Bloch Theory and Quantization Classes

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Abstract

Quantizing the motion of particles on a Riemannian manifold in the presence of a magnetic field poses the problems of existence and uniqueness of quantizations. Both of them are settled since the early days of geometric quantization but there is still some structural insight to gain from spectral theory. Following the work of Asch, Over & Seiler (1994) for the 2–torus we describe the relation between quantization on the manifold and Bloch theory on its covering space for more general compact manifolds.

Introduction

In geometric quantization for symplectic manifolds one is faced with questions of existence and uniqueness (see e.g. Woodhouse, 1980) which do not arise for the common phase space T^*M (with standard symplectic structure) of Hamiltonian mechanics. But, when incorporating magnetic fields (closed 2-forms $b \in \Omega^2(M)$) into the picture one is forced either to choose magnetic potentials ($a \in \Omega^1(M)$ with $da = b$) or to "charge" the standard symplectic structure by the magnetic field (see remark 1 below). In either case, the questions of existence and uniqueness come up now even for the phase space T^*M. Indeed, these questions arise for prequantizations, whereas — given a prequantization — the choice of a quantization is canonical when the phace space is T^*M with a charged symplectic structure.

On the other hand, the cohomological obstructions and degrees of freedom for geometric quantization vanish on the covering space $X := \tilde{M}$. Since the classical Hamiltonian system may be lifted from M to X one may try to quantize on X and push the quantization down to M again. This push down is possible if and only if the system on M is quantizable. But quantization on X is unique, so one may ask which quantizations on M one gets by this procedure, and how to recover the other quantizations on M from that on X. Since the magnetic Schrödinger operator H arising from a quantization on X is periodic (any operator arising from a periodic classical symbol is) one can, in the case of abelian covering group, analyze it using Bloch theory. This gives a decompostion of H into a direct integral of operators (the "fibers" of H) acting on line bundles over M. It turns out that the fibers are unitarily equivalent to magnetic Schrödinger operators arising from quantizations on M, and that the direct integral runs just over all classes of quantizations on M, using a natural integration measure. This follows the ideas of Asch, Over & Seiler (1994) who did the same work for the 2-torus.

Outline

In section 1 we recall the definitions (quantization of a system with magnetic field, equivalence of quantizations) and the appropriate cohomology groups. All of that is known from the standard literature on geometric quantization, so we will not give references to the results individually.

In section 2 we describe the connections between sets of equivalence classes of quantizations, as determined in the previous section, and representions of the fundamental group.

In section 3 we analyze the Bloch decomposition for Schrödinger operators with magnetic fields, identify the fibers of this decomposition (theorem 6) and draw our final conclusions about the relation to quantization (corollary 1).

In the appendix we recall Bloch theory in the geometric context of periodic operators acting on sections of vector bundles.

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1 Equivalence classes of quantizations

Remark 1 (minimal coupling). Lorentz force is described in Newton's equations of classical mechanics using a magnetic field $b \in C^\infty(T\mathbb{R}^3) \simeq \Omega^2(\mathbb{R}^3)$ ("axial vector field"). When trying to incorporate it into the formalism of Lagrange or Hamilton mechanics, one is faced with the necessity (or, at least, utility) of introducing a vector potential $a \in C^\infty(T\mathbb{R}^3) \simeq \Omega^1(\mathbb{R}^3)$ ("polar vector field") such that[1] $b = da$. A Hamiltonian $h \in C^\infty(T\mathbb{R}^3)$ is replaced by $h_a\colon (x,p) \mapsto h(x, p - qa(x))$ (electronic charge q), the so–called *mimimally coupled Hamiltonian*. Doing this for a free particle ($h(x,p) = \frac{1}{2m}|p|^2$, mass m) one gets $h_a(x,p) = \frac{1}{2m}|p - qa(x)|^2$ which suggests using $H_a = \frac{1}{2m}\left(\frac{\hbar}{\imath}\nabla - a\right)^2$ as Hamiltonian in quantum mechanics, where ∇ denotes gradient in $\mathbb{R}^3$. $\nabla - \frac{\imath}{\hbar}a$ may be viewed as connection on the trivial complex line bundle $\mathbb{R}^3 \times \mathbb{C}$. Note especially that the curvature is given by $\operatorname{curv}(\nabla - \frac{\imath}{\hbar}a) = \frac{1}{\hbar}da = \frac{1}{\hbar}b$ if we identify the Lie algebra of $U(1)$ with $\mathbb{R}$ suitably ($-\imath v \mapsto v \in \mathbb{R}$).

In the case of non-exact magnetic fields (on a manifold M with non-trivial $H^2_{dR}(M)$) one can, in general, only find local vector potentials and connections on locally trivial complex line bundles. If everything fits together "nicely" one gets a global connection on a (global) complex line bundle with curvature $\frac{1}{\hbar}b$. This motivates definition 1.

