REPRESENTATION OF FINITE GROUPS AND THE FIRST BETTI NUMBER OF BRANCHED COVERINGS OF A UNIVERSAL BORROMEAN ORBIFOLD

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Abstract. The paper investigates the first homology of the regular branched coverings of universal Borromean orbifold $B_{4,4,4}\backslash\mathbb{H}^3$, whose arithmetic structures are intensively studied in [HLM1]. The action of the group G of the covering transformations on the first homology is studied to obtain a criterion for an irreducible representation of G to be an irreducible component of the first homology with particular enthusiasm on the principal congruent subgroups. The investigation is motivated by a problem of the three dimensional topology due to Thurston and provides a criterion for a class of 3-manifolds to have a finite sheeted covering of positive first Betti number in terms of the group theory.

1. Introduction

1.1 *Motivation* The object in this paper is the first homology with complex coefficient of regular branched coverings of a universal Borromean orbifold, called $B_{4,4,4}\backslash\mathbb{H}^3$ in [HLM1], as a representation space of group G of the covering transformations. The interest in such an object is motivated by the following problem in 3-dimensional topology:

Problem. *Does every aspherical 3-manifold have a finite-sheeted cover of positive first Betti number?*

This problem was raised by Thurston in his paper [Th], which can be one of the crucial steps towards his hyperbolization conjecture of irreducible atoroidal 3-manifolds through his hyperbolization theorem for Haken atoroidal 3-manifolds. We shall sketch how our object is useful to construct a covering of positive first Betti number for a given 3-manifold.

The following elementary lemma shows that the information of the action of G gives us the first Betti number of all branched coverings of $B_{4,4,4}\backslash\mathbb{H}^3$ covered by the regular covering.

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Lemma. *Suppose* Γ *is a cocompact Kleinian group and* Γ_0 *a normal subgroup of finite index in* Γ. *Then we have*

$$H_*(\Gamma\backslash\mathbb{H}^3, \mathbb{C}) \simeq H_*(\Gamma_0\backslash\mathbb{H}^3, \mathbb{C})^{\Gamma/\Gamma_0}$$

where superscript Γ/Γ_0 *denotes the fixed point set by the action of* Γ/Γ_0.

Proof. The proof is a canonical application of the transfer map of homologies. Consider cell decompositions of $\Gamma\backslash\mathbb{H}^3$ and $\Gamma_0\backslash\mathbb{H}^3$ compatible with the covering map $\pi : \Gamma_0\backslash\mathbb{H}^3 \to \Gamma\backslash\mathbb{H}^3$ so that any element of $g \in G := \Gamma/\Gamma_0$ which stabilizes a cell of $\Gamma_0\backslash\mathbb{H}^3$ fixes the cell pointwise. (Such a cell decomposition can be done by successive subdivision of the cells.) Let C_* and C'_* be the associated chain complexes with complex coefficient for $\Gamma_0\backslash\mathbb{H}^3$ and $\Gamma\backslash\mathbb{H}^3$ respectively. Chain map *cor* induced by covering map π maps cell c in $\Gamma_0\backslash\mathbb{H}^3$ to cell $\pi(c)$ in $\Gamma\backslash\mathbb{H}^3$. We can also introduce transfer chain map *res* by

$$res\colon C'_* \ni c \mapsto \sum_{g\in G} g\cdot\tilde{c} \in C_*$$

where $\tilde{c}$ is a cell in $\Gamma_0\backslash\mathbb{H}^3$ which covers c. Clearly the definition does not depend on the choice of $\tilde{c}$. From the definitions we have

$$(1)\ cor \circ res(z) = |G|z, \quad (2)\ res \circ cor(\xi) = \sum_{g\in G} g\cdot\xi$$

for any $z \in C'_*$ and $\xi \in C_*$, hence passing to the homology, for any $z \in H_*(\Gamma\backslash\mathbb{H}^3, \mathbb{C})$ and $\xi \in H_*(\Gamma_0\backslash\mathbb{H}^3, \mathbb{C})$. By equation (1), *res* is injective and *cor* is surjective. Since res is injective, equation (2) implies

$$\mathrm{Ker}(cor) = \mathrm{Ker}\{\xi \mapsto \sum_{g\in G} g\cdot\xi\}.$$

Thus, *cor* induces the isomorphism

$$cor\colon H_*(\Gamma_0\backslash\mathbb{H}^3, \mathbb{C})_G \simeq H_*(\Gamma\backslash\mathbb{H}^3, \mathbb{C})$$

where subscript G denotes the co-invariant of the G-module. The complete reducibility of complex representations of finite groups implies that the co-invariant and the invariant (i.e. fixed point set) are isomorphic. This proves the lemma. Q.E.D.

Now let us recall the definition of universal groups.

Definition. *Kleinian group* Γ *is universal if, for any given closed 3-manifold* M, *there is subgroup* Γ_M *of finite index in* Γ *such that* $\Gamma_M\backslash\mathbb{H}^3$ *is homeomorphic to* M.

See [HLM2] for the universality of Kleinian group $B_{4,4,4}$. We denote by T_Γ the subgroup of Γ generated by all elements of finite order in Γ.

Proposition. *For given closed 3-manifold M, any subgroup Γ_M of universal group $B_{4,4,4}$ associated to M in the definition and each normal subgroup Γ_0 of finite index in $B_{4,4,4}$, we can find a finite-sheeted (unbranched) covering $\tilde{M}_{\Gamma_0}$ of M which satisfies*

$$b_1(\tilde{M}_{\Gamma_0}) \geq \ dim(H_1(\Gamma_0\backslash\mathbb{H}^3, \mathbb{C})^{T_{\Gamma_M}\Gamma_0/\Gamma_0})$$

where $b_1(\cdot)$ denotes the first Betti number.

Proof. Since Γ_0 is normal and of finite index, $\Gamma_1 := T_{\Gamma_M}\Gamma_0$ forms a subgroup of finite index in $B_{4,4,4}$. Set

$$\tilde{M}_{\Gamma 0} := \Gamma_M \cap \Gamma_1\backslash\mathbb{H}^3.$$

By definition of T_{Γ_M} $\tilde{M}_{\Gamma_0}$ is a finite-sheeted unbranched covering of $\Gamma_M\backslash\mathbb{H}^3 = M$. The argument of Lemma shows that the covering map $\Gamma_M \cap \Gamma_1\backslash\mathbb{H}^3 \to \Gamma_1\backslash\mathbb{H}^3$ induces a surjection on the first homology with complex coefficient since we can take a normal subgroup N of finite index in Γ_1 with $N \subset \Gamma_1 \cap \Gamma_M$. Hence $b_1(\tilde{M}_{\Gamma_0}) \geq b_1(\Gamma_1\backslash\mathbb{H}^3)$. Then, again by Lemma, we have

$$b_1(M_{\Gamma_0}) \geq b_1(\Gamma_1\backslash\mathbb{H}^3) = \dim(H_1(\Gamma_0\backslash\mathbb{H}^3, \mathbb{C})^{\Gamma_1/\Gamma_0}).$$

This proves the proposition. Q.E.D.

The purpose of the paper is to investigate which irreducible representation of G is an irreducible component of G-module $H_1(\Gamma_0\backslash\mathbb{H}^3, \mathbb{C})$. If we know all the irreducible components in G-module $H_1(\Gamma_0\backslash\mathbb{H}^3, \mathbb{C})$ and the image of T_{Γ_1} in G, we can verify whether the lower bound in Proposition of the first Betti number is zero or not. In this point of view, Problem reduces to the investigation of T_Γ and finding a nice Γ_0 for given T_Γ so that the action T_Γ has a nontrivial fixed point in $H_1(\Gamma_0\backslash\mathbb{H}^3, \mathbb{C})$. Of course these problems are far from easy. I hope the current work could be a help to specify which Γ_0 can be nice in the sense above.

1.2 *General result* To obtain a general criterion to clarify whether a given irreducible representation of G is a component of G-module $H_1(\Gamma_0\backslash\mathbb{H}^3, \mathbb{C})$, we shall make use of the symmetry of $B_{4,4,4}\backslash\mathbb{H}^3$. $B_{4,4,4}$ admits three normalizer r_1, r_2, r_3 in $Isom(\mathbb{H}^3)$, all of which are reflections with respect to hyperbolic planes, hence are of order 2. These reflections generates finite group $S = \langle r_1, r_2, r_3\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ in $Isom(\mathbb{H}^3)$. If Γ_0 is a normal subgroup of $B_{4,4,4}$ normalized by subgroup S_0 in S (we shall call such a normal subgroup S_0-normal), $H_*(\Gamma_0\backslash\mathbb{H}^3, \mathbb{C})$ naturally carries the action of semidirect product $G \rtimes S_0$. For $S_0 = \langle r_1\rangle$ we shall prove the following criterion.

Theorem A.1. *Suppose Γ_0 is an r_1-normal subgroup of finite index in $B_{4,4,4}$ and ρ is a nontrivial r_1-invariant irreducible representation of G. Then ρ is an irreducible component of G-module $H_1(\Gamma_0\backslash\mathbb{H}^3, \mathbb{C})$, if there holds*

$$\sum_i \alpha_i \chi_{\tilde{\rho}}(g_i r_1) \neq 0$$

where $\chi_{\tilde{\rho}}$ denotes an irreducible character of $G \rtimes \langle r_1\rangle$ which restricts to that of ρ on G ,$\alpha_i \in \mathbb{Q}$ and g_i's are elements of G which are the image in G of elements of finite order in $B_{4,4,4}$.

We also verifies the following theorem of similar type for $r_1r_2r_3$-invariant irreducible representations.

Theorem A.2. *Suppose Γ_0 is an $r_1r_2r_3$-normal subgroup of finite index in $B_{4,4,4}$ and ρ is a nontrivial $r_1r_2r_3$-invariant irreducible representation of G. Then irreducible representation ρ is an irreducible component of G-module $H_1(\Gamma_0\backslash\mathbb{H}^3,\mathbb{C})$, if there holds*

$$\chi_{\bar\rho}(r_1r_2r_3)+\chi_{\bar\rho}(hr_1r_2r_3)\neq 0$$

where $\bar\rho$ is as in Theorem A.1 and $h\in G$ is the image in G of an element of $B_{4,4,4}$.

g_i's and h in the formula are elements such that g_ir_1 and $hr_1r_2r_3$ are of order two or four in $B_{4,4,4}$. This apriori knowledge admits the explicit computation of the generalized character when we have enough information on the structure of finite group G. In any case the role played by the symmetry is essential in the proof. The usage of the symmetry is inspired by the work of Millson [Mi].

1.3 *Computation for congruent subgroups* Since $B_{4,4,4}$ is an arithmetic lattice of $SO(3,1)$ over $\mathbb{Q}(\sqrt{5})$ (cf.[HLM1]) we can consider the congruent subgroups. Since any congruent subgroup $\Gamma_{\mathfrak{m}}$ associated to ideal $\mathfrak{m}$ in ring $\mathfrak{o}$ of integers in $\mathbb{Q}(\sqrt{5})$ is S-normal, we can apply Theorem A's for $\Gamma_{\mathfrak{m}}$.To test the criterions given in 1.2 we shall apply Theorem A.1 and Theorem A.2 to the case of congruent subgroup $\Gamma_{\mathfrak{p}}$ associated to prime ideal $\mathfrak{p}$. As a consequence we obtain the following theorem.

Theorem B. *There exists finite set $P_{B_{4,4,4}}$ of prime ideals of $k=\mathbb{Q}(\sqrt{5})$ such that the followings hold true for prime ideal $\mathfrak{p}\notin P_{B_{4,4,4}}$.*
(1) Suppose that norm q of prime ideal $\mathfrak{p}$ verifies $q\equiv\pm1 \mod 8$. Then any r_1-invariant irreducible representation is an irreducible component of G-module $H_1(\Gamma_{\mathfrak{p}}\backslash\mathbb{H}^3,\mathbb{C})$.
(2) Suppose that $q\equiv\pm3 \mod 8$. Then r_1-invariant irreducible representation ρ is an irreducible components of G-module $H_1(\Gamma_{\mathfrak{p}}\backslash\mathbb{H}^3,\mathbb{C})$ unless ρ is one of exceptional representations in Theorem 6.5.1 and Theorem 7.5.1.

Basing on Theorem B and proposition in 1.1 we can prove the following theorem for a class of finite-sheeted covering of $B_{4,4,4}\backslash\mathbb{H}^3$ to have a positive first Betti number. From our point of view in 1.1 the theorem can be considered to be a partial result for Problem in 1.1 in this particular arithmetic situation.

Theorem C. *Let Γ be a subgroup of finite index in $B_{4,4,4}$. If there exists prime ideal $\mathfrak{p}\notin P_{B_{4,4,4}}$ of $\mathbb{Q}(\sqrt{5})$ such that T_Γ projects to a subgroup of $G=B_{4,4,4}/\Gamma_{\mathfrak{p}}$ which does not contain a noncentral normal subgroup, $\Gamma\backslash\mathbb{H}^3$ has a finite sheeted unbranched covering of positive first Betti number.*

We also prove the following general result by the method developed to prove Theorem B (1).

Theorem D. *Let Γ be a maximal $r_1r_2r_3$-normal, but not maximal normal subgroup of finite index in $B_{4,4,4}$. Then any nontrivial $r_1r_2r_3$-invariant irreducible representation of $G=B_{4,4,4}/\Gamma$ is an irreducible component of $H_1(\Gamma\backslash\mathbb{H}^3,\ \mathbb{C})$.*

1.4 *Plan of the paper* The paper consists of two parts of slightly different nature. The first part (Section 2-4) is devoted to the general argument to prove Theorem A's. In this part we formulate the problem via simple topological observation and the argument invokes the classical general theory of representation of finite groups, in which no arithmeticity comes into the play. The second part (Section 5-8) is focussed on the arithmeticity of $B_{4,4,4}$ and its congruent subgroups to prove Theorem

B and Theorem C. The argument in this part not only invokes the classical theory but also requires some essence from the representation of groups of Lie type.
We shall sketch the contents of each section. We begin with the geometric description of universal group $B_{4,4,4}$ following [HLM1] in Section 2. In Section 3 we describe a natural cell decomposition of $B_{4,4,4}\backslash\mathbb{H}^3$ and its equivalent lift to the regular branched coverings derived from the face decomposition of convex polyhedra. We also describe the chain complex associated to the cell decomposition with the action of the covering transform and the symmetry of $B_{4,4,4}\backslash\mathbb{H}^3$. Section 4 is devoted to the investigation of the chain complex as a representation space of the group action. We derive some formula for the characters of the group action to prove Theorem A's basing on the elementary observation in 4.1 via classical homological algebra. The second part follows the brief introduction in Section 5 of arithmeticity of $B_{4,4,4}$ and its congruent subgroups. We shall see that the structure of $G := B_{4,4,4}/\Gamma_{\mathfrak{p}}$ depends on whether $-a := -(1+\sqrt{5})/2$ is square in local field $F_{\mathfrak{p}} := \mathfrak{o}/\mathfrak{p}$ or not. If it is square, except for finite number of $\mathfrak{p}$'s, quotient group G is isomorphic to the splitting 4-dimensional orthogonal group. In this case G is essentially the direct product of two copies of SL_2 and r_1 permutes the components. On the other hand, if $-a$ is not square in $F_{\mathfrak{p}}$, G is isomorphic to the non-splitting orthogonal group, which is isomorphic to $PSL_2(\mathbb{F}_{q^2})$, and r_1 acts as a Frobenius map. The following two sections, Section 6 and Section 7, are devoted to these two cases, respectively. In these sections we develop the methods to compute the irreducible characters of semidirect product by $\mathbb{Z}_2$ from those of the kernel. Via the method we can compute out $\bar{\rho}$'s in the formula of Theorem A's and Theorem B follows from the direct computation. The method in Section 6 can be applied to much more general situation. This aspect is to be mentioned in Section 9 to prove Theorem D. In Section 8 we shall prove Theorem C. In view of the proposition in 1.1 the problem reduces to the triviality of the fixed point set by the action of subgroups of G. The classical Mackey's formula shows that the triviality of the fixed point set is closely related to the number of double cosets by a pair of subgroups. Then thanks to the classification of subgroups of $SL_2(\mathbb{F}_q)$ due to Dickson [Di] we can rigorously investigate the triviality of the fixed point set by subgroups and Theorem C follows. In the final section I summarize miscellaneous remarks and problems for the further development and prove Theorem D as promised.
1.5 *Global references* Though the paper is motivated by purely topological problem, most of the argument bases on classical theories of algebra such as representation theory of finite groups, theory of groups of Lie type and particularly orthogonal groups which heavily depend on the theory of quadratic forms over finite fields. Since the description of the manuscript is meant for readers with topological background and I am basically geometer, some description of the classically known algebraic objects might be superfluous, e.g. some parts of Section 4, 6.1 and 7.1, which are to be looked over by the readers with algebraic background. For these classical objects and theorems I give a global cross reference here rather than a specific reference for each of them. The most convenient cross reference is the famous textbook [Ja] by Jacobson. The general theory of representation of finite groups, which is required throughout the paper, is completely covered by Chapter 5 in Volume II with reference to Chapter 2 in the same volume. The theory of quadratic forms, which is required in Section 2, Section 5, Section 6 and Section 7 in the paper, is found mostly in Chapter 6, Volume I and the subjects related to the spinorial norm in 4.8, Volume II. For the purely group theoretic argument,

especially for $SL_2(\mathbb{F}_q)$, refer to [Sz]. (3.3 and 3.6 in Volume I for $SL_2(\mathbb{F}_q)$.) For the structure and the representation of groups of Lie type, which is required in Section 6, Section 7 and Section 8, the textbook [Ca] of Carter is the canonical modern reference. Section 6 does not require the theory other than the knowledge of quadratic forms in [Ja] provided the reader accept the validity of the character table in 6.5 of $SL_2(\mathbb{F}_q)$. Some of the basics of the theory are required in 7.3 and all the ingredients are found in Chapter 1 (in particular, 1.17 for Lang-Steinberg theorem) and Chapter 3 in [Ca]. Section 8 deal with the Steinberg character for a very special case. Chapter 6 of [Ca] is devoted to the subject. The lecture note by Digne-Michel [DM] also provides a nice crash course for the subject and includes a brief argument for the character table of $SL_2(\mathbb{F}_q)$ in the last chapter.

