A geometric estimate for a periodic Schrödinger operator whose potential is the curvature of a spherical curve. *

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Abstract

We estimate from below by geometric data the eigenvalues of the periodic Sturm-Liouville operator $-4\frac{d^2}{ds^2} + \kappa^2(s)$ with potential given by the curvature of a closed curve.

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1 Introduction

Let $X^3(c)$ be a 3-dimensional space form of constant curvature $c = 0$ or 1 and admitting a real Killing spinor with respect to some spin structure. Consider a compact, oriented and immersed surface $M^2 \subset X^3(c)$ with mean curvature H. The spin structure of $X^3(c)$ induces a spin structure on M^2. Denote by D the corresponding Dirac operator acting on spinor fields defined over the surface M^2. The first eigenvalue $\lambda_1^2(D)$ of the operator D^2 and the first eigenvalue μ_1 of the Schrödinger operator $\Delta + H^2 + c$ are related by the inequality

$$\lambda_1^2(D) \leq \mu_1(\Delta + H^2 + c).$$

Equality holds if and only if the mean curvature H is constant (see [1], [5]). Moreover, the Killing spinor defines a map $f \mapsto \Phi(f)$ of the space $L^2(M^2)$ of functions into the space $L^2(M^2; S)$ of spinors such that

$$||D(\Phi(f))||_{L^2}^2 = \langle \Delta f + H^2 f + cf, f \rangle_{L^2}.$$

In particular, the mentioned inequality holds for all eigenvalues, i.e.,

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$$\lambda_k^2(D) \leq \mu_k(\Delta + H + c).$$

This inequality was used in order to estimate the first eigenvalue of the Dirac operator defined on special surfaces of Euclidean space (see [1]). On the other hand, in case we know $\lambda_1^2(D)$, the inequality yields a lower bound for the spectrum of the Schrödinger operator $\Delta + H^2 + c$. For example, for any Riemannian metric g on the 2-dimensional sphere S^2 the inequality

$$\lambda_1^2(D) \geq \frac{4\pi}{\mathrm{vol}\,(S^2, g)}$$

holds (see [2], [6]). Consequently, we obtain

$$\frac{4\pi}{\mathrm{vol}\,(M^2, g)} \leq \mu_1(\Delta + H^2)$$

for any surface $M^2 \hookrightarrow \mathbb{R}^3$ of genus zero in Euclidean space $\mathbb{R}^3$. In this note we expose the described idea and, in particular, we estimate the spectrum of special periodic Schrödinger operators where the potential is given by the curvature κ of a spherical curve.

2 The one-dimensional case

First of all, let us consider the 1-dimensional case, i.e., a curve γ of length L in a two-dimensional space form $X^2(c)$. Let Φ be a Killing spinor of length one on $X^2(c)$:

$$\nabla_T \Phi = \frac{1}{2} c \cdot T \cdot \Phi.$$

The restriction $\varphi = \Phi_{|\gamma}$ defines a pair of spinors on γ and the spinor field $\psi = f \cdot \varphi$ satisfies the equation:

$$|D\psi|^2 = |\dot{f}|^2 + f^2 \left(\frac{c}{4} + \frac{1}{4}\kappa_g^2 \right),$$

where κ_g is the curvature of the curve γ in $X^2(c)$. Therefore, we obtain

$$\lambda_k^2(D) \leq \mu_k \left(-\frac{d^2}{ds^2} + \frac{c}{4} + \frac{1}{4}\kappa_g^2 \right).$$

Suppose now that the spin structure on γ induced by the spin structure of $X^2(c)$ is non-trivial. Then we have $\lambda_{k+1}^2(D) = \dfrac{4\pi^2}{L^2}(k + 1/2)^2$ (see [4]) and, in particular, we obtain

$$\frac{4\pi^2}{L^2} \left(k + \frac{1}{2} \right)^2 \leq \mu_{k+1} \left(-\frac{d^2}{ds^2} + \frac{c}{4} + \frac{1}{4}\kappa_g^2 \right).$$

Theorem 1: *Let $\gamma \subset \mathbb{R}^3$ be a plane or spherical curve and denote by $\kappa^2 = c + \kappa_g^2$ the square of its curvature. Suppose that the induced spin structure on γ is non-trivial, i.e., the tangent vector field has an odd rotation number. Then the inequality*

$$\frac{4\pi^2}{L^2} \leq \mu_1\left(-4\frac{d^2}{ds^2} + \kappa^2\right)$$

holds, where μ_1 is the first eigenvalue of the periodic Sturm-Liouville operator on the interval $[0, L]$. Moreover, equality occurs if and only if the curvature is constant.

