Twistorial constructions of spacelike surfaces in Lorentzian 4-manifolds

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Abstract

A construction of J. Eells and S. Salamon [ES85] uses Riemannian twistor theory for the investigation of minimal surfaces in Riemannian 4-manifolds. In this paper the twistor space and the Grassmannian fibre bundle of a Lorentzian 4-space with natural almost optical structures and its induced CR-structures are studied. The twistor spaces of the Lorentzian space forms $\mathbf{R}^4_1$, $\mathbf{S}^4_1$ and $\mathbf{H}^4_1$ are explicitly discussed. Similar as in Riemannian geometry, the given twistor construction is applied to surface theory in Lorentzian 4-spaces. Immersed surfaces with special geometric properties like semi-umbilic and (semi)-stationary spacelike surfaces correspond to holomorphic curves in the twistor space. For the Lorentzian space forms $\mathbf{R}^4_1$, $\mathbf{S}^4_1$ and $\mathbf{H}^4_1$ those surfaces are explicitly constructed and classified.

Contents

1 Introduction

The aim of this paper is the investigation of spacelike surfaces in Lorentzian 4-manifolds with the aid of a so-called twistor construction in 4-dimensional Lorentzian geometry. The idea of the concept, that is presented here, comes from Riemannian twistor theory and its well known applications to the theory of surfaces in 4-dimensional Riemannian manifolds. We give a short description of this Riemannian construction.

Consider an oriented, 4-dimensional Riemannian manifold (M^4, g). The tangent space at any point of the manifold M is isometric to the Euclidean 4-space $\mathbf{R}^4$. The set of complex structures on $\mathbf{R}^4$ can be identified with the homogenous space $\mathbf{GL}(4,\mathbf{R})/\mathbf{GL}(2,\mathbf{C})$. There are two kinds of orthogonal complex structures on $\mathbf{R}^4$, we set $J_o := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{gl}(2,\mathbf{R})$:

$$A_+ := \{J \in \mathbf{SO}(4) | \ \exists\, A \in \mathbf{SO}(4) \text{ with } J = A^{-1} \begin{pmatrix} J_o & 0 \\ 0 & J_o \end{pmatrix} A\},$$

$$A_- := \{J \in \mathbf{SO}(4) | \ \exists\, A \in \mathbf{SO}(4) \text{ with } J = A^{-1} \begin{pmatrix} J_o & 0 \\ 0 & -J_o \end{pmatrix} A\}.$$

Both sets of orthogonal complex structures are naturally identified with the homogenous space $\mathbf{SO}(4)/\mathbf{U}(2)$, which is the 2-sphere $\mathbf{S}^2$. We choose the set A_- and define the twistor space $\mathcal{A}_-(M)$ of M^4 due to Atiyah/Hitchin/Singer [AHS78] as the associated fibre bundle

$$\mathcal{A}_-(M) := SO(M) \times_{\mathbf{SO}(4)} A_- = SO(M) \times_{\mathbf{SO}(4)} \mathbf{S}^2$$

over M^4 consisting of orthogonal complex structures on TM. $SO(M)$ denotes thereby the bundle of positive oriented orthonormal frames on M. The twistor space $\mathcal{A}_-(M)$ admits two natural almost complex structures J_1 and J_2. The first almost complex structure J_1 on $\mathcal{A}_-(M)$ is integrable if and only if the Riemannian 4-manifold M^4 is self-dual. The second complex structure J_2 is never integrable.

Consider now a conformal immersion $f : N^2 \hookrightarrow M^4$ of a Riemannian surface N^2 into the Riemannian 4-space M^4. The image $df(T_nN)$ of the tangent space at any point $n \in N$ is an oriented 2-plane in the Euclidean vector space $T_{f(n)}M$. Any oriented 2-plane V^2 in the oriented, 4-dimensional Euclidean space $\mathbf{R}^4$ gives naturally rise to an orthogonal complex structure on $\mathbf{R}^4$, which is the rotation around the angle $\frac{\pi}{2}$ in positive direction on V and negative direction on $V^\perp$ and which is an element of the set A_-. Hence the 2-plane $df(T_nN)$ in $T_{f(n)}M$ corresponds uniquely to an element J_n in the fibre of the twistor space $\mathcal{A}_-(M)$ over the point $f(n)$. This gives rise to a natural lift of the immersion f to the twistor space $\mathcal{A}_-(M)$:

$$\gamma_f : \begin{array}{ccc} N & \to & \mathcal{A}_-(M) \\ n & \mapsto & J_n \end{array}, \qquad \begin{tikzcd} & \mathcal{A}_-(M) \arrow[d] \\ f : N \arrow[r] \arrow[ur, "\gamma_f"] & M \end{tikzcd} .$$

J. Eells and S. Salamon studied this 'Gauss' lift in [ES85] and proved the following relation for minimal surfaces.

Theorem 1.1 *Conformally immersed minimal surfaces in a Riemannian 4-space M^4 correspond bijectively to non-vertical, J_2-holomorphic curves in the twistor space $\mathcal{A}_-(M)$ over M^4.*

In particular, the lift of a conformally immersed surface to the twistor space is horizontal iff the immersed surface is superminimal. This means that in the case of a self-dual Riemannian 4-space M^4, superminimal surfaces can be constructed by horizontal holomorphic curves in the twistor space, which is a complex manifold. Those constructions of superminimal surfaces have been done by Th. Friedrich in [Fri84] and especially for the hyperbolic four-space in [Fri97]. By this way a well known result is obtained, which says that a superminimal surface in the Euclidean 4-space has locally the form

$$\begin{array}{ccc} \mathbf{C} & \to & \mathbf{C} \times \mathbf{C} \cong \mathbf{R}^4 \\ z & \mapsto & (z, f(z)) \end{array},$$

where f is a holomorphic function. Using the twistor space $\mathbf{P}^3(\mathbf{C})$ of the sphere $\mathbf{S}^4$, R. Bryant proved in [Bry82] the global result that any Riemannian surface admits a conformal, superminimal immersion into $\mathbf{S}^4$.

The idea of twistor theory and its applications to surface theory in Lorentzian geometry is as follows. The twistor space $\mathcal{Z}(M_1^4)$ of an oriented Lorentzian 4-manifold M_1^4 is the bundle of null directions in the tangent space TM over M (comp. [Nur96]):

$$\mathcal{Z}(M_1^4) := SO(M_1^4) \times_{\mathbf{SO}(3,1)} P,$$

where P is the space of null directions in the Minkowski space $\mathbf{R}_1^4$. Instead of almost complex structures, the twistor space $\mathcal{Z}(M_1^4)$ admits natural almost optical structures, which are related to CR-structures.

Consider the conformal immersion f of a Riemannian surface N^2 into an oriented Lorentzian 4-space M_1^4,

$$f : N^2 \hookrightarrow M_1^4.$$

In any point $f(n) \in M_1^4$ of the immersed surface, there is an ordered pair of normal null directions on the tangent space $df(T_nN)$ to the surface in M_1^4. By choosing one of these normal null directions, we obtain a natural lift of the immersion f to the bundle of null directions $\mathcal{Z}(M_1^4)$,

$$\begin{array}{ccc} & & \mathcal{Z}(M_1^4) \\ & \overset{\gamma_f}{\nearrow} & \downarrow \\ f : N^2 & \to & M_1^4 \end{array} .$$

Similar as in 4-dimensional Riemannian geometry, spacelike surfaces in M_1^4 with special geometric properties can be characterized and constructed by holomorphic curves in the almost optical manifold $\mathcal{Z}(M_1^4)$.

The paper is organized in two parts. In the first part, section 2, we recall the basic facts of optical geometry. The twistor space and the Grassmannian fibre bundle of a Lorentzian 4-space with its natural almost optical structures are investigated. The twistor spaces with natural optical structure of the Minkowski space $\mathbf{R}_1^4$, the pseudosphere $\mathbf{S}_1^4$ and the pseudohyperbolic space $\mathbf{H}_1^4$ are discussed. Special attention is given to the underlying CR-manifolds. The induced CR-structure on the restriction $\mathcal{Z}(M_1^4)|_P$ of the twistor space to a spacelike hypersurface P^3 in the Lorentzian 4-space M_1^4 is integrable. Moreover, if the spacelike hypersurface P^3 is totally umbilic in M_1^4, the restriction $\mathcal{Z}(M_1^4)|_P$ of the twistor bundle to P^3 is equivalent as CR-manifold to the unit sphere bundle $S^2(TP)$ over P^3.

In the second part, section 3, we study the second fundamental form of an isometrically immersed spacelike surface (N^2, h) in a Lorentzian 4-space M_1^4. The second fundametal form II decomposes naturally to

$$II = h \otimes H_+ + h \otimes H_- + L_+ + L_-,$$

where H_+ and H_- are the lightlike parts of the mean curvature. It tourns out that the vanishing of components in this decomposition of the second fundamental form is related to the holomorphicity of the lift of the immersion to the twistor space and the Grassmannian fibre bundle. Non-vertical, holomorphic curves in the twistor space over a Lorentzian 4-space M_1^4 projects to semi-umbilic ($L_+ = 0$ or $L_- = 0$) surfaces in M_1^4 and to surfaces that have mean curvature of vanishing length. For conformally flat Lorentzian 4-spaces, we describe locally any semi-umbilic surface. Moreover for the three Lorentzian space forms $\mathbf{R}_1^4$, $\mathbf{S}_1^4$ and $\mathbf{H}_1^4$, we construct explicitly any semi-umbilic surface, which is in addition stationary or has mean curvature of vanishing length.

2 Lorentzian twistor space and optical geometry

2.1 Optical geometry and CR-geometry

In this section, we recall some basic facts about CR-geometry and optical geometry. Special attention is given to the relation between them. Optical geometry is introduced by A. Trautman in [Tra85] and is also treated in [Nur96].

We fix two notations. Let $M_1^n := (M^n, g)$ denote an n-dimensional Lorentzian C^∞-manifold, where 'Lorentzian' means that the smooth metric tensor g is of signature $(n-1, 1)$, i.e. the tangent space (T_mM, g_m) at any point $m \in M$ is isometric to the n-dimensional Minkowski space $\mathbf{R}_1^n := (\mathbf{R}^n, <,>_1^n)$. For any oriented semi-Riemannian manifold (M^n, g) of signature $(n-r, r)$, we denote by $SO(M)$ the $\mathbf{SO}(n-r, r)$-principal fibre bundle of oriented (pseudo)-orthonormal frames over M.

Consider a $2r$-dimensional C^∞-distribution $\mathcal{H} \subset TM^n$, $2r \leq n$, on an n-dimensional C^∞-manifold M. A bundle morphism

$$J : \mathcal{H} \to \mathcal{H}, \quad J^2 = -id,$$

is called an almost complex structure on the subbundle $\mathcal{H} \subset TM$. In case that $\mathcal{H} = TM$, $2r = n$, such a bundle morphism J on $\mathcal{H}$ is called an almost complex structure on the C^∞-manifold M. The almost complex structure J on TM induces a complex structure on M iff J is integrable, i.e. the Nijenhuis tensor vanishes:

$$N(X,Y) := J([X,JY] + [JX,Y]) - [JX,JY] + [X,Y] = 0 \quad \forall X,Y \in \Gamma(TM).$$

In case that an almost complex structure J is given on a distribution $\mathcal{H} \subset TM^{2n-1}$ with codimension 1 in TM, the pair $(\mathcal{H}, J)$ is called an almost CR-structure on the $(2n-1)$-dimensional C^∞-manifold M^{2n-1}. An almost CR-structure $(\mathcal{H}, J)$ on M^{2n-1} is called integrable or just a CR-structure on M iff

$$[JX,Y] + [X,JY] \in \Gamma(\mathcal{H}) \quad \forall X,Y \in \Gamma(\mathcal{H}) \quad \text{and}$$

$$J([X,JY] + [JX,Y]) - [JX,JY] + [X,Y] = 0 \quad \forall X,Y \in \Gamma(\mathcal{H}),$$

i.e. the Nijenhuis tensor vanishes on $\mathcal{H}$.

Remark 2.1 *1. If a subbundle $S \subset \mathcal{H}$ is an involutive distribution such that $J|_S$ maps again into S, the integral manifolds of S are almost complex submanifolds of the almost CR-manifold $(M, \mathcal{H}, J)$.*

2. An almost CR-structure $(\mathcal{H}, J)$ on a Riemannian manifold (M^{2n-1}, g) is orthogonal iff J on $\mathcal{H}$ is orthogonal with respect to g on $\mathcal{H}$.

3. A C^∞-map f between two almost CR-manifolds $(M, \mathcal{H}, J)$ and $(\tilde{M}, \tilde{\mathcal{H}}, \tilde{J})$ is called a CR-map iff

$$df(\mathcal{H}) \subset \tilde{\mathcal{H}}, \quad df \circ J = \tilde{J} \circ df.$$

Example 2.1 *Natural CR-structure on the unit sphere bundle of a 3-dimensional oriented Riemannian manifold*

Consider a 3-dimensional, oriented Euclidean vector space $(V^3, <,>)$. Any unit vector $l \in S^2(V^3)$ gives rise to a 2-dimensional subspace of V^3 with a complex structure on it, namely the 2-dimensional subspace is the orthogonal complement of l,

$$H_l := (\mathbf{R}l)^\perp,$$

and the complex structure on H_l is given by

$$J_l(h_1) = h_2, \quad J_l(h_2) = -h_1,$$

where (l, h_1, h_2) is a positive oriented orthonormal basis in V^3.

Let (N^3, h) be a 3-dimensional, oriented Riemannian manifold. The unit sphere bundle of N^3 is given by

$$\begin{aligned} S^2(TN) &:= SO(N) \times_{\mathbf{SO}(3)} \mathbf{S}^2 = SO(N) \times_{\mathbf{SO}(3)} \mathbf{SO}(3)/\mathbf{SO}(2) \\ &= SO(N)/\mathbf{SO}(2) \subset TN. \end{aligned}$$

Let $\pi : SO(N)/\mathbf{SO}(2) \to N$ denote the natural projection. The Levi-Civita connection of (N^3, h) decomposes the tangent bundle $TS^2(TN)$ into a horizontal and a vertical part:

$$TS^2(TN) = T^V S^2(TN) \oplus T^H S^2(TN).$$

On $S^2(TN)$ exists a natural smooth distribution $\mathcal{H}^{S^2(TN)} \subset TS^2(TN)$ of codimension 1 given in a point $l \in S^2(TN)$ by

$$\mathcal{H}_l^{S^2(TN)} = \pi_*^{-1}(H_l) = T^V S^2(TN) \oplus (T^H S^2(TN) \cap \pi_*^{-1}(H_l)).$$

Let $J^{\mathbf{S}^2}$ denote the standard $\mathbf{SO}(3)$-invariant complex structure on $\mathbf{S}^2$. It exists a natural complex structure on the distribution $\mathcal{H}^{S^2(TN)} \subset TS^2(TN)$ defined by

$$J^{S^2(TN)} = \pi_*^{-1} \circ J_l \circ \pi_* + [e]^{-1} \circ J^{\mathbf{S}^2} \circ [e],$$

where $[e]$ denotes the identification of fibres with $\mathbf{S}^2$ by an orthonormal frame e and $\pi_*^{-1} \circ J_l \circ \pi_*$ is the horizontal lift of J_l on $H_l \subset T_{\pi(l)}N$. It is easy to prove that this CR-structure $(\mathcal{H}^{S^2(TN)}, J^{S^2(TN)})$ on $S^2(TN)$ is always integrable. Furthermore, the unit sphere bundle together with its natural CR-structure is conformally invariant.

In a later section, we will explicitly discuss the unit sphere bundle with natural CR-structure of the three space forms $\mathbf{R}^3$, $\mathbf{S}^3$ and $\mathbf{H}^3$.

An almost optical structure $\mathcal{O} = (\mathcal{K}, \mathcal{L}, J)$ on a $2n$-dimensional C^∞-manifold M consists of distributions $\mathcal{K}$ and $\mathcal{L}$, where $\mathcal{K} \subset \mathcal{L} \subset TM$ are subbundles, $\dim \mathcal{K} = 1$, $\dim \mathcal{L} = 2n-1$, and an almost complex structure J on the quotient bundle $\mathcal{L}/\mathcal{K}$. An almost optical structure $\mathcal{O} = (\mathcal{K}, \mathcal{L}, J)$ is by definition integrable iff

(A)
$$[\Gamma(\mathcal{K}), \Gamma(\mathcal{L})] \subset \Gamma(\mathcal{L}), \qquad \phi_t^{k*} J = J \quad \forall k \in \Gamma(\mathcal{K}),$$

where ϕ_t^k denotes the flow of the field k, that means, locally in every point $m \in M$ the almost optical structure $\mathcal{O}$ may be pushed down to an almost CR-structure $(\mathcal{H}^m, J^m)$ on a locally induced quotient manifold $U_m/{\sim^{\mathcal{K}}}$, where $U_m \subset M$ is a suitable neighborhood of $m \in M$ and any two points $u_1, u_2 \in U_m$ are $\sim^{\mathcal{K}}$-related iff both belong to the same integral curve of the distribution $\mathcal{K}$,

(B) these locally induced CR-structures $(\mathcal{H}^m, J^m)$, $m \in M$, are integrable (comp. [Nur96], [Lei98]).