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2. Description of $B_{4,4,4}$

Following [HLM1] we shall construct universal group $B_{4,4,4}$ in this section. However, since we are going to consider $B_{4,4,4}$ as a lattice in $S(3,1;\mathbb{R})$ rather than in $SL_2(\mathbb{C})$ or $PGL_2(\mathbb{C})$, we shall stick to the Beltrami-Klein model for the hyperbolic 3-space without passing to the Poincare model.

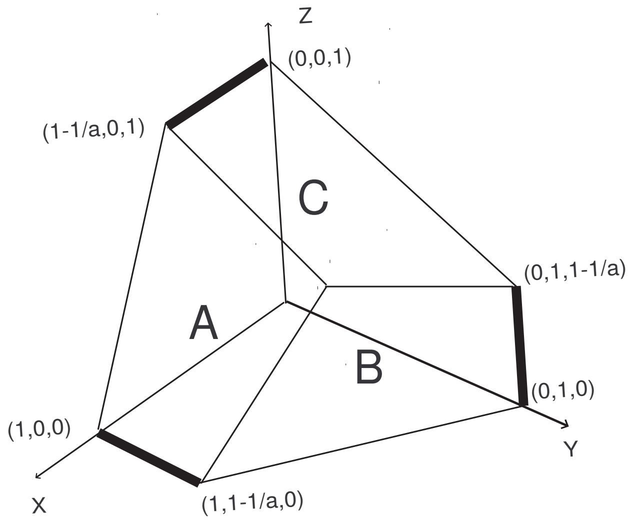

FIGURE 1. Polyhedron P

2.1 *Geometric aspects* Let $f = a^{-1}(x^2+y^2+z^2) - t^2$ be a quadratic form on $\mathbb{R}P^3$. In this paper always $a = (1+\sqrt{5})/2$. Note that a is denoted by A^2 in [HLM1]. The component $\mathfrak{L}_0$ of $\mathbb{R}P^3 \setminus \{f=0\}$ containing $(0:0:0:1)$ can be identified with the canonical Beltrami-Klein model of hyperbolic 3-space by $(x:y:z:t) \mapsto (\zeta:\eta:\theta:\tau) = (\sqrt{a}x : \sqrt{a}y : \sqrt{a}z : t)$. Passing from homogeneous coordinate $(x:y:z:t)$ to $(X,Y,Z) = (\frac{x}{t}, \frac{y}{t}, \frac{z}{t})$, $\mathfrak{L}_0$ can be identified with the interior of ball $X^2+Y^2+Z^2 < a$. Note that these identifications preserve the symmetry of the

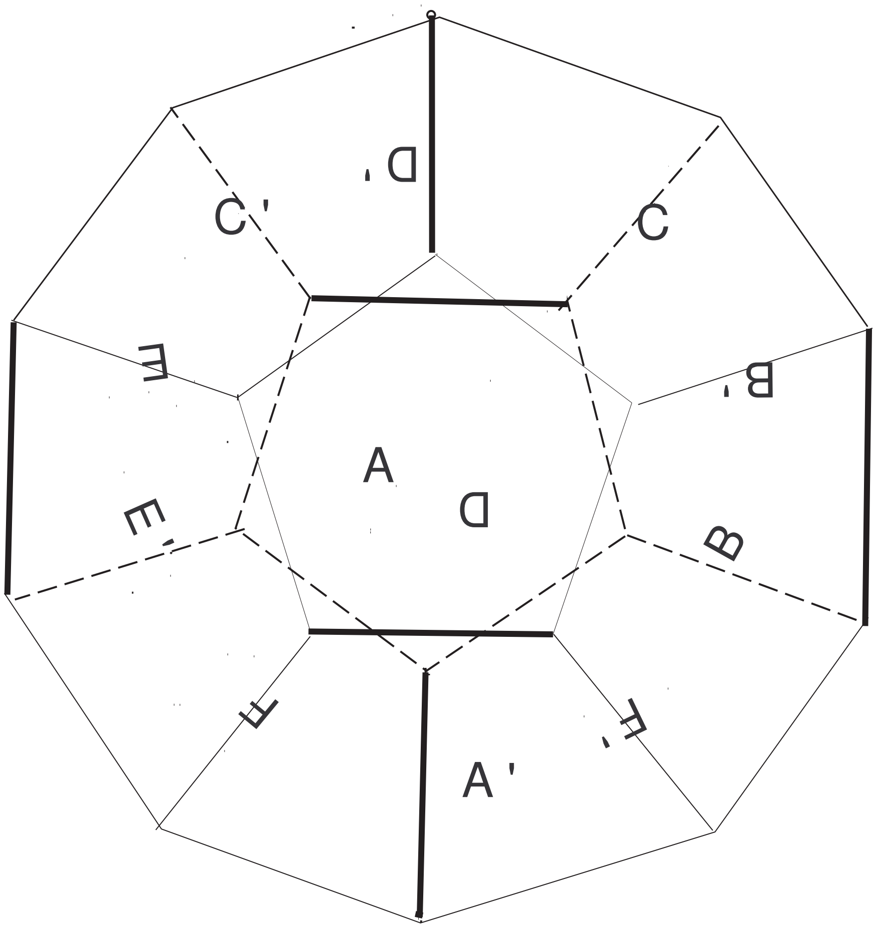

FIGURE 2. Arrangement of Sides of Regular Dodecahedron R

first three variables (x, y, z), (ζ, η, θ) and (X, Y, Z) of these coordinates.

Consider polyhedron P located in our rescaled hyperbolic space $\mathfrak{L}_0$ according to Figure 1.

The reflection with respect to each coordinate plane transfers P to each octant and these eight copies of P forms a regular dodecahedron R in hyperbolic space with all dihedral angles being $\pi/2$. (See [HLM1] Section 3 for the computation of the dihedral angles.) We label each sides of R as in Figure 2.

Consider topological space O obtained by gluing sides X and X' ($X = A, B, .., F$) by the elliptic transformation θ_X of order 4 which takes X to X' with its axis on the common ridge. Since all dihedral angles are $\pi/2$, one can observe that under this gluing pattern the hyperbolic structure locally projects to O around six light-faced ridges and one light-faced vertex. Similarly the hyperbolic structure ramifies around six bold-faced ridges and at its end points in O to form a dihedral singular locus of degree 4. In particular, O is a compact hyperbolic orbifold. $B_{4,4,4}$ is defined to be the cocompact Kleinian group associated to hyperbolic orbifold O. Topological observation shows that O is homeomorphic to the 3-sphere and ramifies over the Borromean ring with degree 4 on all components.

2.2 *Group theoretic aspects*

Lemma 1. *$B_{4,4,4}$ is generated by $\theta_A, \theta_B, .., \theta_F$. $B_{4,4,4}$ is represented by the follow-*

ing relations.

$$\begin{aligned}
&{\theta_X}^4 = \varepsilon, \quad (X = A, B, .., F),\\
&{\theta_B}^{-1}\theta_D\theta_B\theta_A = \varepsilon,\ {\theta_C}^{-1}\theta_E\theta_C\theta_B = \varepsilon,\\
&{\theta_A}^{-1}\theta_F\theta_A\theta_C = \varepsilon,\ {\theta_E}^{-1}\theta_A\theta_E\theta_D = \varepsilon,\\
&{\theta_F}^{-1}\theta_B\theta_F\theta_E = \varepsilon,\ {\theta_D}^{-1}\theta_C\theta_D\theta_F = \varepsilon,
\end{aligned}$$

Proof. By our gluing pattern, the generators are $\theta_A, \theta_B, .., \theta_F$. Their relations are obtained by Poincare's theorem. (see e.g. [Ra] Section 13) The relation on the first line is obtained from the bold-faced ridges in Figure 2 and the other six relations are obtained from the light-faced ones. Q.E.D.

Lemma 2. *Let r_1, r_2 and r_3 be hyperbolic reflections with respect to $YZ-, ZX-$ and $XY-$ planes in Figure 1, respectively.*

(1) The subgroup generated by $r_1, r_2,$ and r_3 in $Isom(\mathbb{H}^3)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

(2) $r_1, r_2,$ and r_3 are normalizers of $B_{4,4,4}$.

Proof. Since the reflections are represented in our rescaled coordinate of hyperbolic space $\mathfrak{L}_0$ as the Euclidean reflections with respect to coordinate planes, one can readily check that they generates a finite group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. This proves (1). For (2), first observe that these reflections stabilize regular dodecahedron R together with the set of the bold-faced ridges. Then by Lemma 1 we can verify that these permute the generators, keeping its relations. Q.E.D.

Lemma 3. *The following relations hold*

$$\begin{aligned}
&{}^{r_1}\theta_A = \theta_D^{-1},\ {}^{r_1}\theta_B = \theta_B^{-1},\ {}^{r_1}\theta_C = \theta_C,\\
&\qquad\qquad {}^{r_1}\theta_E = \theta_E^{-1},\ {}^{r_1}\theta_F = \theta_F,\\
&{}^{r_2}\theta_A = \theta_A,\ {}^{r_2}\theta_B = \theta_E^{-1},\ {}^{r_2}\theta_C = \theta_C^{-1},\\
&{}^{r_2}\theta_D = \theta_D, \qquad\qquad {}^{r_2}\theta_F = \theta_F^{-1},\\
&{}^{r_3}\theta_A = \theta_A^{-1},\ {}^{r_3}\theta_B = \theta_B,\ {}^{r_3}\theta_C = \theta_F^{-1},\\
&{}^{r_2}\theta_D = \theta_D,\ {}^{r_1}\theta_E = \theta_E.
\end{aligned}$$

Proof. Observing Figure 2, we can see that ${}^{r_1}\theta_A\theta_D$ fixes side A. Since Γ acts freely on sides, this verifies the first relation. The other relations are verified similarly. Q.E.D.

3. The cell decomposition and the chain complex with the action

3.1 *Cell decomposition* Let $\mathfrak{F}_i$ be the set of the interiors of i-faces of Polyhedron R. γR ($\gamma \in B_{4,4,4}$) forms a tesselation of $\mathbb{H}^3$. The tesselation and $\mathfrak{F}_i$ naturally induce the regular cell decomposition of $\mathbb{H}^3$. We denote by $\tilde{\mathfrak{F}}_i$ the set of i-cells of the regular cell decomposition.

Lemma 1. *Suppose Γ is a subgroup of $B_{4,4,4}$. Then the projection of i-cells in $\tilde{\mathfrak{F}}_i$ to $\Gamma\backslash\mathbb{H}^3$ gives the i-cells of $\Gamma\backslash\mathbb{H}^3$ to form the cell decomposition of $\Gamma\backslash\mathbb{H}^3$.*

Proof. By our gluing construction of $O = B_{4,4,4}\backslash\mathbb{H}^3$ we can see that no points in $f \in \mathfrak{F}_*$ are identified in O. This shows that elements of $B_{4,4,4}$ stabilize cell $\tilde{f} \in \tilde{\mathfrak{F}}_*$

only if it fixes the cell pointwise. Hence so is Γ. Thus each cell $\tilde{f} \in \tilde{\mathfrak{F}}_*$ is mapped homeomorphically onto a cell in $\Gamma\backslash\mathbb{H}^3$. The collection of these cells together with the pasting map induced on the boundary of the cells in $\tilde{\mathfrak{F}}_i$ clearly give the CW-complex structure on $\Gamma\backslash\mathbb{H}^3$. Q.E.D.

Lemma 2. *The i-cells described in Lemma 1 can be identified with the orbits $\Gamma\tilde{f}$ in $\tilde{\mathfrak{F}}_i$ by the left action of Γ. Moreover if Γ is normal, the action of covering transform $G = B_{4,4,4}/\Gamma$ is compatible with the cell decomposition and the set of i-cells form a G-set with the action given by*

$$\Gamma\tilde{f} \mapsto \Gamma g\tilde{f}$$

where $g \in G$ and $\tilde{f} \in \tilde{\mathfrak{F}}$.

Proof. Immediate from Lemma 1. Q.E.D.

For subgroup Γ of $B_{4,4,4}$ we define $(\mathfrak{F}_i)_\Gamma$ to be the set of Γ-orbits in $\tilde{\mathfrak{F}}_i$. By Lemma 2 this is identified with the set of i-cells of our cell decomposition of $\Gamma\backslash\mathbb{H}^3$. For the concrete description of the cell decomposition we label all the ridges and vertexes in $B_{4,4,4}\backslash\mathbb{H}^3$ in the diagram in Figure 3. In the diagram the index is attached to a representative in $\mathfrak{F}_*$ of the cell in $(\mathfrak{F}_*)_\Gamma$. For later use we also give an orientation to the ridges by arrows.

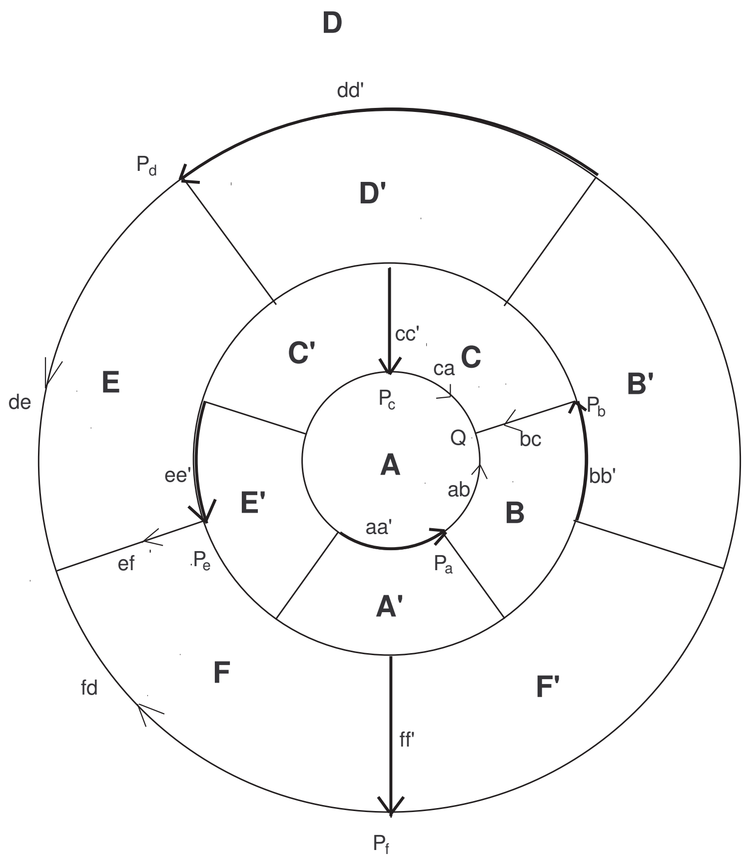

FIGURE 3. Edges and Vertexes

Lemma 3. *Let Γ be a normal subgroup of $B_{4,4,4}$ and $G = B_{4,4,4}/\Gamma$. $(\mathfrak{F}_*)_\Gamma$ is decomposed into G-orbits as follows.*
(0) $(\mathfrak{F}_0)_\Gamma = G(\Gamma Q) \cup \{G(\Gamma P_x);\ x = a, b, ..f\}$.
As a left G-set, each G-orbit is isomorphic to the following quotient.

$$G(\Gamma Q) \simeq G, \quad G(\Gamma P_x) \simeq G/\langle\theta_X\rangle\ (x = a, b, .., f, X = A, B, .., F).$$

(1) $(\mathfrak{F}_1)_\Gamma = \{G(\Gamma xx');\ x = a, b, ..f\} \cup \{G(\Gamma y); y = ab, bc, ca, de, ef, fd\}$.
As a left G-set

$$G(\Gamma xx') \simeq G/\langle\theta_x\rangle\ (x = a, b, .., f),$$
$$G(\Gamma y) \simeq G\ (y = ab, bc, ca, de, ef, fd).$$

(2) $(\mathfrak{F}_2)_\Gamma = \{G(\Gamma X);\ X = A, B, .., F\}$.
Each G-orbit is isomorphic to G as a left G-set.
(3) $(\mathfrak{F}_3)_\Gamma = G(\Gamma R)$.
It is isomorphic to G as a left G-set.