Another aspect of definition 1 is given by the point of view of geometric quantization. It rests on the observation that Hamiltonian mechanics with (closed) magnetic field $b \in \Omega^2(M)$ can be formulated without any magnetic vector potential if one uses a "charged" symplectic form $\omega_b = \omega + \tilde{b}$ on $N := T^*M$, where ω is the canonical symplectic form on T^*M and $\tilde{b}$ the pull-back π^*b of b from M to T^*M by the projection $\pi\colon T^*M \to M$ onto the base points. A prequantization of such a system is given by a Hermitian line bundle $\tilde{L}$ over T^*M with connection (covariant derivative) $\tilde{\nabla}$ such that $\hbar\operatorname{curv}(\tilde{\nabla}) = \omega_b$. A quantization is a prequantization together with a complex polarization[2] P of N. Since our symplectic manifold is a cotangent space with (vertically) charged symplectic form there is a canonical polarization given by the fibration over M with fiber $(T_xM)_\mathbb{C}$, $x \in M$, the *vertical polarization.*

[1] b is divergence free, i.e. closed; since $H^2_{dR}(\mathbb{R}^3) = 0$ b is exact.

[2] A complex polarization of $N = T^*M$ is a complex distribution (i.e. a family $(P_x)_{x\in N}$ of complex subspaces of the complexified tangent space $TN_\mathbb{C}$, locally defined by smooth frames) with the following properties:

1. Every P_x is Lagrangian with respect to the complexified symplectic structure.
2. $\dim P \cap \bar{P} \cap TN$ is constant on N.
3. P is integrable, i.e. closed with respect to Lie brackets.

To be definite: the corresponding distribution is $V_P = (\ker T\pi)_{\mathbb{C}}$. Polarized sections in $\tilde{L}$ with respect to this polarization can be viewed as sections into a complex line bundle L over M with $\pi^* L = \tilde{L}$. Such L exist because the fibers of $\pi\colon T^*M \to M$ are contractible; L can be constructed as pull-back by the 0-section in T^*M. Finally, $\tilde{\nabla}$ induces a connection ∇ on L with curvature b.

In the sequel we choose units with $\hbar = 1, q = 1, 2m = 1$.

Definition 1 (quantization with magnetic field). Let (M, g) be an orientable Riemannian manifold, $b \in \Omega^2(M)$ a closed real-valued 2-form (the *magnetic field*). A *quantization* of the particle motion on (M, g) in the presence of the magntic field b is given by a Hermitian line bundle (L, h, ∇) over M with connection such that $\operatorname{curv}(\nabla) = b$. The *magnetic Schrödinger operator* is defined by the *Bochner-Laplacian*

$$H^{L,\nabla} := \nabla^\dagger \nabla \text{ with domain } \mathcal{D}(H^{L,\nabla}) = C_0^\infty(L) \tag{1}$$

in the Hilbert space $L^2(L)$ of square-integrable sections of L, defined by g and h. Here, $\nabla^\dagger$ is the formal adjoint of ∇.

Remark 2 (self-adjointness). Since $H^{L,\nabla}$ is symmetric and bounded below (by 0) there is a canonical self-adjoint extension given by the Friedrichs extension $H_F^{L,\nabla}$. It is the self-adjoint operator associated to the closure of the symmetric form

$$q(f, g) := (f, H^{L,\nabla} g) = (\nabla f, \nabla, g)$$

with (form) domain $\mathcal{Q}(q) = \mathcal{D}(H^{L,\nabla})$.

Remark 3 (equivalence classes of line bundles). Denote by $\underline{G}_M$ the sheaf of germs of smooth G-valued functions on M. Every complex line bundle L over M is defined by a Čech cocycle $(g_{\alpha\beta}) \in \check{Z}^1(M, \underline{\mathbb{C}^\times}_M)$. Given any $(l_{\alpha\beta}) \in \check{C}^1(M, \underline{\mathbb{C}}_M)$ with $\exp 2\pi\imath l_{\alpha\beta} = g_{\alpha\beta}$ one has $\delta l \in \check{Z}^2(M, \underline{\mathbb{Z}}_M) = \check{Z}^2(M, \mathbb{Z})$. Here δ denotes Čech codifferential. Other choices l' fulfil $l' - l \in \check{C}^1(M, \mathbb{Z})$, so that δl and $\delta l'$ define the same class in $\check{H}^2(M, \mathbb{Z})$, and the mapping

$$j\colon H^1(M, \underline{\mathbb{C}^\times}_M) \to H^2(M, \mathbb{Z}), g \mapsto \delta l$$

is well-defined.

Every line bundle isomorphism from g to g' corresponds to a Čech cochain $(f_\alpha) \in \check{C}^0(M, \underline{\mathbb{C}^\times}_M)$, $g' = g\, \delta f$.

