Actions of finite abelian groups on abelian C^*-algebras $\mathcal{Z}$: Second cohomology and description by C^*-extensions $\mathcal{F} \supset \mathcal{Z}$

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Abstract

The paper discusses the second cohomology $H^2(\mathcal{Z}, \Gamma)$ w.r.t. a modified cocycle equation, given by a finite abelian automorphism group $\Gamma \subset \operatorname{aut} \mathcal{Z}$. A complete description of H^2 is given for the cyclic group $\Gamma \cong Z_N$. The discussion and solution of the case $\Gamma \cong Z_2 \times Z_2$ points to the difficulty to figure out a "composition principle" for direct products. The connection with the "outer" description of $H^2(\mathcal{Z}, \Gamma)$ by extensions $\mathcal{F} \supset \mathcal{Z}$ is pointed out.

1 Introduction

Let $\mathcal{Z}$ be an abelian C^*-algebra and $\Gamma \subset \operatorname{aut} \mathcal{Z}$ a (discrete) abelian automorphism group, equipped with the following

1 PROPERTY. *Either $\gamma_1, \gamma_2 \in \Gamma$ coincide or they are disjoint,, i.e.* $(\gamma_1, \gamma_2) = \{0\}$.

By $(\gamma_1, \gamma_2) \subset \mathcal{Z}$ we denote the set of all $A \in \mathcal{Z}$ such that $A\gamma_1(Z) = \gamma_2(Z)A$ for all $Z \in \mathcal{Z}$.

Suggested by similar procedures in the theory of superselection structures (see e.g. Doplicher/Roberts [1]) the problem arises to describe all extensions $\mathcal{F} \supset \mathcal{Z}$, up to $\mathcal{Z}$-isomorphisms, where $\mathcal{F}$ is a C^*-algebra with $\mathcal{Z}' \cap \mathcal{F} = \mathcal{Z}$ and the compact group $\mathcal{G} := \hat{\Gamma}$ is faithfully represented by a compact group $\alpha_\mathcal{G} \subset \operatorname{aut} \mathcal{F}$, where $\operatorname{aut} \mathcal{F}$ is equipped with the topology of pointwise norm convergence, such that $\{\mathcal{F}, \alpha_\mathcal{G}\}$ is a so-called Hilbert C^*-system, i.e.

$$\operatorname{spec} \alpha_\mathcal{G} = \Gamma,$$

to each $\gamma \in \Gamma$ there corresponds a unitary $U_\gamma \in \Pi_\gamma \mathcal{F}$ (Π_γ the spectral projection of γ), $U_\iota = 1 \in \mathcal{Z}$, such that

$$\gamma(A) = U_\gamma A U_\gamma^{-1}, \quad A \in \mathcal{Z},\ \gamma \in \Gamma,$$

$\mathcal{Z} = \Pi_\iota \mathcal{F}$ is the fixed point algebra of $\alpha_\mathcal{G}$.

It is not hard to see that Property 1 is a necessary condition for the existence of such an extension. We mention that Property 1 can be formulated using the corresponding homeomorphism

$$\gamma \leftrightarrow \sigma$$

given by

$$(\gamma(Z))(\mu) = Z(\sigma^{-1}(\mu)), \quad Z \in \mathcal{Z},\ \mu \in \operatorname{spec} \mathcal{Z}.$$

2 LEMMA. *Two automorphisms $\gamma_1, \gamma_2 \in \operatorname{aut} \mathcal{Z}$ are disjoint, i.e. $(\gamma_1, \gamma_2) = \{0\}$ iff the (closed) set of points $\mu \in \operatorname{spec} \mathcal{Z}$such that $\sigma_{\gamma_1}(\mu) = \sigma_{\gamma_2}(\mu)$ does not contain any inner points.*