Remark 2.2 *1. On a Lorentzian manifold $M_1^{2n} := (M^{2n}, g)$, the optical structure $\mathcal{O} = (\mathcal{K}, \mathcal{L}, J)$ is called orthogonal if $\mathcal{K}$ is a null subbundle, $\mathcal{L} = \mathcal{K}^\perp$ is the orthogonal complement and J is orthogonal with respect to the Riemannian metric on $\mathcal{L}/\mathcal{K}$ induced by g.*

2. A C^∞-map $f : M \to \tilde{M}$ between two almost optical manifolds $(M, \mathcal{K}, \mathcal{L}, J)$ and $(\tilde{M}, \tilde{\mathcal{K}}, \tilde{\mathcal{L}}, \tilde{J})$ is called an optical map iff

$$df(\mathcal{K}) \subset \tilde{\mathcal{K}}, \quad df(\mathcal{L}) \subset \tilde{\mathcal{L}}, \quad df \circ J = \tilde{J} \circ df.$$

3. An almost CR-manifold $(N^{2n-1}, \mathcal{H}, J)$ gives rise to a canonical almost optical structure on the manifold $M := \mathbf{R} \times N$. This almost optical structure on M is defined by

$$\mathcal{K}^M := T\mathbf{R}, \quad \mathcal{L}^M := T\mathbf{R} \oplus \mathcal{H}, \quad J^M \cong J : \; \mathcal{L}/\mathcal{K} \cong \mathcal{H} \to \mathcal{L}/\mathcal{K} \cong \mathcal{H}.$$

4. If an almost optical structure $(\mathcal{K}, \mathcal{L}, J)$ on M is integrable or just satisfies condition A, the locally induced almost CR-structure on $U_m/{\sim^{\mathcal{K}}}$, $m \in M$, is given by

$$\mathcal{H}^m := \tilde{\pi}_*(\mathcal{L}), \quad J^m := \tilde{\pi}_* \circ J \circ \tilde{\pi}_*^{-1},$$

where $\tilde{\pi} : U_m \to U_m/{\sim^{\mathcal{K}}}$ is the natural projection. The open neighborhood $U_m \subset M$ of $m \in M$ may be chosen in such a way that $(U_m, \mathcal{K}|_{U_m}, \mathcal{L}|_{U_m}, J|_{U_m})$ is optical diffeomorphic to $\mathbf{R} \times U_m/{\sim^{\mathcal{K}}}$ with the natural almost optical structure given as above.

5. *Let $N^{2n-1} \subset M^{2n}$ be a submanifold of codimension 1 in the almost optical manifold $(M, \mathcal{K}, \mathcal{L}, J)$ such that*
$$TN \oplus \mathcal{K}|_N = TM|_N.$$
It follows that $\mathcal{H}^N := TN \cap \mathcal{L}|_N \subset TN$ is a distribution in TN with codimension 1 and $\mathcal{H}^N$ is naturally identified with $\mathcal{L}/\mathcal{K}|_N$. Therefore, J induces an almost complex structure J^N on $\mathcal{H}^N \subset TN$. The pair $(\mathcal{H}^N, J^N)$ is a naturally induced almost CR-structure on $N^{2n-1} \subset M^{2n}$. We say that $(N, \mathcal{H}^N, J^N)$ is an almost CR-submanifold of the almost optical manifold $(M, \mathcal{K}, \mathcal{L}, J)$.

6. *Let $(N^{2n-1}, \mathcal{H}^N, J^N)$ be an almost CR-submanifold of the almost optical manifold $(M^{2n}, \mathcal{K}, \mathcal{L}, J)$. If $(\mathcal{K}, \mathcal{L}, J)$ is integrable or just satisfies condition A, the almost CR-structure $(\mathcal{H}^N, J^N)$ in $m \in N \subset M$ is locally equivalent to the naturally induced almost CR-structure $(\mathcal{H}^m, J^m)$ on the quotient manifold $U_m/{\sim^{\mathcal{K}}}$ in the point $\{m\} \in U_m/{\sim^{\mathcal{K}}}$. In case that condition A is not satisfied, the almost CR-structure that is induced on a submanifold*
$$N^{2n-1} \subset (M^{2n}, \mathcal{K}, \mathcal{L}, J), \quad TN \oplus \mathcal{K}|_N = TM|_N,$$
depends on the imbedding of N in M.

7. *Let $(\mathcal{K}, \mathcal{L}, J)$ be an almost orthogonal optical structure on a Lorentzian manifold M_1^{2n}. For any submanifold $N^{2n-1} \subset M_1^{2n}$ of codimension 1, whose induced metric $g|_{TN}$ is positive definite (spacelike), it holds*
$$TN \cap \mathcal{K}|_N = \{0\}.$$
Therefore, the almost orthogonal optical structure $(\mathcal{K}, \mathcal{L}, J)$ on M induces an almost orthogonal CR-structure $(\mathcal{H}^N, J^N)$ on N.

8. *Let $N^{2r} \subset (M^{2n}, \mathcal{K}, \mathcal{L}, J)$, $r < n$, be a submanifold such that*
$$TN \cap \mathcal{K}|_N = 0 \quad \text{and} \quad TN \subset \mathcal{L}|_N.$$
Then, the tangent bundle TN is naturally identified with a subbundle of $\mathcal{L}/\mathcal{K}|_N$. We may write
$$TN \subset \mathcal{L}/\mathcal{K}|_N.$$
If in addition J on $\mathcal{L}/\mathcal{K}$ ristricts to an almost complex structure J^N on TN, we may say that (N, J^N) is an almost complex submanifold of the almost optical manifold $(M, \mathcal{K}, \mathcal{L}, J)$.

9. *An 1-dimensional distribution $\mathcal{K}$ on a manifold M is called regular if there is a smooth differentiable structure on the quotient set $M/{\sim^{\mathcal{K}}}$ such that $\tilde{\pi} : M \to M/{\sim^{\mathcal{K}}}$ is a C^∞-submersion. If $(M, \mathcal{K}, \mathcal{L}, J)$ is an optical manifold, $M/{\sim^{\mathcal{K}}}$ admits globally a natural CR-structure.*

Until now, we have defined complex, CR- and optical structures. We want to consider mappings between them.

Definition 2.1 *Let (Q, J^Q) denote an almost complex manifold, $(N, \mathcal{H}^N, J^N)$ an almost CR-manifold and $(M, \mathcal{K}^M, \mathcal{L}^M, J^M)$ an almost optical manifold.*

1. *A C^∞-map $f : Q \to N$ is called holomorphic if*
$$df(TQ) \subset \mathcal{H}^N \quad \text{and} \quad df \circ J^Q = J^N \circ df.$$

2. *A C^∞-map $g : Q \to M$ is called holomorphic if*
$$dg(TQ) \subset \mathcal{L}^M, \quad dg(TQ) \cap \mathcal{K} = 0, \quad dg \circ J^Q = J^M \circ \hat{\pi} \circ dg,$$
where $\hat{\pi} : \mathcal{L} \to \mathcal{L}/\mathcal{K}$ is the natural projection.

3. A C^∞-map $h : N \to M$ is called holomorphic if

$$dh(\mathcal{H}^N) \subset \mathcal{L}^M, \quad dh(\mathcal{H}^N) \cap \mathcal{K}^M = 0 \quad \text{and} \quad dh \circ J^N = J^M \circ \hat{\pi} \circ df.$$

In the notations of the previous definition, we have

Proposition 2.1 *1. If the mappings $f : Q \to N$ and $h : N \to M$ are holomorphic, the map $g := h \circ f : Q \to M$ is also holomorphic.*

2. If $f : Q \to M$ is holomorphic and $f(Q) \subset N \subset M$, where N is an almost CR-submanifold of M, then the map $f : Q \to N$ is holomorphic. In case that $(\mathcal{K}^M, \mathcal{L}^M, J^M)$ on M satisfies condition A and $f : Q \to M$ is holomorphic, at least locally the map $\tilde{\pi} \circ f$ is holomorphic, i.e. for an open subset $U \subset Q$ with $f(U) \subset U_m$, $m \in U_m \subset M$, the map

$$\tilde{\pi} \circ f : U \subset Q \to U_m/{\sim^{\mathcal{K}}}$$

is holomorphic.

3. Let $(\mathcal{K}^M, \mathcal{L}^M, J^M)$ on M be integrable and let $k \in \Gamma(\mathcal{K}^M)$ be a vector field on M. If the flow ϕ_t^k of k is defined on $]-\epsilon, \epsilon[\times U$, where $\epsilon > 0$ and $U \subset M$ is an open subset, and if $g : Q \to M$ is holomorphic, then the map

$$g_t := \phi_t^k \circ g|_{g^{-1}(U)} : g^{-1}(U) \subset Q \to M$$

is holomorphic for any $t \in]-\epsilon, \epsilon[$.

The proof of this is obvious. In a way one can say, that a holomorphic map $g : (Q, J^Q) \to (M^{2n}, \mathcal{K}^M, \mathcal{L}^M, J^M)$ into an optical manifold corresponds to a holomorphic map $f : (Q, J^Q) \to (N^{2n-1}, \mathcal{H}^N, J^N)$ up to an arbitrary flow of the holomorphic map g along the integral curves of the distribution $\mathcal{K}^M$.

2.2 Twistor space of a Lorentzian manifold M_1^4

The twistor space of an oriented Lorentzian 4-manifold M_1^4 is defined to be the fibre bundle of null directions in the tangent space TM_1^4 (comp. [Nur96], [Lei98]). The twistor space admits natural almost optical structures. The integrability, the conformal invariance and the underlying CR-hypersurfaces of these optical structures are investigated in this section. We begin with the discussion of the fibre type of the twistor bundle.

Consider the Minkowski space $\mathbf{R}_1^4$ and denote by P the set of null directions in $\mathbf{R}_1^4$. The space P is a submanifold of the projective space $\mathbf{P}^3(\mathbf{R})$ and is defined to be the fibre type of the twistor space of an oriented Lorentzian 4-manifold. There are several characterizations of the fibre P. For the first, the fibre P may be written as homogenous space. The Lorentzian group $\mathbf{SO}(3,1)$ acts transitively on P. Let H_+ and H_- denote the isotropy groups of this action resp. in $o = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $o = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$. We obtain $P \cong \mathbf{SO}(3,1)/H_+ \cong \mathbf{SO}(3,1)/H_-$. The Lie algebra of the group $\mathbf{SO}(3,1)$ is given by

$$\mathfrak{o}(3,1) = \{A \in \mathfrak{gl}(4, \mathbf{R}) : \; A \begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix} A^\perp = 0\}.$$

The elements

$$\{E_{ij} : \; i, j = 1, \ldots, 4, \; i < j\}, \qquad E_{ij} := \begin{pmatrix} \cdot & -\epsilon_j \\ \epsilon_i & \cdot \end{pmatrix},$$

form a basis of $\mathfrak{o}(3,1)$. The Lie algebras $\mathfrak{h}_+$ and $\mathfrak{h}_-$ of H_+ resp. H_- are then given by

$$\mathfrak{h}_+ = Span\{E_{12}, E_{13} + E_{23}, E_{14} + E_{24}, E_{34}\}, \quad \mathfrak{h}_- = Span\{E_{12}, E_{13} - E_{23}, E_{14} - E_{24}, E_{34}\}.$$

It is

$$\mathfrak{o}(3,1) = \mathfrak{m}_+ \oplus \mathfrak{h}_+ = \mathfrak{m}_- \oplus \mathfrak{h}_-,$$

$$\mathfrak{m}_+ = Span\{E_{13} - E_{23}, E_{14} - E_{24}\}, \quad \mathfrak{m}_- = Span\{E_{13} + E_{23}, E_{14} + E_{24}\}.$$

These decompositions of the Lie algebra $\mathfrak{o}(3,1)$ are not reductive.

The space P of null directions in $\mathbf{R}^4_1$ is also naturally identified with the positive resp. negative projective spinor modul. For this let $\Delta_+ \cong \mathbf{C}^2$ and $\Delta_- \cong \mathbf{C}^2$ denote the positive resp. negative spinor modul (comp. [BFGK 91]). To any spinor $\psi_\pm \in \Delta_\pm$ corresponds the $\mathbf{R}$-linear mapping

$$\begin{array}{rccc} \hat{\psi}_\pm : & \mathbf{R}^4_1 & \to & \Delta_\pm \ , \\ & x & \mapsto & x \cdot \psi_\pm \end{array}$$

where $x \cdot \psi_\pm$ is the Clifford product. The kernel $\ker \hat{\psi}_\pm$ of this mapping is a null direction in $\mathbf{R}^4_1$. Obviously, for $\phi_\pm = c \cdot \psi_\pm$, $0 \neq c \in \mathbf{C}$, it holds

$$\ker \hat{\phi}_\pm = \ker \hat{\psi}_\pm.$$

Moreover, the mapping

$$\begin{array}{rccc} \iota : & \mathbf{P}(\Delta_\pm) & \to & P \quad , \\ & [\psi_\pm] & \mapsto & \ker \hat{\psi}_\pm, \quad \psi_\pm \in [\psi_\pm] \end{array}$$

is a diffeomorphism. The natural complex structures $J^{\mathbf{P}(\Delta_\pm)}$ on $P \cong \mathbf{P}(\Delta_\pm) \cong \mathbf{P}^1(\mathbf{C})$ are invariant by the $\mathbf{SO}(3,1)$-action on P and are given on $\mathfrak{m}_\pm \cong T_o P$ by

$$J^{\mathbf{P}(\Delta_+)}(E_{13} - E_{23}) = -(E_{14} - E_{24}), \quad J^{\mathbf{P}(\Delta_+)}(E_{14} - E_{24}) = E_{13} - E_{23},$$

$$J^{\mathbf{P}(\Delta_-)}(E_{13} + E_{23}) = -(E_{14} + E_{24}), \quad J^{\mathbf{P}(\Delta_-)}(E_{14} + E_{24}) = E_{13} + E_{23}.$$

The identification of $\mathbf{P}(\Delta_+)$ and $\mathbf{P}(\Delta_-)$ via the space of null directions P is anti-holomorphic.

A further description of P is the identification with the sphere $\mathbf{S}^2$ by

$$\begin{array}{rccc} \iota : & P & \leftrightarrow & \mathbf{S}^2 \quad . \\ & \left[\begin{pmatrix} 1 \\ a \\ b \\ c \end{pmatrix}\right] & \mapsto & \begin{pmatrix} a \\ b \\ c \end{pmatrix} \end{array}$$

More generally, any unit timelike vector $T \in \mathbf{R}^4_1$, $g(T,T) = -1$, gives an identification of the space of null directions P and the 2-dimensional sphere $S^2(T^\perp) \cong \mathbf{S}^2$ in the orthogonal complement $T^\perp \cong \mathbf{R}^3$:

$$\begin{array}{rccc} \iota : & P & \leftrightarrow & \mathbf{S}^2 \quad . \\ & \mathbf{R}(T+S) & \mapsto & S \end{array}$$

The positive resp. negative twistor space of a 4-dimensional, oriented Lorentzian manifold $M^4_1 := (M^4, g)$ is defined to be the positive resp. negative projective spinor bundle

$$\mathcal{Z}_+(M^4_1) := P(S_+) = SO(M^4_1) \times_{\mathbf{SO}(3,1)} \mathbf{P}(\Delta_+),$$

$$\mathcal{Z}_-(M^4_1) := P(S_-) = SO(M^4_1) \times_{\mathbf{SO}(3,1)} \mathbf{P}(\Delta_-).$$

Let $\pi : \mathcal{Z}_\pm(M) \to M$ be the natural projection. We may these bundles also interpret as the bundle of null directions in TM^4_1,

$$\mathcal{Z}_\pm(M) = SO(M) \times_{\mathbf{SO}(3,1)} P = SO(M) \times_{\mathbf{SO}(3,1)} \mathbf{SO}(3,1)/H_\pm.$$

On the twistor space $\mathcal{Z}_+(M)$ are given the natural almost optical structures

$$\mathcal{O}^+_+ = (\mathcal{K}_+, \mathcal{L}_+, J^+_+) \quad \text{and} \quad \mathcal{O}^-_+ = (\mathcal{K}_+, \mathcal{L}_+, J^-_+).$$

They are defined pointwise as follows. Let $[\psi_+] \in \mathcal{Z}_+(M)$ be an arbitrary point. For a suitable orthonormal basis $s = (s_1, \ldots, s_4) \in SO(M)$ we may write

$$[\psi_+] = [s, \left[\left(\begin{smallmatrix} 1 \\ 1 \\ 0 \\ 0 \end{smallmatrix} \right) \right]] \in \mathcal{Z}_+(M).$$

To $[\psi_+] \in \mathcal{Z}_+(M)$ corresponds the orthogonal optical structure

$$\mathcal{O}^{[\psi+]} = (K^{[\psi+]}, L^{[\psi+]}, J^{[\psi+]})$$

in $T_{\pi([\psi_+])}M$ that is defined by

$$K^{[\psi+]} = \mathbf{R}(s_1 + s_2) \subset T_{\pi([\psi_+])}M,$$

$$L^{[\psi+]} = Span\{s_1 + s_2, s_3, s_4\} \subset T_{\pi([\psi_+])}M \quad \text{and}$$

$$J^{[\psi+]}(s_3 + K^{[\psi+]}) = s_4 + K^{[\psi+]}, \qquad J^{[\psi+]}(s_4 + K^{[\psi+]}) = -s_3 + K^{[\psi+]}.$$