$H^2(M, \mathbb{Z})$ parametrizes the set of equivalence classes of complex line bundles: The short exact sequence of sheaves

$$\begin{gathered} 0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{C}}_M \xrightarrow{\exp 2\pi\imath\cdot} \underline{\mathbb{C}^\times}_M \to 0, \text{ where} \\ \exp 2\pi\imath\cdot : \mathbb{C} \ni z \mapsto \exp(2\pi\imath z) \in \mathbb{C}^\times, \end{gathered} \tag{2}$$

induces the following long exact sequence in Čech cohomology[3]:

$$\begin{array}{cccc} 0 \to H^0(M,\underline{\mathbb{Z}}_M) \to & H^0(M,\underline{\mathbb{C}}_M) \to & H^0(M,\underline{\mathbb{C}^\times}_M) \to & \\ \to H^1(M,\underline{\mathbb{Z}}_M) \to & H^1(M,\underline{\mathbb{C}}_M) \to & H^1(M,\underline{\mathbb{C}^\times}_M) \xrightarrow{j} & \\ \| & \| & & \\ H^1(M,\mathbb{Z}) & 0 & & \\ \xrightarrow{j} H^2(M,\underline{\mathbb{Z}}_M) \to & H^2(M,\underline{\mathbb{C}}_M) \to & \dots & \\ \| & \| & & \\ H^2(M,\mathbb{Z}) & 0 & & \end{array} \tag{3}$$

So $H^i(M,\underline{\mathbb{C}^\times}_M) \overset{j}{\simeq} H^{i+1}(M,\mathbb{Z})$ for every $i \geq 1$, and the joining homomorphism j is just the mapping described before. The class in $H^2(M,\mathbb{Z})$ characterizing L is called the *first Chern class* $c_1(L)$ of L.

Every Hermitian line bundle (L,h) is defined by a $(g_{\alpha\beta}) \in \check{Z}^1(M,\underline{S^1}_M)$, every Hermitian line bundle isomorphism (i.e. every isometry) by some $(f_\alpha) \in \check{C}^0(M,\underline{S^1}_M)$, $g' = g\,\delta f$. Using the short exact sequence

$$0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{R}}_M \xrightarrow{\exp 2\pi\imath\cdot} \underline{S^1}_M \to 0 \tag{4}$$

and the corresponding long exact sequence in Čech cohomology one gets again $H^i(M,\underline{S^1}_M) \overset{j}{\simeq} H^{i+1}(M,\mathbb{Z})$ for $i \geq 1$, and j comes from the mapping $\delta \circ \frac{\log\cdot}{2\pi\imath}$ on cochains as before.

Finally we recall that the group structure induced on $H^1(M,\underline{\mathbb{C}^\times}_M)$ and $H^1(M,\underline{S^1}_M)$ by the coefficient groups is just the tensor product of line bundles.

Remark 4 (integral de Rham class). The short exact sequence of groups

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{R} \xrightarrow{\exp 2\pi\imath\cdot} S^1 \to 0 \tag{5}$$

[3]Every fine sheaf is acyclic.

induces the long exact sequence of cohomology groups

$$
\begin{array}{llll}
0 \to H^0(M,\mathbb{Z}) \xrightarrow{H^0(i)} H^0(M,\mathbb{R}) \to H^0(M,S^1) \to & \\
& & & \searrow \\
& & & \quad 0 \\
\to H^1(M,\mathbb{Z}) \xrightarrow{H^1(i)} H^1(M,\mathbb{R}) \to H^1(M,S^1) \to & \\
\nearrow & & & \\
0 & & & \\
\xrightarrow{i} H^2(M,\mathbb{Z}) \xrightarrow{H^2(i)} H^2(M,\mathbb{R}) \to \quad \ldots
\end{array}
\tag{6}
$$

A de Rham class is called *integral* if it is contained in the range of $H^*(i)$.

Remark 5 (curvature and Chern class). For every line bundle with connection one has $H^*(i)(c_1(L)) = [-\frac{1}{2\pi}\operatorname{curv}(\nabla)]$, using the identifaction $-\imath\mathbb{R} \simeq \mathbb{R}$ as in the introduction. This can be seen for example using Deligne cohomology with coefficients in $\mathbb{R}(2) := (2\pi\imath)^2\mathbb{R}$ (see Brylinski, 1993, Chap. 1 for these notions): Let $\mu = \delta\left(\frac{\log g}{2\pi\imath}\right) \in \check{Z}^2(M,\mathbb{Z})$ as in remark 3 represent $c_1(L)$ for some choice of logarithms $\log_{\alpha\beta}$. This defines a cocycle in $\check{Z}^2(M,\mathbb{R}(2)^\infty_D)$ given by $(-(2\pi\imath)^2\mu, -2\pi\imath\log g, -2\pi a)$, and from a proposition on Deligne cohomology groups $H^p(M,\mathbb{R}(p)^\infty_D)$ (ibidem, Lemma 1.5.4) one gets $-(2\pi\imath)^2 H^*(i)([\mu]) = -2\pi[da] \in H^2(M,\mathbb{R})$ using the Čech–de Rham isomorphism.

This connection between curvature and Chern class immmediately implies

Theorem 1 (existence of quantizations). *A system with magnetic field (M,g,b) is quantizable if and only if the de Rham class of $\frac{1}{2\pi}b$ is integral.*

Definition 2 (equivalence of quantizations). Two quantizations given by (L,h,∇) and (L',h',∇') are called *equivalent* if there is a Hermitian line bundle isomorphism $\Phi\colon L \to L'$ intertwining the connections:

$$\forall s \in C^\infty(L) : \forall X \in C^\infty(TM) : \Phi \circ \nabla_X s = \nabla'_X(\Phi \circ s) \tag{7}$$

Remark 6 (unitary equivalence). If (L,h,∇) and (L',h',∇') are two quantizations equivalent via Φ, then

$$
\begin{aligned}
U_\Phi : L^2(L) &\to L^2(L'), \\
s &\mapsto U_\Phi s := \Phi \circ s,
\end{aligned}
$$

defines a unitary operator intertwining the magnetic Schrödinger operators:

$$U_\Phi H^{L,\nabla} = H^{L',\nabla'} U_\Phi$$

Conversely, if U_Φ is unitary then Φ is a Hermitian isomorphism. Equation 7 is just the intertwining property for first order operators defined as quantizations of vector fields.

