from on-shell properties (or distacne to polarization free objects), is particularly suited for high energy particle physics where the main measurable quantities are not correlation functions of fields but rather the S-matrix and perhaps formfactors of some distinguished fields (Noether currents) both closely related to modular concepts associated with wedge localization. The situation is different in statical mechanics and condensed matter physics; in that case one has to work harder for the calculation of correlation functions (which often can be measured by neutron scattering). Our nonperturbative approach contains the message to separate the construction of a nontrivial field theory and the understanding of its physical content from the technically complicated construction of its field coordinates.

A somewhat related problem is the question of how to avoid an explicit for-

mula for the PFG operators in terms of incoming operators. Such a formula is a priory not known (i.e. cannot be written down on the basis of commutation relations which involve the S-matrix data) for nondiagonal factorizing models. A glance at the $Z^{\#}(\theta)$ operators reveals that if the annihilators annihilate the vacuum state, the Z-F algebra determines n-particle Hilbert space inner products (even without knowing a relation to incoming fields). For the interpretation in terms of formfactors it is sufficient to be able to relate the vector *states* generated by the $Z^{\#}(\theta)$ to the incoming/outgoing states. The construction of the double cone algebras can be done in the related total space without use of a Fock space structure because the expansion formulas (25) are defined as soon as the iterative action of the $Z^{\#}(\theta)'s$ on the vacuum is known. Having the local algebras within that space, one only has to remind oneself that the Haag-Ruelle scattering theory uses just locality and the energy-momentum spectrum. Therefore one may *compute the incoming fields* and then invert the relation in favor of writing the Z's in terms of $a^{\#}_{in}$.

The second question concerning the applicability of these ideas outside of factorizing models should be subdivided according to space-time dimension $= 1+1$, or $> 1+1$. In the first case the mass shell is still parametrized in terms of rapidities but they do not furnish a uniformization variable for the S-matrix which retains its rich creation/annihilation threshold analytic structure. The prerequisites for quantum wedge locality hold since crossing symmetry remains still a valid concept. Of course this does not yet give a blueprint for the construction of Z and $M(\theta)$. As in the factorizing case this cannot be fully covariant; at least the pointlike parity transformation must be violated. This reduced covariance of the PFG operators is related to their weaker localization and not with broken symmetries in the final QFT).

As a curious side remark we mention that the central idea of this paper, namely the existence of PFG operators as the best compromise between the of-shell localization of fields with the on-shell particle structure, has some vague resemblance with certain formal simplifications in the so-called light-cone or light-front quantization. Closing one's eye to the fact that the canonical nature of the approach (as in all quantization approaches) is not compatible with realistic ideas about interactions[9] (apart from superrenormalizable interactions), one realizes that formally such a quantization suppresses vacuum or one-particle polarization phenomena [23] which is of course precisely the characteristic property of the PFG's. However whereas in the light-front approach the relation of the so obtained field operators to the original local fields seems to have been irrevocably lost, and with it the possibility if physically interpreting the theory (apart from the one particle spectral informations), the PFG'ss are auxiliary operators which have a conceptually clear and mathematically precise relation within LQP. The message from our modular approach is that via "quantum localization" (formation of algebras and intersections) one should be able to recover the local variables from the ones obtained by light cone quantization. For

[9]In fact genuine 2-dim. conformal fields, which are sums over products of anomalous dimension (spin) chiral fields from the two light cones, already expose this short-distance problem in the light ray restriction of pointlike fields.

people who do not believe in strange coincidences we may add that the Galilei group which appeared as a Poicaré subgroup in light cone quantization [23] had an independent outing in form of an 8-parametric subgroup in the modular approach in connection with so called modular intersections (see the first paper in [4]).

As (apart from superrenormalizable theories, where the fields are "locally normal" with repect to the canonical free fields) the only consistent remainder of the standard canonical quantization formalism is the Einstein causality, the semilocal PFG' fields of this paper could have a similar relation to the light cone formalism fields. In view of the popularity which ideas of noncommutative geometry have enjoyed within part of the physics community, the present message, that the nonperturbative construction of local fields requires very noncommutative intermediate steps based on modular theory, should have an attractive appeal.

The higher dimensional cases are additionally complicated by the fact that wedge-adapted rapidities do not furnish a complete mass shell parametrization and that even if one succeeds to construct PFG's, the compactly localized double cone algebras cannot be obtained by intersecting only two wedge algebras with the translated opposite (i.e. J-transformed) wedge algebras. Here a modular revival of the idea inherent in the old dual model, namely a relation between a higher dimensional (on-shell) S-matrix and a low dimensional d=1+1 (off-shell) field theory would be desirable.

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