The almost optical structure $\mathcal{O}_+^+ = (\mathcal{K}_+, \mathcal{L}_+, J_+^+)$ is given in $T_{[\psi_+]}\mathcal{Z}_+(M)$ by

$$\mathcal{O}^+_{+[\psi_+]} = \pi_*^{-1} \circ \mathcal{O}^{[\psi+]} \circ \pi_* - [s]^{-1} \circ J^{\mathbf{P}(\Delta_+)} \circ [s],$$

where $\pi_*^{-1} \circ \mathcal{O}^{[\psi+]} \circ \pi_*$ denotes the horizontal lift of the optical structure $\mathcal{O}^{[\psi+]}$ in $T_{\pi([\psi_+])}M$ to $T^H_{[\psi_+]}\mathcal{Z}_+(M)$. The almost optical structure $\mathcal{O}_+^-$ is given in $T_{[\psi_+]}\mathcal{Z}_+(M)$ by

$$\mathcal{O}^-_{+[\psi_+]} = \pi_*^{-1} \circ \mathcal{O}^{[\psi+]} \circ \pi_* + [s]^{-1} \circ J^{\mathbf{P}(\Delta_+)} \circ [s].$$

Analogously, we may define the almost optical structures $\mathcal{O}_-^+ = (\mathcal{K}_-, \mathcal{L}_-, J_-^+)$ and $\mathcal{O}_-^- = (\mathcal{K}_-, \mathcal{L}_-, J_-^-)$ on $\mathcal{Z}_-(M)$. Under the identification

$$\Phi : \mathcal{Z}_+(M) \cong SO(M) \times_{\mathbf{SO}(3,1)} P \cong \mathcal{Z}_-(M)$$

of the positive and negative twistor space, the almost optical structures $\mathcal{O}_+^+$ and $\mathcal{O}_-^+$ are related by

$$\Phi^*(\mathcal{K}_-, \mathcal{L}_-, J_-^+) = (\mathcal{K}_+, \mathcal{L}_+, -J_+^+).$$

Theorem 2.1 *(comp. [Nur96], [Lei98]) Let M_1^4 be an oriented 4-dimensional Lorentzian manifold and let $\mathcal{Z}_+(M)$ and $\mathcal{Z}_-(M)$ be its twistor spaces.*

1. *The almost optical structures $\mathcal{O}_+^+$ on $\mathcal{Z}_+(M)$ and $\mathcal{O}_-^+$ on $\mathcal{Z}_-(M)$ are integrable iff M_1^4 is conformally flat.*

2. *The almost optical structures $\mathcal{O}_+^-$ and $\mathcal{O}_-^-$ are never integrable.*

Remark 2.3 1. *The almost optical structure $\mathcal{O}_+^+$ is integrable in $[\psi_+] = [s, \left[\left(\begin{smallmatrix} 1 \\ 1 \\ 0 \\ 0 \end{smallmatrix} \right) \right]] \in \mathcal{Z}_+(M)$ iff $\mathbf{R}(s_1 + s_2)$ is a principal null direction (comp. [O'N93], [Nur96]). In terms of the Riemannian curvature tensor R, the integrability condition in this point is equivalent to*

$$R_{1413} + R_{2413} + R_{1423} + R_{2423} = 0,$$

$$R_{1414} + R_{2414} + R_{1424} + R_{2424} - R_{1313} - R_{2313} - R_{1323} - R_{2323} = 0.$$

2. *The almost optical structures $\mathcal{O}_+^-$ and $\mathcal{O}_-^-$ are not only non-integrable, even more they don't induce locally any almost CR-structure.*

We consider the conformal invariance of the twistor spaces with its almost optical structures. Let $\tilde{g} = \exp(2\rho)g$, $\rho : M^4 \to \mathbf{R}$ a smooth function, be a conformally equivalent metric to g on M^4. The twistor spaces $\mathcal{Z}_\pm(M, g)$ and $\mathcal{Z}_\pm(M, \tilde{g})$ are naturally identified by

$$\begin{array}{cccc} \Theta : & \mathcal{Z}_\pm(M,g) & \cong & \mathcal{Z}_\pm(M,\tilde{g}) \quad , \\ & [s \cdot A, \mathbf{R}(u_1 \pm u_2)] & \leftrightarrow & [\exp(-\rho)s \cdot A, \mathbf{R}(u_1 \pm u_2)] \end{array}$$

where $\{u_i : i = 1, \ldots, 4\}$ denotes the standard basis in $\mathbf{R}^4_1$.

Theorem 2.2 *Let (M^4, g) be an oriented Lorentzian 4-manifold and let $\tilde{g} = \exp(2\rho)g$ be a conformally equivalent metric to g. The identification Θ of the twistor spaces $(\mathcal{Z}_\pm(M^4, g), \mathcal{O}^+_\pm)$ and $(\mathcal{Z}_\pm(M^4, \tilde{g}), \mathcal{O}^+_\pm)$ is an optical diffeomorphism.*

Proof: We point out the proof for the positive twistor space $\mathcal{Z}_+(M)$. Consider a smooth lightlike curve in (M^4, g). There is an unique lift of this curve to the twistor space $\mathcal{Z}_+(M)$ in the way, that any point of the curve is lifted to the null direction in $\mathcal{Z}_+(M)$, which is the tangent space to the curve in this point. This lift of a smooth lightlike curve in (M^4, g) to $\mathcal{Z}_+(M)$ is horizontal if and only if the curve is a lightlike geodesic. The construction of the distribution $\mathcal{K}_+$ on $\mathcal{Z}_+(M)$ implies that the lift of a smooth lightlike curve to the twistor space $\mathcal{Z}_+(M)$ is an integral curve of $\mathcal{K}_+$ iff the lift is horizontal, or equivalently, iff the curve is a lightlike geodesic. It is well known that lightlike geodesics are invariant to conformal change of the Lorentzian metric. But this just means that the distribution $\mathcal{K}_+$ is invariant under conformal change, i.e.

$$\Theta^* \mathcal{K}^{\tilde{g}}_+ = \mathcal{K}^g_+.$$

As a direct consequence, we obtain that the optical flag

$$\mathcal{K}_+ \subset \mathcal{L}_+ \subset T\mathcal{Z}_+(M)$$

is conformally invariant. It remains to show that the almost complex structure on the screen space $\mathcal{L}_+/\mathcal{K}_+$ is conformally invariant. For this purpose, let

$$s = (s_1, s_2, s_3, s_4) : U \subset M \to SO(M, g)$$

be a local frame and denote by $\tilde{s} = \exp(-\rho)s$ the conformally changed frame in $SO(M, \tilde{g})$. In the trivialization $U \times \mathbf{P}(\Delta_+)$ of $\mathcal{Z}_+(M)$ given by s, the horizontal lift s_i^H of s_i to $U \times \mathbf{P}(\Delta_+)$ is given by

$$s_i^H = s_i + g(\nabla_{s_i}(s_1 + s_2), s_3)\frac{E_{13} - E_{23}}{2} + g(\nabla_{s_i}(s_1 + s_2), s_4)\frac{E_{14} - E_{24}}{2}.$$

For the conformal change of the Levi-Civita connection holds (comp. [Bes87])

$$\tilde{\nabla}_X Y = \nabla_X Y + d\rho(X)Y + d\rho(Y)X - g(X, Y)grad(\rho).$$

For the horizontal lift $\tilde{s}_i^H$ of $\tilde{s}_i$ with respect to the connection $\tilde{\nabla}$, we have

$$\begin{aligned} \tilde{s}_i^H = \exp(-\rho) \cdot [s_i^H \quad &+ \quad [g((s_1 + s_2)(\rho) \cdot s_i, s_3) - g(s_3(\rho) \cdot s_i, s_1 + s_2)] \frac{E_{13} - E_{23}}{2} \\ &+ \quad [g((s_1 + s_2)(\rho) \cdot s_i, s_4) - g(s_4(\rho) \cdot s_i, s_1 + s_2)] \frac{E_{14} - E_{24}}{2}], \end{aligned}$$

in particular

$$\tilde{s}_3^H = \exp(-\rho) \cdot \left(s_3^H + (s_1 + s_2)(\rho)\frac{E_{13} - E_{23}}{2} \right) \quad \text{and}$$

$$\tilde{s}_4^H = \exp(-\rho) \cdot \left(s_4^H + (s_1 + s_2)(\rho)\frac{E_{14} - E_{24}}{2} \right).$$

It follows

$$J_+^{+g}(\tilde{s}_3^H + \mathcal{K}_+) = \tilde{s}_4^H + \mathcal{K}_+ = J_+^{+\tilde{g}}(\tilde{s}_3^H + \mathcal{K}_+) \quad \text{and}$$
$$J_+^{+g}(\tilde{s}_4^H + \mathcal{K}_+) = -\tilde{s}_3^H + \mathcal{K}_+ = J_+^{+\tilde{g}}(\tilde{s}_4^H + \mathcal{K}_+),$$

i.e. J_+^+ is conformally invariant □

The proof also shows that the almost optical structures $\mathcal{O}_+^-$ and $\mathcal{O}_-^-$ are not conformally invariant.

We are now interested in the underlying CR-hypersurfaces of $(\mathcal{Z}_\pm(M), \mathcal{O}_\pm^+)$. Let $T \in \Gamma(TM)$, $g(T,T) = -1$, be a timelike unit vector field on M_1^4. The choice of such a field T is equivalent to a $\mathbf{SO}(3)$-reduction $SO_T(M)$ of the frame bundle $SO(M)$ over M, i.e. equivalent to an imbedding

$$\iota : SO_T(M) \hookrightarrow SO(M)$$

of a $\mathbf{SO}(3)$-principal fibre bundle over M in $SO(M)$. In particular, the twistor bundle may then be written as

$$\mathcal{Z}_+(M) = SO_T(M) \times_{\mathbf{SO}(3)} P \cong SO_T(M) \times_{\mathbf{SO}(3)} \mathbf{S}^2.$$

The twistor fibre $\pi^{-1}(m)$ over a point $m \in M_1^4$ is identified via $T_m \in T_m M$ with the sphere

$$S^2(T_m^\perp) \subset T_m^\perp \subset T_m M,$$

i.e. we can think of the twistor bundle as the bundle of 2-dimensional unit spheres, which are orthogonal to the timelike vector field $T \in \Gamma(TM)$.

Theorem 2.3 *Let $M_1^4 := (M^4, g)$ be an oriented, 4-dimensional Lorentzian manifold and let $\mathcal{Z}_+(M_1^4)$ be the positive twistor space of M_1^4. Further, let $N^3 \subset M_1^4$ be an oriented, 3-dimensional spacelike submanifold of M_1^4, i.e. the restriction $g|_N$ is positive definite on N^3.*

1. *The restriction $\mathcal{Z}_+(M)|_N$ of the twistor bundle to N^3 is in a natural way identified with the unit sphere bundle $S^2(TN)$ over N,*

$$\Psi : \mathcal{Z}_+(M)|_N \cong S^2(TN).$$

2. *The restricted twistor bundle $\mathcal{Z}_+(M)|_N$ over N admits an almost CR-structure $(\mathcal{H}_+, J_+)$ that is naturally induced by the almost optical structure $\mathcal{O}_+^+$ on $\mathcal{Z}_+(M)$. The almost CR-structure $(\mathcal{H}_+, J_+)$ on $\mathcal{Z}_+(M)|_{N^3}$ is always integrable.*

3. *The induced almost CR-structure $(\mathcal{H}_+, J_+)$ on $\mathcal{Z}_+(M)|_N$ and the natural CR-structure $(\mathcal{H}^{S^2(TN)}, J^{S^2(TN)})$ on the unit sphere bundle $S^2(TN)$ over the Riemannian 3-manifold N are equivalent under the natural identification Ψ if and only if $N^3 \subset M_1^4$ is a totally umbilic hypersurface.*

Remark: An analogous theorem holds for the negative twistor space $(\mathcal{Z}_-(M), \mathcal{O}_-^+)$.

Proof: 1. The spacelike submanifold N is oriented, i.e. there is an unique timelike unit normal field T on N such that (T, s_2, s_3, s_4) is positive oriented if $(s_2, s_3, s_4) \in SO(N)$. The field T induces a $\mathbf{SO}(3)$-reduction of the restricted frame bundle $SO(M)|_N$. Then, we have

$$\begin{aligned} \mathcal{Z}_+(M)|_N &= SO(M)|_N \times_{\mathbf{SO}(3,1)} P \cong SO_T(M)|_N \times_{\mathbf{SO}(3)} S^2(T^\perp) \\ &\cong SO(N) \times_{\mathbf{SO}(3)} \mathbf{S}^2, \end{aligned}$$

which is the natural identification Ψ of the restricted twistor space with the unit sphere bundle $S^2(TN)$ over N.

2. The submanifold $\mathcal{Z}_+(M)|_N \subset \mathcal{Z}_+(M)$ has codimension 1 and obviously, it holds

$$T(\mathcal{Z}_+(M)|_N) \cap \mathcal{K}_+ = \{0\},$$

i.e.

$$T(\mathcal{Z}_+(M)|_N) \oplus \mathcal{K}_+|_{\mathcal{Z}_+(M)|_N} = (T\mathcal{Z}_+(M))|_{\mathcal{Z}_+(M)|_N}.$$

The almost optical structure $\mathcal{O}_+^+$ induces the almost CR-structure $(\mathcal{H}_+, J_+)$ on $\mathcal{Z}_+(M)|_N$ by

$$\mathcal{H}_+ = T(\mathcal{Z}_+(M)|_N) \cap \mathcal{L}_+ \quad \text{and}$$

$$J_+ \cong J_+^+ \text{ on } \mathcal{H}_+ \cong \mathcal{L}_+/\mathcal{K}_+|_{\mathcal{Z}_+(M)|_N}.$$

We need to prove the integrabilitiy of $(\mathcal{H}_+, J_+)$. To any point $n \in N \subset M$ exists an open neighborhood $U \subset M$ and a local frame

$$s = (s_1, s_2, s_3, s_4) : U \subset M \to SO(M)$$

such that $T = s_1$ on $U \cap N$ and

$$(s_2, s_3, s_4) : U \cap N \to SO(N)$$

is an orthonormal frame on the Riemannian manifold $U \cap N$. Any orthonormal basis on $U \cap N$ may be written as

$$(s_2, s_3, s_4) \cdot A = ((s \cdot A)_2, (s \cdot A)_3, (s \cdot A)_4), \quad A \in \mathbf{SO}(3).$$

An arbitrary element $p \in \mathcal{Z}_+(M)|_{N\cap U}$ may be written for a suitable $A \in \mathbf{SO}(3)$ as $p = [s \cdot A, \left[\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right]] \in \mathcal{Z}_+(M)|_{N\cap U}$. It is

$$\Psi(p) = \Psi\left(\mathbf{R}(s_1 + (s \cdot A)_2)\right) = (s \cdot A)_2 \in S^2(TN).$$

In $p \in \mathcal{Z}_+(M)|_{N\cap U}$ holds

$$\pi_*\left(\mathcal{L}_{+p} \cap T_p^H(\mathcal{Z}_+(M)|_N)\right) = \left(\mathbf{R}T_{\pi(p)} \oplus \mathbf{R}(T_{\pi(p)} + (s_{\pi(p)} \cdot A)_2)\right)^\perp$$

$$= Span\{(s_{\pi(p)} \cdot A)_3, (s_{\pi(p)} \cdot A)_4\} = \pi_*\left(\mathcal{H}_{\Psi(p)}^{S^2(TN)} \cap T_{\Psi(p)}^H S^2(TN)\right).$$

It follows for any point $p \in \mathcal{Z}_+(M)|_N$

$$\Psi_*(\mathcal{H}_{+p}) = \Psi_*(T\mathcal{Z}_+(M)|_N \cap \mathcal{L}_+) = \mathcal{H}_{\Psi(p)}^{S^2(TN)}.$$

Hence, the distributions $\mathcal{H}_+$ and $\mathcal{H}^{S^2(TN)}$ are the same under the identification Ψ. A local frame s as above gives trivializations of $\mathcal{Z}_+(M)|_N$ and $S^2(TN)$:

$$\begin{array}{ccc} \mathcal{Z}_+(M)|_{N\cap U} & \cong & S^2(TN)|_{N\cap U} \\ \updownarrow & & \updownarrow \\ (N \cap U) \times \mathbf{SO}(3,1)/H_+ & \cong & (N \cap U) \times \mathbf{SO}(3)/\mathbf{SO}(2) \end{array}$$

The horizontal lifts of s_i, $i = 2, 3, 4$, are given in the trivialization of $\mathcal{Z}_+(M)|_{U\cap N}$ by

$$\begin{aligned} s_i^H &= s_i - \sum_{k<l} \epsilon_k \epsilon_l \cdot g(\nabla_{s_i} s_k, s_l) E_{kl} H_+ \\ &= s_i + g(\nabla_{s_i}(s_1 + s_2), s_3) \frac{E_{13} - E_{23}}{2} H_+ + g(\nabla_{s_i}(s_1 + s_2), s_3) \frac{E_{14} - E_{24}}{2} H_+. \end{aligned}$$

The horizontal lifts of s_i, $i = 2,3,4$, with respect to the Levi-Civita connection of N are given in the trivialization of $S^2(TN)|_{U\cap N}$ by

$$\begin{aligned} t_i^H &= s_i - \sum_{1<k<l} g(\nabla_{s_i} s_k, s_l) E_{kl}\mathbf{SO}(2) \\ &= s_i - g(\nabla_{s_i} s_2, s_3) E_{23}\mathbf{SO}(2) - g(\nabla_{s_i} s_2, s_4) E_{24}\mathbf{SO}(2). \end{aligned}$$

The almost complex structure J_+ on $\mathcal{H}_+$ is given on $\mathcal{H}^{S^2(TN)}$ in $TS^2(TN)$ via the identification Ψ by

$$J_+(t_3^H) = t_4^H - 2g(\nabla_{s_3} s_1, s_4) E_{23}\mathbf{SO}(2) + (g(\nabla_{s_3} s_1, s_3) - g(\nabla_{s_4} s_1, s_4)) E_{24}\mathbf{SO}(2),$$

$$J_+(t_4^H) = -t_3^H + 2g(\nabla_{s_3} s_1, s_4) E_{24}\mathbf{SO}(2) + (g(\nabla_{s_3} s_1, s_3) - g(\nabla_{s_4} s_1, s_4)) E_{23}\mathbf{SO}(2),$$

$$J_+(E_{23}\mathbf{SO}(2)) = E_{24}\mathbf{SO}(2), \qquad J_+(E_{24}\mathbf{SO}(2)) = -E_{23}\mathbf{SO}(2).$$

An easy calculation shows that the Nijenhuis tensor of the almost CR-structure $(\mathcal{H}_+, J_+)$ on $S^2(TN) \cong \mathcal{Z}_+(M)|_N$ is defined and vanishes always.
3. Moreover, the above formulas show that J_+ and $J^{S^2(TN)}$ on $\mathcal{H}^{S^2(TN)}$ are identical iff

$$g(\nabla_{s_3} s_1, s_3) = g(\nabla_{s_4} s_1, s_4) \quad \text{and}$$

$$g(\nabla_{s_3} s_1, s_4) = -g(\nabla_{s_4} s_1, s_3)$$

for any orthonormal basis (s_2, s_3, s_4) on N. But this condition just means that the second fundamental form II of N in M looks like

$$II = g \otimes H,$$

where H is the mean curvature, i.e. the hypersurface N in M is totally umbilic. □

The almost CR-structure on the restriction $\mathcal{Z}_+(M)|_N$ of the twistor bundle to a spacelike oriented hypersurface N that is induced by the almost optical structure $\mathcal{O}_+^-$ on $\mathcal{Z}_+(M)$ is never integrable.