Proof. We prove only (0). The others are proved similarly. By our construction of $O := B_{4,4,4}\backslash\mathbb{H}^3$ in Section 2 the action of $B_{4,4,4}$ decomposes $\tilde{\mathfrak{F}}_0$ into the orbits $B_{4,4,4}Q$ and $B_{4,4,4}Px(x = a, b, .., f)$. Decomposing these $B_{4,4,4}$-orbits into Γ-orbits, we have the first half of the statement (0) by Lemma 2. Since the isotropy groups of Q and Px's by $B_{4,4,4}$ are $\{e\}$ and $\langle\theta_X\rangle$, respectively, we have the identifications as a left G-set

$$\{\Gamma gQ\}_{g\in G} \simeq \Gamma\backslash B_{4,4,4}Q \simeq \Gamma\backslash B_{4,4,4} \simeq G$$

and

$$\{\Gamma gP_x\}_{g\in G} \simeq \Gamma\backslash B_{4,4,4}P_x \simeq \Gamma\backslash B_{4,4,4}/\langle\theta_X\rangle \simeq G/\langle\theta_X\rangle.$$

This proves the second half of (0). Q.E.D.

3.2 *Description of the chain complex* To describe the chain complex, we shall consider the oriented cells. First we shall give orientations below to all cells in $\mathfrak{F}_*$ which appear in Lemma 3.1.3 as representatives of G-orbits of cells.
(1) Each 1-cell indexed in Figure 3 is given the orientation indicated by the arrow in the diagram.
(2) 2-cell X (X=A,B,..,F) is oriented by the outer normal of $\bar{R}$ and the canonical orientation of rescaled hyperbolic space $\mathfrak{L}_0$ in 2.1.
(3) 3-cell $\bar{R}$ is oriented by the canonical orientation of rescaled hyperbolic space $\mathfrak{L}_0$ in 2.1.

Since an element of $B_{4,4,4}$ stabilize a cell only if it fixes each point of the cell, the orientations given by (1),(2) and (3) are transferred consistently by the action of $B_{4,4,4}$ to all cells in $\tilde{\mathfrak{F}}_*$. Hence the orientation naturally projects to an orientation of $(\mathfrak{F}_*)_\Gamma$ for any subgroup Γ of $B_{4,4,4}$. For cell $c \in (\mathfrak{F}_*)_\Gamma$ we denote by $[c]$ the oriented cell according to the orientation given as above. Thus we can define the i-th chain module as usual by

$$C_i := \bigoplus_{c\in(\mathfrak{F}_i)_\Gamma} \mathbb{C}\,[c]$$

and its boundary operator $\partial_i : C_i \to C_{i-1}$ in usual manner. Clearly the chain complex (C_*, ∂_*) is a G-chain complex. If Γ is S_0-normal, the complex is a $G \rtimes S_0$-chain complex since the left action of S_0 permutes the cells by Lemma 2.2.2 (2) and Lemma 3.1.2.

By Lemma 3.1.3 and our convention on the orientation of cells above, 0-chain module C_0 is described as G-module as follows.

$$C_0 \simeq \mathbb{C}[G] \cdot v_Q \oplus \bigoplus_x \mathbb{C}[G/\langle \theta_X \rangle] \cdot v_x := C_0' \oplus C_0''$$

where (x, X) runs over $(a, A), (b, B), .., (f, F)$ and $v_Q = [\Gamma Q]$, $v_x = [\Gamma P_x]$. We decompose C_0 into two summands, $C_0' \simeq \mathbb{C}[G] \cdot v_Q$ and $C_0'' \simeq \bigoplus_x \mathbb{C}[G/\langle \theta_x \rangle] \cdot v_x$. Similarly 1-chain module C_1 is described as follows.

$$C_1 = \bigoplus_x \mathbb{C}[G/\langle \theta_X \rangle] \cdot e_x \oplus \bigoplus_y \mathbb{C}[G] \cdot e_y := C_1' \oplus C_1''$$

where x runs over $a, b, .., f$, y over ab, bc, ca, de, ef, fd, $e_x = [\Gamma x x']$ and $e_y = [\Gamma y]$. We also decompose C_1 into the first six summands C_1' and the second six summands C_1''.

2- and 3-chain modules are described in the similar notation as

$$C_2 = \bigoplus_x \mathbb{C}[G] \cdot s_X, \quad C_3 = \mathbb{C}[G] \cdot c_R.$$

For the concrete computation, we use the "coordinate notions". e.g.,

$$x = (\alpha_A, \alpha_B, .., \alpha_F) \in C_0''$$

which means

$$x = \sum_X \alpha_X \cdot v_x$$

with $\alpha_X \in \mathbb{C}[G/\langle \theta_X \rangle]$. For $\alpha \in \mathbb{C}[G/\langle \theta \rangle]$, $g \in G$ and $r \in \mathrm{Aut}(G)$ we abbreviate to ${}^r\alpha g^{-1}$ the element of $\mathbb{C}[G/\langle g^r \theta g^{-1} \rangle]$ obtained by the action of r followed by the right multiplication of G.

Lemma 1. *Suppose Γ is r_1-normal. Then the action of r_1 on C_* is described as follows.*

$$\begin{aligned}
&C_0' \ni \alpha \mapsto {}^{r_1}\alpha\theta_B \in C_0', \\
&C_0'' \ni (\alpha_A, \alpha_B, .., \alpha_F) \longmapsto \\
&\qquad\qquad ({}^{r_1}\alpha_A\theta_B, {}^{r_1}\alpha_B, {}^{r_1}\alpha_F\theta_A, {}^{r_1}\alpha_D\theta_E, {}^{r_1}\alpha_E, {}^{r_1}\alpha_C\theta_D) \in C_0'', \\
&C_1' \ni (\alpha_a, \alpha_b, .., \alpha_f) \mapsto (-{}^{r_1}\alpha_d, {}^{r_1}\alpha_b, -{}^{r_1}\alpha_c, -{}^{r_1}\alpha_a, {}^{r_1}\alpha_e, -{}^{r_1}\alpha_f) \in C_1', \\
&C_1'' \ni (\alpha_{ab}, \alpha_{bc}, \alpha_{ca}, \alpha_{de}, \alpha_{ef}, \alpha_{fd}) \longmapsto \\
&\quad ({}^{r_1}\alpha_{ab}\theta_B, {}^{r_1}\alpha_{bc}\theta_B, {}^{r_1}\alpha_{fd}\theta_A\theta_C, {}^{r_1}\alpha_{de}\theta_E, {}^{r_1}\alpha_{ef}\theta_E, {}^{r_1}\alpha_{ca}\theta_D\theta_F) \in C_1'', \\
&C_2 \ni (\alpha_A, \alpha_B, .., \alpha_F) \longmapsto \\
&\qquad\qquad ({}^{r_1}\alpha_D\theta_A, {}^{r_1}\alpha_B\theta_B, -{}^{r_1}\alpha_C, {}^{r_1}\alpha_A\theta_D, {}^{r_1}\alpha_E\theta_E, -{}^{r_1}\alpha_F) \in C_2, \\
&C_3 \ni \alpha \mapsto -{}^{r_1}\alpha \in C_3.
\end{aligned}$$

Moreover if ρ is a r_1-invariant irreducible representation of G these actions restrict to the homogeneous components of ρ.

Proof. We shall prove the formula for C_1'. The action of r_1 on C_1' can be computed as follows.

$$r_1(\alpha_a \cdot e_a + \alpha_b \cdot e_b + .. + \alpha_f \cdot e_f) = {}^{r_1}\alpha_a \cdot r_1 e_a + {}^{r_1}\alpha_b \cdot r_1 e_b + .. + {}^{r_1}\alpha_f \cdot r_1 e_f.$$

Applying reflection r_1 to these ridges in Figure 2, we can see that aa' is mapped to dd' with opposite direction. Hence $r_1 e_a = -e_d$. Similar observations gives us the formula of type $e_* = \pm e_*$ to show the formula for C_1'. Similar proof verifies all the formulae. The last statement is obvious from the definition of homogeneous components. Q.E.D.

Lemma 2. *Suppose Γ is $r_1r_2r_3$-normal. The actions of $r_1r_2r_3$ on six term modules C_0'', C_1', C_1'' and C_2 permute the components of pairs $A \leftrightarrow D$, $B \leftrightarrow E$ and $C \leftrightarrow F$. The actions on C_0' and C_3 are given by*

$$\begin{aligned} C_0' \ni \alpha &\mapsto {}^{r_1r_2r_3}\alpha\theta_E^{-1}\theta_F^{-1}\theta_A \in C_0', \\ C_3 \ni \alpha &\mapsto -{}^{r_1r_2r_3}\alpha \in C_3. \end{aligned}$$

If ρ is a $r_1r_2r_3$-invariant irreducible representation of G, these actions restrict to the homogeneous components of ρ.

Proof. The statement for the six term modules is readily verified by the observation on Figure 2. The action on C_0' is computed as follows.

$$r_1r_2r_3 \cdot \alpha v_Q = {}^{r_1r_2r_3}\alpha^{r_2r_3}\theta_B{}^{r_3}\theta_C\theta_A.$$

Hence the formula for C_0' follows from Lemma 2.2.3. The formula for C_3 and the last statement is immediate. Q.E.D.

4. Character formulae

Notations Irreducible representations are usually denoted by greek letters $\rho, \sigma, \tau, ..$ and $\chi_\rho, \chi_\sigma, \chi_\tau, ..$ denote the corresponding irreducible characters. For G-module M and irreducible representation ρ of G, M_ρ denotes the homogeneous component of M associated to ρ. By calligraphic letter $\mathcal{M}$ we denote the corresponding character of G-module M. ρ^* and χ^* denote the cogridients of representation ρ and character χ, respectively. Res and Ind denote the restriction and the induction operator on the representations, respectively. $\langle *, * \rangle_G$ denotes the usual inner product of class functions on finite group G.

4.1 *Preliminaries* Let (C_*, ∂_*) be the chain complex described in 3.2.

Lemma 1. *Suppose Γ is S_0-normal and of finite index. For any irreducible representation $\bar{\rho}$ of $G \rtimes S_0$ chain complex (C_*, ∂_*) restricts to $G \rtimes S_0$-subcomplex $(C_{*,\bar{\rho}}, \partial_*|_{C_{*,\bar{\rho}}})$ and*

$$H_*(\Gamma \backslash \mathbb{H}^3, \mathbb{C})_{\bar{\rho}} \simeq H_*(C_{*,\bar{\rho}}, \partial_*|_{C_{*,\bar{\rho}}}).$$

Proof. Since $G \rtimes S_0$ is finite, $\mathbb{C}[G \rtimes S_0]$-modules are completely reducible. Then the lemma follows from the definition of homogeneous components. Q.E.D.

Lemma 2. *Suppose Γ is normal and of finite index in $B_{4,4,4}$. For any irreducible representation ρ of G*

$$H_1(\Gamma\backslash\mathbb{H}^3,\mathbb{C})_\rho \simeq H_2(\Gamma\backslash\mathbb{H}^3,\mathbb{C})_\rho$$

as G-module.

Proof. Since $\Gamma\backslash\mathbb{H}^3$ is a closed oriented 3-manifold and the action of G on the homology comes from that of the orientation preserving isometries, the Poincare duality induces the isomorphism of G-module

$$H_1(\Gamma\backslash\mathbb{H}^3,\mathbb{C})^* \simeq H_2(\Gamma\backslash\mathbb{H}^3,\mathbb{C})$$

where the right upper asterisque on the left hand side denotes the dual space as G-module. Hence for any irreducible representation ρ of G we have

$$(H_1(\Gamma\backslash\mathbb{H}^3,\mathbb{C})_\rho)^* \simeq H_2(\Gamma\backslash\mathbb{H}^3,\mathbb{C})_{\rho^*}. \tag{4.1}$$

On the other hand, since the homology and the action of G is defined over $\mathbb{R}$, the homology admits the complex conjugate which commutes with the action of G. Hence the complex conjugate induces an isomorphism of $\mathbb{R}$-vector spaces

$$H_2(\Gamma\backslash\mathbb{H}^3,\mathbb{C})_{\rho^*} \simeq H_2(\Gamma\backslash\mathbb{H}^3,\mathbb{C})_\rho. \tag{4.2}$$

Since the structure of homogeneous components as $\mathbb{C}[G]$-module is determined by its dimension, the result follows from (4.1) and (4.2). Q.E.D.

If ρ is a S_0-invariant irreducible representation of G and M is a $G \rtimes S_0$-module, then S_0 stabilizes homogeneous component M_ρ of G-module $Res_G^{G\rtimes S_0} M$. Hence M_ρ carries the action of $G \rtimes S_0$ and we denote by $\bar{\mathcal{M}}_\rho$ the associated character of $G \rtimes S_0$.

Proposition 1. *Suppose that Γ is S_0-normal and nontrivial irreducible representation ρ of G is S_0-invariant. Then ρ is an irreducible component of $H_1(\Gamma\backslash\mathbb{H}^3,\mathbb{C})$ if the generalized character*

$$\bar{\mathcal{E}}_\rho := \sum_i (-1)^i \bar{\mathcal{C}}_{i,\rho}$$

of $G \rtimes S_0$ is not trivial.

Proof. Since $\Gamma\backslash\mathbb{H}^3$ is a connected closed 3-manifold,

$$\mathcal{H}_0(\Gamma\backslash\mathbb{H}^3,\mathbb{C})_\rho = \mathcal{H}_3(\Gamma\backslash\mathbb{H}^3,\mathbb{C})_\rho = 0$$

for $\rho \neq 1_G$. Hence the alternated sum $\bar{\mathcal{E}}_\rho$ is equal to the generalized character $\bar{\mathcal{H}}_1(\Gamma\backslash\mathbb{H}^3,\mathbb{C})_\rho - \bar{\mathcal{H}}_2(\Gamma\backslash\mathbb{H}^3,\mathbb{C})_\rho$ by Lemma 1. If it admits an action of $G \rtimes S_0$ with non-vanishing character, one of $H_1(\Gamma\backslash\mathbb{H}^3,\mathbb{C})_\rho$ or $H_2(\Gamma\backslash\mathbb{H}^3,\mathbb{C})_\rho$ is at least nontrivial. Applying Lemma 2, the proposition follows. Q.E.D.

Remark. As the alternated sum $\bar{\mathcal{E}}_\rho$ is equal to $\bar{\mathcal{H}}_1(\Gamma\backslash\mathbb{H}^3,\mathbb{C})_\rho - \bar{\mathcal{H}}_2(\Gamma\backslash\mathbb{H}^3,\mathbb{C})_\rho$, the argument in Lemma 2 shows that $\bar{\mathcal{E}}_\rho$ can be non-vanishing only on elements of $G \rtimes S_0 \setminus G$ which comes from the orientation-reversing isometry.