Remark 7 (local form of the gauge). We choose a cochain $f \in \check{C}^0(M, \underline{S^1}_M)$ representing the isomorphism Φ, i.e. $\varphi'_\alpha \circ \Phi \circ {\varphi_\alpha}^{-1} = \mathrm{id}_M \times f_\alpha$, and cocycles (g, a) and (g', a') for (L, h, ∇) and (L', h', ∇') with respect to bundle charts $\varphi_\alpha \colon L|_{U_\alpha} \to U_\alpha \times \mathbb{C}$ and $\varphi'_\alpha \colon L'|_{U_\alpha} \to U_\alpha \times \mathbb{C}$. Then one easily calculates

$$\imath(a'_\alpha - a_\alpha) = f_\alpha^{-1} df_\alpha = d \log f_\alpha. \tag{8}$$

Remark 8 (2-term complex). The "second half" of the condition for the Deligne cocycle in remark 5, i.e. $-\imath(\delta a)_{\alpha\beta} = -d \log g_{\alpha\beta}$, can be viewed as cochain condition in the 2-term complex of sheaves

$$K := \begin{cases} K^0 := \underline{S^1}_M \\ \quad\quad \big\downarrow {\scriptstyle \imath d \log \cdot} \\ K^1 := \underline{\Omega^1}_M \end{cases} \tag{9}$$

Here $\underline{\Omega^1}_M$ denotes the sheaf of (real-valued) 1-forms on M. $(g, -a)$ defines a cocycle, hence it defines a class in the hypercohomology $H^1(M, K)$ of K; in (Brylinski, 1993, chapter 2) it is shown that this class does not depend on the choice of line bundle isomorphism $g_{\alpha\beta}$ and connection forms a_α; moreover, it parametrizes isomorphism classes of line bundles with connection:

Theorem 2 (quantization classes). *The set of Hermitian isomorphism classes of Hermitian line bundles with connection on a Riemannian manifold M is given by the hypercohomology group $H^1(M, K)$ of the complex of sheaves K (9).*

Since we are interested in quantizations for a given magnetic field, we will elaborate on isomorphism classes for fixed L and b:

Theorem 3 (quantization classes for fixed line bundle). *Let (M, g, b) be a quantizable system with magnetic field and L a complex line bundle over M with $H^*(i)\,(c_1(L)) = [-\frac{1}{2\pi} b]$. Then the set of equivalence classes of quantizations (L, h, ∇) of (M, g, b) for fixed (L, h) is given by $H^1(M, \mathbb{R})/H^1(M, \mathbb{Z})$.*

Proof. The set of Hermitian connections is parametrized by $\Omega^1(M)$ since two Hermitian connections differ by an imaginary 1-form $-\imath\eta$. Because $\operatorname{curv}(\nabla) = \operatorname{curv}(\nabla - \imath\eta) = \operatorname{curv}(\nabla) + d\eta$ we have $d\eta = 0$, so $\eta = dk_\alpha$ for a suitable bundle atlas and $k_\alpha \in \check{C}^0(M, \underline{\mathbb{R}})$. Two quantizations (L, h, ∇) and $(L, h, \nabla' = \nabla + \imath\eta)$ are equivalent if and only if there is a Hermitian line bundle isomorphism with

$$\imath(a'_\alpha - a_\alpha) = f_\alpha^{-1} df_\alpha$$

(see (8)). Therefore $\eta = a' - a = -\imath d \log f$. On the other hand, using the Bockstein homomorphism $j = \delta \circ \frac{\log}{2\pi\imath} : H^0(M, \underline{S^1}_M) \to H^1(M, \mathbb{Z})$ one has

$$g' = g\delta f = g \Leftrightarrow \delta f = 1 \Rightarrow j([f]) \in H^1(M, \mathbb{Z}),$$

and such f exist if and only if η is integral. So the sequence

$$0 \to H^1(M, \mathbb{Z}) \to H^1(M, \mathbb{R}) \to \Omega^{1,\text{closed}}(M)/\sim \to 0$$

is exact; here two closed 1-forms η_1, η_2 are equivalent ("$\sim$") if the connections $\nabla - \imath\eta_1$ and $\nabla - \imath\eta_2$ are equivalent. □

Definition 3 (Jacobi torus). $J(M) := H^1(M, \mathbb{R})/H^1(M, \mathbb{Z})$ is called the *Jacobi torus* of M. The metric on M induces a metric on $H^1(M, \mathbb{R})$ and $H^1(M, \mathbb{Z})$ via

$$(\eta, \omega) := \int_M \eta \wedge *\omega.$$

$J(M)$ carries the quotient topology.

Definition 4 (flat line bundle). A line bundle is called *flat* if there is a bundle atlas with locally constant transition functions.