Examples

A. The twistor space $\mathcal{Z}_+(\mathbf{R}_1^4)$ of the Minkowski space and the optical structure $\mathcal{O}_+^+$
Let $\mathbf{R}_1^4 = (\mathbf{R}^4, <,>_1^4)$ be the flat Minkowski space. The twistor space $\mathcal{Z}_+(\mathbf{R}_1^4)$ of $\mathbf{R}_1^4$ is given as

$$\begin{array}{ccccccc} \mathbf{R}_1^4 \times \mathbf{P}(\Delta_+) & \cong & \mathbf{R}_1^4 \times P & \cong & \mathbf{R}_1^4 \times \mathbf{SO}(3,1)/H_+ & \cong & \mathbf{R}_1^4 \times \mathbf{S}^2 \\ (x, [\psi_+]) & \leftrightarrow & (x, ker\hat{\psi}_+) & \leftrightarrow & (x, AH_+) & \leftrightarrow & (x, \begin{pmatrix} a \\ b \\ c \end{pmatrix}) \\ & & = (x, \left[\begin{pmatrix} 1 \\ a \\ b \\ c \end{pmatrix}\right]) & & & & \\ & & = (x, \left[A\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right]) & & & & \end{array}$$

Let $\{u_i : i = 1, \ldots, 4\}$ be the standard basis in $T_x\mathbf{R}_1^4 \cong \mathbf{R}_1^4$ and let $\{u_i^A\}$ be the orthonormal basis transformed by $A \in \mathbf{SO}(3,1)$, $(u_i^A) = (u_1, \ldots, u_4) \cdot A$. The optical structure $\mathcal{O}_+^+$ on $\mathcal{Z}_+(\mathbf{R}_1^4) = \mathbf{R}_1^4 \times \mathbf{SO}(3,1)/H_+$ is given in $[\psi_+] = (x, AH_+) \in \mathcal{Z}_+(\mathbf{R}_1^4)$ by

$$\mathcal{K}_{[\psi_+]} = \mathbf{R}(u_1^A + u_2^A), \qquad \mathcal{L}_{[\psi_+]} = Span\{u_1^A + u_2^A, u_3^A, u_4^A\},$$

$$J_+^+|_{T^V\mathcal{Z}_+(\mathbf{R}_1^4)} \cong -J^{\mathbf{P}(\Delta_+)}, \qquad \begin{array}{l} J_+^+(u_3^A + \mathcal{K}_{[\psi_+]}) = u_4^A + \mathcal{K}_{[\psi_+]} \\ J_+^+(u_4^A + \mathcal{K}_{[\psi_+]}) = -u_3^A + \mathcal{K}_{[\psi_+]} \end{array}$$

The twistor space $(\mathcal{Z}_+(\mathbf{R}_1^4), \mathcal{O}_+^+)$ is optical diffeomorphic to $\mathbf{R} \times (\mathbf{R}^3 \times \mathbf{S}^2)$ with the optical structure

$$(T\mathbf{R}, T\mathbf{R} \oplus H^{S^2(T\mathbf{R}^3)}, J^{S^2(T\mathbf{R}^3)}),$$

where $(\mathbf{R}^3 \times \mathbf{S}^2, H^{S^2(T\mathbf{R}^3)}, J^{S^2(T\mathbf{R}^3)})$ is the unit sphere bundle over the Euclidean space $\mathbf{R}^3$ with the natural induced CR-structure on it (comp. Example 2.1).

B. The twistor space $\mathcal{Z}_+(\mathbf{S}_1^4)$ of the pseudosphere $\mathbf{S}_1^4$ and the optical structure $\mathcal{O}_+^+$

The hypersurface

$$\mathbf{S}_1^4 := \{x \in \mathbf{R}_1^5 : \langle x, x \rangle_1^5 = 1\}$$

in $\mathbf{R}_1^5$ is with the induced metric of $\mathbf{R}_1^5$ an oriented Lorentzian 4-space of constant sectional curvature 1 and is called the 4-dimensional pseudosphere (comp. [O'N83]). The pseudosphere $\mathbf{S}_1^4$ is a symmetric space:

$$\begin{array}{ccc} \mathbf{S}_1^4 & \cong & \mathbf{SO}(4,1)/\mathbf{SO}(3,1), \\ A\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \leftrightarrow & A\mathbf{SO}(3,1) \end{array}$$

where

$$\begin{array}{rccc} \iota: & \mathbf{SO}(3,1) & \hookrightarrow & \mathbf{SO}(4,1) \\ & (A_{ij}) & \mapsto & \begin{pmatrix} A_{11} & 0 & A_{1i} \\ 0 & 1 & 0 \\ A_{i1} & 0 & A_{ij} \end{pmatrix} \end{array}$$

is the imbedding of the isotropy group $\mathbf{SO}(3,1)$.

The twistor space $\mathcal{Z}_+(\mathbf{S}_1^4)$ is a homogenous space given as

$$\mathcal{Z}_+(\mathbf{S}_1^4) = \mathbf{SO}(4,1)/H_+$$

and is diffeomorphic to

$$\mathbf{R} \times S^2(T\mathbf{S}^3) \cong \mathbf{R} \times \mathbf{SO}(4)/\mathbf{SO}(2) \cong \mathbf{R} \times \mathbf{S}^3 \times \mathbf{S}^2,$$

where $S^2(T\mathbf{S}^3) \cong \mathbf{SO}(4)/\mathbf{SO}(2)$ is the unit sphere bundle over the sphere $\mathbf{S}^3 \cong \mathbf{SO}(4)/\mathbf{SO}(3)$. The optical structure $\mathcal{O}_+^+ = (\mathcal{K}_+, \mathcal{L}_+, J_+^+)$ on $\mathcal{Z}_+(\mathbf{S}_1^4)$ is $\mathbf{SO}(4,1)$-equivariant and is given as follows. For the Lie algebra $\mathfrak{o}(4,1) = Span\{E_{ij} : 1 \leq i < j \leq 5\}$ of $\mathbf{SO}(4,1)$, we have the decomposition

$$\mathfrak{o}(4,1) = \mathfrak{m}_+ \oplus \mathfrak{h}_+ \oplus \mathfrak{b},$$

where $\mathfrak{o}(3,1) = \mathfrak{m}_+ \oplus \mathfrak{h}_+$ and $\mathfrak{b} = Span\{E_{12}, E_{23}, E_{24}, E_{25}\}$. The subspace $\mathfrak{b}$ is $Ad(H_+)$-invariant and on $\mathfrak{b}$ exists an $Ad(H_+)$-equivariant optical structure $(\mathfrak{k}, \mathfrak{l}, \underline{J})$ defined by

$$\mathfrak{k} := \mathbf{R}(-E_{12} + E_{23}), \qquad \mathfrak{l} := Span\{E_{24}, E_{25}\} \oplus \mathfrak{k},$$

$$\begin{array}{rccc} \underline{J}: & \mathfrak{l}/\mathfrak{k} & \to & \mathfrak{l}/\mathfrak{k}. \\ & E_{24} + \mathfrak{k} & \mapsto & E_{25} + \mathfrak{k} \\ & E_{25} + \mathfrak{k} & \mapsto & -E_{24} + \mathfrak{k} \end{array}$$

The twistor bundle $\mathcal{Z}_+(\mathbf{S}_1^4)$ decomposes in a horizontal and a vertical part:

$$T\mathbf{SO}(4,1)/H_+ = T^H\mathbf{SO}(4,1)/H_+ \oplus T^V\mathbf{SO}(4,1)/H_+.$$

The horizontal bundle is given by

$$T^H\mathbf{SO}(4,1)/H_+ = \mathbf{SO}(4,1)\times_{Ad(H_+)}\mathfrak{b}.$$

The distributions $\mathcal{K}_+$ and $\mathcal{L}_+$ of the optical structure $\mathcal{O}_+^+$ are defined by

$$\mathcal{K}_+ = \mathbf{SO}(4,1)\times_{Ad(H_+)}\mathfrak{k},\qquad \mathcal{L}_+ = (T^V\mathbf{SO}(4,1)/H_+)\oplus(\mathbf{SO}(4,1)\times_{Ad(H_+)}\mathfrak{l}).$$

The complex structure J_+^+ on $\mathcal{L}_+/\mathcal{K}_+$ is given by

$$J_+^+|_{T^V\mathbf{SO}(4,1)/H_+}\cong -J^{\mathbf{P}(\Delta_+)}$$

and

$$J_+^+([A,l+\mathfrak{k}]) = [A,\underline{J}(l+\mathfrak{k})],\quad A\in\mathbf{SO}(4,1),\ l\in\mathfrak{l},$$

on $T^H\mathbf{SO}(4,1)/H_+$.

The distribution $\mathcal{K}_+$ is regular on $\mathcal{Z}_+(\mathbf{S}_1^4)$ and the underlying CR-manifold of $(\mathcal{Z}_+(\mathbf{S}_1^4),\mathcal{O}_+^+)$ is equivalent to the unit sphere bundle of $\mathbf{S}^3$ with natural CR-structure. We describe this CR-structure on $\mathbf{SO}(4)/\mathbf{SO}(2)\cong S^2(T\mathbf{S}^3)$. Let $\{E_{ij}\}$ be the standard basis in $\mathfrak{o}(4)$ and

$$\begin{array}{cccc}\iota: & \mathbf{SO}(2) & \hookrightarrow & \mathbf{SO}(4)\\ & \begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix} & \mapsto & \exp tE_{34}\end{array}$$

the imbedding of $\mathbf{SO}(2)$. Let

$$\mathfrak{h} := Span\{E_{13},E_{14},E_{23},E_{24}\}\subset\mathbf{SO}(4)$$

be a subspace of $\mathfrak{o}(4)$. The subspace $\mathfrak{h}$ is $Ad(\mathbf{SO}(2))$-invariant and the linear map

$$\underline{J}:\mathfrak{h}\to\mathfrak{h},\qquad \begin{array}{ll}E_{13}\mapsto E_{14}, & E_{23}\mapsto E_{24},\\ E_{14}\mapsto -E_{13}, & E_{24}\mapsto -E_{23},\end{array}$$

is an $Ad(\mathbf{SO}(2))$-equivariant complex structure on $\mathfrak{h}$. The canonical CR-structure on $\mathbf{SO}(4)/\mathbf{SO}(2)$ is given by

$$\mathcal{H}^{S^2(TS^3)} = \mathbf{SO}(4)\times_{Ad(\mathbf{SO}(2))}\mathfrak{h},$$

$$\begin{array}{cccc}J^{S^2(TS^3)}: & \mathcal{H}^{S^2(TS^3)} & \to & \mathcal{H}^{S^2(TS^3)}.\\ & [A,h] & \mapsto & [A,\underline{J}h]\end{array}$$

C. The twistor space $\mathcal{Z}_+(\mathbf{H}_1^4)$ of $\mathbf{H}_1^4$ and the optical structure $\mathcal{O}_+^+$

The hypersurface

$$\mathbf{H}_1^4 := \{x\in\mathbf{R}_2^5 : < x,x >_2^5 = -1\}$$

in $\mathbf{R}_2^5$ is with the induced metric of $\mathbf{R}_2^5$ an oriented Lorentzian 4-manifold of constant sectional curvature -1 and is called the 4-dimensional pseudohyperbolic space. The pseudohyperbolic space $\mathbf{H}_1^4$ is a symmetric space:

$$\mathbf{H}_1^4\cong\mathbf{SO}(3,2)/\mathbf{SO}(3,1),$$

where

$$\begin{array}{cccc}\iota: & \mathbf{SO}(3,1) & \hookrightarrow & \mathbf{SO}(3,2)\\ & A & \mapsto & \begin{pmatrix}1 & 0\\ 0 & A\end{pmatrix}\end{array}$$

is the imbedding of the isotropy group.

The twistor space $\mathcal{Z}_+(\mathbf{H}_1^4)$ is a homogenous space given as

$$\mathcal{Z}_+(\mathbf{S}_1^4) = \mathbf{SO}(3,2)/H_+$$

and is diffeomorphic to

$$\mathbf{S}^1 \times S^2(T\mathbf{H}^3) \cong \mathbf{S}^1 \times \mathbf{SO}_o(3,1)/\mathbf{SO}(2) \cong \mathbf{S}^1 \times \mathbf{R}^3 \times \mathbf{S}^2,$$

where $S^2(T\mathbf{H}^3) \cong \mathbf{SO}_o(3,1)/\mathbf{SO}(2)$ is the unit sphere bundle over the hyperbolic space $\mathbf{H}^3 \cong \mathbf{SO}_o(3,1)/\mathbf{SO}(3)$. The optical structure $\mathcal{O}_+^+ = (\mathcal{K}_+, \mathcal{L}_+, J_+^+)$ on $\mathcal{Z}_+(\mathbf{H}_1^4)$ is $\mathbf{SO}(3,2)$-equivariant. We describe $\mathcal{O}_+^+$ as follows. The Lie algebra of $\mathbf{SO}(3,2)$ decomposes to

$$\mathfrak{o}(3,2) = \mathfrak{m}_+ \oplus \mathfrak{h}_+ \oplus \mathfrak{b},$$

where the subspace $\mathfrak{b} = Span\{E_{12}, E_{13}, E_{14}, E_{15}\}$ is $Ad(H_+)$-invariant. The subspaces

$$\mathfrak{k} := \mathbf{R}(E_{12} + E_{13}), \qquad \mathfrak{l} := Span\{E_{12} + E_{13}, E_{14}, E_{15}\}$$

of $\mathfrak{b}$ and the complex structure

$$\begin{array}{rccc} \underline{J}: & \mathfrak{l}/\mathfrak{k} & \rightarrow & \mathfrak{l}/\mathfrak{k}. \\ & E_{14} + \mathfrak{k} & \mapsto & E_{15} + \mathfrak{k} \\ & E_{15} + \mathfrak{k} & \mapsto & -E_{14} + \mathfrak{k} \end{array}$$

form together an $Ad(H_+)$-equivariant optical structure $(\mathfrak{k}, \mathfrak{l}, \underline{J})$ on $\mathfrak{b}$. The optical structure $\mathcal{O}_+^+$ on $\mathcal{Z}_+(H_1^4) = \mathbf{SO}(3,2)/H_+$ is then given by

$$\mathcal{K}_+ := \mathbf{SO}(3,2) \times_{Ad(H_+)} \mathfrak{k}, \qquad \mathcal{L}_+ = (T^V \mathbf{SO}(3,2)/H_+) \oplus (\mathbf{SO}(3,2) \times_{Ad(H_+)} \mathfrak{l})$$

and

$$J_+^+ : \mathcal{L}_+/\mathcal{K}_+ \rightarrow \mathcal{L}_+/\mathcal{K}_+, \qquad \begin{array}{c} J_+^+|_{T^V \mathbf{SO}(3,2)/H_+} \cong -J^{\mathbf{P}(\Delta_+)} \\ J_+^+([A, l + \mathfrak{k}]) = [A, \underline{J}(l + \mathfrak{k})] \end{array},$$

where $A \in \mathbf{SO}(3,2)$ and $l \in \mathfrak{l}$. The locally induced CR-structures of $\mathcal{O}_+^+$ are equivalent to the natural CR-structure on the unit sphere bundle of $\mathbf{H}^3$. We describe this CR-structure on $\mathbf{SO}_o(3,1)/\mathbf{SO}(2) \cong S^2(T\mathbf{H}^3)$. Let

$$\begin{array}{rccc} \iota: & \mathbf{SO}(2) & \hookrightarrow & \mathbf{SO}_o(3,1) \\ & A & \mapsto & \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} \end{array}$$

be the imbedding of $\mathbf{SO}(2)$ in $\mathbf{SO}_o(3,1)$. The pair $(\mathfrak{h}, \underline{J})$ defined by

$$\mathfrak{h} := Span\{E_{13}, E_{14}, E_{23}, E_{24}\} \subset \mathfrak{o}(3,1) \quad \text{and}$$

$$\underline{J} : \mathfrak{h} \rightarrow \mathfrak{h}, \qquad \begin{array}{ll} E_{13} \mapsto E_{14}, & E_{14} \mapsto -E_{13}, \\ E_{23} \mapsto E_{24}, & E_{24} \mapsto -E_{23}, \end{array}$$

is an $Ad(\mathbf{SO}(2))$-equivariant CR-structure on $\mathfrak{c} = Span\{E_{13}, E_{14}, E_{23}, E_{24}, E_{34}\} \subset \mathfrak{o}(3,1)$. The CR-structure $(\mathcal{H}^{S^2(T\mathbf{H}^3)}, J^{S^2(T\mathbf{H}^3)})$ is then given by

$$\mathcal{H}^{S^2(T\mathbf{H}^3)} = \mathbf{SO}_o(3,1) \times_{Ad(\mathbf{SO}(2))} \mathfrak{h},$$

$$\begin{array}{rccc} J^{S^2(TH^3)}: & \mathcal{H}^{S^2(TH^3)} & \rightarrow & \mathcal{H}^{S^2(TH^3)}. \\ & [A, h] & \mapsto & [A, \underline{J}h] \end{array}, \quad A \in \mathbf{SO}_o(3,1), \ h \in \mathfrak{h}.$$