Heuristically it is obvious that the extension problem is connected with a cohomological problem: If the unitaries U_γ are given "abstractly", then the introduction of a product structure

$$U_{\gamma_1} \circ U_{\gamma_2} := \omega(\gamma_1, \gamma_2) U_{\gamma_1 \gamma_2} \quad \omega(\gamma_1, \gamma_2) \in \mathcal{Z} \text{ a unitary}$$

and the requirement of its associativity implies a (modified) cocycle equation, namely

$$\omega(\gamma_1, \gamma_2)\omega(\gamma_1\gamma_2, \gamma_3) = \gamma_1(\omega(\gamma_2, \gamma_3))\omega(\gamma_1, \gamma_2\gamma_3), \quad \gamma_1, \gamma_2, \gamma_3 \in \Gamma$$

together with the "boundary condition"

$$\omega(\iota, \gamma) = \omega(\gamma, \iota) = 1, \quad \gamma \in \Gamma.$$

We introduce

$$(\partial\omega)(\gamma_1, \gamma_2, \gamma_3) := \frac{\omega(\gamma_1, \gamma_2)\omega(\gamma_1\gamma_2, \gamma_3)}{\gamma_1(\omega(\gamma_2, \gamma_3))\omega(\gamma_1, \gamma_2\gamma_3)}$$

where we use for convenience the "fractional" notation $A^{-1} = \frac{1}{A}$.
Then a cocycle ω is defined by

3 DEFINITION. *A mapping $\Gamma \times \Gamma \ni (\gamma_1, \gamma_2) \to \omega(\gamma_1, \gamma_2) \in \mathcal{Z}$ is called a cocycle if $\omega(\gamma_1, \gamma_2)$ is unitary for all $\gamma_1, \gamma_2 \in \Gamma$,*

$$(\partial\omega)(\gamma_1, \gamma_2, \gamma_3) = 1, \quad \gamma_1, \gamma_2, \gamma_3 \in \Gamma,$$

and

$$\omega(\iota, \gamma) = \omega(\gamma, \iota) = 1, \quad \gamma \in \Gamma.$$

Obviously, the set of all cocycles forms a group Ω w.r.t. pointwise multiplication

$$(\omega_1 \circ \omega_2)(\gamma_1, \gamma_2) := \omega_1(\gamma_1, \gamma_2) \cdot \omega_2(\gamma_1, \gamma_2).$$

Further we need the concept of coboundaries.

4 DEFINITION. *A mapping* $\Gamma \ni \gamma \to Z(\gamma) \in \mathcal{Z}$ *is called a 1-cycle if* $Z(\gamma)$ *is unitary for all* $\gamma \in \Gamma$ *and if* $Z(\iota) = 1$. *The expression*

$$(\partial Z)(\gamma_1, \gamma_2) := \frac{Z(\gamma_1)\gamma_1(Z(\gamma_2))}{Z(\gamma_1\gamma_2)}. \tag{1}$$

is then called a coboundary.

An easy calculation yields

$$\partial(\partial Z) = 1$$

and $(\partial Z)(\iota, \gamma) = (\partial Z)(\gamma, \iota) = 1$ for all $\gamma \in \Gamma$, i.e. each coboundary ∂Z is a cocycle and the set of all coboundaries ∂Z forms a subgroup $\Omega_0 \subset \Omega$ of the group of all cocycles because $\partial(Z_1 Z_2) = \partial Z_1 \cdot \partial Z_2$.

5 DEFINITION. *The factor group*

$$\Omega/\Omega_0 =: H^2(\mathcal{Z}, \Gamma)$$

is called the second cohomology of the pair $\{\mathcal{Z}, \Gamma\}$.

The problem arises to describe $H^2(\mathcal{Z}, \Gamma)$. An "outer" characterization of H^2 is connected with the construction of the extensions $\mathcal{F} \supset \mathcal{Z}$ mentioned before. The "inner" characterization means simply the pure group-theoretical description of H^2, i.e. to describe the cocycles, the coboundaries and the corresponding factor group.