Lemma 1 (classes of flat line bundles). *The group (w.r.t. tensor product) of classes of flat line bundles on a manifold M is isomorphic to the grouup $H^1(M, S^1)$.*

Proof. Flat line bundles are just locally constant line bundles. Thus a line bundle cocycle is a Čech 1-cocycle with values in the locally constant S^1-valued functions. Čech coboundaries are exactly the isomorphisms of flat line bundles so that the set of classes of flat line bundles corresponds to the set of classes of Čech 1-cocycles. Finally, the cocycle of a tensor product is given by the product of the cycles of the factors. □

Theorem 4 (quantization classes). *For a Riemannian manifold* (M, g) *with quantizable magnetic field b the set of equivalence classes of quantizations* (L, h, ∇) *corresponds to* $H^1(M, S^1)$.

Proof. For a given choice (L_1, h_1, ∇_1) of a quantization every quantization (L_2, h_2, ∇_2) is – modulo equivalence – given by

$$\begin{aligned}
(L_2, h_2, \nabla_2) &\simeq (L_1 \otimes L_{12}, h_1 \otimes h_{12}, \nabla_1 \otimes \mathrm{id}_{L_{12}} + \mathrm{id}_{L_1} \otimes \nabla_{12}) \text{ with} \\
L_{12} &= L_1^* \otimes L_2, \\
h_{12} &= \overline{h_1} \otimes h_2, \\
\nabla_{12} &= \nabla_{1^*} \otimes \mathrm{id}_{L_2} + \mathrm{id}_{L_1^*} \otimes \nabla_2.
\end{aligned}$$

Therefore the characterization of flat line bundle following lemma 1 gives the set of quantization classes. □

2 Connections

First we will identify the Jacobi torus with the connected component of the unit in the group of one-dimensional unitary representations of the fundamental group of of M:

Lemma 2 (Jacobi torus).

$$H^1(M, \mathbb{R})/H^1(M, \mathbb{Z}) \simeq \left(\widehat{\pi_1(M)}\right)_0 \tag{10}$$

Proof. For every manifold M, $H := H_1(M, \mathbb{Z})$ is the abelization of $\Gamma := \pi_1(M)$ so that $\hat{H} = \hat{\Gamma}$. As in (Katsuda & Sunada, 1987) we define the mapping

$$\begin{aligned}
\Omega^{1,\text{closed}}(M) \ni \omega &\mapsto \chi_\omega \in \hat{H}, \\
\chi_\omega(\gamma) &:= \exp\left(2\pi\imath \int_{c(\gamma)} \omega\right),
\end{aligned} \tag{11}$$

for a closed path $c(\gamma)$ representing the class γ. The integral does not depend on the choice of path since ω is closed. On exact forms, the integral over closed paths vanishes so that we obtain a well-defined mapping

$$H^1(M, \mathbb{R}) \ni [\omega] \mapsto \chi_\omega \in \hat{H}. \tag{12}$$

It is a homomorphism of groups because $\chi_\omega(\gamma)\chi_{\omega'}(\gamma) = \chi_{\omega+\omega'}(\gamma)$. The kernel consists of the (classes of) closed 1-forms ω for which $\int_c \omega$ is integral for all closed paths c, i.e. just (classes of) integral 1-forms.

(11) is continuous for every γ and thus defines a continuous mapping into $\hat{H}$. Since $H^1(M,\mathbb{R})$ is connected the range of (12) is connected, and it contains the trivial character as image of the zero class. $\square$

Lemma 3 (torsion torus). *The isomorphism*

$$\widehat{\pi_1(M)} \simeq H^1(M, S^1) \tag{13}$$

can be realized geometrically by association of flat line bundles:

$$\chi \mapsto F_\chi = \tilde{M} \times_\chi \mathbb{C} \tag{14}$$

Proof. Equality follows from the universal coefficient theorem (see e.g. Bott & Tu, 1995, chapter 15)

$$H^1(M, S^1) = \mathrm{Hom}(H_1(M,\mathbb{Z}), S^1) \oplus \mathrm{Ext}(H_0(M,\mathbb{Z}), S^1),$$

since $H_0(M,\mathbb{Z})$ is free ($\Rightarrow \mathrm{Ext}(H_0(M,\mathbb{Z}), S^1)$ trivial) and $\pi_1(M)$ has the same one-dimensional representaions as its abelization $H_1(M,\mathbb{Z})$.

By lemma 1 $H^1(M,S^1)$ is the set of classes of flat line bundles with respect to "flat equivalence". On the other hand, flat vector bundles are just the vector bundles which are associated to a representation of the fundamental group. Therefore, flat line bundles correspond to bundles associated to one-dimensional representations of the fundamental group:

$$H^1(M,S^1) \simeq \{\tilde{M} \times_\chi \mathbb{C} \mid \chi \in \widehat{\pi_1(M)}\}/\sim$$

On $\tilde{M} \times_\chi \mathbb{C}$ the natural flat connection is given by restriction of the canonical connection d of the trivial bundle $\tilde{M} \times \mathbb{C}$.