2.3 Grassmannian fibre bundle of a Lorentzian manifold M_1^4

The Grassmannian fibre bundle of an oriented Riemannian 4-manifold M^4 is the bundle of oriented planes in the tangent bundle TM^4. Each one of the positive and negative Riemannian twistor space may be seen as one 'half' of the Grassmannian fibre bundle. In 4-dimensional Lorentzian geometry this point of view of the Grassmannian is also suitable up to an slight modification

Let $G(3,1)$ denote the set of oriented spacelike planes in the Minkowski space $\mathbf{R}_1^4$. We call $G(3,1)$ the Grassmannian of $\mathbf{R}_1^4$. The Grassmannian $G(3,1)$ can also be viewed as the set of decomposable 2-forms with lenght 1:

$$G(3,1) \cong \{E \in \Lambda^2 \mathbf{R}_1^{4*} | < E, E >= 1,\ E \wedge E = 0\}.$$

The group $\mathbf{SO}(3,1)$ acts transitively on $G(3,1)$ and the isotropy group of this action at the point $o = Span\{u_3, u_4\}$ is $T = \mathbf{SO}(1,1) \times \mathbf{SO}(2)$, where

$$\begin{array}{rccc} \iota : & T & \hookrightarrow & \mathbf{SO}(3,1) \\ & (A,B) & \mapsto & \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \end{array}$$

describes the imbedding of T in $\mathbf{SO}(3,1)$. It is $G(3,1) \cong \mathbf{SO}(3,1)/T$ a symmetric space. The Lie algebra $\mathfrak{o}(3,1)$ decomposes to $\mathfrak{o}(3,1) = \mathfrak{m} \oplus \mathfrak{t}$, where $\mathfrak{m} = Span\{E_{13}, E_{14}, E_{23}, E_{24}\}$ and $\mathfrak{t} = \{E_{12}, E_{34}\}$ is the Lie algebra of T. The tangent bundle of $G(3,1)$ is the associated bundle

$$TG(3,1) = \mathbf{SO}(3,1) \times_{Ad(T)} \mathfrak{m}.$$

Let $B(X,Y) := -\frac{1}{4} tr(adX \circ adY),\ X, Y \in \mathfrak{o}(3,1)$, be the Killing form on $\mathfrak{o}(3,1)$. The restriction $B|_{\mathfrak{m}}$ of the Killing form to $\mathfrak{m}$ is an $Ad(T)$-invariant scalar product on $\mathfrak{m}$ and induces a $\mathbf{SO}(3,1)$-invariant metric b^G of signature $(2,2)$ on the Grassmannian $G(3,1)$. The Grassmannian $G(3,1)$ admits also the $\mathbf{SO}(3,1)$-equivariant complex structure J_1, which is induced by the $Ad(T)$-equivariant complex structure

$$\begin{array}{ll} J^{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{m}, & J^{\mathfrak{m}}(E_{13}) = E_{14},\ \ J^{\mathfrak{m}}(E_{14}) = -E_{13}, \\ & J^{\mathfrak{m}}(E_{23}) = E_{24},\ \ J^{\mathfrak{m}}(E_{24}) = -E_{23} \end{array}$$

on $\mathfrak{m}$. We denote $J_2 := -J_1$, which is a further complex structure on $G(3,1)$. Consider again the twistor fibres $P = \mathbf{SO}(3,1)/H_{\pm}$. We have the following natural projections:

$$\begin{array}{rccc} \alpha_+ : & G(3,1) & \to & P = \mathbf{SO}(3,1)/H_+, \\ & E = Span\{h_3, h_4\} = AT & \mapsto & \mathbf{R}(h_1 + h_2) = AH_+ \end{array}$$

$$\begin{array}{rccc} \alpha_- : & G_{1,1}(3,1) & \to & P = \mathbf{SO}(3,1)/H_-, \\ & E = Span\{h_3, h_4\} = AT & \mapsto & \mathbf{R}(h_1 - h_2) = AH_- \end{array}$$

where $\{h_1, h_2, h_3, h_4\}$ is a positive oriented orthonormal basis in $\mathbf{R}_1^4$. Furthermore, we have the diffeomorphism

$$\begin{array}{rcccc} \alpha := \alpha_+ \times \alpha_- : & G(3,1) & \to & P \times P \backslash diag & \subset P \times P. \\ & Span\{h_3, h_4\} & \mapsto & (\mathbf{R}(h_1 + h_2), \mathbf{R}(h_1 - h_2)) & \end{array}$$

In words, an oriented spacelike plane in $\mathbf{R}_1^4$ is uniquely determined by its two ordered normal null directions. The projections

$$\alpha_+ : (G(3,1), J_2) \to (P, J^{\mathbf{P}(\Delta_+)}), \qquad \alpha_- : (G(3,1), J_2) \to (P, J^{\mathbf{P}(\Delta_-)})$$

are holomorphic.

Remark 2.4 *1. The Lie algebra $\mathfrak{o}(4) \cong \Lambda^2\mathbf{R}^4$ is semisimple. Therefore, the Grassmannian of the Euclidean space $\mathbf{R}^4$ is isometric to the Riemannian product of the positive and the negative twistor space fibre:*

$$G_2(4) \cong \mathbf{S}^2 \times \mathbf{S}^2.$$

The Lie algebra $\mathfrak{o}(3,1) \cong \Lambda^2\mathbf{R}^4_1$ of the Lorentz group is simple. The Grassmannian $G(3,1)$ of $\mathbf{R}^4_1$ isn't the product of the positive and negative twistor space fibre P, it holds only

$$G(3,1) \cong P \times P \backslash diag \subset \; P \times P \cong \mathbf{S}^2 \times \mathbf{S}^2.$$

It doesn't exist a natural $\mathbf{SO}(3,1)$-invariant metric on P.

Let M^4_1 be an oriented Lorentzian manifold. We call

$$\mathcal{G}(M) := SO(M) \times_{\mathbf{SO}(3,1)} G(3,1) = SO(M)/T$$

the Grassmannian fibre bundle of M^4_1. The Grassmannian fibre bundle projects naturally to the twistor spaces. Let $e : U \to SO(M)$ and $s = e \cdot A : U \to SO(M),\ A \in \mathbf{SO}(3,1)$, be local orthonormal frames on $U \subset M$. The projections are given by

$$\begin{array}{rrcl} \alpha_+ : & \mathcal{G}(M) & \to & \mathcal{Z}_+(M) \cong SO(M) \times_{\mathbf{SO}(3,1)} P \quad , \\ & [e, AT] = Span\{s_3, s_4\} & \mapsto & [e, AH_+] = \mathbf{R}(s_1 + s_2) \end{array}$$

$$\begin{array}{rrcl} \alpha_- : & \mathcal{G}(M) & \to & \mathcal{Z}_-(M) \cong SO(M) \times_{\mathbf{SO}(3,1)} P \quad . \\ & [e, AT] = Span\{s_3, s_4\} & \mapsto & [e, AH_-] = \mathbf{R}(s_1 - s_2) \end{array}$$

Moreover, we have the imbedding

$$\begin{array}{rccl} \alpha : & \mathcal{G}(M) & \hookrightarrow & \mathcal{Z}_+(M)[\times]\mathcal{Z}_-(M) \quad , \\ & x & \mapsto & (\alpha_+(x), \alpha_-(x)) \end{array}$$

where $[\times]$ denotes the fiberwise product over M.

On the Grassmannian fibre bundle exist the natural almost optical structures

$$\mathcal{O}^+_G = (\mathcal{K}_G, \mathcal{L}_G, J^+_G) \quad \text{and} \quad \mathcal{O}^-_G = (\mathcal{K}_G, \mathcal{L}_G, J^-_G).$$

They are given as follows. Let $p \in \mathcal{G}(M)$ be a point. In a suitable orthonormal basis $s = (s_1, s_2, s_3, s_4)$, we have $p = [s, eT] \in \mathcal{G}(M)$. It is

$$\mathcal{O}^p = (\mathbf{R}(s_1 + s_2), (s_1 + s_2)^\perp, J^p) \ ,$$

$$J^p(s_3 + \mathbf{R}(s_1 + s_2)) = s_4 + \mathbf{R}(s_1 + s_2), \quad J^p(s_4 + \mathbf{R}(s_1 + s_2)) = -s_3 + \mathbf{R}(s_1 + s_2),$$

an optical structure on $T_{\pi(p)}M$. The almost optical structure $\mathcal{O}^+_G$ is given in $T_p\mathcal{G}(M)$ by

$$(\mathcal{O}^+_G)_p = \pi_*^{-1} \circ \mathcal{O}^p \circ \pi_* - [s]^{-1} \circ J_2 \circ [s]$$

and $\mathcal{O}^-_G$ is given in $T_p\mathcal{G}(M)$ by

$$(\mathcal{O}^-_G)_p = \pi_*^{-1} \circ \mathcal{O}^p \circ \pi_* - [s]^{-1} \circ J_1 \circ [s].$$

Because of $\mathcal{O}^p = \mathcal{O}^{\alpha_+(p)}$ and $\alpha_{+*} \circ J_{1,2} = \mp J^{\mathbf{P}(\Delta_+)} \circ \alpha_{+*}$, the projections

$$\alpha_+ : (\mathcal{G}(M), \mathcal{O}^+_G) \to (\mathcal{Z}_+(M), \mathcal{O}^+_+), \qquad \alpha_+ : (\mathcal{G}(M), \mathcal{O}^-_G) \to (\mathcal{Z}_+(M), \mathcal{O}^-_+)$$

are optical maps.

Theorem 2.4 *Let M^4_1 be an oriented 4-dimensional Lorentzian manifold and let $\mathcal{G}(M)$ be its Grassmannian fibre bundle.*

1. *The almost optical structure $\mathcal{O}_G^-$ on $\mathcal{G}(M)$ is never integrable.*
2. *The almost optical structure $\mathcal{O}_G^+$ on $\mathcal{G}(M)$ is integrable in a point $p = [s, eT] \in \mathcal{G}(M)$ if and only if*

$$R_{1414} + R_{2414} - R_{1313} - R_{2313} = 0, \quad R_{1424} + R_{2424} - R_{1323} - R_{2323} = 0,$$

$$2R_{1314} + R_{2314} + R_{2413} = 0, \quad 2R_{2324} + R_{1423} + R_{1324} = 0.$$

3. *The almost optical structure $\mathcal{O}_G^+$ is integrable on $\mathcal{G}(M)$ iff M has constant sectional curvature.*

Remark: $\mathcal{O}_G^+$ on $\mathcal{G}(M)$ is not conformally invariant.

Sketch of the proof: The integrability conditions in a point may be obtained as in the proof of Theorem 2.1 (comp. [Lei98]). Obviously, a space with constant sectional curvature satisfies these conditions in any point. On the other side, it can easily be seen that the Riemannian curvature tensor of a space, which hasn't constant sectional curvature, doesn't satisfy these conditions in a point □

The almost optical structure $\mathcal{O}_G^+$ induces an almost CR-structure on the restriction $\mathcal{G}(M)|_N$ of the Grassmannian fibre bundle to an oriented spacelike hypersurface N. As in Theorem 2.3 2., this almost CR-structure is always integrable.

To the end of this section, we define a parametrization of natural metrics on the Grassmannian fibre bundle $\mathcal{G}(M)$. Let $0 \neq \lambda \in \mathbf{R}$ be a parameter and b^G the $\mathbf{SO}(3,1)$-invariant metric tensor on the Grassmannian $G(3,1)$. The tensor

$$g_\lambda := \pi^* g + \lambda \cdot [s]^*(b^G),$$

where $\pi : \mathcal{G}(M) \to M$ is the natural projection and $s \in SO(M)$, is for any λ a metric of signature $(5,3)$ on $\mathcal{G}(M)$.

3 Surface theory and Lorentzian twistor construction

3.1 Immersed spacelike surfaces

In this section, we investigate the second fundamental form of isometrically immersed surfaces with Riemannian metric in a Lorentzian 4-space.

Let (N, h) be an oriented 2-dimensional Riemannian manifold and let dN be the volume form in the orientation of N. The Hodge operator $*$ is defined by $\nu \wedge *\xi = h(\nu, \xi) dN$. There is an orthogonal complex structure J^N on N such that $*\omega = -\omega \circ J^N$ for any 1-form ω on N. Locally, it exist complex coordinates (x_1, x_2) of the Riemannian surface (N, J^N). For a suitable C^∞-function F, we have

$$h = F \cdot (dx_1 \circ dx_1 + dx_2 \circ dx_2).$$

The Bochner-Laplace operator on $C^\infty(N)$ is defined by $\Delta = *d*d$. For any C^∞-function f on N, we hace locally

$$\Delta f = \frac{f_{x_1x_1} + f_{x_2x_2}}{F}.$$

Let $M_1^4 := (M^4, g)$ be an oriented Lorentzian 4-space and let $f : N^2 \hookrightarrow M_1^4$ be an isometric immersion. It is $f(N) \subset M_1^4$ a spacelike surface. A local Darboux frame is a map

$$e = (e_1, \ldots, e_4) : U \subset N \to SO(M)|_N$$

such that (e_3, e_4) is locally a positive oriented orthonormal frame in TN. Let $TN^\perp$ denote the normal bundle of the immersion f over N. The metric g on TM induces a metric $h^\perp$ of signature $(1,1)$ on $TN^\perp$. The normal bundle decomposes into two null line bundles

$$TN^\perp = TN^\perp_+ \oplus TN^\perp_-,$$

where these line bundles over N are locally given by

$$TN^\perp_+ = \mathbf{R}(e_1 + e_2), \qquad TN^\perp_- = \mathbf{R}(e_1 - e_2)$$

with respect to a Darboux frame e. We call $TN^\perp_+$ and $TN^\perp_-$ the bundles of positive resp. negative oriented normal null directions over N. The second fundamental form of the immersion f is define by

$$II = -g(\nabla.\cdot, e_1)e_1 + g(\nabla.\cdot, e_2)e_2 = h^\alpha_{ij} \cdot e^*_i \circ e^*_j \otimes e_\alpha, \qquad \alpha = 1,2,\ i,j = 3,4,$$

where $h^\alpha_{ij} = \epsilon_\alpha g(\nabla_{e_i} e_j, e_\alpha) = \epsilon_\alpha \omega_{j\alpha}(e_i)$. The $\omega_{j\alpha}$ denote some components of the local connection form of the Levi-Civita connection on M^4_1. It is $h^\alpha_{ij} = h^\alpha_{ji}$. Here and also later on we use Einstein summation convention. The mean curvature vector H of the isometric immersion f is given by

$$\begin{aligned} H &= \frac{1}{2} tr II = \frac{1}{2} g^{ij} h^\alpha_{ij} e_\alpha := H_+ + H_- \\ &= \frac{h^1_{33} + h^1_{44} + h^2_{33} + h^2_{44}}{4}(e_1 + e_2) + \frac{-h^1_{33} - h^1_{44} + h^2_{33} + h^2_{44}}{4}(e_2 - e_1), \end{aligned}$$

where $H_- \in TN^\perp_-$, $H_+ \in TN^\perp_+$. Let $t = \frac{-ie^*_3 + e^*_4}{\sqrt{2}} \in \mathbf{C} \otimes T^*M$. It is

$$\begin{aligned} II &= h \otimes H + Re\left[\left(\tfrac{1}{2}(-h^1_{33} + h^1_{44} - h^2_{33} + h^2_{44}) + i(h^1_{34} + h^2_{34})\right) \cdot t \circ t\right] \otimes (e_1 + e_2) \\ &\quad + Re\left[\left(\tfrac{1}{2}(h^1_{33} - h^1_{44} - h^2_{33} + h^2_{44}) + i(-h^1_{34} + h^2_{34})\right) \cdot t \circ t\right] \otimes (e_2 - e_1) \\ &:= h \otimes H_+ + h \otimes H_- + L_+ + L_- \end{aligned}$$

with $L_- \in Sym^2(TN) \otimes TN^\perp_-$, $L_+ \in Sym^2(TN) \otimes TN^\perp_+$. This is the decomposition of the second fundamental form II into the four components of a certain irreducible decomposition of the $\mathbf{SO}(2) \times \mathbf{SO}(1,1)$-modul $Sym^2(TN) \otimes TN^\perp = \mathcal{H}_+ \oplus \mathcal{H}_- \oplus \mathcal{L}_+ \oplus \mathcal{L}_-$, $\dim \mathcal{H}_+ = \dim \mathcal{H}_- = 1$, $\dim \mathcal{L}_+ = \dim L_- = 2$. The second fundamental form II and the decomposition of II is independently of the conformal class of the surface N defined.