Of course, the problem of solving the equation $\partial\omega = 1$ with the boundary condition $\omega(\iota, \gamma) = \omega(\gamma, \iota) = 1$ for a given group Γ is independent of assuming Property 1. However, as it is mentioned before, if the aim is to construct extensions $\mathcal{F} \supset \mathcal{Z}$ with $\mathcal{Z}' \cap \mathcal{F} = \mathcal{Z}$ then it is necessary to assume Property 1.

2 The case that Γ is isomorphic to Z_N

In this case the solution of the cocycle equation, i.e. the parametrization of the manifold of solutions, and the description of the coboundaries can be given explicitly.

Let $\Gamma := \{\gamma, \gamma^2, ..., \gamma^N = \iota\} \cong Z_N$, $\gamma \in \operatorname{aut}\mathcal{Z}$. For example, if $\mathcal{Z} := C(S^1)$ and γ corresponds to the rotation σ by $\frac{2\pi}{N}$ then Property 1 is satisfied.

In the following we use the convenient notation $A_n = A_m$ if $n \equiv m (\operatorname{mod} N)$, $A_0 = 1$ and

$$\omega(\gamma^n, \gamma^m) =: \omega(n, m), \quad 0 \le n, m \le N-1.$$

6 THEOREM. *The set of all cocycles* $\omega(\cdot, \cdot)$ *is given as follows: Let* $A_1, A_2, ..., A_{N-1}$ *be unitaries* $A_j \in \mathcal{Z}$ *which satisfy the condition*

$$\gamma(\prod_{j=1}^{N-1} A_j) = \prod_{j=1}^{N-1} A_j,$$

i.e. $\prod_{j=1}^{N-1} A_j$ *is invariant under the group* Γ. *Then each cocycle* $\omega(\cdot,\cdot)$ *is given by*

$$\omega(1,1) := A_1, \omega(2,1) := A_2, ..., \omega(N-1,1) := A_{N-1}$$

and

$$\omega(n,l) := \frac{\prod_{\mu=0}^{l-1} A_{n+\mu}}{\gamma^n(\prod_{\mu=0}^{l-1} A_\mu)}, \quad l = 2,3,...,N;\ n = 1,2,...,N-1. \tag{2}$$

A cocycle ω *is a coboundary, i.e.* $\omega = \partial Z$ *with a 1-cycle* $Z(\cdot)$, $Z(n) := D_n$, $n = 1,2,...,N-1$, $D_0 := 1$ *iff*

$$A_n := \frac{D_n \gamma^n(D_1)}{D_{n+1}}, \quad n = 1,2,...,N-1,$$

such that

$$\prod_{j=1}^{N-1} A_j = D_1 \gamma(D_1) \gamma^2(D_1) ... \gamma^{N-1}(D_1). \tag{3}$$

7 REMARK. Obviously

$$\{D\gamma(D)\gamma^2(D)...\gamma^{N-1}(D),\ D \in \mathcal{Z}\,\text{unitary}\} \subseteq \{Z \in \mathcal{Z} : \gamma(Z) = Z, Z\,\text{unitary}\},$$

i.e. the set of all coboundaries corresponds to the "special set" of unitary invariants from $\mathcal{Z}$ w.r.t. Γ, given by the left hand side.

Proof of Theorem 6. The cocycle equation reads

$$\omega(n,m)\omega(n+m,l) = \gamma^n(\omega(m,l))\omega(n,m+l). \tag{4}$$

Assume that the values $\omega(n,1)$, $n = 1,2,...,N-1$ are known (recall that $\omega(0,1) = \omega(N,1) = 1$). Then, putting $m = 1$, $l = 1$ in (4), the values $\omega(n,2)$ can be calculated because $m + l = 2$. Further, putting $m = 2$, $l = 1$ in (4) one gets the values $\omega(n,3)$. Pursuing this process one obtains by induction the formula

$$\omega(n,l) = \frac{\prod_{\mu=0}^{l-1} \omega(n+\mu,1)}{\gamma^n(\prod_{\mu=1}^{l-1} \omega(\mu,1))}, \quad l = 2,3,...,\ n = 1,2,...N-1. \tag{5}$$