4.2 *Formulae for $\bar{\mathcal{E}}$* Let $r \in \mathrm{Aut}(G)$ and ρ a r-invariant irreducible representation of G. For $\theta, g, h \in G$ with ${}^{hr}\theta \in \langle\theta\rangle$ we set

$$\begin{aligned}
&\varphi^r_\theta(g,h)_\rho : \ \mathbb{C}[G/\langle\theta\rangle]_\rho \ni \alpha \mapsto g\,{}^r\alpha h^{-1} \in \mathbb{C}[G/\langle\theta\rangle]_\rho,\\
&T^r_\theta(g,h)_\rho := \mathrm{Trace}(\varphi^r_\theta(g,h)_\rho).
\end{aligned}$$

We omit the upperscript r when it is obvious in the context and the subscript θ is not indicated if $\theta = \varepsilon$.

Lemma 1. *Suppose Γ is r_1-normal and $\rho \neq 1_G$ is a r_1-invariant irreducible representation of G. Then we have*

$$\begin{aligned}\bar{\mathcal{E}}_\rho(gr_1) = &- T^{r_1}(g,\theta_E^{-1})_\rho - T^{r_1}(g,\varepsilon)_\rho + T^{r_1}_{\theta_A}(g,\theta_B^{-1})_\rho \\ &+ T^{r_1}_{\theta_D}(g,\theta_E^{-1})_\rho + T^{r_1}_{\theta_C}(g,\varepsilon)_\rho + T^{r_1}_{\theta_F}(g,\varepsilon)_\rho.\end{aligned}$$

Proof. By Lemma 3.2.1 we have

$$\begin{aligned}\bar{\mathcal{C}}'_{0,\rho}(gr_1) =&T(g,\theta_B^{-1})_\rho, \\ \bar{\mathcal{C}}''_{0,\rho}(gr_1) =&T_{\theta_A}(g,\theta_B^{-1})_\rho + T_{\theta_D}(g,\theta_E^{-1})_\rho + T_{\theta_B}(g,\varepsilon)_\rho + T_{\theta_E}(g,\varepsilon)_\rho, \\ \bar{\mathcal{C}}'_{1,\rho}(gr_1) =&T_{\theta_B}(g,\varepsilon)_\rho + T_{\theta_E}(g,\varepsilon)_\rho + T_{\theta_C}(g,\varepsilon)_\rho + T_{\theta_F}(g,\varepsilon)_\rho, \\ \bar{\mathcal{C}}''_{1,\rho}(gr_1) =&2\left\{T(g,\theta_B^{-1})_\rho + T(g,\theta_E^{-1})_\rho\right\}, \\ \bar{\mathcal{C}}_{2,\rho}(gr_1) =&T(g,\theta_B^{-1})_\rho + T(g,\theta_E^{-1})_\rho - 2T(g,\varepsilon)_\rho, \\ \bar{\mathcal{C}}_{3,\rho}(gr_1) =&- T(g,\varepsilon)_\rho.\end{aligned}$$

Hence the result follows immediately from the definition of $\bar{\mathcal{E}}_\chi$ and the direct computation. Q.E.D.

Lemma 2. *Suppose Γ is $r_1r_2r_3$-normal and $\rho \neq 1_G$ is $r_1r_2r_3$-invariant. Then*

$$\bar{\mathcal{E}}_\rho(gr_1r_2r_3) = T^{r_1r_2r_3}(g,\theta_A^{-1}\theta_F\theta_E)_\rho - T^{r_1r_2r_3}(g,\varepsilon)_\rho.$$

Proof. This follows immediately from Lemma 3.2.2. Q.E.D.

4.3 *Character formulae for $T_*(*,*)_\rho$*

Lemma 1. *Let K be a finite group. Suppose that $r \in \mathrm{Aut}(K)$ is of order two and ρ is a r-invariant irreducible representation of K. Then there exist exactly two irreducible representations $\bar{\rho}$ and $\bar{\rho}'$ of $K \rtimes \langle r\rangle$ which restricts to ρ on K. These satisfy*

$$\chi_{\bar{\rho}}(x) + \chi_{\bar{\rho}'}(x) = 0$$

for $x \in K \rtimes \langle r\rangle \setminus K$.

Proof. Every irreducible representation σ of $K \rtimes \langle r\rangle$ which satisfies

$$\langle \mathrm{Res}_K^{K\rtimes\langle r\rangle}\, \sigma, \rho\rangle \neq 0$$

is a component of $\mathrm{Ind}_K^{K\rtimes\langle r\rangle}\,\rho$. Since ρ is r-invariant, Cliford's theorem shows

$$\mathrm{Res}_K^{K\rtimes\langle r\rangle}\,\mathrm{Ind}_K^{K\rtimes\langle r\rangle}\,\rho = 2\rho.$$

Hence $\mathrm{Ind}_K^{K\rtimes\langle r\rangle}\,\rho$ is reducible and decomposed into two irreducible representations which restricts to ρ on K. This proves the first statement. The second statement follows from $\bar{\rho} + \bar{\rho}' = \mathrm{Ind}_K^{K\rtimes\langle r\rangle}\,\rho$. Q.E.D.

Definition. *Let K and r be as in Lemma 1. The sharp operator is the involution on the space of complex valued class functions on $K \rtimes \langle r \rangle$ defined by*

$$\sharp_r \chi(x) := \begin{cases} \chi(x) & \text{if } x \in K, \\ -\chi(x) & \text{if } x \in K \rtimes \langle r \rangle \setminus K \end{cases}$$

for class function χ.

Mackey's criterion and Lemma 1 show that every irreducible representation of $K \rtimes \langle r \rangle$ is either induced from an irreducible representation of K which is not r-invariant or is of the form described in Lemma 1. Hence the sharp operator stabilizes the set of irreducible characters of $K \rtimes \langle r \rangle$ and induces an involution of the set of irreducible representations of $K \rtimes \langle r \rangle$, which we also denote by $\sharp_r$. When r is obvious from the context, we omit the subscript of $\sharp$.

Lemma 2. *Let r and ρ be as in Lemma 1. The action of r and the bi-action of $K \times K$ on $\mathbb{C}[K]$ induces the action of $(K \times K) \rtimes \langle r \rangle$ on $\mathbb{C}[K]_\rho$ where r acts on $K \times K$ by $(g, h) \mapsto ({}^r g, {}^r h)$.*

Proof. For $\alpha \in \mathbb{C}[K]$ and $(g, h) \in K \times K$, we have

$${}^r\{(g, h) \cdot \alpha\} = {}^r(g\alpha h^{-1}) = {}^r g {}^r \alpha {}^r h^{-1} = ({}^r g, {}^r h) \cdot {}^r \alpha.$$

Since ρ is r-invariant the action restricts to the homogeneous component. This proves the Lemma. Q.E.D.

Lemma 3. *Let r and ρ be as in Lemma 1 with $K = G$. Then*

$$T^r(g, h) = \chi_{\bar\rho}(gr)\chi^*_{\bar\rho}(hr).$$

Proof. We denote by σ the representation of $(G \times G) \rtimes \langle r \rangle$ given by $\mathbb{C}[G]_\rho$. Since $\mathbb{C}[G]_\rho$ is isomorphic to $\rho \times \rho^*$ as $G \times G$-module, Lemma 1 and Lemma 2 show that σ is isomorphic to one of two irreducible representations of $(G \times G) \rtimes \langle r \rangle$ which restricts to $\rho \times \rho^*$. Since $(G \times G) \rtimes \langle r \rangle$ is considered to be a normal subgroup of $(G \rtimes \langle r \rangle) \times (G \rtimes \langle r \rangle)$ with index 2, one of the irreducible representations is defined by

$$\tau := \mathrm{Res}^{(G \rtimes \langle r \rangle) \times (G \rtimes \langle r \rangle)}_{(G \times G) \rtimes \langle r \rangle}(\bar\rho \times \bar\rho^*)$$

and the other is $\sharp\tau$ by Lemma 1. We shall prove that $\sigma = \tau$.

Recall that $\mathbb{C}[G]_\rho$ is a simple component of $\mathbb{C}$-algebra $\mathbb{C}[G]$. Since the action of r induces a $\mathbb{C}$-algebra automorphism of $\mathbb{C}[G]$ together with the conjugation by elements of G, the idempotent associated to r-invariant representation ρ is fixed by these actions. Hence we have

$$\langle \mathrm{Res}^{(G \times G) \rtimes \langle r \rangle}_H \sigma, 1_H \rangle_H \neq 0 \tag{4.3}$$

where H is the image of the diagonal embedding of $G \rtimes \langle r \rangle$ into $(G \rtimes \langle r \rangle) \times (G \rtimes \langle r \rangle)$, hence a subgroup of $(G \times G) \rtimes \langle r \rangle$. Since

$$\begin{aligned} 1 = \langle \bar\rho, \bar\rho \rangle_{G \rtimes \langle r \rangle} &= \frac{1}{|G \rtimes \langle r \rangle|} \sum_{x \in G \rtimes \langle r \rangle} |\chi_{\bar\rho(x)}|^2 \\ &= \frac{1}{2|G|} \left\{ \sum_{x \in G} |\chi_\rho(x)|^2 + \sum_{x \in G} |\chi_{\bar\rho}(xr)|^2 \right\} \\ &= \frac{1}{2} \left\{ \langle \rho, \rho \rangle_G + \frac{1}{|G|} \sum_{x \in G} |\chi_{\bar\rho}(xr)|^2 \right\}, \end{aligned}$$

we have

$$1 = \frac{1}{|G|}\sum_{x\in G} |\chi_{\bar\rho}(xr)|^2. \tag{4.4}$$

By definition of $\sharp\tau$ and (4.4),

$$\begin{aligned}\langle \mathrm{Res}_L^{(G\times G)\rtimes\langle r\rangle}\tau', 1_L\rangle_L =& \frac{1}{|H|}\left\{\sum_{x\in G}|\chi_\rho(x)|^2 - \sum_{x\in G}|\chi_{\bar\rho}(xr)|^2\right\}\\ =& \frac{1}{2}\left\{\langle\rho,\rho\rangle_G - \frac{1}{|G|}\sum_{x\in G}|\chi_{\bar\rho}(xr)|^2\right\} = 0.\end{aligned}$$

Hence by (4.3) $\sigma \neq \sharp\tau$. Thus $\sigma = \tau$. In particular, this proves the lemma. Q.E.D.

For $\theta \in G$ and irreducible representation ρ of G we set

$$M_{\theta,\rho} := \{\alpha \in \mathbb{C}[G]_\rho; \alpha\cdot\theta = \alpha\}.$$

If $g, h \in G,\ r \in \mathrm{Aut}(G)$ with ${}^{hr}\theta \in \langle\theta\rangle$ and ρ is r-invariant, we define isomorphism $\psi_\theta^r(g,h)_\rho$ by

$$\psi_\theta^r(g,h)_\rho : M_{\theta,\rho} \ni \alpha \mapsto g\ {}^r\alpha h^{-1} \in M_{\theta,\rho}.$$

Lemma 4. *For any $\theta \in G$ and irreducible representation ρ of G, the projection $\pi : \mathbb{C}[G] \to \mathbb{C}[G/\langle\theta\rangle]$ induces the isomorphism of G-modules*

$$\pi|_{M_{\theta,\rho}} : M_{\theta,\rho} \simeq \mathbb{C}[G/\langle\theta\rangle]_\rho.$$

If $g, h \in G,\ r \in \mathrm{Aut}(G)$ with ${}^{hr}\theta \in \langle\theta\rangle$ and ρ is r-invariant, the following diagram commutes

$$\begin{tikzcd} M_{\theta,\rho} \arrow[r, "\psi_\theta^r(g,h)"] \arrow[d, "\pi|_{M_{\theta,\rho}}"'] & M_{\theta,\rho} \arrow[d, "\pi|_{M_{\theta,\rho}}"] \\ \mathbb{C}[G/\langle\theta\rangle]_\rho \arrow[r, "\varphi_\theta^r(g,h)"] & \mathbb{C}[G/\langle\theta\rangle]_\rho. \end{tikzcd}$$

Proof. Let n be the order of θ. The right action of θ decomposes $\mathbb{C}[G]$ into eigenspaces

$$M_\theta(\omega) := \{\alpha \in \mathbb{C}[G]_\rho; \alpha\cdot\theta = \omega\alpha\}$$

where ω is an n-th root of unity. Picking representative $\bar{x}$ from each coset $x\langle\theta\rangle$, we define $v_{\omega,x}$ by

$$v_{\omega,x} := \sum_{i=o}^{n-1}\omega^i\bar{x}\theta^i.$$

It is readily verified that $M_\theta(\omega)$ is spanned by basis $\{v_{\omega,x}\}_{x\in G/\langle\theta\rangle}$. Since

$$\pi(v_{\omega,x}) = (\sum_{i=0}^{n-1}\omega^i)\cdot x,$$

$M_\theta(\omega)$ is killed by π if $\omega \neq 1$. Since $\dim M_\theta(1) = \sharp(G/\langle\theta\rangle) = \dim\mathbb{C}[G/\langle\theta\rangle]$, π induces an isomorphism of $M_\theta(1)$ onto $\mathbb{C}[G/\langle\theta\rangle]$. Since π is G-morphism, this restricts to the homogeneous components. This proves the first statement. The second statement follows immediately from the definition of φ's in 4.2 and ψ's. Q.E.D.

Lemma 5. *Let r and ρ be as in Lemma 1 and $\theta, g, h \in G$ with ${}^{hr}\theta \in \langle\theta\rangle$. Then*

$$T_\theta^r(g,h) = \frac{1}{n}\chi_{\bar\rho}(gr)\sum_{i=0}^{n-1}\chi_{\bar\rho}^*(\theta^i hr)$$

where n is the order of θ.

Proof. In view of Lemma 4 we shall compute the trace of action of $(g,h)r \in (G \times G) \rtimes \langle r\rangle$ on $M_{\theta,\rho}$ to compute $T_\theta(g,h)$. Let H be a subgroup of $(G \times G) \rtimes \langle r\rangle$ generated by $a := (g,h)r$ and $b := (\varepsilon, \theta)$. By assumption that ${}^{hr}\theta \in \langle\theta\rangle$ we have

$$ab = (g,h)(1, {}^r\theta)r = (1, {}^{hr}\theta)(g,h)r = b^\delta a$$

for some δ. Let $\bar H$ denote the semidirect product $\langle b\rangle \rtimes \langle a\rangle$ with the action of a on $\langle b\rangle$ given by $b \mapsto b^\delta$. Then H is a homomorphic image of $\bar H$ by the natural epimorphism, say ξ. Cliford's theorem gives the following isomorphism as $\bar H$-module,

$$\xi^* Res_H^{(G\times G)\rtimes\langle r\rangle} M_{\theta,\rho} \simeq \sum_l \langle \xi^* Res_H^{(G\times G)\rtimes\langle r\rangle}\sigma, l\rangle_{\bar H} l$$

where l runs over all the irreducible representations of $\bar H$ which factors through $\bar H \to \bar H/\langle b\rangle \simeq \langle a\rangle$ and σ denotes the representation of $(G \times G) \rtimes \langle r\rangle$ given by $\mathbb{C}[G]_\rho$. Since l's are all linear, we can compute the character explicitly. In fact we have

$$\begin{aligned}
T_\theta^r(g,h) &= \sum_l \langle p^* Res_H^{(G\times G)\rtimes\langle r\rangle}\mathbb{C}[G]_\rho, l\rangle_{\bar H} l(a)\\
&= \frac{1}{|\bar H|}\sum_{i,j,k}\sigma(p(b^i a^j))\exp(-\frac{2\pi}{m}kj)\exp(\frac{2\pi}{m}k)\\
&= \frac{1}{|\bar H|}\sum_{i,j}\sigma(p(b^i a^j))\sum_k \exp(\frac{2\pi}{m}k(1-j))\\
&= \frac{m}{|\bar H|}\sum_i \sigma(p(b^i a))
\end{aligned}$$

where m is the order of a. Hence by Lemma 3,

$$T_\theta^r(g,h) = \frac{1}{n}\chi_{\bar\rho}(gr)\sum_i \chi_{\bar\rho}^*(\theta^i hr).$$

This proves the Lemma. Q.E.D.

4.4 *Proof of Theorem A*

Theorem 1. *Let Γ be an r_1-normal subgroup of finite index in $B_{4,4,4}$ and ρ a nontrivial r_1-invariant irreducible representation of $G = B_{4,4,4}/\Gamma$. If*

$$-\chi_{\bar\rho}(\theta_E^{-1}r) - \chi_{\bar\rho}(r) + \frac{1}{4}\sum_{i=0}^{3}\{\chi_{\bar\rho}(\theta_A^i\theta_B^{-1}r) + \chi_{\bar\rho}(\theta_D^i\theta_E^{-1}r) + \chi_{\bar\rho}(\theta_C^i r) + \chi_{\bar\rho}(\theta_F^i r)\} \neq 0$$

for an irreducible representation $\bar{\rho}$ of $G \rtimes \langle r_1 \rangle$ which restricts to ρ on G, then ρ is an irreducible component of G-module $H_1(\Gamma\backslash\mathbb{H}^3, \mathbb{C})$.