Definition 3.1 *The isometric (conformal) immersion $f : (N,h) \to (M,g)$ is called*

$$\text{(positive) semi} - \text{stationary} \Leftrightarrow H_- = 0,$$

$$\text{(negative) semi} - \text{stationary} \Leftrightarrow H_+ = 0,$$

$$+\text{isotropic} \Leftrightarrow H_- = L_- = 0 \Leftrightarrow II(V,W) \in TN^\perp_+,$$

$$-\text{isotropic} \Leftrightarrow H_+ = L_+ = 0 \Leftrightarrow II(V,W) \in TN^\perp_-,$$

$$\text{stationary} \Leftrightarrow H = 0,$$

$$\text{totally umbilic} \Leftrightarrow II = g \otimes H,$$

$$\text{totally geodesic} \Leftrightarrow II = 0.$$

Remark 3.1 *Consider an isometric immersion $f:(N,h)\to(M,g)$. Let $\tilde{g}:=\exp(2\rho)g$ be a conformally equivalent metric to g on M^4. Denote by $\tilde{II}$ the second fundamental form of the isometric immersion $f:(N,f^*\tilde{g})\to(M,\tilde{g})$. The comparison of the covariant derivatives gives (see [Bes87])*

$$\tilde{\nabla}_X Y=\nabla_X Y+d\rho(X)Y+d\rho(Y)X-g(X,Y)grad\rho$$

and this yields

$$\tilde{II}=II-g\otimes\mathcal{N}grad\rho.$$

If the metric h is positive definite, we have

$$\tilde{II}=f^*\tilde{g}\otimes[\exp(-2\rho)\cdot(H-\mathcal{N}grad\rho)]+L_+ +L_-.$$

This shows that the vanishing of the components $L_+,\ L_-$ is invariant under conformal change of the metric g on M^4. In particular, the property of an immersion to be totally umbilic is conformally invariant, whereas the stationary condition isn't conformally invariant.

3.2 Gauss lift of an immersed surface

We define now the Gauss lifts of a spacelike immersed surface to the twistor space and the Grassmannian fibre bundle. Geometric properties of an immersed surface are related to the holomorphicity of its Gauss lifts. Moreover, we describe the surfaces that have a harmonic Gauss lift to the Grassmannian.

Let (N^2,J^N) be a Riemannian surface and $f:N^2\to M_1^4$ a conformal immersion into an oriented Lorentzian 4-manifold. It is

$$df(T_nN)\in G(T_{f(n)}M,g_{(f(n))})\cong G(3,1)$$

for any $n\in N$ an oriented Euclidean 2-plane in $T_{f(n)}M$, i.e. an element in the Grassmannian $\mathcal{G}(M)$ over M. The smooth mapping

$$\begin{array}{rccc} v_f: & N & \to & \mathcal{G}(\mathcal{M})\ , \\ & n & \mapsto & df(T_nN) \end{array} \qquad \begin{array}{ccc} & & \mathcal{G}(M) \\ & \overset{v_f}{\nearrow} & \downarrow \\ f:N & \to & M \end{array}$$

is called the Gauss map or Gauss lift of the immersion f. We define also the projected Gauss maps into the twistor spaces over M_1^4:

$$\gamma_{f\pm}:=\alpha_\pm\circ v_f:N\to\mathcal{Z}_\pm(M)\ , \qquad \begin{array}{ccc} & & \mathcal{Z}_\pm(M) \\ & \overset{\gamma_{f\pm}}{\nearrow} & \downarrow \\ f:N & \to & M \end{array}\ .$$

The image of a point $n\in N$ under the mappings $\gamma_{f\pm}$ is the positive resp. negative normal null direction on $df(T_nN)$ in $T_{f(n)}M_1^4$.

Definition 3.2 *Let $f:N^2\to M_1^4$ be a conformal immersion of a Riemannian surface into an oriented Lorentzian 4-space. The immersion f is called.*

$$\begin{array}{l}
\mathcal{O}_G^+\text{–holomorphic if } v_f:(N,J^N)\to(\mathcal{G}(M),\mathcal{O}_G^+)\text{ is holomorphic,}\\
\mathcal{O}_G^-\text{–holomorphic if } v_f:(N,J^N)\to(\mathcal{G}(M),\mathcal{O}_G^-)\text{ is holomorphic,}\\
\mathcal{O}_+^\pm\text{–holomorphic if } \gamma_{f+}:(N,J^N)\to(\mathcal{Z}_+(M),\mathcal{O}_+^\pm)\text{ is holomorphic,}\\
\mathcal{O}_-^\pm\text{–holomorphic if } \gamma_{f-}:(N,J^N)\to(\mathcal{Z}_-(M),\mathcal{O}_-^\pm)\text{ is holomorphic.}
\end{array}$$

The conformal immersion $f:N^2\to M_1^4$ is called $\pm$horizontal if the lift $\gamma_{f\pm}$ to $\mathcal{Z}_\pm(M)$ is horizontal, that means

$$d\gamma_{f\pm}(T_nN)\subset T^H_{f(n)}\mathcal{G}(M)\quad\forall n\in N.$$

Consider the differentials of the lifts v_f and $\gamma_{f\pm}$. The horizontal parts dv_f^H and $d\gamma_{f\pm}^H$ are simply the horizontal lifts of df. For the vertical part dv_f^V holds

$$dv_f^V = -\omega_{13}E_{13}T + \omega_{23}E_{23}T - \omega_{14}E_{14}T + \omega_{24}E_{24}T,$$

and the vertical parts $d\gamma_{f+}^V$ and $d\gamma_{f-}^V$ are given by

$$d\gamma_{f+}^V = \frac{-\omega_{13} - \omega_{23}}{2}(E_{13} - E_{23})H_+ + \frac{-\omega_{14} - \omega_{24}}{2}(E_{14} - E_{24})H_+,$$

$$d\gamma_{f-}^V = \frac{-\omega_{13} + \omega_{23}}{2}(E_{13} + E_{23})H_- + \frac{-\omega_{14} + \omega_{24}}{2}(E_{14} + E_{24})H_-.$$

Proposition 3.1 *Let $f : N^2 \to M_1^4$ be a conformal immersion. The following relations hold:*

1. *f is $\mathcal{O}_+^-$-holomorphic $\Leftrightarrow$ $H_- = 0 \Leftrightarrow f$ is positive semi-stationary*
   ($\Leftrightarrow *\omega_{13} + *\omega_{23} + \omega_{14} + \omega_{24} = 0 \Leftrightarrow -h_{33}^1 + h_{33}^2 - h_{44}^1 + h_{44}^2 = 0$)

   f is $\mathcal{O}_-^-$-holomorphic $\Leftrightarrow$ $H_+ = 0 \Leftrightarrow f$ is negative semi-stationary
   ($\Leftrightarrow *\omega_{13} - *\omega_{23} + \omega_{14} - \omega_{24} = 0 \Leftrightarrow h_{33}^1 + h_{33}^2 + h_{44}^1 + h_{44}^2 = 0$)

2. *f is $\mathcal{O}_+^+$-holomorphic $\Leftrightarrow$ $L_- = 0$*
   ($\Leftrightarrow *\omega_{13} + *\omega_{23} - \omega_{14} - \omega_{24} = 0 \Leftrightarrow -h_{33}^1 + h_{33}^2 + h_{44}^1 - h_{44}^2 = 0,\ h_{34}^1 - h_{34}^2 = 0$)
   f is $\mathcal{O}_-^+$-holomorphic $\Leftrightarrow$ $L_+ = 0$
   ($\Leftrightarrow *\omega_{13} - *\omega_{23} - \omega_{14} + \omega_{24} = 0 \Leftrightarrow -h_{33}^1 - h_{33}^2 + h_{44}^1 + h_{44}^2 = 0,\ h_{34}^1 + h_{34}^2 = 0$)

3. *f is $\mathcal{O}_G^+$-holomorphic $\Leftrightarrow$ f is totally umbilic*
   ($*\omega_{13} = \omega_{14},\ *\omega_{23} = \omega_{24} \Leftrightarrow h_{33}^1 - h_{44}^1 = 0,\ h_{33}^2 - h_{44}^2 = 0,\ h_{34}^1 = h_{34}^2 = 0$)

   f is $\mathcal{O}_G^-$-holomorphic $\Leftrightarrow$ f is stationary $\Leftrightarrow$ f is harmonic
   ($\Leftrightarrow *\omega_{13} = -\omega_{14},\ *\omega_{23} = -\omega_{24} \Leftrightarrow h_{33}^1 + h_{44}^1 = 0,\ h_{33}^2 + h_{44}^2 = 0$)

4. *f is $+$horizontal $\Leftrightarrow$ $H_- = L_- = 0 \Leftrightarrow$ f is $+$isotropic,*

   f is $-$horizontal $\Leftrightarrow$ $H_+ = L_+ = 0 \Leftrightarrow$ f is $-$isotropic.

Remark 3.2 *We may also say that $\pm$isotropic conformal immersions are super-semi-stationary. This is analogous to the notation for surfaces in a Riemannian 4-manifold that have a horizontal Gauss lift to the Riemannian twistor space. Those surfaces satisfy a stronger condition then minimality and are called superminimal.*

For characterizing an isometric immersion, whose corresponding Gauss lift is harmonic, we have the following

Theorem 3.1 *Let $f : (N^2, h) \to (M^4, g)$ be an isometric (conformal) immersion of an oriented Riemannian 2-manifold into an oriented 4-dimensional Lorentzian space form, whose constant sectional curvature is S. For $\lambda S \neq 12$, the Gauss lift $v_f : (N, h) \to (\mathcal{G}(M), g_\lambda)$ is harmonic iff $H = 0$. In case that $\lambda S = 12$, $v_f : (N, h) \to (\mathcal{G}(M), g_\lambda)$ is harmonic iff $\nabla H = 0$.*

Proof: On the frame bundle $SO(M)$, we have a natural parametrization of metrics,

$$g_\lambda = \pi^* g + \lambda B, \quad \lambda \in \mathbf{R}\backslash\{0\},$$

such that $\tilde{\pi} : (SO(M), g_\lambda) \to (\mathcal{G}(M), g_\lambda)$ is a Riemannian submersion. Let $e = (e_1, \ldots, e_4) : U \subset N \to SO(M)$ be a Darboux frame, (e_3, e_4) tangential to N. Moreover, we may assume that the frame (e_3, e_4) is parallel in $n \in U$ with respect to ∇^N and that (e_1, e_2) is parallel in $n \in U$ with respect to the connection $\nabla^{TN^\perp}$ on $TN^\perp$,

$$\nabla^N e_i(n) = 0, \quad \nabla^{N^\perp} e_\alpha(n) = 0, \qquad i = 3, 4, \; \alpha = 1, 2.$$

For the Levi-Civita connection ∇^λ of $(SO(M), g_\lambda)$, we have

$$\nabla^\lambda_{e_i^H} e_j^H = (\nabla_{e_i} e_j)^H + \frac{1}{2} Z[e_i^H, e_j^H], \quad \nabla^\lambda_{E_{mn}} E_{kl} = \frac{1}{2}[E_{mn}, E_{kl}],$$

$$\nabla^\lambda_{e_i^H} E_{mn} = \frac{\lambda}{2} \sum_{k=1}^{4} \epsilon_k \cdot g(E_{mn}, [e_k^H, e_i^H]) e_k^H = \nabla^\lambda_{E_{mn}} e_i^H,$$

where e_i^H denote the horizontal lift of e_i, $i = 1, \ldots, 4$, to $SO(M)$. For the tension field τ_e of the local frame e in $n \in N$ holds

$$\begin{aligned}
\tau_e \;=\; & tr(\nabla de) = \nabla^\lambda_{de(e_3)} de(e_3) - de(\nabla_{e_3} e_3) + \nabla^\lambda_{de(e_4)} de(e_4) - de(\nabla_{e_4} e_4) \\
=\; & (\nabla^M_{e_3} e_3)^H + \sum_{k<l} \epsilon_k \epsilon_l \cdot e_3(g(\nabla_{e_3} e_k, e_l)) E_{kl} + 2 \sum_{k<l} \epsilon_k \epsilon_l \cdot g(\nabla_{e_3} e_k, e_l) \nabla_{e_3} E_{kl} \\
+\; & \frac{1}{2} \cdot \sum_{k<l, m<n} \epsilon_k \epsilon_l \epsilon_m \epsilon_n \cdot g(\nabla_{e_3} e_k, e_l) \cdot g(\nabla_{e_3} e_m, e_n) \cdot [E_{kl}, E_{mn}], \\
+\; & (\nabla^M_{e_4} e_4)^H + \sum_{k<l} \epsilon_k \epsilon_l \cdot e_4(g(\nabla_{e_4} e_k, e_l)) E_{kl} + 2 \sum_{k<l} \epsilon_k \epsilon_l \cdot g(\nabla_{e_4} e_k, e_l) \nabla_{e_4} E_{kl} \\
+\; & \frac{1}{2} \cdot \sum_{k<l, m<n} \epsilon_k \epsilon_l \epsilon_m \epsilon_n \cdot g(\nabla_{e_4} e_k, e_l) \cdot g(\nabla_{e_4} e_m, e_n) \cdot [E_{kl}, E_{mn}], \\
=\; & H^H + \sum_{k<l} \epsilon_k \epsilon_l \cdot (e_3(g(\nabla_{e_4} e_k, e_l)) + e_4(g(\nabla_{e_4} e_k, e_l)))\, E_{kl} \\
-\; & \lambda \sum_{k<l} g(\nabla_{e_3} e_k, e_l) \cdot \sum_m \epsilon_m R_{m3kl} \cdot e_m^H - \lambda \sum_{k<l} g(\nabla_{e_4} e_k, e_l) \cdot \sum_m \epsilon_m R_{m4kl} \cdot e_m^H \\
+\; & terms\ in\ E_{12}, E_{34}.
\end{aligned}$$

By the assumption of constant sectional curvature of M_1^4, we obtain for the tension field of v_f:

$$\begin{aligned}
\tau \;=\; & tr(\nabla dv_f) = d\tilde{\pi}(\tau_e) = H^H - \frac{\lambda S}{12} H^H \\
& + e_3(h^1_{33} + h^1_{44}) E_{13} + e_4(h^1_{33} + h^1_{44}) E_{14} - e_3(h^2_{33} + h^2_{44}) E_{23} - e_4(h^2_{33} + h^2_{44}) E_{24}.
\end{aligned}$$

Thus v_f is harmonic iff $H = 0$ or $\lambda S = 12, \; \nabla H = 0$. □

3.3 Twistorial construction of spacelike surfaces

The Proposition 3.1 of the previous section relates properties of an immersed spacelike surface to properties of its Gauss lift. In the following, the reconstruction of semi-stationary and semi-umbilic surfaces in Lorentzian 4-spaces by holomorphic curves in the Lorentzian twistor space is established. We can give a whole classification of semi-umbilic surfaces in conformally flat Lorentzian 4-spaces. In particular, we classify every isotropic surface in the Lorentzian space forms $\mathbf{R}^4_1$, $\mathbf{S}^4_1$ and $\mathbf{H}^4_1$.

Theorem 3.2 *Let (N^2, J^N) be a Riemannian surface and let M_1^4 be an oriented Lorentzian 4-manifold.*

1. *If*
$$\gamma : (N, J^N) \to (\mathcal{Z}_+(M), \mathcal{O}_+^-)$$
is a holomorphic map into the positive twistor space over M_1^4 such that $d\gamma \neq 0$ on N and γ is non-vertical, that means $d\gamma(T_nN) \not\subset T^V_{\gamma(n)}\mathcal{Z}_+(M)$ $\forall n \in N$, then the projection
$$f := \pi \circ \gamma : N^2 \to M_1^4$$
is a conformal immersion, which is positive semi-stationary ($H_- = 0$).

2. *There is a bijective correspondence between $\mathcal{O}_+^-$-holomorphic, non-vertical curves in $\mathcal{Z}_+(M)$ and conformally immersed positive semi-stationary surfaces in M_1^4.*

Remark 3.3 1. *There is also a bijective correspondence between $\mathcal{O}_+^+$-holomorphic, non-vertical curves in $\mathcal{Z}_+(M)$ and conformally immersed surfaces in M_1^4 with $L_- = 0$ (semi-umbilic).*

2. *Conformally immersed harmonic (stationary) surfaces in M_1^4 correspond bijectively to $\mathcal{O}_G^-$-holomorphic, non-vertical curves in the Grassmannian $\mathcal{G}(M)$ over M.*

3. *Let*
$$f : (N^2, h) \to (P^3, g)$$
be a conformal immersion of an oriented surface into a 3-dimensional oriented Riemannian manifold P. There is an unique lift, which we also call the Gauss lift of the immersion f to the unit sphere bundle $S^2(TP)$ over P. For $n \in N$, let (e_1, e_2) be a positive oriented orthonormal basis of $df(T_nN) \subset T_{f(n)}P$ and let $l_{f(n)}$ be the unit vector such that $(l_{f(n)}, e_1, e_2)$ is a positive oriented orthonormal basis in $T_{f(n)}P$. The Gauss lift of f is
$$\begin{array}{rccc} \gamma_f : & N & \to & S^2(TP). \\ & n & \mapsto & l_{f(n)} \end{array}$$
There is a bijective correspondence between non-vertical, holomorphic curves in the CR-manifold
$$(S^2(TP), H^{S^2(TP)}, J^{S^2(TP)})$$
and totally umbilic surfaces in P.