However, the "boundary condition" $\omega(n,N) = 1$ requires that

$$1 = \omega(n,N) = \frac{\prod_{\mu=0}^{N-1} \omega(n+\mu,1)}{\gamma^n(\prod_{\mu=0}^{N-1} \omega(\mu,1))}.$$

Because of the periodicity we have

$$\prod_{\mu=0}^{N-1} \omega(n+\mu,1) = \prod_{\mu=0}^{N-1} \omega(\mu,1),$$

i.e. putting in particular $n = 1$ we obtain

$$\prod_{\mu=1}^{N-1} \omega(\mu, 1) = \gamma(\prod_{\mu=1}^{N-1} \omega(\mu, 1))$$

and this implies

$$\omega(n, N) = 1, \quad n = 1, 2, ..., N-1.$$

The condition is a trivial one for $n = N$ because of $\gamma^N = \iota$.

Therefore we have the first result: If $\omega(\cdot, \cdot)$ is a solution of the cocycle equation then ω has necessarily the form (5) and the boundary conditions are satisfied.

Secondly, with ω given by (2) we calculate $\partial\omega$. According to the cocycle equation $\partial\omega = 1$ we have to verify that

$$\frac{\prod_{\mu=0}^{m-1} A_{n+\mu}}{\gamma^n(\prod_{\mu=0}^{m-1} A_\mu)} \cdot \frac{\prod_{\mu=0}^{l-1} A_{n+m+\mu}}{\gamma^{n+m}(\prod_{\mu=0}^{l-1} A_\mu)} = \gamma^n \left(\frac{\prod_{\mu=0}^{l-1} A_{m+\mu}}{\gamma^m(\prod_{\mu=0}^{l-1} A_\mu)} \right) \cdot \frac{\prod_{\mu=0}^{m+l-1} A_{n+\mu}}{\gamma^n(\prod_{\mu=0}^{m+l-1} A_\mu)}.$$

We have

$$\prod_{\mu=0}^{m-1} A_{n+\mu} \cdot \prod_{\mu=0}^{l-1} A_{n+m+\mu} = \prod_{\mu=0}^{m+l-1} A_{n+\mu},$$

i.e. it remains to show

$$\frac{1}{\gamma^n(\prod_{\mu=0}^{m-1} A_\mu)} = \frac{\gamma^n(\prod_{\mu=0}^{l-1} A_{m+\mu})}{\gamma^n(\prod_{\mu=0}^{m+l-1} A_\mu)}$$

or

$$1 = \gamma^n \left(\frac{\prod_{\mu=0}^{m-1} A_\mu \cdot \prod_{\mu=0}^{l-1} A_{m+\mu}}{\prod_{\mu=0}^{m+l-1} A_\mu} \right)$$

which is an identity. Furthermore we have

$$\omega(0, l) = 1, \quad l = 1, 2, ..., N-1$$

and

$$\omega(n, 0) = \omega(n, N) = \frac{\prod_{\mu=o}^{N-1} A_{n+\mu}}{\gamma^n(\prod_{\mu=0}^{N-1} A_\mu)} = \frac{\prod_{\mu=0}^{N-1} A_\mu}{\gamma^n(\prod_{\mu=0}^{N-1} A_\mu)} = 1.$$

For the 1-cycles $Z(\cdot)$ with $Z(n) =: D_n$, $n = 1, 2, ..., N-1$, $Z(0) = Z(N) = 1$ we have, according to the formula (1)

$$(\partial Z)(n, 1) = \frac{D_n \gamma^n(D_1)}{D_{n+1}}$$

hence in this case the parameters A_n are given by

$$A_n = \frac{D_n \gamma^n(D_1)}{D_{n+1}}, \quad n = 1, 2, ..., N-1.$$

Therefore we obtain

$$\prod_{j=1}^{N-1} A_j = \prod_{j=1}^{N-1} \frac{D_j \gamma^j(D_1)}{D_{j+1}} = \prod_{j=0}^{N-1} \gamma^j(D_1). \tag{6}$$