Proof. We shall apply Lemmas in 4.3 to $r = r_1$. Observe that if θ_* degenerates in G to an element of order 2 or 1, the character summation formula in Lemma 4.3.5 still remains valid with $n = 4$ since it makes the summation twice or four times while the multiple is divided out by the factor 1/4. Computing out $\bar{\mathcal{E}}_\rho(gr)$ by Lemma 4.2.1, Lemma 4.3.3 and Lemma 4.3.5, we have

$$\bar{\mathcal{E}}_\rho(gr) = \chi_{\bar{\rho}}(gr)\Big\{ - \chi^*_{\bar{\rho}}(\theta_E^{-1}r) - \chi^*_{\bar{\rho}}(r) + \frac{1}{4}\sum_{i=0}^{3} \{\chi^*_{\bar{\rho}}(\theta_A^i\theta_B^{-1}r) + \chi^*_{\bar{\rho}}(\theta_D^i\theta_E^{-1}r) + \chi^*_{\bar{\rho}}(\theta_C^i r) + \chi^*_{\bar{\rho}}(\theta_F^i r)\}\Big\}.$$

Suppose $\chi_{\bar{\rho}}(gr) = 0$ for any $g \in G$. This implies $\chi_{\bar{\rho}} = \chi_{\sharp\bar{\rho}}$ by the second statement of Lemma 4.3.1, hence $\bar{\rho} = \sharp_{r_1}\bar{\rho}$. But this contradicts to the first statement of the lemma that there are exactly two such representations and $\bar{\rho}$ and $\sharp_{r_1}\bar{\rho}$ are those. Hence $\chi_{\bar{\rho}}(gr) \neq 0$ for some $g \in G$. Then the result follows from Proposition 4.1.1. Q.E.D.

Theorem 2. *Let Γ be a $r_1r_2r_3$-normal subgroup of finite index in $B_{4,4,4}$ and ρ a nontrivial $r_1r_2r_3$-invariant irreducible representation of G. If*

$$\chi_{\bar{\rho}}(\theta_A^{-1}\theta_F\theta_E r_1r_2r_3) + \chi_{\bar{\rho}}(r_1r_2r_3) \neq 0$$

for an irreducible representation $\bar{\rho}$ of $G \rtimes \langle r_1r_2r_3 \rangle$ which restricts to ρ on G, ρ is an irreducible component of G-module $H_1(\Gamma\backslash\mathbb{H}^3, \mathbb{C})$.

Proof. Computing out $\bar{\mathcal{E}}_\rho(gr)$ by Lemma 4.2.2 and Lemma 4.3.3 with $r = r_1r_2r_3$ and applying the same argument to Theorem 1 we obtain the theorem. Q.E.D.

5. Arithmetic lattices and congruent subgroups

5.1 *General notions* Let F be a field of characteristic$\neq 2$ and f a non-degenerate quadratic form on F^4. Set

$$O_f(F) := \{g \in \mathrm{GL}_4(F);\ g \cdot f = f\}$$

where g acts f by $g \cdot f(x, y) = f(g^{-1}x, g^{-1}y)$. We denote $O_f(F)$ by O_f if the coefficient is obvious in the context. It is immediate that $\det g = \pm 1$ for $g \in O_F(f)$. For $\xi \in F^4$ with $f(\xi) \neq 0$ we can associate reflection r_ξ defined by

$$r_\xi(u) = u - \frac{2\langle u, \xi\rangle}{f(\xi)}\xi$$

for $u \in F^4$ where $\langle *, * \rangle$ is the bilinear form associated to f. It is readily verified that r_ξ is an element of $O_f(F)$ and of determinant -1. Hence we have a normal subgroup of index two

$$SO_f(F) := \{g \in O_f(F);\ \det g = 1\}.$$

O_f has more $\mathbb{Z}_2$-quotients in general. Namely,

Definition. *Spinorial norm Sp_f is the homomorphism of $O_f(F)$ to $F^*/(F^*)^2$ which takes reflection r_ξ to $f(\xi)$ mod $(F^*)^2$. We denote by $\Omega_f(F)$ the intersection of $SO_f(F)$ and the kernel of Sp_f*

Remark. Since every element of $O_f(F)$ is represented as a product of reflections by Cartan-Dieudonne theorem, the definition determines the unique homomorphism. For the existence of the homomorphism, see [Ja];Volume II, Section 4.8.

5.2 *Arithmetic lattices* Let k be a number field, $\mathfrak{o}$ the ring of integers in k and f a non-degenerate quadratic form on k^4. Set

$$O_f(\mathfrak{o}) := \{g \in O_f(k);\ \text{all entries of } g \text{ are integers}\}.$$

Suppose that k has a real infinite place, say v_0, and quadratic form f_{v_0} is of type (3.1), that is, there exists $h \in \mathrm{GL}_4(\mathbb{R})$ such that

$$h \cdot f_{v_0} = f_0 := x^2 + y^2 + z^2 - t^2$$

where f_{v_0} is the quadratic form obtained from f by the base field extension via $v_0 : k \to \mathbb{R}$. Then we can embed $O_f(k)$ into $O_{f_0}(\mathbb{R})$ by

$$\lambda : O_f(k) \ni \alpha \mapsto h\alpha_{v_0} h^{-1} \in O_{f_0}(\mathbb{R})$$

where α_{v_0} is an element of $O_{f_{v_0}}(\mathbb{R})$ obtained from α by the base field extension via v_0. Since the kernel of spinorial norm $Sp_{f_0}(\mathbb{R})$ in $O_{f_0}(\mathbb{R})$ is the stabilizer of the interior of positive light cone in the Lorenzian space of type (3,1), we have

$$\ker Sp_{f_0}(\mathbb{R}) = Isom(\mathbb{H}^3),\ \ \Omega_{f_0}(\mathbb{R}) = Isom_0(\mathbb{H}^3).$$

The following is derived from the classical theorem due to Siegel.

Theorem (Siegel). *Suppose k is a totally real number field and f is a non-degenerate anisotropic quadratic form on k^4. If f is definite at all infinite places except for v_0 and of type (3,1) at v_0,*

$$\Gamma_{v_0} := \lambda(O_f(\mathfrak{o})) \cap SO_{f_0}(\mathbb{R})$$

is a cocompact Kleinian group.

Definition. *Γ is an arithmetic lattice of $O_{f_0}(\mathbb{R})$ if Γ is commensurable with Γ_{v_0} for some k, f and v_0 which verifies the assumption of Siegel's Theorem above.*

To see the arithmetic property of $B_{4,4,4}$, we shall describe the hyperbolic space as the positive hyperbolic leaf

$$H := \left\{(x,y,z,t);\ a^{-1}(x^2+y^2+z^2) - t^2 = 1,\ t > 0\right\}$$

in Lorentz metric space L spanned by (x,y,z,t)-coordinate in 2.1.

Lemma 1. *Sides A, B and C in Figure 1 are on planes*

$$
\begin{aligned}
F_A &:= \left\{(x,y,z,t) \in H; x + a^{-1}z - t = 0\right\}, \\
F_B &:= \left\{(x,y,z,t) \in H; y + a^{-1}x - t = 0\right\}, \\
F_B &:= \left\{(x,y,z,t) \in H; z + a^{-1}y - t = 0\right\},
\end{aligned}
$$

respectively. Planes F_A, F_B and F_C are perpendicular in L to

$$\xi_A = (a,0,1,1),\ \xi_B = (1,a,0,1),\ \xi_C = (0,1,a,1),$$

respectively.

Proof. See [HLM1] Section 3.

Lemma 2. *Let $r_X(X = A, B, C)$ be the reflection with respect to plane F_X. Then*

$$\theta_A = r_3 r_A,\ \theta_B = r_1 r_B,\ \theta_C = r_2 r_C.$$

Proof. Observe Figure 1 or 2. Q.E.D.

Lemma 3. *$B_{4,4,4}$ is arithmetic and a subgroup of finite index in Γ_{v_0} with $k = \mathbb{Q}(\sqrt{5})$ and $f_{v_0} = a^{-1}(x^2+y^2+z^2) - t^2$ with $a = (1+\sqrt{5})/2$.*

Proof. First note that k-rational point of L plays the role of k^4 in the definition of λ. Hence we shall consider θ_* and r_* to be elements both of $O_{f_{v_0}}(\mathbb{R}) \subset \mathrm{GL}_4(\mathbb{R})$ and $O_{f_0}(\mathbb{R})$ via λ. (x,y,z,t)-coordinate allows us the rigorous computations as follows. With this abbreviation r_A can be computed

$$r_A(u) = u - \frac{2\langle u, \xi_A\rangle}{f(u)}\xi_A = u - a\langle u, \xi_A\rangle\xi_A.$$

Since a and all entries in ξ_A are integers by Lemma 1, the computation shows $r_A \in O_f(\mathfrak{o})$. Similarly $r_A, r_B \in O_f(\mathfrak{o})$. Since obviously $r_1, r_2, r_3 \in O_f(\mathfrak{o})$, we have $\theta_A, \theta_B, \theta_C \in \Gamma_{v_0}$ by Lemma 2. Lemma 2.2.1 and Lemma 2.2.3 show that $B_{4,4,4} \subset \Gamma_{v_0}$. Since $B_{4,4,4}$ is cocompact by the construction and so is Γ_{v_0} by Theorem of Siegel, $B_{4,4,4}$ is of finite index, hence arithmetic. Q.E.D.

5.3 *Congruent subgroups* For ideal $\mathfrak{m}$ of $\mathfrak{o}$ we define congruent subgroup $O_f(\mathfrak{m})$ by

$$O_f(\mathfrak{m}) := \{g \in O_f(\mathfrak{o}); \gamma \equiv 1 \bmod \mathfrak{m}\}.$$

Clearly $O_f(\mathfrak{m})$ is an normal subgroup of $O_f(\mathfrak{o})$. Set

$$
\begin{aligned}
\Gamma'_{v_0} &:= \lambda(O_f(\mathfrak{o}) \cap \Omega_f(k)), \\
\Gamma_{\mathfrak{m}} &:= \lambda(O_f(\mathfrak{m})) \cap \Omega_{f_0}(\mathbb{R}), \\
\Gamma'_{\mathfrak{m}} &:= \lambda(O_f(\mathfrak{m}) \cap \Omega_f(k)) := \Gamma'_{v_0} \cap \Gamma_{\mathfrak{m}}.
\end{aligned}
$$

Note that Γ'_{v_0} and $\Gamma'_{\mathfrak{m}}$ are of finite index in Γ_{v_0} and $\Gamma_{\mathfrak{m}}$, respectively since those groups are finitely generated by its cocompactness as Kleinian groups and the spinorial norm maps those groups to the abelian group any non-trivial element of which are of order two. Suppose that $\mathfrak{m}$ is of form

$$\mathfrak{m} = \prod_i \mathfrak{p}_i$$

with all $\mathfrak{p}_i$'s being distinct. Reducing the entries modulo $\mathfrak{m}$ we have injections

$$\begin{aligned}\iota_{\mathfrak{m}} :&\Gamma_{v_0}/\Gamma_{\mathfrak{m}} \longrightarrow \prod_i SO_{f_{\mathfrak{p}_i}}(\mathfrak{o}/\mathfrak{p}_i)\\ \iota'_{\mathfrak{m}} :&\Gamma'_{v_0}/\Gamma'_{\mathfrak{m}} \longrightarrow \prod_i \Omega_{f_{\mathfrak{p}_i}}(\mathfrak{o}/\mathfrak{p}_i)\end{aligned}$$

where $f_{\mathfrak{p}}$ denotes the quadratic form reduced from f modulo $\mathfrak{p}$.

Lemma 1. *Let Γ be a subgroup of finite index in Γ_{v_0}. Then there exists finite set P_Γ of prime ideals of $\mathfrak{o}$ containing all prime ideals dividing 2 such that for prime ideal $\mathfrak{p} \notin P_\Gamma$ the injection $\iota'_{\mathfrak{p}}$ restricts to an isomorphism of $\Gamma'_{v_0} \cap \Gamma/\Gamma'_{\mathfrak{m}} \cap \Gamma$ to $\Omega_{f_{\mathfrak{p}}}(\mathfrak{o}/\mathfrak{m})$.*

Proof. Let P'_Γ be the set of primes for which $\iota'_{\mathfrak{p}}$ fails to be surjective on $\Gamma'_{v_0}\cap\Gamma/\Gamma'_{\mathfrak{m}}\cap\Gamma$ and which is coprime to the discriminant of f and 2.

Set $\mathfrak{m} = \prod_{i=1}^N \mathfrak{p}_i$ for distinct $\mathfrak{p}_i \in P'_\Gamma$. By definition of P'_Γ there exists proper subgroup H_i of $\Omega_{f_{\mathfrak{p}_i}}(\mathfrak{p})$ such that

$$\iota'_{\mathfrak{m}}(\Gamma'_{v_0} \cap \Gamma/\Gamma'_{\mathfrak{m}} \cap \Gamma) \subset \prod_i H_i.$$

Since Kneser's strong approximation theorem for simply connected group Ω_f shows that $\iota'_{\mathfrak{m}}$ is surjective (See [Mi];Proposition 3.2 for the detail of the argument.),

$$|\Gamma'_{v_0}/\Gamma \cap \Gamma'_{v_0}| \geq \frac{\Gamma'_{v_0}/\Gamma'_{\mathfrak{m}}}{|\Gamma'_{v_0} \cap \Gamma/\Gamma'_{\mathfrak{m}} \cap \Gamma|} \geq \prod_i |\Omega_{f_{\mathfrak{p}_i}}(\mathfrak{p}_i)/H_i| \geq 2^N \tag{5.1.}$$

Hence P'_Γ must be a finite set. The lemma follows with $P_\Gamma := P'_\Gamma \cup \{\mathfrak{p};\ \mathfrak{p}|\ 2\}$ Q.E.D.

Lemma 2. *Let k and f be as in Lemma 5.2.3. Suppose $\mathfrak{p} \notin P_{B_{4,4,4}}$.*
(1) $\iota_{\mathfrak{p}}(B_{4,4,4}) = \Omega_{f_{\mathfrak{p}}}(\mathfrak{o}/\mathfrak{p})$ if and only if $|\mathfrak{o}/\mathfrak{p}| \equiv \pm 1 \mod 8$.
(2) $\iota_{\mathfrak{p}}(B_{4,4,4}) = SO_{f_{\mathfrak{p}}}(\mathfrak{o}/\mathfrak{p})$ if and only if $|\mathfrak{o}/\mathfrak{p}| \equiv \pm 3 \mod 8$.

Proof. Since $F_{\mathfrak{p}} := \mathfrak{o}/\mathfrak{p}$ is a finite field, $|SO_{f_{\mathfrak{p}}}(F_{\mathfrak{p}})/\Omega_{f_{\mathfrak{p}}}(F_{\mathfrak{p}})| = |F^*_{\mathfrak{p}}/(F^*_{\mathfrak{p}})^2| = 2$. Hence Lemma 1 shows that the image $\iota_{\mathfrak{p}}(B_{4,4,4})$ coincides with $\Omega_{f_{\mathfrak{p}}}$ or $SO_{f_{\mathfrak{p}}}$. Thus the second statement follows from the first. By Lemma 5.2.2

$$Sp_f(\theta_A) = Sp_f(r_A)Sp_f(r_3) = 2a^{-1} \cdot a^{-1} = 2a^{-2}.$$

Hence reducing the spinorial norm modulo $\mathfrak{p}$, we have

$$Sp_{f_{\mathfrak{p}}}(\theta_A) = 2a^{-2} = 2 \mod (F^*_{\mathfrak{p}})^2.$$

It is classical that 2 is a square in $F_{\mathfrak{p}}$ if and only if $|F_{\mathfrak{p}}| \equiv \pm 1 \mod 8$. The same argument applies to $\theta_B, \theta_C, \theta_D = {}^{r_1}\theta_A^{-1}, \theta_E = {}^{r_1}\theta_B^{-1}$ and $\theta_F = {}^{-1}\theta_C^{-1}$. Then the first statement follows from Lemma 2.2.1. This proves the lemma. Q.E.D.