If P is an oriented spacelike hypersurface of the oriented Lorentzian 4-space M_1^4, then the Gauss lifts of
$$f : N^2 \hookrightarrow P^3 \subset M_1^4$$
to $S^2(TP)$ and to $\mathcal{Z}_+(M)$ are identical under the identification
$$\Psi : S^2(TP) \cong \mathcal{Z}_+(M)|_P.$$
In case that $P^3 \subset M_1^4$ is a totally umbilic hypersurface, the Gauss lift of $f : N^2 \hookrightarrow P^3 \subset M_1^4$ to $(\mathcal{Z}_+(M), \mathcal{O}_+^+)$ is holomorphic iff the Gauss lift to $(S^2(TP), H^{S^2(TP)}, J^{S^2(TP)})$ is holomorphic, i.e. iff $N^2 \subset P^3$ is totally umbilic.

Proof of the theorem: Let
$$\gamma : (N, J^N) \to (\mathcal{Z}_+(M), \mathcal{O}_+^-)$$
be a holomorphic curve, $d\gamma \neq 0$ on N, that is non-vertical. Obviously, the projected map $f := \pi \circ \gamma$ is an immersion. Because of γ is holomorphic, $d\gamma(TN) \subset \mathcal{L}_+$ follows, and therefore
$$\pi_*(d\gamma(T_nN)) \subset L^{\gamma(n)} \quad \forall n \in N.$$

Moreover, it is

$$d\gamma \circ J^N = J_+^- \circ \hat{\pi} \circ d\gamma,$$

which means that

$$df \circ J_n^N = J^{\gamma(n)} \circ pr_{\gamma(n)} \circ df : T_nN \to L^{\gamma(n)}/K^{\gamma(n)} \quad \forall n \in N,$$

where $pr_{\gamma(n)} : L^{\gamma(n)} \to L^{\gamma(n)}/K^{\gamma(n)}$ is the natural projection. $J^{\gamma(n)}$ is orthogonal with respect to $g_{f(n)}$. This means that the immersion $f : N^2 \to M_1^4$ is conformal. If we prove that the Gauss lift γ_f of the immersion $f = \pi \circ \gamma$ is equal to the original map γ, then the theorem follows from Proposition 3.1. The element $\gamma(n) \in \mathcal{Z}_+(M),\ n \in N$, may be seen as a null direction in $T_{f(n)}M$, which is orthogonal to $df(T_nN) \subset T_{f(n)}M$. It holds that the optical structure $\mathcal{O}^{\gamma(n)}$ in $T_{f(n)}M$ is equal to

$$(\gamma(n), \gamma(n) \oplus df(T_nN), df \circ J^N \circ df^{-1})$$

and therefore the Gauss lift of f in any point $n \in N$ is equal to $\gamma(n)$. $\square$

The statements in Remark 3.3 can be proved in the same way as Theorem 3.2. Parts of Proposition 3.1 and Theorem 3.2 are also proved in [Bob98]. For the third point of Remark 3.3 remember also Theorem 2.3.

One way of finding holomorphic curves in an almost optical manifold is that of finding holomorphic curves in a CR-hypersurface of the almost optical manifold. For example, if $P^3 \subset M_1^4$ is a spacelike hypersurface in a Lorentzian 4-manifold, then the unit sphere bundle $S^2(TP)$ with CR-structure $(\mathcal{H}_+, J_+)$ is a CR-hypersurface in $(\mathcal{Z}_+(M), \mathcal{O}_+^+)$ (comp. Theorem 2.3). A non-vertical holomorphic curve in $(S^2(TP), \mathcal{H}_+, J_+)$ is then a non-vertical holomorphic curve in $(\mathcal{Z}_+(M), \mathcal{O}_+^+)$ and projects to a semi-umbilic surface in M_1^4. This fact is described in Remark 3.3 for the special case, when $P^3 \subset M_1^4$ is totally umbilic.

We remember to the fact that the existence of one holomorphic curve in an integrable optical manifold gives rise to a whole class of holomorphic curves by an arbitrary flow of the original curve along the 1-dimensional distribution in the definition of the optical structure (comp. Proposition 2.1).

Theorem 3.3 *Let M_1^4 be an oriented, conformally flat Lorentzian 4-manifold and $\mathcal{Z}_+(M)$ its positive twistor space. Further, let $k \in \Gamma(\mathcal{K}_+)$ denote a vector field in the distribution $\mathcal{K}_+$ on $\mathcal{Z}_+(M)$ and let ϕ_t^k denote its flow.*

1. *If $f : N^2 \hookrightarrow M_1^4$ is a conformally immersed Riemannian surface with $L_- = 0$ (semi-umbilic), then the map*
$$f_t := \pi \circ \phi_t^k \circ \gamma_f : N^2 \to M_1^4$$
is at least locally for small t's a conformal immersion of a surface in M_1^4, which is still semi-umbilic ($L_- = 0$).

2. *Assume that M_1^4 is not only conformally flat, but has constant sectional curvature. If $f : N^2 \hookrightarrow M_1^4$ is an isometric (conformal) immersed +isotropic surface in M_1^4, the map*
$$f_t := \pi \circ \phi_t^k \circ \gamma_f : N^2 \to M_1^4$$
is at least locally for small t's an isometric (conformal) immersion of a +isotropic surface in M_1^4.

Proof:

1. If $f : N^2 \hookrightarrow M_1^4$ is a conformally immersed surface with $L_- = 0$, it follows that
$$\gamma_f : (N, J^N) \to (\mathcal{Z}_+(M), \mathcal{O}_+^+)$$

is holomorphic. Because M_1^4 is conformally flat $\mathcal{O}_+^+$ on $\mathcal{Z}_+(M)$ is integrable and therefore

$$\phi_t^k \circ \gamma_f : (N, J^N) \to (\mathcal{Z}_+(M), \mathcal{O}_+^+)$$

is locally and for small t's a holomorphic map. The map $f_t = \pi \circ \phi_t^k \circ \gamma_f$ is then by Remark 3.3 1. a conformal immersion of a positive semi-umbilic surface.

2. This proof will be done below explicitly by the investigation of +isotropic surfaces in the three Lorentzian space forms $\mathbf{R}_1^4$, $\mathbf{S}_1^4$ and $\mathbf{H}_1^4$.

□

The theorem may be interpreted as follows. Let N^2 be any oriented spacelike surface in M_1^4. The normal bundle $TN^\perp$ over N^2 in M_1^4 decomposes to the bundle of positive and negative normal null directions on the surface N^2 in M_1^4. Any positive oriented normal null vector on N^2 defines a null geodesic that starts from the surface N^2. The distribution $\mathcal{K}_+$ on $\mathcal{Z}_+(M)$ is the lightlike geodesic spray of M. Hence, the theorem says that any smooth deformation of a semi-umbilic ($L_- = 0$) surface N^2 along its positive oriented normal null geodesics is also positive semi-umbilic. If the ambient Lorentzian manifold M_1^4 has constant sectional curvature it holds moreover, that any deformation of a +isotropic surface N^2 in M_1^4 along its positive oriented normal null geodesics remains +isotropic.

Corollary 3.1 *1. Let M_1^4 be conformally flat. If $P^3 \subset M_1^4$ is totally umbilic and any complete lightlike geodesic intersect the hypersurface P^3, then any positive semi-umbilic surface $N^2 \subset M_1^4$ is locally a deformation of a totally umbilic surface $\tilde{N}^2 \subset P^3 \subset M_1^4$ along the positive oriented normal null geodesics on $\tilde{N}^2$.*

2. If M_1^4 has constant sectional curvature and $P^3 \subset M_1^4$ is totally geodesic and any complete lightlike geodesic intersect the hypersurface P^3, then any +isotropic surface $N^2 \subset M_1^4$ is locally a deformation of a totally geodesic surface $\tilde{N}^2 \subset P^3 \subset M_1^4$ along the positive oriented normal null geodesics on $\tilde{N}^2$.

Proof: 1. For any $n \in N^2$ exists an open neighborhood $U \subset N^2$ and a null field $k \in \Gamma(TM)$, which is tangential to the positive oriented normal null geodesics that start from U an intersect the hypersurface P^3, such that $\phi_{t_1}^k$ is a diffeomorphism between U and

$$\tilde{N}^2 := \phi_{t_1}^k(U) \subset P^3$$

for a suitable t_1. From Theorem 3.3 1., it follows that $\tilde{N}^2 \subset M_1^4$ is positive semi-umbilic and because P^3 is totally umbilic in M_1^4, it follows by Remark 3.3 3. that $\tilde{N}^2 \subset P^3$ is totally umbilic. The open set $U \subset N^2$ is the deformation of the totally umbilic surface $\tilde{N}^2$ in P^3.

2. Let $P^3 \subset M_1^4$ be totally geodesic and let $f : \tilde{N}^2 \hookrightarrow P^3 \subset M_1^4$ be a +isotropic surface in M_1^4. The second fundamental form II of $f : \tilde{N}^2 \hookrightarrow M_1^4$ maps only to TP^3, i.e. II maps only to null directions if $f : \tilde{N}^2 \hookrightarrow P^3$ is totally geodesic. Now, the assertion follows as in 1. by deformation along the positive oriented normal null geodesics on N^2 and Theorem 3.3 2. □

Examples

A. Positive semi-umbilic and +isotropic surfaces in the Minkowski space $\mathbf{R}_1^4$

Consider the flat Minkowski space $\mathbf{R}_1^4 = (\mathbf{R}^4, <,>_1^4)$. The complete totally umbilic spacelike hypersurfaces in $\mathbf{R}_1^4$ are the Euclidean and hyperbolic 3-spaces. The Euclidean 3-spaces in $\mathbf{R}_1^4$ are totally geodesic. We choose the Euclidean 3-space

$$\begin{array}{ccc} P^3 = \mathbf{R}^3 & \hookrightarrow & \mathbf{R}_1^4 \\ (y_1, y_2, y_3) & \mapsto & (0, y_1, y_2, y_3) \end{array}$$

in the Minkowski space $\mathbf{R}^4_1$. Any maximal lightlike geodesic in $\mathbf{R}^4_1$ intersects $P^3 = \mathbf{R}^3$.

The complete totally umbilic surfaces in the Euclidean 3-space $\mathbf{R}^3$ are the 2-spheres and the planes. The conformal immersion

$$\begin{array}{rccl} j: & \mathbf{S}^2 & \hookrightarrow & \mathbf{R}^4_1 \\ & (y_1, y_2, y_3) & \mapsto & (0, \theta y_1, \theta y_2, \theta y_3), \quad y_1^2 + y_2^2 + y_3^2 = 1, \ 0 < \theta \in \mathbf{R}, \end{array}$$

of the 2-sphere is totally umbilic. The null vector field

$$\frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial x_3} + y_3 \frac{\partial}{\partial x_4} \in \Gamma(T\mathbf{R}^4_1|_{j(\mathbf{S}^2)})$$

is normal to $j(\mathbf{S}^2) \subset \mathbf{R}^4_1$ and positive oriented. The positive oriented normal null geodesics that start from the sphere $j(\mathbf{S}^2)$ are given by

$$\gamma_{(a,b,c)}(t) = (t, (t+\theta)a, (t+\theta)b, (t+\theta)c), \quad t \in \mathbf{R}, \ a^2 + b^2 + c^2 = 1.$$

Any deformation of $j(\mathbf{S}^2) \subset \mathbf{R}^4_1$ along the positive oriented normal null geodesics has the form

$$\begin{array}{rccl} j_\lambda: & \mathbf{S}^2 & \hookrightarrow & \mathbf{R}^4_1, \\ & (y_1, y_2, y_3) & \mapsto & (\lambda, (\lambda+\theta) y_1, (\lambda+\theta) y_2, (\lambda+\theta) y_3) \end{array}$$

where λ is a smooth function on $\mathbf{S}^2$. By Theorem 3.3 1., it follows that these conformal immersions of $\mathbf{S}^2$ are complete, semi-umbilic ($L_- = 0$) surfaces in $\mathbf{R}^4_1$.

The isometric immersion

$$\begin{array}{rccl} i: & (\mathbf{R}^2, <,>) & \hookrightarrow & (\mathbf{R}^4_1, <,>^4_1) \\ & (z_1, z_2) & \mapsto & (0, 0, z_1, z_2) \end{array}$$

of the Euclidean plan is totally geodesic. The positive oriented normal null geodesics that start from the plane $i(\mathbf{R}^2)$ are given by

$$\gamma_{(a,b)}(t) = (t, t, a, b), \quad t \in \mathbf{R}.$$

Any immersed surface that is the deformation of $i(\mathbf{R}^2) \subset \mathbf{R}^4_1$ along the positive oriented normal null geodesics on $i(\mathbf{R}^2)$ has the form

$$\begin{array}{rccl} i_\lambda: & \mathbf{R}^2 & \to & \mathbf{R}^4_1 \\ & (z_1, z_2) & \mapsto & (\lambda(z_1, z_2), \lambda(z_1, z_2), z_1, z_2), \end{array}$$

where λ is a smooth function on $\mathbf{R}^2$. The immersion i_λ is isometric and $i_\lambda(\mathbf{R}^2) \subset \mathbf{R}^4_1$ is a complete, semi-umbilic ($L_- = 0$) surface in $\mathbf{R}^4_1$ for any function λ on $\mathbf{R}^2$. From Corollary 3.1, it follows that any positive semi-umbilic surface in $\mathbf{R}^4_1$ is locally an immersion of the form j_λ or i_λ up to an isometry of the Minkowski space $\mathbf{R}^4_1$. The property that a surface is positive semi-umbilic is independent of the conformal class of the ambient Lorentzian 4-space. Therefore, the immersions i_λ and j_λ describe locally any positive semi-umbilic surface in a conformally flat Lorentzian 4-space.

In fact, the immersions of the form i_λ into $\mathbf{R}^4_1$ are not only positive semi-umbilic, but even +isotropic. We calculate explicitly the second fundamental form of $i_\lambda(\mathbf{R}^2)$ in $\mathbf{R}^4_1$. A Darboux frame field on $i_\lambda(\mathbf{R}^2)$ is given in the standard basis $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4})$ of $T\mathbf{R}^4_1$ by

$$e_1 = \begin{pmatrix} 1 + \frac{1}{2}[(\frac{\partial \lambda}{\partial z_1})^2 + (\frac{\partial \lambda}{\partial z_2})^2] \\ \frac{1}{2}[(\frac{\partial \lambda}{\partial z_1})^2 + (\frac{\partial \lambda}{\partial z_2})^2] \\ \frac{\partial \lambda}{\partial z_1} \\ \frac{\partial \lambda}{\partial z_2} \end{pmatrix}, \qquad e_2 = \begin{pmatrix} -\frac{1}{2}[(\frac{\partial \lambda}{\partial z_1})^2 + (\frac{\partial \lambda}{\partial z_2})^2] \\ 1 - \frac{1}{2}[(\frac{\partial \lambda}{\partial z_1})^2 + (\frac{\partial \lambda}{\partial z_2})^2] \\ -\frac{\partial \lambda}{\partial z_1} \\ -\frac{\partial \lambda}{\partial z_2} \end{pmatrix},$$

$$e_3 = \begin{pmatrix} \frac{\partial\lambda}{\partial z_1} \\ \frac{\partial\lambda}{\partial z_1} \\ 1 \\ 0 \end{pmatrix}, \qquad e_4 = \begin{pmatrix} \frac{\partial\lambda}{\partial z_2} \\ \frac{\partial\lambda}{\partial z_2} \\ 0 \\ 1 \end{pmatrix}.$$

The vector field $\frac{1}{2}(e_1 + e_2)$ is a parallel and positive oriented normal null field on $i_\lambda(\mathbf{R}^2)$, which already implies that $i_\lambda(\mathbf{R}^2)$ is $+$isotropic in $\mathbf{R}^4_1$. The second normal null field on $i_\lambda(\mathbf{R}^2)$ is

$$\frac{1}{2}(e_2 - e_1) = \begin{pmatrix} -1 - \frac{1}{2}[(\frac{\partial\lambda}{\partial z_1})^2 + (\frac{\partial\lambda}{\partial z_2})^2] \\ 1 - \frac{1}{2}[(\frac{\partial\lambda}{\partial z_1})^2 + (\frac{\partial\lambda}{\partial z_2})^2] \\ -2\frac{\partial\lambda}{\partial z_1} \\ -2\frac{\partial\lambda}{\partial z_2} \end{pmatrix}.$$

For the covariant derivatives, we obtain:

$$\nabla_{df(\frac{\partial}{\partial z_1})} df(\frac{\partial}{\partial z_1}) = \frac{\partial^2\lambda}{\partial z_1 \partial z_1} \cdot \frac{\partial}{\partial x_1} + \frac{\partial^2\lambda}{\partial z_1 \partial z_1} \cdot \frac{\partial}{\partial x_2},$$

$$\nabla_{df(\frac{\partial}{\partial z_2})} df(\frac{\partial}{\partial z_2}) = \frac{\partial^2\lambda}{\partial z_2 \partial z_2} \cdot \frac{\partial}{\partial x_1} + \frac{\partial^2\lambda}{\partial z_2 \partial z_2} \cdot \frac{\partial}{\partial x_2}.$$

The components of the second fundamental form are

$$h^1_{33} = \frac{\partial^2\lambda}{\partial z_1 \partial z_1}, \qquad h^1_{44} = \frac{\partial^2\lambda}{\partial z_2 \partial z_2},$$

$$h^2_{33} = \frac{\partial^2\lambda}{\partial z_1 \partial z_1}, \qquad h^2_{44} = \frac{\partial^2\lambda}{\partial z_2 \partial z_2},$$

$$h^1_{34} = \frac{\partial^2\lambda}{\partial z_1 \partial z_2}, \qquad h^2_{34} = \frac{\partial^2\lambda}{\partial z_1 \partial z_2},$$

and the second fundamental form II of i_λ is then given by

$$\begin{aligned} II \quad = \quad & <,> \otimes (\ \Delta\lambda\ ) \cdot \frac{e_1 + e_2}{2} \\ + \ & <,> \otimes \left[\frac{1}{2}(e_4^* \circ e_4^* - e_3^* \circ e_3^*) \cdot (-\frac{\partial^2\lambda}{\partial z_1 \partial z_1} + \frac{\partial^2\lambda}{\partial z_2 \partial z_2}) \ + \ \frac{\partial^2\lambda}{\partial z_1 \partial z_2} \cdot (e_3^* \circ e_4^*)\right] \cdot (e_1 + e_2) \end{aligned}$$

Theorem 3.4 *Any complete +isotropic surface in the flat Minkowski space $\mathbf{R}^4_1$ is isometric to $\mathbf{R}^2$ and up to an isometry of $\mathbf{R}^4_1$ a +isotropic surface has (locally) the form*

$$\begin{array}{rccc} i_\lambda : & \mathbf{R}^2 & \rightarrow & \mathbf{R}^4_1, \\ & (z_1, z_2) & \mapsto & (\lambda(z_1,z_2), \lambda(z_1,z_2), z_1, z_2) \end{array}$$

where λ is a smooth function on $\mathbf{R}^2$.