Note that $D_1\gamma(D_1)\gamma^2(D_1 \cdots \gamma^{N-1}(D_1)$ automatically satisfies the invariance condition required. That is, the unitary parameters $A_1, A_2, ..., A_{N-1}$ for the coboundaries satisfy the special boundary condition of the form (3). □

3 The case that Γ is isomorphic to $Z_2 \times Z_2$

For the group $Z_2 \times Z_2$ we choose the denotation $Z_2 \times Z_2 = \{e, a, b, c\}$ where $a^2 = b^2 = e$, $ab = ba = c$. The corresponding automorphisms are denoted by $\gamma_a, \gamma_b, \gamma_c \in \text{aut}\, \mathcal{Z}$. Thus we have $\gamma_a \circ \gamma_b = \gamma_c$, $\gamma_a^2 = \gamma_b^2 = \gamma_c^2 = \iota$.

In this case the cocycle equation yields 27 equations for 9 terms, namely for $\omega(a,a), \omega(b,b), \omega(c,c)$ and $\omega(x,y)$ with $x \neq y$. A first inspection of these equations shows that one obtains 5 different types of equations:

$$\text{I} \quad \omega(a,a) = \gamma_a(\omega(a,a)),$$

and two further equations where a is replaced by b and c. This implies

$$\gamma_b(\omega(a,a)) = \gamma_c(\omega(a,a))$$

and two further equations obtained by cyclic permutation of a, b, c.

$$\text{II} \quad \omega(a,a) = \gamma_a(\omega(a,b)) \cdot \omega(a,c),$$

$$\text{III} \quad \omega(b,a) \cdot \omega(c,a) = \gamma_a(\omega(a,a)),$$

$$\text{IV} \quad \omega(a,b) \cdot \omega(c,a) = \gamma_a(\omega(b,a)) \cdot \omega(a,c)$$

and

$$\text{V} \quad \omega(b,c) \cdot \omega(a,a) = \gamma_b(\omega(c,a)) \cdot \omega(b,b),$$

together with further equations obtained by permutation of a, b, c.

8 LEMMA. *The equations of type (V) are implied by those of type (I),(II) and (III).*

Proof. Choose an equation of type (III), for example

$$\omega(b,a)\cdot\omega(c,a)=\gamma_b(\omega(a,a)).$$

Applying γ_b we get

$$\gamma_b(\omega(b,a))\cdot\gamma_b(\omega(c,a))=\omega(a,a).$$

Multiplication with $\omega(b,b)$ yields

$$\gamma_b(\omega(b,a))\cdot\gamma_b(\omega(c,a))\cdot\omega(b,b)=\omega(a,a)\cdot\omega(b,b).$$

Now we have $\omega(b,b)=\gamma_b(\omega(b,a))\cdot\omega(b,c)$, because this is an equation of type (III). Inserting this in the foregoing equation on the right hand side we get

$$\gamma_b(\omega(c,a))\cdot\omega(b,b)=\omega(b,c)\cdot\omega(a,a)$$

and this is an equation of type (V). □

This means that it is sufficient to consider the equations of type (I),(II),(III) and (IV). However, if the terms $\omega(a,a),\omega(b,b),\omega(c,c)$ are fixed it turns out that the equations of type (II) and (III) determine uniquely all 6 terms $\omega(x,y)$ with $x\neq y$ if one of them, say $\omega(a,b)$ is given.