Lemma 3. *Let $k, \mathfrak{p}$ and f be as in Lemma 2 and d a non-square element of $F_{\mathfrak{p}} := \mathfrak{o}/\mathfrak{p}$. Quadratic form $f_{\mathfrak{p}}$ belongs to the (unique) cogridient class of isotropic quadratic forms or that of anisotropic ones according to $-a$ is a square in $F_{\mathfrak{p}}$ or not.*

Proof. Since quadratic forms over finite field F determined up to cogridient by its discriminant, a quadratic form over F^4 is isotropic if and only if its discriminant is trivial. Reduction modulo $\mathfrak{p}$ shows that the discriminant of $f_{\mathfrak{p}}$ is $-a^3 = -a \mod (F_{\mathfrak{p}}^*)^2$. This proves the lemma. Q.E.D.

6. Isotropic case

In Section 6 and Section 7 f always denotes the quadratic form $a^{-1}(x^2+y^2+z^2)-t^2$ over $k = \mathbb{Q}(\sqrt{5})$ and we assume that $\mathfrak{p} \notin P_{B_{4,4,4}}$. Throughout this section we assume that $-a = -(1+\sqrt{5})/2$ is a square in $F_{\mathfrak{p}}$.

6.1 *Structure of the split 4-orthogonal group* Suppose q is a power of an odd rational prime. Let $f_{q,0}$ be the isotropic quadratic form over $V := \bigoplus_{i=1}^{4} \mathbb{F}_q \cdot u_i$ define by

$$f_{q,0} := u_1u_2 + u_3u_4.$$

Note that the element of the basis of V and the associated coordinate are denoted by the same u_i as usual. Define the canonical totally isotropic subspaces of V by

$$V_1 := \mathbb{F}_q \cdot u_1 \oplus \mathbb{F}_q \cdot u_2,\ V_2 := \mathbb{F}_q \cdot u_3 \oplus \mathbb{F}_q \cdot u_4,$$
$$W_1 := \mathbb{F}_q \cdot u_1 \oplus \mathbb{F}_q \cdot u_4,\ W_2 := \mathbb{F}_q \cdot u_2 \oplus \mathbb{F}_q \cdot u_3$$

and the mutually orthogonal canonical hyperbolic subspaces by

$$H_1 := \mathbb{F}_q \cdot u_1 \oplus \mathbb{F}_q \cdot u_3,\ H_2 := \mathbb{F}_q \cdot u_2 \oplus \mathbb{F}_q \cdot u_4.$$

Set

$$S_1 := \left\{ g \in O_{f_{q,0}}(\mathbb{F}_q); g(V_i) = V_i,\ i = 1,2, g|_{V_1} \in SL_2(V_1) \right\},$$
$$S_2 := \left\{ g \in O_{f_{q,0}}(\mathbb{F}_q); g(W_i) = W_i,\ i = 1,2, g|_{W_1} \in SL_2(W_1) \right\}.$$

Lemma 1. *The restrictions*

$$\mathrm{Res}_{V_1} : S_1 \ni g \mapsto g|_{V_1} \in SL_2(V_1),$$
$$\mathrm{Res}_{W_1} : S_1 \ni g \mapsto g|_{W_1} \in SL_2(W_1),$$

induce isomorphisms.

Proof. With respect to (u_1,u_2,u_3,u_4)-coordinate quadratic form $f_{q,0}$ is represented by symmetric matrix

$$M = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}.$$

Since elements of S_1 stabilize $V_i (i = 1,2)$, they are of the form

$$s = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\ X \in SL_2(\mathbb{F}_q), \quad Y \in GL_2(\mathbb{F}_q).$$

It follows from $S_1 \subset O_{f_{q,0}}(\mathbb{F}_q)$ that $sM^ts^{-1} = M$. Hence $s \in S_1$ if and only if $Y = {}^tX^{-1}$. This shows the statement for Res_{V_1}. Let r_0 be the reflection with respect to $u_2 - u_4$. r_0 permutes u_2 and u_4 and fixes H_1. Clearly we have $r_0(V_i) = W_i$ and $S_2 = {}^{r_0}S_1$. Hence $\mathrm{Res}_{W_1} \circ r_0 = r_0 \circ \mathrm{Res}_{V_1}$ where r_0's denote the obvious isomorphisms of groups induced by the actions of r_0 on $O_{f_{q,0}}$ and V. This shows the statement for Res_{V_2}. Q.E.D.

Lemma 2. *S_1 and S_2 commute.*

Proof. Since $S_2 = {}^{r_0}S_1$, element s_i of S_i $(i = 1, 2)$ can be represented in the matrix form as

$$s_1 = \begin{pmatrix} X & 0 \\ 0 & {}^tX^{-1} \end{pmatrix}, \ X \in SL_2(\mathbb{F}_q)$$
$$s_2 = \begin{pmatrix} aI & bJ \\ -cJ & dI \end{pmatrix}, \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{F}_q),$$

where

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The direct computation shows

$$s_1s_2 = \begin{pmatrix} aX & bXJ \\ -c\,{}^tX^{-1} & d\,{}^tX^{-1} \end{pmatrix}, \ s_2s_1 = \begin{pmatrix} aX & bJ\,{}^tX^{-1} \\ -cJX & d\,{}^tX^{-1} \end{pmatrix}.$$

Since $JXJ^{-1} = J^{-1}XJ = {}^tX^{-1}$ for $X \in SL_2(\mathbb{F}_q)$, $s_1s_2 = s_2s_1$. Q.E.D.

Lemma 3. *S_1 and S_2 are subgroups of $\Omega_{f_{q,0}}(\mathbb{F}_q)$ and $S_1 \cap S_2 = Z(S_1) = Z(S_2)$ where $Z(*)$ denotes the center of the group.*

Proof. By Lemma 2, $S_1 \simeq S_2 \simeq SL_2(\mathbb{F}_q)$. Since $SL_2(\mathbb{F}_q)$ has no $\mathbb{Z}_2$-quotient, the determinant and the spinorial norm are trivial on S_1 and S_2. Hence $S_1, S_2 \subset \Omega_{f_{q,0}}(\mathbb{F}_q)$. This proves the first statement. The second statement follows immediately from the matrix representation of elements of S_1 and S_2. Q.E.D.

Lemma 4. *The sequence*

$$1 \longrightarrow S_1 \cap S_2 \longrightarrow S_1 \times S_2 \longrightarrow \Omega_{f_{q,0}}(\mathbb{F}_q) \longrightarrow 1$$

is exact. In particular

$$\Omega_{f_{q,0}}(\mathbb{F}_q) \simeq SL_2(\mathbb{F}_q) \times SL_2(\mathbb{F}_q)/ \pm (I, I).$$

Proof. Lemma 2 and Lemma 3 show the exactness except for the surjectivity of the right arrow. The surjectivity follows from the comparison of the order of $\Omega_{f_{q,0}}(\mathbb{F}_q)$ and $SL_2(\mathbb{F}_q) \times SL_2(\mathbb{F}_q)/ \pm (1, 1)$. The orders of both groups can be computed elementary and are equal to $q^2(q^2 - 1)^2/2$. Q.E.D.

Lemma 5. *$SO_{f_{q,0}}(\mathbb{F}_q)$ normalizes S_1 and S_2. The image of $SO_{f_{q,0}}(\mathbb{F}_q)/\Omega_{f_{q,0}}(\mathbb{F}_q)$ in $\mathrm{Out}(SL_2(\mathbb{F}_q))$ is isomorphic to $\mathbb{Z}_2$. If we identify $S_i(i=1,2)$ with $SL_2(\mathbb{F}_q)$ by restriction morphisms in Lemma 1 and the canonical coordinates (u_1,u_2) or (u_1,u_4), the actions of the nontrivial element on S_1 and S_2 are represented by the same automorphism*

$$SL_2(\mathbb{F}_q) \ni S \mapsto T_d\ {}^*ST_d^{-1} \in SL_2(\mathbb{F}_q)$$

where

$$T_d := \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \text{ with } d \notin (\mathbb{F}_q^*)^2.$$

and ${}^*S$ denotes the transpose of inverse of matrix S,i.e. ${}^*S := {}^tS^{-1}$ in the canonical notion.

Proof. Lemma 3 shows that S_1 and S_2 are normal. Since $SO_{f_{q,0}}(\mathbb{F}_q)/\Omega_{f_{q,0}}(\mathbb{F}_q) \simeq \mathbb{Z}_2$, the first statement follows just by verifying it for an element of $SO_{f_{q,0}}(\mathbb{F}_q) \setminus \Omega_{f_{q,0}}(\mathbb{F}_q)$. Let $t_{\alpha,\beta}$ be the element of $SO_{f_{q,0}}$ which maps

$$u_1 \mapsto \alpha u_3,\ u_2 \mapsto \beta u_4,\ u_3 \mapsto \alpha^{-1}u_1,\ u_4 \mapsto \beta^{-1}u_2.$$

Since $t_{\alpha,\beta}$ is the composition of two reflections with respect to αu_1+u_3 and βu_2+u_4, $t_{\alpha,\beta} \in \Omega_{f_{q,0}}(\mathbb{F}_q)$ iff $\alpha\beta \in (\mathbb{F}_q^*)^2$. Writing down $t_{\alpha,\beta}$ in the matrix form, we can compute the action as

$$\begin{aligned} t_{1,d}s_1t_{1,d} &= \begin{pmatrix} 0 & T_d^{-1} \\ T_d & 0 \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & {}^*X \end{pmatrix} \begin{pmatrix} 0 & T_d^{-1} \\ T_d & 0 \end{pmatrix} \\ &= \begin{pmatrix} T_d^{-1\,*}XT_d & 0 \\ 0 & T_dXT_d^{-1} \end{pmatrix}. \end{aligned}$$

This proves the first statement and the second statement for S_1. The statement can be similarly verified for S_2. Q.E.D.

Lemma 6. *Elements of $O_{f_{q,0}}(\mathbb{F}_q) \setminus SO_{f_{q,0}}(\mathbb{F}_q)$ permute S_1 and S_2.*

Proof. Since $SO_{f_{q,0}}$ stabilizes S_1 and S_2 by Lemma 5 and $O_{f_{q,0}}/SO_{f_{q,0}} \simeq \mathbb{Z}_2$, we only have to verify the statement for single element r_0. This is an immediate consequence of the definitions of S_1, S_2 and r_0. Q.E.D.

6.2 *Description of irreducible representations* Let $q = |F_\mathfrak{p}|$. Then by Lemma 5.3.3, there exists isomorphism μ of

$$L_\mathfrak{p} := F_\mathfrak{p}\ x \oplus F_\mathfrak{p}\ y \oplus F_\mathfrak{p}\ z \oplus F_\mathfrak{p}\ t$$

onto V in 6.1 such that $\mu^*f_{q,0} = f_\mathfrak{p}$. Clearly μ^* induces isomorphisms , $O_{f_{q,0}} \simeq O_{f_\mathfrak{p}}$, $SO_{f_{q,0}} \simeq SO_{f_\mathfrak{p}}$ and $\Omega_{f_{q,0}} \simeq \Omega_{f_\mathfrak{p}}$. Identifying $S_1 \subset \Omega_{f_{q,0}}$ with $SL_2(\mathbb{F}_q)$ by Lemma 6.1.1 and the canonical coordinate (u_1,u_2), we have an inclusion

$$i_1 :\ SL_2(\mathbb{F}_q) \hookrightarrow \Omega_{f_{q,0}} \xrightarrow{\mu} \Omega_{f_\mathfrak{p}}.$$

We can also define an inclusion of $SL_2(\mathbb{F}_q)$ into $\Omega_{f_\mathfrak{p}}$ which factors through S_2 by $i_2 := r_1 \circ i_1$ since r_1 is a reflection and permutes $\mu(S_1)$ and $\mu(S_2)$ by Lemma 6.1.6. We denote by $\mathrm{Irr}(G)$ the set of complex irreducible representations of finite group G.

Lemma 1. *(i) Let p be the morphism defined by*

$$p:\ SL_2(\mathbb{F}_q)\times SL_2(\mathbb{F}_q)\ni(s_1,s_2)\mapsto\ i_1(s_1)i_2(s_2)\in\Omega_{f_{\mathfrak{p}}}.$$

Then the pull back

$$\mathrm{Irr}(\Omega_{f_{\mathfrak{p}}})\ni\rho\mapsto p^*\rho\in\mathrm{Irr}(SL_2(\mathbb{F}_q)\times SL_2(\mathbb{F}_q))\simeq\mathrm{Irr}(SL_2(\mathbb{F}_q))\times\mathrm{Irr}(SL_2(\mathbb{F}_q))$$

induce a one-to-one corresponding of $\mathrm{Irr}(\Omega_{f_{\mathfrak{p}}})$ *with*

$$\mathfrak{I}:=\{(\rho_1,\rho_2)\in\mathrm{Irr}(SL_2(\mathbb{F}_q))\times\mathrm{Irr}(SL_2(\mathbb{F}_q));\rho_1(-I)\times\rho_2(-I)=id\}.$$

(ii) The action of r_1 on $\mathrm{Irr}(\Omega_{f_{\mathfrak{p}}})$ is represented in $\mathfrak{I}$ as the permutation of the components. In particular, r_1-invariant irreducible representations of $\Omega_{f_{\mathfrak{p}}}$ correspond to elements of the form $(\rho_1,\rho_1)\in\mathfrak{I}$.

Proof. Since morphism p is surjective by Lemma 6.1.4, pull back p^* injects $\mathrm{Irr}(\Omega_{f_{\mathfrak{p}}})$ to $\mathrm{Irr}(SL_2(\mathbb{F}_q)\times SL_2(\mathbb{F}_q))$. For given $\rho_1\times\rho_2\in\mathrm{Irr}(SL_2(\mathbb{F}_q)\times SL_2(\mathbb{F}_q))$ to be in the image of p^* it is necessary and sufficient that $\rho_1\times\rho_2$ reduces to the identity on the kernel $\pm(I,I)$ of p. This proves (i). (ii) is readily verified by definitions of i_1 and i_2. Q.E.D.

Lemma 2. *The action of the non-trivial element of $SO_{f_{\mathfrak{p}}}/\Omega_{f_{\mathfrak{p}}}$ on $\mathrm{Irr}(\Omega_{f_{\mathfrak{p}}})$ induces the action on $\mathfrak{I}$ defined by*

$$\mathfrak{I}\ni(\rho_1,\rho_2)\mapsto({}^{*d}\rho_1,{}^{*d}\rho_2)\in\mathfrak{I}$$

where $*d$ is the automorphism of $SL_2(\mathbb{F}_q)$ given in Lemma 6.1.5.

Proof. We have made a slight arrangement of the identification of S_2 with $SL_2(\mathbb{F}_q)$ in this subsection, i.e. we did it via r_0 in 6.1 and via r_1 in this subsection. This changes the action on $SL_2(\mathbb{F}_q)$ into $x(*d)x^{-1}$ where x is the automorphism induced from $\mu^{-1}r_1\mu r_0$. But the automorphism represents the same outer automorphism as T regardless x is inner or not. Noting this point, the statement is a consequence of Lemma 6.1.5. Q.E.D.

Lemma 3. *There exists an element of order two in $SO_{f_{\mathfrak{p}}}\setminus\Omega_{f_{\mathfrak{p}}}$ denoted by t such that t commutes with r_1 and tr_1 is a reflection.*

Proof. Let W be the three dimensional subspace of $L_{\mathfrak{p}}$ fixed by reflection r_1. Since W is a three dimensional, non-degenerate metric space over $\mathbb{F}_q$, W is of Witt index 1. Hence uniqueness of the Witt index shows that there are mutually orthogonal hyperbolic subspaces H and H' such that $H\subset W$ and $r_1|_{H'}$ is a reflection. Picking reflection r_1' of H with $Sp_{f_{\mathfrak{p}}}(r_1)Sp_{f_{\mathfrak{p}}}(r_1')\neq 1$, we have $r_1r_2'\in SO_{\mathfrak{p}}\setminus\Omega_{f_{\mathfrak{p}}}$ of the required property. Q.E.D.