Remark: No geometric lower bound for the Sturm-Liouville operator $-4\frac{d^2}{ds^2} + \kappa^2$ with potential defined by the square of the curvature $\kappa(s)$ of a closed curve γ in Euclidean space seems to be known. We conjecture that the estimate given in Theorem 1 holds for any closed curve in $\mathbb{R}^3$. Let us compare this inequality with the well-known Fenchel-Milnor inequality

$$2\pi \leq \oint_\gamma \kappa.$$

Thus, by the Cauchy-Schwarz inequality we obtain

$$\frac{4\pi^2}{L^2} \leq \frac{1}{L}\oint_\gamma \kappa^2.$$

Moreover, using the test function $f \equiv 1$, we have

$$\mu_1\left(-4\frac{d^2}{ds^2} + \kappa^2\right) \leq \frac{1}{L}\oint_\gamma \kappa^2.$$

Suppose that γ is a simple curve in $\mathbb{R}^3$ and denote by ρ the minimal number of generators of the fundamental group $\pi_1(\mathbb{R}^3 \backslash \gamma)$. Then we have

$$2\pi\rho \leq \oint_\gamma \kappa.$$

In the spirit of this remark one should be able to prove the stronger inequality

$$\frac{4\pi^2}{L^2}\rho^2 \leq \mu_1\left(-4\frac{d^2}{ds^2} + \kappa^2\right)$$

in case of a simple curve in $\mathbb{R}^3$.

Examples: We calculated the eigenvalue μ_1 for some classical curves in $\mathbb{R}^3$:

a.) *The lemniscate:* $x = \sin(t), \quad y = \cos(t)\sin(t)$.

$$4\pi^2/L^2 = 1.06193, \quad \mu_1 = 3.7315, \quad \frac{1}{L}\oint_\gamma \kappa^2 = 4.36004.$$

b.) *The trefoil:* $x = \sin(3t)\cos(t), \quad y = \sin(3t)\sin(t)$.

$$4\pi^2/L^2 = 0.221, \quad \mu_1 = 5.21, \quad \frac{1}{L}\oint_\gamma \kappa^2 = 8.16.$$

c.) *Viviani's curve:* $x = 1 + \cos(t), \quad y = \sin(2t), \quad z = 2\sin(t).$

$$4\pi^2/L^2 = 0.169071, \quad \mu_1 = 0.5335, \quad \frac{1}{L}\oint_\gamma \kappa^2 = 0.567803.$$

d.) *Torus knot:* $x = (8 + 3\cos(5t))\cos(2t), \quad y = (8 + 3\cos(5t))\sin(2t),$
$z = 5\sin(5t).$

$$4\pi^2/L^2 = 0.00146034, \quad \mu_1 = 0.03232, \quad \frac{1}{L}\oint_\gamma \kappa^2 = 0.0333803.$$

e.) *The spherical spiral:* $x = \cos(t)\cos(4t), \quad y = \cos(t)\sin(4t), \quad z = \sin(t).$

$$4\pi^2/L^2 = 0.127036, \quad \mu_1 = 1.744, \quad \frac{1}{L}\oint_\gamma \kappa^2 = 4.93147.$$

3 The two-dimensional Schrödinger operator

We generalize this inequality to the case of the two-dimensional periodic Schrödinger operator

$$P_{A,L} = -\left(1 + \frac{A^2}{L^2}\right)\frac{\partial}{\partial t^2} - 4\frac{\partial^2}{\partial s^2} - \frac{4A}{L}\frac{\partial}{\partial t}\frac{\partial}{\partial s} + \kappa^2(s)$$

defined on $[0, 2\pi] \times [0, L]$:

Theorem 2: *Let $\gamma \subset S^2 \subset \mathbb{R}^3$ be a closed, simple curve of length L bounding a region of area A, and denote by κ its curvature. Then the spectrum of the two-dimensional periodic Schrödinger operator $P_{A,L}$ is bounded by*

$$\frac{4\pi^2}{L^2} \le \mu_1(P_{A,L}).$$

Equality holds if and only if the curvature of γ is constant.

In general, let us consider a Riemannian manifold (Y^n, g) of dimension n as well as an S^1-principal fibre bundle $\pi : P \to Y^n$ over Y^n. Denote by $\vec{V}$ the vertical vector field on P induced by the action of the group S^1 on the total space P, i.e.,

$$\vec{V}(p) = \frac{d}{dt}\left(p \cdot e^{it}\right)_{t=0} \quad , \quad p \in P.$$

A connection Z in the bundle P defines a decomposition of the tangent bundle $T(P) = T^v(P) \oplus T^h(P)$ into its vertical and horizontal subspace. We introduce a Riemannian metric g^* on the total space P, requiring that

a) $g^*(\vec{V}, \vec{V}) = 1,$

b) $g^*(T^v, T^h) = 0,$

c) the differential $d\pi$ maps $T^h(P)$ isometrically onto $T(Y^n)$.