9 LEMMA. *Let $\omega(a,a),\omega(b,b),\omega(c,c)$ satisfy the equations of type (I). Then if $\omega(a,b)$ is given then all other terms $\omega(x,y)$, $x\neq y$,are uniquely determined by the equations of types (II) and (III).*

Proof. Start with $\omega(a,b)$. Then by equations of type II and III one can express successively the other terms $\omega(x,y)$ with $x\neq y$ by $\omega(a,b)$. Thus we obtain a circular sequence

$$(a,b)\leftarrow(a,c)\leftarrow(b,c)\leftarrow(b,a)\leftarrow(c,a)\leftarrow(c,b)\leftarrow(a,b)$$

where one has to choose mutually equations of type II and III. For example, $\omega(c,a)$ can be expressed by $\omega(c,b)$ via type II by

$$\omega(c,a)=\frac{\omega(c,c)}{\gamma_c(\omega(c,b))}.$$

$\omega(c,b)$ can be expressed by $\omega(a,b)$ via type III by

$$\omega(c,b)=\frac{\gamma_a(\omega(b,b))}{\omega(a,b)}$$

hence

$$\omega(c,a)=\frac{\omega(c,c)}{\omega(b,b)}\cdot\gamma_c(\omega(a,b))$$

follows. Furthermore, $\omega(b,a)$ can be expressed by $\omega(c,a)$ via type III by

$$\omega(b,a)=\frac{\gamma_b(\omega(a,a))}{\omega(c,a)}=\frac{\gamma_b(\omega(a,a))}{\omega(c,c)}\cdot\gamma_c(\omega(c,b))=\frac{\gamma_b(\omega(a,a))}{\omega(c,c)}\cdot\gamma_c\left(\frac{\gamma_a(\omega(b,b))}{\omega(a,b)}\right)=$$

$$\frac{\gamma_b(\omega(a,a)\cdot\omega(b,b))}{\omega(c,c)}\cdot\frac{1}{\gamma_c(\omega(a,b))}.$$

Finally, $\omega(a,c)$ is given by $\omega(a,b)$ via type II by

$$\omega(a,c)=\frac{\omega(a,a)}{\gamma_a(\omega(a,b))}$$

Note that the expression of $\omega(a,b)$ by $\omega(a,b)$ along the circular sequence does not lead to a boundary condition but to an identity.□

However, the equations of type (IV) are responsible for boundary conditions and these are the only ones. This boundary condition arises if the expressions for $\omega(c,a)$, $\omega(b,a)$, $\omega(a,c)$ in terms of $\omega(a,b)$, calculated before, are inserted into the corresponding equation of type (IV). That is, we obtain the following expression:

$$\omega(a,b)\cdot\frac{\omega(c,c)}{\omega(b,b)}\cdot\gamma_c(\omega(a,b))=\gamma_a\left(\frac{\gamma_b(\omega(a,a))\omega(b,b)}{\omega(c,c)}\cdot\frac{1}{\gamma_c(\omega(a,b))}\right)\cdot\frac{\omega(a,a)}{\gamma_a(\omega(a,b))},\quad(7)$$

which can be written in a more transparent form given in the next theorem which is the final result.

10 THEOREM. *The set of all cocycles is determined by* 4 *unitary parameters*:

$$A:=\omega(a,a),\ B:=\omega(b,b),\ C:=\omega(c,c)$$

and $F:=\omega(a,b)$, *which satisfy the following boundary conditions*:

$$A=\gamma_a(A),\ B=\gamma_b(B),\ C=\gamma_c(C)\quad(8)$$

and

$$F\cdot\gamma_a(F)\cdot\gamma_b(F)\cdot\gamma_c(F)=\frac{AB}{C}\cdot\frac{\gamma_b(A)\gamma_c(B)}{\gamma_a(C)}.\quad(9)$$

The parameters of the coboundaries defined by the 1-cycle $\{Z(a),Z(b),Z(c)\}$ *as before by formula* (1), *are given by*

$$F:=\frac{Z(a)\gamma_a(Z(b))}{Z(c)}$$

$$A:=Z(a)\gamma_a(Z(a)),\ B:=Z(b)\gamma_b(Z(b)),\ C:=Z(c)\gamma_c(Z(c)).$$

Proof. The equations (8) are the cocycle equations of type I. The equations (9) are obtained by transforming equation (7) and they represent the equations of type