On the other hand, given a flat line bundle one gets back the character χ as holonomy of the connections around closed paths[4]. Since connection forms are invariant under flat equivalence the holonomy gives a well-defined mapping of $H^1(M,S^1)$ into $\widehat{\pi_1(M)}$ which obviously is inverse to the mapping "associating to $\tilde{M}$". $\square$

[4]For a flat connection on a complex line bundle L parallel transport around a closed path depends only on the homotopy class of the path and therefore defines a unitary representation ρ of $\pi_1(M)$. Thus parallel transport gives a line bundle isomorphism $L \simeq \tilde{M} \times_\rho \mathbb{C}$.

Remark 9 (torsion torus). By lemma 2 the Jacobi torus is just $\left(\widehat{\pi_1(M)}\right)_0$. Decomposing Γ into free (finitely generated) and (finite) torsion parts one sees that characters in $\left(\widehat{\pi_1(M)}\right)_0$ are just the ones vanishing on the torsion part. The subsequence

$$0 \to H^1(M,\mathbb{Z}) \xrightarrow{H^1(i)} H^1(M,\mathbb{R}) \xrightarrow{H^1(\exp 2\pi\imath\cdot)} H^1(M,S^1)$$

of the exact sequence (6) shows that the Jacobi torus is embedded in $H^1(M,S^1)$ and does not contain torsion elements. Therefore $H^1(M,S^1)$ is the "torsive version" of the Jacobi torus, hence its name.

3 Periodic magnetic fields

From now on we assume the existence of a free isometric properly discontinuous action of a discrete group Γ on the Riemannian manifold X. We assume the action to be cocompact in the sense that the quotient manifold $M := \Gamma\backslash X$ is compact. Furthermore, let $b \in \Omega^2(X)$ be a quantizable periodic magnetic field so that

$$\begin{aligned} db &= 0, \\ \left[\frac{1}{2\pi}b\right] &\in H^*(i)\left(H^2(X,\mathbb{Z})\right) \subset H^2(X,\mathbb{R}), \\ b &= \pi^* b_M \text{ for a } b_M \in \Omega^2(M), \end{aligned}$$

where

$$\pi : X \to M$$

is the projection. The main point is that integrality of $\frac{1}{2\pi}b_M$ is not automatic: For a two-dimensional manifold Xintegrality of $\frac{1}{2\pi}b_M$ means integrality of the magnetic flux $\int_M b_M$ through one elementary lattice cell, whereas $\frac{1}{2\pi}b$ is integral automatically if e.g. $H^2(X,\mathbb{Z}) = 0$.

Theorem 5 (periodic magnetic Schrödinger operator). *If $\frac{1}{2\pi}b_M$ is integral then there exists a quantization (L,h,∇) on X such that the corresponding magnetic Schrödinger operator $H^{L,\nabla}$ is Γ-periodic.*

Proof. If $\frac{1}{2\pi}b_M$ is integral there is a Hermitian line bundle L' over M with connection ∇' by theorem 1, so that $\mathrm{curv}(\nabla') = b_M$. L' and ∇' can be pulled back via π from M to X, giving a line bundle $L = \pi^* L'$ over X with connection ∇ and curvature b.

The Γ-action on X induces a Γ-action on L: Let $\alpha_\gamma : X \to X$ be the action of $\gamma \in \Gamma$ on X. Then $\pi \circ \alpha_\gamma = \pi$, and therefore

$$\alpha_\gamma^* L = \alpha_\gamma^* \pi^* L' = \pi^* L' = L.$$

Thus, $L = \{(x, l) \in X \times L' \mid l \in L'_{\pi(x)}\}$ carries a natural Γ-action by acting on the first component, using the action on X.

Since ∇ is lifted by π it is automatically Γ-periodic: because of the proper discontinuity of the Γ-action every finite covering of M by open sets induces a locally finite covering of X by Γ-invariant open sets, and the connection forms of M can be pulled back to periodic forms on X. □

Usually one adds a smooth, periodic function V (the "electric potential") to get the full Schrödinger operator. The resulting operator is periodic and elliptic, therefore we can apply appropriate analytic methods. Especially, $H^{L,\nabla}$ is essentially self-adjoint. If Γ is abelian we have the Bloch decomposition:

Theorem 6 (direct integral). *For abelian Γ the fibers of $\overline{H^{L,\nabla}}$ are given by*

$$\begin{aligned}
\left(\overline{H^{L,\nabla}}\right)_\chi &= \overline{H_\chi^{L,\nabla}} \text{ with} && (15)\\
\mathcal{D}\left(H_\chi^{L,\nabla}\right) &= C^\infty(L_\chi) = C^\infty(L)^{\Gamma,\chi}, && (16)\\
L_\chi &= L' \otimes F_\chi, && (17)\\
H_\chi^{L,\nabla} &= H^{L,\nabla}|_{C^\infty(L)^{\Gamma,\chi}} && (18)\\
&= H^{L_\chi,\nabla_\chi}, && (19)
\end{aligned}$$

where

$$\nabla_\chi = (\nabla' \otimes \mathrm{id} + \mathrm{id} \otimes d)|_{C^\infty(L)^{\Gamma,\chi}} \qquad (20)$$

In other words: every fiber of the magnetic Schrödinger operator $H^{L,\nabla}$ is a magnetic Schrödinger operator of type H^{L_χ,∇_χ}.