A closed curve $\gamma : [0, L] \to Y^n$ of length L defines a torus $H(\gamma) := \pi^{-1}(\gamma) \subset P$ and we want to study the isometry class of this flat torus in P. Let $\alpha = e^{i\Theta} \in S^1$ be the holonomy of the connection Z along the closed curve γ. Consider a horizontal lift $\hat{\gamma} : [0, L] \to P$ of the curve γ. Then

$$\hat{\gamma}(L) = \hat{\gamma}(0) e^{i\Theta}$$

holds. Consequently, the formula

$$\Phi(t, s) = \hat{\gamma}(s) e^{-i\Theta s/L} e^{it}$$

defines a parametrization $\Phi : [0, 2\pi] \times [0, L] \to H(\gamma)$ of the torus $H(\gamma)$. Since

$$\frac{\partial \Phi}{\partial t} = \vec{V} \quad , \quad \frac{\partial \Phi}{\partial s} = dR_{e^{it}e^{-i\Theta s/L}}(\dot{\hat{\gamma}}(s)) - \frac{\Theta}{L}\vec{V},$$

we obtain

$$g^*\left(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial t}\right) = 1, \quad g^*\left(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial s}\right) = -\frac{\Theta}{L}, \quad g^*\left(\frac{\partial \phi}{\partial s}, \frac{\partial \phi}{\partial s}\right) = 1 + \frac{\Theta^2}{L^2},$$

i.e., the torus $H(\gamma)$ is isometric to the flat torus $(\mathbb{R}^2/\Gamma_o, g^*)$, where Γ_o is the orthogonal lattice $\Gamma_o = 2\pi \cdot \mathbb{Z} \oplus L \cdot \mathbb{Z}$ and the metric g^* has the non-diagonal form

$$g^* = \begin{pmatrix} 1 & -\frac{\Theta}{L} \\ -\frac{\Theta}{L} & 1 + \frac{\Theta^2}{L^2} \end{pmatrix}.$$

Using the transformation

$$x = -\frac{\Theta}{L}s + t \quad , \quad y = s,$$

we see that $H(\gamma)$ is isometric to the flat torus $(\mathbb{R}^2/\Gamma,\ dx^2 + dy^2)$, where the lattice Γ is generated by the two vectors

$$v_1 = \begin{pmatrix} 2\pi \\ 0 \end{pmatrix} \quad , \quad v_2 = \begin{pmatrix} \Theta \\ L \end{pmatrix}.$$

In case the closed curve $\gamma : [0, L] \to Y^n$ is the oriented boundary of an oriented compact surface $M^2 \subset Y^n$, we can calculate the holonomy $\alpha = e^{i\Theta}$ along the curve γ. Indeed, let Ω^Z be the curvature form of the connection Z. Ω^Z is a 2-form defined on the manifold Y^n with values in the Lie algebra of the group S^1, i.e., with values in $i \cdot \mathbb{R}^1$. The parameter Θ is given by the integral

$$\Theta = i \int_{M^2} \Omega^Z.$$

Let us consider the Hopf fibration $\pi : S^3 \to S^2$, where

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$$

is the 3-dimensional sphere of radius 1. The connection Z is given by the formula

$$Z = \frac{1}{2}\{\bar{z}_1 dz_1 - z_1 d\bar{z}_1 + \bar{z}_2 dz_2 - z_2 d\bar{z}_2\}$$

and its curvature form ($\omega = z_1/z_2$)

$$\Omega^Z = -\frac{d\omega \wedge d\bar{\omega}}{(1+|\omega|^2)^2} = -\frac{i}{2} dS^2$$

essentially coincides with one half of the volume form of the unit sphere S^2 of radius 1. However, the differential $d\pi : T^h(S^3) \to T(S^2)$ multiplies the length of a vector by two, i.e., the Hopf fibration is a Riemannian submersion in the sense described before if we fix the metric of the sphere $S^2(\frac{1}{2}) = \{x \in \mathbb{R}^3 : |x| = \frac{1}{2}\}$ on S^2. Consequently, in case of a closed simple curve $\gamma \subset S^2$ bounding a region of area A, the Hopf torus $H(\gamma) \subset S^3$ is isometric to the flat torus $\mathbb{R}^2/\Gamma$ and the lattice Γ is generated by the two vectors

$$v_1 = \begin{pmatrix} 2\pi \\ 0 \end{pmatrix} \quad , \quad v_2 = \begin{pmatrix} A/2 \\ L/2 \end{pmatrix}.$$