Fix $t_1\in SO_{f_{\mathfrak{p}}}\setminus\Omega_{\mathfrak{p}}$ with the properties in Lemma 3. By Lemma 6.1.5, t_1 induces actions on $SL_2(\mathbb{F}_q)$ through i_1 and i_2. It follows from the commutativity of r_1 and t_1 that these actions coincide. Via the common action set

$$\widehat{SL_2}(\mathbb{F}_q):=SL_2(\mathbb{F}_q)\rtimes\langle t_1\rangle.$$

We have an epimorphism

$$\hat{p}:\ (SL_2(\mathbb{F}_q)\times SL_2(\mathbb{F}_q))\rtimes\langle t_1\rangle\ni(s_1,s_2)\delta\mapsto i_1(s_1)i_2(s_2)\delta\in SO_{f_{\mathfrak{p}}}$$

where t_1 acts on $(SL_2(\mathbb{F}_q)\times SL_2(\mathbb{F}_q))$ by $(s_1,s_2)\mapsto({}^ts_1,{}^ts_2)$ and $\delta=\varepsilon$ or t_1. $(SL_2(\mathbb{F}_q)\times SL_2(\mathbb{F}_q))\rtimes\langle t_1\rangle$ is a normal subgroup of index two in $\widehat{SL_2}(\mathbb{F}_q)\times\widehat{SL_2}(\mathbb{F}_q)$ via inclusion $\hat{\imath}:\ (s_1,s_2)\delta\mapsto(s_1\delta,s_2\delta)$.

Lemma 4. *(i) $Irr(SO_{f_{\mathfrak{p}}})$ is the disjoint union of the following subsets.*

$$M_1 := \left\{\hat\rho \in \mathrm{Irr}(SO_{f_{\mathfrak{p}}});\ \mathrm{Res}_{\Omega_{f_{\mathfrak{p}}}}^{SO_{f_{\mathfrak{p}}}} \hat\rho \textit{ is irreducible.}\right\},$$
$$M_2 := \left\{\mathrm{Ind}_{\Omega_{f_{\mathfrak{p}}}}^{SO_{f_{\mathfrak{p}}}} \rho;\ \rho \in \mathrm{Irr}(\Omega_{f_{\mathfrak{p}}}) \textit{ and } \rho \neq {}^{t_1}\rho\right\}.$$

(ii) M_1 has the canonical one-to-one corresponding with $\hat{\mathfrak{I}}/\langle \sharp_{t_1}\rangle$ where

$$\hat{\mathfrak{I}} := \{(\hat\rho_1, \hat\rho_2) \in \mathrm{Irr}(\widehat{SL_2}(\mathbb{F}_q)) \times \mathrm{Irr}(\widehat{SL_2}(\mathbb{F}_q));$$
$$\mathrm{Res}_{SL_2}^{\widehat{SL}_2} \hat\rho_1,\ \mathrm{Res}_{SL_2}^{\widehat{SL}_2} \hat\rho_2 \in \mathrm{Irr}(SL_2(\mathbb{F}_q)),\ \textit{and } \hat\rho_1(-I) \times \hat\rho_2(-I) = id.\}$$

and the $\sharp_{t_1}$-operator acts on $\hat{\mathfrak{I}}$ by $(\hat\rho_1, \hat\rho_2) \mapsto (\sharp_{t_1}\hat\rho_1, \sharp_{t_1}\hat\rho_2)$. The corresponding is obtained by the composition of pull-back $\hat{\mathfrak{p}}$ and the extension of the representation over inclusion $\hat{i}$. In particular, the character of irreducible representation $\hat\rho$ of $SO_{f_{\mathfrak{p}}}$ corresponding to $(\hat\rho_1, \hat\rho_2) \in \hat{\mathfrak{I}}$ is given by

$$\chi_{\hat\rho}(i_1(s_1)i_2(s_2)\delta) = \chi_{\hat\rho_1}(s_1\delta)\chi_{\hat\rho_2}(s_2\delta)$$

for $s_1 \in S_1$, $s_2 \in S_2$ and $\delta = \varepsilon$ or t_1.

Proof. (i) follows from Lemma 4.3.1 and the Mackey's criterion. Similarly to Lemma 1, (ii) is proved according to the recipe in the statement. We shall sketch the strategy.
The pull back via $\hat{\mathfrak{p}}$ induces the one-to-one corresponding of M_1 with

$$\hat{\mathfrak{I}}' := \{\check\rho \in \mathrm{Irr}((SL_2(\mathbb{F}_q) \times SL_2(\mathbb{F}_q)) \rtimes \langle t_1\rangle);$$
$$\check\rho(-I, -I) = id,\ \mathrm{Res}_{SL_2 \times SL_2}^{(SL_2 \times SL_2) \rtimes \langle t_1\rangle} \check\rho \text{ is irreducible.}\}.$$

Then we shall show that the representations in $\hat{\mathfrak{I}}'$ extends to $\widehat{SL}_2(\mathbb{F}_q) \times \widehat{SL}_2(\mathbb{F}_q)$ over inclusion $\hat{i}$ and the extended representations lie in $\hat{\mathfrak{I}}$.
Suppose $\check\rho \in \hat{\mathfrak{I}}'$. Then $\mathrm{Res}_{SL_2 \times SL_2}^{(SL_2 \times SL_2) \rtimes \langle t_1\rangle} \check\rho = \rho_1 \times \rho_2$ for some $\rho_1, \rho_2 \in \mathrm{Irr}(SL_2(\mathbb{F}_q))$. By the Mackey's criterion and Lemma 6.1.5, both ρ_1 and ρ_2 are t_1-invariant. Thus by Lemma 4.3.1, we can find $\hat\rho_1$ and $\hat\rho_2$ in $\mathrm{Irr}(\widehat{SL}_2(\mathbb{F}_q))$ which restrict to ρ_1 and ρ_2, respectively. Again by Lemma 4.3.1 one of $\mathrm{Res}_{(SL_2 \times SL_2) \rtimes \langle t_1\rangle}^{\widehat{SL}_2 \times \widehat{SL}_2}(\hat\rho_1 \times \hat\rho_2)$ or $\mathrm{Res}_{(SL_2 \times SL_2) \rtimes \langle t_1\rangle}^{\widehat{SL}_2 \times \widehat{SL}_2}(\hat\rho_1 \times \sharp\hat\rho_2)$ coincides with $\check\rho$. This shows that representations in $\hat{\mathfrak{I}}'$ extend to $\widehat{SL}_2 \times \widehat{SL}_2$. By the argument, clearly the resulting pair of representations belongs to $\hat{\mathfrak{I}}$ and satisfies the character formula in the statement. Conversely for each $(\hat\rho_1, \hat\rho_2) \in \hat{\mathfrak{I}}$ associate the irreducible representation in M_1 via the character formula. This defines the left inverse of the composition of $\hat{\mathfrak{p}}^*$ followed by the extension over i. The corresponding is two-to-one because of the two possibilities of the extension. By Lemma 4.3.1, the ambiguity is divided out by the action of $\sharp_{t_1}$-operator. Q.E.D.

Lemma 5. *The set of r_1-invariant irreducible representations of $SO_{f_{\mathfrak{p}}}$ is the disjoint union of the following subsets.*

$$I_1 := \left\{\hat\rho \in M_1;\ \mathrm{Res}_{\Omega_{f_{\mathfrak{p}}}}^{SO_{f_{\mathfrak{p}}}} \hat\rho \textit{ is } r_1\textit{-invariant.}\right\},$$
$$I_2 := \left\{Ind_{\Omega_{f_{\mathfrak{p}}}}^{SO_{f_{\mathfrak{p}}}} \hat\rho \in M_2;\ \rho \textit{ is } r_1\textit{-invariant.}\right\},$$
$$I_3 := \left\{Ind_{\Omega_{f_{\mathfrak{p}}}}^{SO_{f_{\mathfrak{p}}}} \hat\rho \in M_2;\ \rho \textit{ is } r_1t_1\textit{-invariant.}\right\}.$$

Proof. For $\hat{\rho} \in M_1$ to be r_1-invariant, obviously it is necessary that $\mathrm{Res}_{\Omega_{f_{\mathfrak{p}}}}^{SO_{f_{\mathfrak{p}}}} \hat{\rho}$ is r_1-invariant. Conversely suppose that $\mathrm{Res}_{\Omega_{f_{\mathfrak{p}}}}^{SO_{f_{\mathfrak{p}}}} \hat{\rho}$ is r_1-invariant. Then $p^* \mathrm{Res}_{\Omega_{f_{\mathfrak{p}}}}^{SO_{f_{\mathfrak{p}}}} \hat{\rho} = \rho_1 \times \rho_1$ for some $\rho_1 \in SL_2(\mathbb{F}_q)$ by Lemma 1 (ii). It follows from Lemma 4 (ii) that there exists $\hat{\rho}_1 \in \mathrm{Irr}(\widehat{SL}_2(\mathbb{F}_q))$ such that $\mathrm{Res}_{SL_2}^{\widehat{SL}_2} \hat{\rho}_1 = \rho_1$ and $\chi_{\hat{\rho}}(i_1(s_1)i_2(s_2)t) = \pm\chi_{\hat{\rho}_1}(s_1 t)\chi_{\hat{\rho}_1}(s_2 t)$. Hence the definition of i_2 and the commutativity of t and r_1 show

$$\begin{aligned} {}^{r_1}\chi_{\hat{\rho}}(i_1(s_1)i_2(s_2)t) =& \chi_{\hat{\rho}}({}^{r_1}i_1(s_1){}^{r_1}i_2(s_2)t) \\ =& \pm\, \chi_{\hat{\rho}_1}(s_2 t)\chi_{\hat{\rho}_1}(s_1 t) = \chi_{\hat{\rho}}(i_1(s_1)i_2(s_2)t). \end{aligned}$$

This verifies that $\hat{\rho}$ is r_1-invariant. The r_1-invariance of induced representations in M_2 is clearly equivalent to that of its restriction to $\Omega_{f_{\mathfrak{p}}}$. Since

$$\mathrm{Res}_{\Omega_{f_{\mathfrak{p}}}}^{SO_{f_{\mathfrak{p}}}} \mathrm{Ind}_{\Omega_{f_{\mathfrak{p}}}}^{SO_{f_{\mathfrak{p}}}} \rho = \rho + {}^{t_1}\rho,$$

the condition is equivalent to ${}^{r_1}\rho = \rho$ or ${}^{r_1 t_1}\rho = \rho$. Since ${}^{t_1}\rho \neq \rho$ by definition of M_2, these two cases are exclusive. This proves the lemma. Q.E.D.

Lemma 6. *The action of $r_1r_2r_3$ on irreducible representations of $\Omega_{f_{\mathfrak{p}}}$ and $SO_{f_{\mathfrak{p}}}$ is the same as that of r_1. In particular, $r_1r_2r_3$-invariance of an irreducible representation of G is equivalent to its r_1-invariance.*

Proof. It follows immediately from that r_2r_3 induces the inner automorphism of G. Q.E.D.

6.3 *Conjugacy classes* By $Tr_k(g)$ we denote the trace of endmorphism g of L and by $Tr_{\mathfrak{p}}(g)$ the trace of endmorphism g of $L_{\mathfrak{p}}$.

Lemma 1. *(i) $Tr_k(\theta_X) = 2$, $Tr_k(\theta_X^2) = 0$ for $X = A, B, .., F$.*
(ii) $Tr_k(r_1) = Tr_k(\theta_E^{-1} r_1) = 2$.
(iii) $Tr_k(r_1r_2r_3) = Tr_k(\theta_A^{-1}\theta_F\theta_E r_1r_2r_3) = -2$.

Proof. (i) Since θ_X is elliptic with its axis on xx' the action of θ_X on L fixes two-dimensional subspace F which is spanned by xx'. Since f is anisotropic over k (by its definiteness at the other real place), $(F, f|_F)$ is non-degenerate. Hence θ_X stabilize F and $F^\perp$. Since θ_X is of order four and of determinant 1, $Tr_k(\theta|_{F^\perp}) = 0$. Thus

$$Tr_k(\theta_X) = Tr_k(\theta_X|_F) + Tr_k(\theta_X|_{F^\perp}) = 2.$$

The similar proof applies to θ_X^2.
(ii) Since both θ_E and r_1 fix a subspace of L spanned by ee', the argument in (i) applies to this case, noting that the determinants of r_1 and $\theta_E r_1$ are -1.
(iii) By Lemma 2.2.3 and Lemma 5.2.2, $\theta_A^{-1}\theta_F\theta_E r_1r_2r_3 = r_Ar_Br_C$. Hence both $r_1r_2r_3$ and $\theta_A^{-1}\theta_F\theta_E r_1r_2r_3$ are compositions of three reflections with respect to mutually orthogonal hyperbolic planes. This implies the statement. Q.E.D.

Recall that the semisimple conjugacy classes of $s \in SL_2(\mathbb{F}_q)$ is determined uniquely by trace $Tr_{\mathbb{F}_q} s$ as an endmorphism of $\mathbb{F}_q^2$. By $\mathfrak{T}_\alpha$, we denote the semisimple conjugacy class represented by diagonal matrix

$$\begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix}$$

over algebraic closure $\bar{\mathbb{F}}_q$. It is directly verified that $\mathfrak{T}_\alpha = \mathfrak{T}_\beta$ iff $\alpha = \beta$ or $\alpha\beta = 1$. In particular, $SL_2(\mathbb{F}_q)$ contains only one conjugacy class of order four and two conjugacy classes of order eight. We denote these classes by $\mathfrak{T}_4$, $\mathfrak{T}_8$ and $-\mathfrak{T}_8$, respectively. The non-semisimple conjugacy classes are represented by

$$u_1 := \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, \ u_d := \begin{pmatrix} 1 & d \\ & 1 \end{pmatrix}, \ -u_1 := \begin{pmatrix} -1 & -1 \\ & -1 \end{pmatrix}, \ -u_d := \begin{pmatrix} -1 & -d \\ & -1 \end{pmatrix}$$

where d is a non-square of $\mathbb{F}_q^*$. We denote these conjugacy classes by $\mathfrak{U}_1$, $\mathfrak{U}_d$, $-\mathfrak{U}_1$ and $-\mathfrak{U}_d$, respectively.

Lemma 2. *Let $s_1, s_2 \in SL_2(\mathbb{F}_q)$.*

$$Tr_{\mathbb{F}_q}(i_1(s_1)i_2(s_2)) = Tr_{\mathbb{F}_q}(s_1)Tr_{\mathbb{F}_q}(s_2).$$

Proof. The computation in Lemma 6.1.2 shows that the formula holds true provided i_1 and i_2 are replaced by the identification of $SL_2(\mathbb{F}_q)$ with S_1 and S_2 in 6.1. As we mentioned in the proof of Lemma 6.2.2, the difference of these identification, if it is not inner, is represented by the outer automorphism $s \mapsto T_d{}^tS^{-1}T_d^{-1}$, which does not change the trace. This proves the lemma. Q.E.D.

Lemma 3. *Let p be the epimorphism in Lemma 6.2.1.*
(i) θ_X^2 $(X = A, B, .., F)$ is conjugate to $p(s, s)$ with $s \in \mathfrak{T}_4$.
(ii) Suppose $q \equiv \pm 1 \mod 8$. θ_X is conjugate to $p(s, s)$ with $s \in \mathfrak{T}_8$.

Proof. Since $Tr_k(\theta_X^2) = 0$ by Lemma 1, $Tr_{\mathfrak{p}}(\theta_X^2) \neq Tr_{\mathfrak{p}}(I)$ since $\mathfrak{p}$ is coprime to 2. Thus θ_X^2 is an element of order two in $\Omega_{f_{\mathfrak{p}}}$. By Lemma 6.2.5, $(s_1, s_2) \in SL_2(\mathbb{F}_q) \times SL_2(\mathbb{F}_q)$ projects to an element of order two if either $(s_1, s_2) \in \mathfrak{T}_4 \times \mathfrak{T}_4$ or $(s_1, s_2) = \pm(I, -I)$. The latter case is excluded since

$$Tr_{\mathfrak{p}}(p(I, -I)) = Tr_{\mathbb{F}_{\mathfrak{p}}}(I)Tr_{\mathbb{F}_{\mathfrak{p}}}(-I) = -4 \neq 0 = Tr_{\mathfrak{p}}(\theta_X^2)$$

by Lemma 2. This proves (i). Similarly by Lemma 1, θ_X is an element of order four in $\Omega_{f_{\mathfrak{p}}}$ provided $q \equiv \pm 1 \mod 8$. $(s_1, s_2) \in SL_2(\mathbb{F}_q) \times SL_2(\mathbb{F}_q)$ projects to an element of order four if both $s_1$and s_2 are of order eight or $(s_1, s_2) \in \mathfrak{T}_4 \times \{\pm I\} \cup \{\pm I\} \times \mathfrak{T}_4$. The latter case is excluded since

$$Tr_{\mathfrak{p}}(p(s_1, \pm I)) = Tr_{\mathbb{F}_q}(s_1)Tr_{\mathbb{F}_q}(\pm I) = 0$$

by Lemma 2. Since there are two conjugacy classes of order eight in $SL_2(\mathbb{F}_q)$, we still have two possibilities of conjugacy classes of θ_X, $p(\mathfrak{T}_8, \mathfrak{T}_8)$ or $p(\mathfrak{T}_8, -\mathfrak{T}_8)$. Again by Lemma 1, the latter is excluded since

$$Tr_{\mathfrak{p}}(p(\mathfrak{T}_8, -\mathfrak{T}_8)) = -Tr_{\mathfrak{p}}(\mathfrak{T}_8)^2 = -2 \neq 2 = Tr_{\mathfrak{p}}(\theta_X).$$

This proves (ii). Q.E.D.