Proof. By 7 $H^{L,\nabla}$ is decomposable, and equations (15)–(18) follow immediately. Equation (19) follows for $\chi = 1$ from Leibniz's rule for connections since $F_\chi = M \times \mathbb{C}$ in this case. For all χ (20) defines, as we have seen in the proof of lemma 3, a connection for the quantization class characterized by χ following theorem 4. Moreover, the explicit form shows that

$$\nabla|_{C^\infty(L)^{\Gamma,\chi}} = \nabla_\chi$$

since ∇_χ does not depend on χ explicitly. Therefore

$$H^{L,\nabla}|_{C^\infty(L)^{\Gamma,\chi}} = H^{L_\chi,\nabla_\chi},$$

and the proof is completed by (18). $\square$

Corollary 1 (Bloch theory and quantization). *Let (M, g, b) be a quantizable system with magnetic field over a compact manifold M. Then the corresponding system $(\tilde{M}, \tilde{g}, \tilde{b})$ is (up to equivalence) uniquely quantizable on the universal covering space. Moreover, if $\Gamma := \pi_1(M)$ is abelian then the magnetic Schrödinger operator $H^{\tilde{L},\tilde{\nabla}}$ on $L^2(\tilde{M})$ is decomposable over $\hat{\Gamma}$, and the fibers occuring are just the equivalence classes of quantizations of (M, g, b):*

$$\overline{H^{\tilde{L},\tilde{\nabla}}} = \int^{\oplus}_{\widehat{\pi_1(M)}} \overline{H^{L_\chi,\nabla_\chi}}\, d\chi \tag{21}$$

"unique quantization above" = "sum over all quantizations downstairs"

Proof. The system $(\tilde{M}, \tilde{g}, \tilde{b})$ is periodic and quantizable by construction. Since obviously $H^1(\tilde{M}, S^1) = \{1\}$ the quantization is unique up to equivalence. For abelian Γ we can apply theorem 6 from which, together with theorem 4, we get the conclusion. $\square$

Remark 10 (non-abelian fundamental group). Even if $\pi_1(M)$ is non-abelian one may choose a homology covering space X of M such that the covering group is abelian ($H_1(M, \mathbb{Z})$) and $\pi_1(X)$ is finite (the torsion part). Now there is a finite number of classes of quantizations on X, the set of classes of quantizations on M has a finite number of components. Bloch analyzing a quantization on X now gives all quantizations on M belonging to one component of $\widehat{\pi_1(M)}$, generalizing the previous corollary.

Appendix on Bloch theory

In this appendix we recall the basic elements of Bloch theory for periodic operators in the geometric context of vector bundles. The standard reference for the theory of direct integrals is (Dixmier, 1957), for Bloch theory in Euclidean space see (Reed & Simon, 1978).

Our general assumptions are: X is an oriented smooth Riemannian manifold without boundary, Γ a discrete abelian group acting on X freely, isometrically, and properly discontinuously. Furthermore, we assume the action to be cocompact in the sense that the quotient $M := X/\Gamma$ is compact.

Next, let E be a smooth Hermitian vector bundle over X.

Definition 5 (periodic operator). Assume there is an isometric lift γ_* of the action of γ fom X to E in the following sense:

$$\gamma_* : E_x \to E_{\gamma x} \text{ for } x \in X, \gamma \in \Gamma. \tag{22}$$

This defines an action T_γ on the sections: For $s \in C_c^\infty(E)$ we define

$$(T_\gamma s)(x) := \gamma_* s(\gamma^{-1}x) \text{ for } x \in X, \gamma \in \Gamma. \tag{23}$$

$(T_\gamma)_{\gamma\in\Gamma}$ induces a unitary representation of Γ in $L^2(E)$.

A differential operator D on $\mathcal{D}(D) := C_c^\infty(E)$ is called periodic if, on $\mathcal{D}(D)$, we have:

$$\forall \gamma \in \Gamma : [T_\gamma, D] = 0 \tag{24}$$

Lemma 4 (associated bundle). *E is the lift π^*E' of a Hermitian vector bundle E' over M by the projection $\pi : X \to M$. E and X are Γ-principal fiber bundles over E' resp. M.*

To every Γ-principal fiber bundle and every character $\chi \in \hat{\Gamma}$ we associate a line bundle. This gives the relations depicted in the following diagram ("$\rightsquigarrow$" denotes association of line bundles.):

$$\begin{tikzcd}
 & \mathbb{C}^N \arrow[d] & \mathbb{C}^N \arrow[d] & & & \mathbb{C}^N \arrow[d] & \mathbb{C}^N \arrow[d] \\
\Gamma \arrow[r, hook] & E \arrow[r] \arrow[d] & E' \arrow[d] & \rightsquigarrow & \mathbb{C} \arrow[r, hook] & E_\chi \arrow[r] \arrow[d] & E' \arrow[d] \\
\Gamma \arrow[r, hook] & X \arrow[r] & M & \rightsquigarrow & \mathbb{C} \arrow[r, hook] & F_\chi \arrow[r] & M
\end{tikzcd}$$

principal fiber bundels and associated line bundles

In this situation we have $E_\chi \simeq E' \otimes F_\chi$.