Proof: Any $+$isotropic surface is semi-umbilic ($L_- = 0$). The immersions of the form j_λ are obviously never $+$isotropic. Hence, locally any $+$isotropic surface is of the form i_λ. We need to prove the global statement. The surface $i_\lambda(\mathbf{R}^2) \subset \mathbf{R}^4_1$ is isometric to $\mathbf{R}^2$ for any function λ on $\mathbf{R}^2$. These surfaces are complete. Any other $+$isotropic surface isn't a deformation of a complete Euclidean plan $\mathbf{R}^2$ in $\mathbf{R}^4_1$ and therefore isn't complete $\square$

Remark 3.4 *1. If we consider $\mathbf{R}^2$ as the complex numbers, we may write any complete +isotropic surface in $\mathbf{R}^4_1$ as*

$$\begin{array}{rcc} \mathbf{C} & \rightarrow & \mathbf{R}^4_1, \\ z & \mapsto & (\lambda(z), \lambda(z), z) \cdot A^\perp + v \end{array}$$

where $A \in \mathbf{SO}(3,1)$ and $v \in \mathbf{R}^4_1$. This is similar to the case of superminimal surfaces in the Euclidean 4-space, which can be described as a graph of a holomorphic function f:

$$\begin{array}{rcl} \mathbf{C} & \rightarrow & \mathbf{C} \times \mathbf{C} \cong \mathbf{R}^4 \quad . \\ z & \mapsto & (f(z), z) = (Re(f), Im(f), z_1, z_2) \end{array}$$

2. *If the function $\hat{\lambda}$ on $\mathbf{R}^2$ is a solution of the Laplace equation $\Delta\hat{\lambda} = 0$, the surface $i_{\hat{\lambda}}(\mathbf{R}^2)$ in $\mathbf{R}^4_1$ is complete, stationary and positive semi-umbilic. Any such surface has this form up to an isometry of the Minkowski space $\mathbf{R}^4_1$. For $\lambda = z_1^2 + z_2^2 = |z|^2$, the immersion i_λ has the second fundamental form*

$$II = <,> \otimes (e_1 + e_2),$$

i.e. this immersion is totally umbilic and +isotropic, but not totally geodesic.

3. *The image of the immersion*

$$\begin{array}{cccc} \kappa : & \mathbf{R}_+ \times \mathbf{R} & \hookrightarrow & \mathbf{R}^4_1 \\ & (r, \alpha) & \mapsto & (\alpha, \alpha, r\cos\alpha, r\sin\alpha) \end{array}$$

is a +isotropic surface in $\mathbf{R}^4_1$ that is the universal covering of $i(\mathbf{R}^2\backslash\{0\}) \subset \mathbf{R}^4_1$. This surface isn't complete.

4. *The conformal immersion*

$$\begin{array}{cccl} j_{\lambda_1} : & \mathbf{S}^2\backslash\{(1,0,0)\} & \hookrightarrow & \mathbf{R}^4_1, \\ & (y_1, y_2, y_3) & \mapsto & (\lambda_1, (\lambda_1+1)y_1, (\lambda_1+1)y_2, (\lambda_1+1)y_3) \end{array}$$

with $\lambda_1 = \frac{1}{\sqrt{y_2^2+y_3^2}}$ is positive semi-umbilic. The surface $j_{\lambda_1}(\mathbf{S}^2\backslash\{(1,0,0)\}) \subset \mathbf{R}^4_1$ is also complete.

B. Positive semi-umbilic and +isotropic surfaces in the pseudosphere $\mathbf{S}^4_1$

Consider the pseudosphere $\mathbf{S}^4_1$. Let A denote the subset

$$\{x \in \mathbf{S}^4_1 \subset \mathbf{R}^5_1 : \ x_3 = 1\}$$

of $\mathbf{S}^4_1$. The conformal diffeomorphism

$$\begin{array}{cccc} \Xi : & \mathbf{S}^4_1\backslash A & \leftrightarrow & \mathbf{R}^4_1 \\ & (t,a,b,c,d) & \mapsto & \frac{1}{1-b}(t,a,0,c,d) \\ & \frac{2}{1+\|x\|^2}(x_1, x_2, \frac{\|x\|^2-1}{2}, x_4, x_5) & \leftarrow & x = (x_1, x_2, 0, x_4, x_5) \end{array}$$

is the stereographic projection of the pseudosphere $\mathbf{S}^4_1$ in the point $(0,0,1,0,0) \in \mathbf{S}^4_1$ to the Minkowski space $\mathbf{R}^4_1$. The induced metric of constant sectional curvature 1 on $\mathbf{R}^4$ is

$$\frac{4}{(1+\|x\|^2)^2} \cdot <,>^4_1 = \exp(2\rho) \cdot <,>^4_1,$$

where $\rho = \ln\frac{2}{1+\|x\|^2}$. The image of the isometric imbedding

$$\begin{array}{cccc} \iota : & \mathbf{S}^3 \subset \mathbf{R}^4 & \hookrightarrow & \mathbf{S}^4_1 \quad \subset \mathbf{R}^5_1 \\ & y & \mapsto & (0, y) \end{array}$$

is a complete, totally geodesic spacelike hypersurface in $\mathbf{S}^4_1$. Any maximal lightlike geodesic in $\mathbf{S}^4_1$ intersect this hyperpshere. The complete totally umbilic surfaces in $\mathbf{S}^3$ are the 2-spheres

$$\begin{array}{cccc} i^c : & \mathbf{S}^2 & \hookrightarrow & \mathbf{S}^3 \subset \mathbf{R}^4, \\ & (y_1, y_2, y_3) & \mapsto & (c, \sqrt{1-c^2}\cdot y_1, \sqrt{1-c^2}\cdot y_2, \sqrt{1-c^2}\cdot y_3)\cdot A^\perp, \end{array}$$

where $|c| \leq 1$ and $A \in \mathbf{SO}(4)$. The surface $i(\mathbf{S}^2) := i^0(\mathbf{S}^2) \subset \mathbf{S}^3$ is totally geodesic. The conformal immersion

$$\begin{array}{rccc} i^c_\lambda : & \mathbf{S}^2 & \hookrightarrow & \mathbf{S}^4_1 \\ & y = (y_1, y_2, y_3) & \mapsto & (\lambda, \lambda\sqrt{1-c^2} + c, (\sqrt{1-c^2} - \lambda c)y) \end{array}$$

is by Theorem 3.3 1. for any smooth function λ on $\mathbf{S}^2$ semi-umbilic ($L_- = 0$). And by Corollary 3.1 1., it follows that any positive semi-umbilic surface in $\mathbf{S}^4_1$ is locally up to an isometry of $\mathbf{S}^4_1$ an immersion of the form i^c_λ. Consider the isometric immersion of the form

$$\begin{array}{rccl} i_\lambda := i^0_\lambda : & \mathbf{S}^2 & \hookrightarrow & \mathbf{S}^4_1 \subset \mathbf{R}^5_1 \\ & z & \mapsto & (\lambda, \lambda, z), \quad \|z\| = 1. \end{array}$$

This surface in $\mathbf{S}^4_1$ is the image of the surface $i_{\tilde{\lambda}}(\mathbf{R}^2)$ in $\mathbf{R}^4_1$ with

$$\tilde{\lambda} := \exp(-\rho) \cdot \lambda = \frac{1 + z_1^2 + z_2^2}{2} \cdot \lambda$$

under the inverse of the stereographic projection Ξ:

$$\begin{array}{rccl} \Xi^{-1} \circ i_{\tilde{\lambda}} : & \mathbf{R}^2 & \hookrightarrow & \mathbf{S}^4_1 \\ & (z_1, z_2) & \mapsto & \frac{2}{1+z_1^2+z_2^2}(\tilde{\lambda}, \tilde{\lambda}, \frac{z_1^2+z_2^2-1}{2}, z_1, z_2) \\ & & & = (\lambda, \lambda, \frac{z_1^2+z_2^2-1}{1+z_1^2+z_2^2}, \frac{2z_1}{1+z_1^2+z_2^2}, \frac{2z_2}{1+z_1^2+z_2^2}) \end{array}$$

We use this to calculate the second fundamental form $\tilde{II}$ of the isometric immersion $i_\lambda : \mathbf{S}^2 \hookrightarrow \mathbf{S}^4_1$. It holds (comp. Remark 3.1)

$$\begin{array}{rcl} \tilde{II} & = & <,>^{\mathbf{S}^2} \otimes [\exp(-2\rho)(H - \mathcal{N} grad(\rho))] \ + \ L_+ \\ & = & <,>^{\mathbf{S}^2} \otimes [\Delta^{\mathbf{S}^2}\lambda + 2\lambda] \cdot \frac{1}{2}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) \ + \ L_+. \end{array}$$

This proves that $i_\lambda : \mathbf{S}^2 \hookrightarrow \mathbf{S}^4_1$ is a +isotropic immersion into $\mathbf{S}^4_1$. The immersions of the form i^c_λ, $c \neq 0$, are never +isotropic. We have the

Theorem 3.5 *Any complete +isotropic surface in the pseudosphere $\mathbf{S}^4_1$ is isometric to $\mathbf{S}^2$ and up to an isometry of $\mathbf{S}^4_1$ a +isotropic surface has (locally) the form*

$$\begin{array}{rccl} i_\lambda : & \mathbf{S}^2 & \hookrightarrow & \mathbf{S}^4_1 \\ & y & \mapsto & (\lambda, \lambda, y), \quad \|y\| = 1, \end{array}$$

where λ is a smooth function on $\mathbf{R}^2$.

Proof: We need only to prove the global statement. The surface $i_\lambda(\mathbf{S}^2) \subset \mathbf{S}^4_1$ is compact and complete. Converserly, any +isotropic surface, which isn't a deformation of the complete sphere $i(\mathbf{S}^2) \subset \mathbf{S}^4_1$ can not be complete □

Remark 3.5 *1. If $\hat{\lambda}$ is a spherical function to the eigenvalue -2 of the Laplace operator $\Delta^{\mathbf{S}^2}$ on $\mathbf{S}^2$, the surface $i_{\hat{\lambda}}(\mathbf{S}^2) \subset \mathbf{S}^4_1$ is complete, stationary and semi-umbilic.*

2. It is not true that all +isotropic surfaces in $\mathbf{R}^4_1$ and $\mathbf{S}^4_1$ are identified under stereographic projection. For example the +isotropic surfaces of the form

$$\begin{array}{rccc} i_{\frac{\lambda}{1+\lambda}} : & \mathbf{S}^2 & \hookrightarrow & \mathbf{S}^4_1 \\ & y & \mapsto & (\frac{\lambda}{1+\lambda}, y_1, \frac{\lambda}{1+\lambda}, y_2, y_3) \end{array}$$

are projected to surfaces of the form

$$\begin{array}{rccc} \Xi \circ i_{\frac{\lambda}{1+\lambda}}: & \mathbf{S}^2 & \hookrightarrow & \mathbf{R}^4_1, \\ & y & \mapsto & (\lambda, (\lambda+1)y_1, (\lambda+1)y_2, (\lambda+1)y_3), \end{array}$$

which are only semi-umbilic.

C. Positive semi-umbilic and +isotropic surfaces in the pseudohyperbolic space $\mathbf{H}^4_1$

Consider the pseudohyperbolic 4-space

$$\mathbf{H}^4_1 := \{x \in \mathbf{R}^5_2 : < x, x >^5_2 = -1\} \subset \mathbf{R}^5_2.$$

The image of the imbedding

$$\begin{array}{rccl} \iota: & \mathbf{H}^3 \subset \mathbf{R}^4_1 & \hookrightarrow & \mathbf{H}^4_1 \subset \mathbf{R}^5_2 \\ & (y_1, y_2, y_3, y_4) & \mapsto & (y_1, 0, y_2, y_3, y_4), \quad y_1 > 0, \ \|y\| = -1, \end{array}$$

of the 3-dimensional hyperbolic space $\mathbf{H}^3$ into $\mathbf{H}^4_1$ is a complete spacelike hypersurface in $\mathbf{H}^4_1$, but there are lightlike geodesics in $\mathbf{H}^4_1$ that don't intersect $\iota(\mathbf{H}^3) \subset \mathbf{H}^4_1$. Up to an isometry of $\mathbf{H}^3$, which is an element of $\mathbf{SO}_o(3,1)$, any totally umbilic surface in $\mathbf{H}^3$ has one of the following two forms:

$$\begin{array}{rccc} j: & \mathbf{S}^2 & \hookrightarrow & \mathbf{H}^3, \\ & (z_1, z_2, z_3) & \mapsto & (c, \sqrt{c^2-1}\cdot z_1, \sqrt{c^2-1}\cdot z_2, \sqrt{c^2-1}\cdot z_3), \quad c \geq 1, \end{array}$$

$$\begin{array}{rccl} i: & \mathbf{H}^2 & \hookrightarrow & \mathbf{H}^3. \\ & (z_1, z_2, z_3) & \mapsto & (z_1, 0, z_2, z_3) \end{array}$$

The deformations of the surfaces $j(\mathbf{S}^2) \subset \mathbf{H}^4_1$ and $i(\mathbf{H}^2) \subset \mathbf{H}^4_1$ along the positive oriented normal null geodesics, which start on $j(\mathbf{S}^2)$ resp. $i(\mathbf{H}^2)$, look as follows:

$$\begin{array}{rccl} j_\lambda: & \mathbf{S}^2 & \hookrightarrow & \mathbf{H}^4_1 \\ & z = (z_1, z_2, z_3) & \mapsto & (\lambda, \lambda\sqrt{c^2-1} + c, (\sqrt{c^2-1} + \lambda c)z), \quad c \geq 1, \end{array}$$

and

$$\begin{array}{rccc} i_\lambda: & \mathbf{H}^2 & \hookrightarrow & \mathbf{H}^4_1, \\ & (z_1, z_2, z_3) & \mapsto & (z_1, \lambda, \lambda, z_2, z_3) \end{array}$$

where λ is a smooth function resp. on $\mathbf{S}^2$ and $\mathbf{H}^2$. Any positive semi-umbilic surface in $\mathbf{H}^4_1$ is locally up to an isometry of $\mathbf{H}^4_1$ an immersion of the form j_λ or i_λ. For the second fundamental form of immersions of the form i_λ, we have

$$\tilde{II} = <,>^{\mathbf{H}^2} \otimes \left[\Delta^{\mathbf{H}^2}\lambda - 2\lambda\right] \cdot \frac{1}{2}\left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right) \ + \ L_+.$$

The immersions of the form i_λ are isometric and +isotropic, whereas the immersions j_λ are only semi-umbilic. If $\hat\lambda$ is a solution of the equation $\Delta^{\mathbf{H}^2}\lambda - 2\lambda = 0$, the immersion $i_{\hat\lambda}$ is even stationary.

Theorem 3.6 *Any complete +isotropic surface in $\mathbf{H}^4_1$ is isometric to $\mathbf{H}^2$ and up to an isometry of $\mathbf{H}^4_1$ a +isotropic surface has (locally) the form*

$$\begin{array}{rccl} i_\lambda: & \mathbf{H}^2 & \hookrightarrow & \mathbf{H}^4_1 \\ & (y_1, y_2, y_3) & \mapsto & (y_1, \lambda, \lambda, y_2, y_3), \quad y_1 > 0, \ -y_1^2 + y_2^2 + y_3^2 = -1, \end{array}$$

where λ is a smooth function on $\mathbf{H}^2$.

The classification of the +isotropic surface in the three space forms $\mathbf{R}^4_1$, $\mathbf{S}^4_1$ and $\mathbf{H}^4_1$ prove the second part of Theorem 3.3, which is a local statement.

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