The mean curvature H of the torus $H(\gamma) \subset S^3$ coincides with the geodesic curvature κ_g of the curve $\gamma \subset S^2 \subset \mathbb{R}^3$ (see [7], [8]). We apply now the inequality

$$\lambda_1^2(D) \leq \mu_1(\Delta + H^2 + 1)$$

to the Hopf torus $H(\gamma) \subset S^3$. Then we obtain the estimate

$$\lambda_1^2(D) \leq \mu_1\left(P_{A,L}\right),$$

where D is the Dirac operator on the flat torus $\mathbb{R}^2/\Gamma$ with respect to the induced spin structure. All spin structures of a 2-dimensional torus are classified by pairs $(\varepsilon_1, \varepsilon_2)$ of numbers $\varepsilon_i = 0, 1$. If γ is a simple curve in S^2, the induced spin structure on the Hopf torus $H(\gamma)$ is non-trivial and given by the pairs $(\varepsilon_1, \varepsilon_2) = (0, 1)$. The spectrum of the Dirac operator for all flat tori is well-known (see [4]): The dual lattice Γ^* is generated by

$$v_1^* = \begin{pmatrix} \dfrac{1}{2\pi} \\ \\ -\dfrac{A}{2\pi L} \end{pmatrix} \quad , \quad v_2^* = \begin{pmatrix} 0 \\ \\ \dfrac{2}{L} \end{pmatrix}$$

and the eigenvalues of D^2 are given by

$$\begin{aligned} \lambda^2(k,l) &= 4\pi^2 \left|\left| \, k v_1^* + \left(l + \frac{1}{2}\right) v_2^* \right|\right|^2 = \\ &= k^2 + \frac{4\pi^2}{L^2}\left((2l+1) - k\frac{A}{2\pi}\right)^2. \end{aligned}$$

We minimize $\lambda^2(k,l)$ on the integral lattice $\mathbb{Z}^2$. The isoperimetric inequality $4\pi A - A^2 \leq L^2$ and $A \leq \mathrm{vol}\,(S^2) = 4\pi$ yield the result that $\lambda^2(k,l)$ attends its minimum at $(k,l) = (0,1)$, i.e.,

$$\frac{4\pi^2}{L^2} \leq \lambda^2(k,l).$$

Remark 1: We replace the Hopf fibration by the S^1-principal fibre bundle of Chern class $m \geq 0$. The corresponding total space is the Lens space $L(m,1)$ and we have the commutative diagram

$$\begin{tikzcd} S^3 \arrow[rr] \arrow[dr, "\pi"'] & & L(m,1) \arrow[dl, "\pi_m"] \\ & S^2 & \end{tikzcd}$$

Let $H_m(\gamma) \subset L(m,1)$ be the Hopf torus. $H_m(\gamma)$ is isometric to $\mathbb{R}^2/\Gamma_m$, where the lattice Γ_m is generated by the vectors

$$v_1 = \begin{pmatrix} 2\pi/m \\ 0 \end{pmatrix} \quad , \quad v_2 = \begin{pmatrix} A/2 \\ L/2 \end{pmatrix}.$$

Moreover, the Lens space $L(m,1)$ admits a unique spin structure with a Killing spinor (see [3]). Even in case of $m \neq 1$, the induced spin structure on $H_m(\gamma)$ is described by the parameters $(\varepsilon_1, \varepsilon_2) = (0,1)$. Since the local geometry of $H_m(\gamma)$ in $L(m,1)$ essentially coincides with the geometry of $H(\gamma)$ in S^3, we obtain the inequality

$$\frac{4\pi^2}{L^2} = \min_{(k,l)} \left\{ k^2 m^2 + \frac{4\pi^2}{L^2}\left((2l+1) - k\frac{mA}{2\pi}\right)^2 \right\} \leq \mu_1\left(-4\frac{d^2}{ds^2} + \kappa^2\right).$$

Consequently, the investigation of the two-dimensional Schrödinger operator in case of $m \neq 1$ yields the same result for the Sturm-Liouville operator as above.

Remark 2: Suppose now that equality holds for some curve $\gamma \subset S^2$. We consider the corresponding Hopf torus $H(\gamma) \subset S^3$ and then we obtain

$$\lambda_1^2(D) = \mu_1(\Delta + H^2 + 1).$$

Therefore, the mean curvature $H = \kappa$ is constant, i.e., γ is a curve on S^2 of constant curvature κ. Consequently, γ is a circle in a 2-dimensional plane. Denote by r its radius. Then

$$\kappa^2 = \frac{1}{r^2} \quad , \quad L = 2\pi r \quad , \quad A = 2\pi(1 - \sqrt{1-r^2}),$$

and the inequality

$$\frac{4\pi^2}{L^2} \leq \kappa^2$$

is an equality for all $r \neq 0$.

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