We define for every character $\chi \in \hat{\Gamma}$ a mapping $\Phi_\chi : C_c^\infty(E) \ni s \mapsto \tilde{s}_\chi \in C^\infty(E)$ by

$$\tilde{s}_\chi(x) := \sum_{\gamma\in\Gamma} \chi(\gamma)\gamma_* s(\gamma^{-1}x). \tag{25}$$

Since

$$
\begin{aligned}
\tilde{s}_\chi(\gamma' x) &= \sum_{\gamma\in\Gamma} \chi(\gamma)\gamma_* s(\gamma^{-1}\gamma' x) \\
&= \sum_{\gamma\in\Gamma} \chi(\gamma'\gamma'^{-1}\gamma)(\gamma'\gamma'^{-1}\gamma)_* s\left((\gamma'^{-1}\gamma)^{-1}x\right) \\
&= \chi(\gamma')\gamma'_* \tilde{s}_\chi(x)
\end{aligned}
$$

we have

$$
\tilde{s}_\chi \in C^\infty(E)^{\Gamma,\chi} = \{r \in C^\infty(E) \mid \forall_{\gamma\in\Gamma} T_\gamma r = \chi(\gamma) r\}
$$

which defines a section $s_\chi \in C^\infty(E_\chi)$. One easily calculates

$$
\int_{\hat{\Gamma}} \|s_\chi\|^2_{L^2(E_\chi)} d\chi = \|s\|^2_{L^2(E)}
$$

which shows that Φ is an isometry from $L^2(E)$ to $\int_{\hat{\Gamma}}^{\oplus} L^2(E_\chi)\, d\chi$. The inverse is given by

$$
\Phi^* \colon (s_\chi)_{\chi\in\hat{\Gamma}} \mapsto \int_{\hat{\Gamma}} \tilde{s}_\chi(x)\, d\chi.
$$

This shows

Lemma 5 (direct integral). *The mapping defined by* (25) *can be extented continuously to a unitary*

$$
\Phi : L^2(E) \to \int_{\hat{\Gamma}}^{\oplus} L^2(E_\chi)\, d\chi. \tag{26}
$$

For the direct integral of Hilbert spaces $H = \int_{\hat{\Gamma}}^{\oplus} H_\chi d\chi$ the set of decomposable bounded operators $L^\infty(\hat{\Gamma}, \mathcal{L}(H))$ is given by the commutant $(L^\infty(\hat{\Gamma}, \mathbb{C}))'$ in $\mathcal{L}(H)$. Since commutants are weakly closed and $C(\hat{\Gamma}, \mathbb{C})$ is weakly dense in $L^\infty(\hat{\Gamma}, \mathbb{C})$ one has $(L^\infty(\hat{\Gamma}, \mathbb{C}))' = (C(\hat{\Gamma}, \mathbb{C}))'$. Therefore, in order to determine the decomposable operators one has to determine the action of $C(\hat{\Gamma})$ on $L^2(E)$. This is easily done using the explicit form of Φ:

Proposition 1 ($C(\hat{\Gamma})$-action). *$f \in C(\hat{\Gamma})$ acts on $s \in C_c^\infty(E)$ by*

$$
M_f s := \Phi^* f \Phi s, \tag{27}
$$

and one has

$$(M_f s)(x) = \sum_{\gamma\in\Gamma} \hat{f}(\gamma^{-1}) T_\gamma s(x), \text{ where} \tag{28}$$

$$\hat{f}(\gamma) := \int_{\hat{\Gamma}} f(\chi)\bar{\chi}(\gamma)\, d\chi \tag{29}$$

is the Fourier transform of f. M_f is a bounded operator with norm $\|f\|_\infty$.

Corollary 2 (decomposable operators). *Conjugation by Φ defines an isomorphism between decomposable bounded operators on $\int_{\hat{\Gamma}}^{\oplus} L^2(E_\chi)\, d\chi$ and Γ-periodic bounded operators on $L^2(E)$.*

An unbounded operator is decomposable if and only if its (bounded) resolvent is decomposable. For a periodic symmetric elliptic operator D we have a domain of definition $\mathcal{D}(D) = C_c^\infty(X)$ on which D is essentially self-adjoint. This domain is invariant for D as well as for the Γ-action, and one has $[D, \gamma] = 0$ for all $\gamma \in \Gamma$. Thus all bounded functions of D commute with the Γ-action, and one has:

Theorem 7 (decomposition of periodic operators). *The closure $\bar{D}$ of every periodic symmetric elliptic operator D is decomposable with respect to the direct integral $\int_{\hat{\Gamma}}^{\oplus} L^2(E_\chi)\, d\chi$. A core for the domain of $\bar{D}_\chi$ is given by $C^\infty(E_\chi)$, and the action of D_χ on $C^\infty(E_\chi) \simeq C^\infty(E)^{\Gamma,\chi}$ is just the action of D as differential operator on $C^\infty(E)^{\Gamma,\chi}$. We have $\bar{D}_\chi = \overline{D_\chi}$, where $D_\chi := D|_{C^\infty(E)^{\Gamma,\chi}}$, and the closures are to be taken as operators in $L^2(E_\chi)$.*

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