(IV). As pointed out above, the equations of type (II) and (III) determine the other terms $\omega(x,y)$, $x \neq y$. □

11 REMARK. Similarly as in the case of the cyclic group the difference between cocycles and coboundaries is expressed simply by the inclusion

$$\{Z\gamma_a(Z),\ Z \in \mathcal{Z}\text{unitary}\} \subseteq \{Z \in \mathcal{Z}\text{ unitary},\ \gamma_a(Z) = Z\}.$$

It seems that the "composition principle" of the cocycles in this case from those for the case Z_2, discussed before, remains hidden.

4 Appendix

Proof of Lemma 2. First note that $(\gamma_1, \gamma_2) = \{0\}$ is valid iff $(\iota, \gamma_1^{-1} \circ \gamma_2) = \{0\}$ holds. Therefore, it is sufficient to prove Lemma 2 for the case (ι, γ) and this means that one has to prove: $(\iota, \gamma) = \{0\}$ iff the (closed) set $\operatorname{fix}\sigma$ of all fixed points of $\sigma := \sigma_\gamma$ does not contain any inner point.

Now, let $W \in (\iota, \gamma)$ with $W \neq 0$. Then there is $\mu_0 \in \operatorname{spec}\mathcal{Z}$ such that $W(\mu_0) \neq 0$, hence there is a neighborhood $\mathcal{U} = \mathcal{U}(\mu_0)$ with $W(\mu) \neq 0$ for $\mu \in \mathcal{U}$. Then one has

$$WZ = \gamma(Z)W = W\gamma(Z), \quad Z \in \mathcal{Z}$$

or

$$W(\mu)Z(\mu) = W(\mu)Z(\sigma^{-1}(\mu)), \quad \mu \in \operatorname{spec} Z, \quad Z \in \mathcal{Z}.$$

This implies

$$Z(\mu) = Z(\sigma^{-1}(\mu)), \quad \mu \in \mathcal{U}, \quad Z \in \mathcal{Z}.$$

However, $\mathcal{Z}$ separates the points of $\operatorname{spec}\mathcal{Z}$, therefore $\mu = \sigma^{-1}(\mu)$ or

$$\mu = \sigma(\mu), \quad \mu \in \mathcal{U}$$

follows, i.e. μ_0 is an inner point of $\operatorname{fix}\sigma$.

Second, let $(\iota, \gamma) = \{0\}$, i.e. $WZ = W\gamma(Z)$ for all $Z \in \mathcal{Z}$ implies $W = 0$ or

$$W(\mu)Z(\mu) = W(\mu)Z(\sigma^{-1}(\mu)), \quad \mu \in \operatorname{spec}\mathcal{Z}, \quad Z \in \mathcal{Z}$$

implies $W = 0$.

Now assume that there is an inner point $\mu_0 \in \operatorname{fix}\sigma \subset \operatorname{spec}\mathcal{Z}$, i.e. there is a neighborhood $\mathcal{U} = \mathcal{U}(\mu_0)$ such that $\mu \in \mathcal{U}$ implies $\sigma(\mu) = \mu$. Then choose $W \in \mathcal{Z}$ such that $W \neq 0$ but $\operatorname{supp} W(\cdot) \subset \mathcal{U}$. Then

$$W(\mu)Z(\mu) = W(\mu)Z(\sigma^{-1}(\mu)), \quad Z \in \mathcal{Z}, \quad \mu \in \operatorname{spec}\mathcal{Z}$$

holds. Namely, if μ is not a member of $\operatorname{supp} W$ then $W(\mu) = 0$, i.e. the equation holds. If $\mu \in \operatorname{supp} W$ then $\sigma^{-1}(\mu) = \mu$ and then the equation is true, too.

5 References

[1] S.Doplicher and J.Roberts, Commun.Math.Phys. 131, 51-107 (1990)