Lemma 4. *(i) Element $i_1(s_1)i_2(s_2)r_1 \in \Omega_{f_\mathfrak{p}} \rtimes \langle r_1\rangle$ is conjugate to $i_1(s)r_1$ in $\Omega_{f_\mathfrak{p}} \rtimes \langle r_1\rangle$ to $i_1(s)r_1$ for any element $s \in SL_2(\mathbb{F}_q)$ which is conjugate to s_1s_2.*
(ii) Element $i_1(s_1)i_2(s_2)r_1t_1 \in O_{f_\mathfrak{p}}$ is conjugate to $i_1(s)r_1t_1$ in $O_{f_\mathfrak{p}}$ for any element $s \in SL_2(\mathbb{F}_q)$ which is conjugate to ${s_1}^{t_1}s_2$.

Proof. (i) By Lemma 2 and the definition of i_2, we have

$$^{i_2(s_2)^{-1}}(i_1(s_1)i_2(s_2)r_1) = i_1(s_1s_2)r_1.$$

Similarly, for any $g \in SL_2(\mathbb{F}_q)$, element $i_1(g)i_2(g)$ acts on $i_1(s_1s_2)r_1$ by

$$^{i_1(g)i_2(g)}(i_1(s_1s_2)r_1) = i_1(gs_1s_2g^{-1})r_1.$$

Thus $i_1(s_1)i_2(s_2)r_1$ can be conjugated to $i_1(s)r_1$ for any s in the conjugacy class of s_1s_2. This proves (i). (ii) is similarly proved, replacing the latter conjugation by $i_1(g)i_2({}^{t_1}g)$. Q.E.D.

Lemma 5. *Suppose that $q \equiv \pm 1 \mod 8$.*
(i) $\theta_E^{-1}r_1$ is conjugate to r_1 in $\Omega_{f_\mathfrak{p}} \rtimes \langle r_1\rangle$.
(ii) $\theta_C^i r_1$ and $\theta_F^i r_1$ $(i = 1, 2, 3)$ are conjugate to $i_1(\mathfrak{T}_4^i)r_1$ in $\Omega_{f_\mathfrak{p}} \rtimes \langle r_1\rangle$.
(iii) $\theta_A^i\theta_B^{-1}r_1$ and $\theta_D^i\theta_E^{-1}r_1$ $(i = 1, 2, 3)$ are conjugate to $i_1(\mathfrak{T}_4^i)r_1$ in $\Omega_{f_\mathfrak{p}} \rtimes \langle r_1\rangle$.

Proof. (i) By Lemma 3(ii), $\theta_E = i_1(s_1)i_2(s_2)$ with $s_1,\ s_2 \in \mathfrak{T}_8$. By Lemma 2.2.3, we have

$$^{r_1}\theta_E = i_1(s_1)i_2(s_2) = i_1(s_1^{-1})i_2(s_2^{-1}) = \theta_E^{-1}.$$

Thus $s_1s_2 = \pm I$. Since $s_1,\ s_2 \in \mathfrak{T}_8$, The minus signature is not admissible. Then, the result follows from Lemma 4.
(ii) Set $\theta_C = i_1(s_1)i_2(s_2)$ as in (i). By Lemma 2.2.3 we have

$$^{r_1}\theta_C = i_1(s_1)i_2(s_2) = i_1(s_1)i_2(s_2) = \theta_C.$$

Hence $s_1 = s_2 \in \mathfrak{T}_8$. Then the result follows from Lemma 4. The proof for θ_F is the same.
(iii) First verify that θ_D and $\theta_E^{-1}r_1$ commute by Lemma 2.2.1 and Lemma 2.2.3. Then, conjugating $\theta_E^{-1}r_1$ with some $g \in \Omega_{f_\mathfrak{p}}$ to r_1 by (i), it falls into the same situation as (ii), i.e. ${}^g\theta_D$ and r_1 commute and ${}^g\theta_D = i_1(s_1)i_2(s_2)$ with $s_1, s_2 \in \mathfrak{T}_8$. We can apply the proof of (ii) for this situation. The proof for $\theta_A^i\theta_B r_1$ is similar. Q.E.D.

Lemma 6. *Suppose that $q \equiv \pm 3 \mod 8$.*
(i) $\theta_E^{-1}r_1$ is conjugate to r_1t_1 in $O_{f_\mathfrak{p}}$.
(ii) $\theta_C^{\pm 1}r_1$ and $\theta_F^{\pm 1}r_1$ are conjugate to $i_1(\mathfrak{T}_4)r_1t_1$ in $O_{f_\mathfrak{p}}$.
(iii) $\theta_C^2 r_1$ and $\theta_F^2 r_1$ are conjugate to $i_1(-I)r_1$ in $O_{f_\mathfrak{p}}$.
(iv) $\theta_A^{\pm 1}\theta_B^{-1}r_1$ and $\theta_D^{\pm 1}\theta_E^{-1}r_1$ are conjugate to $i_1(\mathfrak{T}_4)r_1$ in $O_{f_\mathfrak{p}}$.
(v) $\theta_A^2\theta_B^{-1}r_1$ and $\theta_D^2\theta_E^{-1}r_1$ are conjugate to $i_1(-I)r_1t_1$ in $O_{f_\mathfrak{p}}$.

Proof. (i) Since $\theta_E \in SO_{f_\mathfrak{p}} \subset \Omega_{f_\mathfrak{p}}$, $\theta_E = i_1(s_1)i_2(s_2)t_1$ for some $s_1,\ s_2 \in SL_2(\mathbb{F}_q)$. By Lemma 2.2.3, we have

$$^{r_1}\theta_E = i_1(s_1)i_2(s_2)t_1 = i_1({}^{t_1}s_1^{-1})i_2({}^{t_1}s_2^{-1})t_1 = \theta_E,$$

hence, ${}^{t_1}s_1s_2 = \pm I$. Lemma 4(ii) shows that $\theta_E^{-1}r_1$ is conjugate to $i_1(\pm I)r_1t_1$. Since r_1t_1 is a reflection by Lemma 6.2.3, we have

$$2 = Tr_k(\theta_E^{-1}r_1) = Tr_k(i_1(\pm I)r_1t_1) = \pm 2.$$

Hence the minus signature is not admissible.
(ii) Set $\theta_C = i_1(s_1)i_2(s_2)t_1$. Relation ${}^{r_1}\theta_C = \theta_C$ in Lemma 2.2.3 shows that $s_1 = \pm s_2$. Then by Lemma 4

$$\theta_C^{\pm 1}r_1 = i_1(s_1^{\pm 1})i_2(\pm s_1^{\pm 1})r_1t_1$$

conjugates to $i_1(\pm(s_1{}^{t_1}s_1))r_1t_1$. On the other hand, since we have

$$\theta_C^2 = i_1(s_1{}^{t_1}s_1)i_2(s_1{}^{t_1}s_1),$$

the argument of Lemma 5(ii) shows that $s_1{}^{t_1}s_1 \in \mathfrak{T}_4$. This proves the statement for θ_C and similarly for θ_F.
(iii) The proof of Lemma 5(ii) applies to this case. (iv) and (v) can be proved similarly to Lemma 5 (iii), making use of (i),(ii) and (iii). Q.E.D.

Since $r_1r_2r_3$ acts on G as the conjugation of the corresponding element of $O_{f_\mathfrak{p}}$, we may identify $G \rtimes \langle r_1r_2r_3\rangle$ with $G \rtimes \langle r_1\rangle$ by $g \cdot r_1r_2r_3 \mapsto gr_2r_3 \cdot r_1$. In the sequel we shall adopt the identification of $G \rtimes \langle r_1r_2r_3\rangle$ with $G \rtimes \langle r_1\rangle$

Lemma 7. *(i) Suppose that $q \equiv \pm 1 \mod 8$. Then $r_1r_2r_3$ and $\theta_A^{-1}\theta_F\theta_E r_1r_2r_3$ are conjugate to $i_1(-I)r_1$ in $\Omega_{f_\mathfrak{p}} \rtimes \langle r_1\rangle$.*
(ii) Suppose that $q \equiv \pm 3 \mod 8$. Then $r_1r_2r_3$ is conjugate to $i_1(-I)r_1$ and $\theta_A^{-1}\theta_F\theta_E r_1r_2r_3$ is conjugate to $i_1(-I)r_1t_1$ in $O_{f_\mathfrak{p}}$.

Proof. Observe that r_2r_3 always has trivial spinorial norm and are commutes with r_1. Hence the argument in Lemma 5(ii) shows that $r_1r_2r_3$ is conjugate to $i_1(\pm I)r_1$. The result for $r_1r_2r_3$ follows from Lemma 1(iii). Similarly, observing that the relation

$$\theta_A^{-1}\theta_F\theta_E r_1r_2r_3\theta_A^{-1}\theta_F\theta_E r_1r_2r_3 = \varepsilon$$

holds by Lemma 2.2.3, we obtain the result for $\theta_A^{-1}\theta_F\theta_E r_1r_2r_3$ by Lemma 1(iii) and the arguments in Lemma 5 and Lemma 6.

6.4 *Characters of $\bar{\rho}$*

Lemma 1. *(i) Let ρ be a r_1-invariant irreducible representation of $\Omega_{f_\mathfrak{p}}$ which is pulled back to $(\rho_1, \rho_1) \in \mathfrak{I}$ by p^*. Two representations $\bar{\rho}_+$ and $\bar{\rho}_-$ of $\Omega_{f_\mathfrak{p}} \rtimes \langle r_1\rangle$ which restrict to ρ are given by*

$$\chi_{\bar{\rho}_\pm}(i_1(s_1)i_2(s_2)r_1) = \pm\chi_{\rho_1}(s_1s_2).$$

(ii) Let ρ be a r_1t_1-invariant irreducible representation of $\Omega_{f_\mathfrak{p}}$ which is pulled back to $(\rho_1, {}^{t_1}\rho_1) \in \mathfrak{I}$ by p^*. Two representations $\check{\rho}_+$ and $\check{\rho}_-$ of $\Omega_{f_\mathfrak{p}} \rtimes \langle r_1t_1\rangle$ which restrict to ρ are given by

$$\chi_{\check{\rho}_\pm}(i_1(s_1)i_2(s_2)r_1t_1) = \pm\chi_{\rho_1}(s_1{}^{t_1}s_2).$$

Proof. (i) Let V be the virtual representation space representing ρ_1. Define the action of r_1 on $V \otimes V$ by

$$r_1 : V \otimes V \ni v_1 \otimes v_2 \mapsto v_2 \otimes v_1 \in V \otimes V.$$

It is immediately verified that this defines the action of $(SL_2(\mathbb{F}_q) \times SL_2(\mathbb{F}_q)) \rtimes \langle r_1 \rangle$ where r_1 acts on $SL_2(\mathbb{F}_q) \times SL_2(\mathbb{F}_q)$ by $(g, h) \mapsto (h, g)$. Hence projecting the action via p, we have irreducible representation $\bar{\rho}_+$ of $\Omega_{f_\mathfrak{p}} \rtimes \langle r_1 \rangle$. To compute the character of $\bar{\rho}_+$ on $i_1(s_1)i_2(s_2)r_1$, we may conjugate the element to $i_1(s_1s_2)r_1$ by Lemma 6.3.4. Taking basis $\{v_\alpha\}_\alpha$ of V which consists of eigenvectors v_α of s_1s_2, we can compute the action of $i_1(s_1s_2)r_1$ as follows,

$$i_1(s_1s_2)r_1(v_\alpha \otimes v_\beta) = i_1(s_1s_2)(v_\beta \otimes v_\alpha) = \lambda_\beta v_\beta \otimes v_\alpha$$

where λ_β is the eigenvalue associated to v_β . Hence

$$\chi_{\bar{\rho}_+}(i_1(s_1)i_2(s_2)r_1) = \chi_{\bar{\rho}_+}(i_1(s_1s_2)r_1) = \sum_\alpha \lambda_\alpha = \rho_1(s_1s_2).$$

The other character is obtained from Lemma 4.3.1.
(ii) For r_1t_1-invariant representation we define the action of $(g, h) \in SL_2 \times SL_2$ on $V \otimes V$ by $v_1 \otimes v_2 \mapsto gv_1 \otimes {}^{t_1}hv_2$ and let r_1t_1 act on it by the permutation of the components. Then the argument of (i) applies. Q.E.D.

Lemma 2. *Suppose that $\hat{\rho}$ is a r_1-invariant irreducible representation of $SO_{f_\mathfrak{p}}$ and $\hat{\rho} \in I_1$ in the classification of Lemma 6.2.5. Let ρ be the restriction of $\hat{\rho}$ to $\Omega_{f_\mathfrak{p}}$, which is r_1-invariant.*
(i) If $\hat{\rho}$ corresponds to $(\hat{\rho}_1, \hat{\rho}_1) \in \hat{\mathfrak{I}}$, two irreducible representations $\bar{\hat{\rho}}_+$ and $\bar{\hat{\rho}}_-$ of $SO_{f_\mathfrak{p}} \rtimes \langle r_1 \rangle = O_{f_\mathfrak{p}}$ restricting to $\hat{\rho}$ satisfy

$$\mathrm{Res}^{O_{f_\mathfrak{p}}}_{\Omega_{f_\mathfrak{p}} \rtimes \langle r_1 \rangle} \bar{\hat{\rho}} = \bar{\rho}_\pm, \ \mathrm{Res}^{O_{f_\mathfrak{p}}}_{\Omega_{f_\mathfrak{p}} \rtimes \langle r_1 t \rangle} \bar{\hat{\rho}} = \check{\rho}_\pm.$$

where $\bar{\rho}_\pm$ and $\check{\rho}_\pm$ are the representations associated to ρ described in Lemma 1 (ii) If $\hat{\rho}$ corresponds to $(\hat{\rho}_1, \sharp_{t_1}\hat{\rho}_1) \in \hat{\mathfrak{I}}$, two irreducible representations $\bar{\hat{\rho}}_+$ and $\bar{\hat{\rho}}_-$ of $SO_{f_\mathfrak{p}} \rtimes \langle r_1 \rangle = O_{f_\mathfrak{p}}$ restricting to $\hat{\rho}$ satisfy

$$\mathrm{Res}^{O_{f_\mathfrak{p}}}_{\Omega_{f_\mathfrak{p}} \rtimes \langle r_1 \rangle} \bar{\hat{\rho}} = \bar{\rho}_\pm, \ \mathrm{Res}^{O_{f_\mathfrak{p}}}_{\Omega_{f_\mathfrak{p}} \rtimes \langle r_1 t \rangle} \bar{\hat{\rho}} = \check{\rho}_\mp.$$

Proof. (i) Let V be a virtual representation space of $\hat{\rho}_1$. Define the action of r_1 on $V \otimes V$ by $v_1 \otimes v_2 \mapsto v_2 \otimes v_1$. This induces the action of $(\widehat{SL_2} \times \widehat{SL_2}) \rtimes \langle r_1 \rangle$. Restricting it to the normal subgroup of index 2 and projecting it by $\hat{p}$, we have a virtual representation space of $SO_{f_\mathfrak{p}} \rtimes \langle r_1 \rangle$ which restricts to $\hat{\rho}$ and denote the associated representation by $\bar{\hat{\rho}}_+$. Restricting $\bar{\hat{\rho}}_+$ to $\Omega_{f_\mathfrak{p}} \rtimes \langle r_1 \rangle$, we obtain the same representation space as in Lemma 1, $\bar{\rho}_+$. For $\Omega_{f_\mathfrak{p}} \rtimes \langle r_1t_1 \rangle$, we compute the character of r_1t_1. Taking basis $\{v_\alpha\}_\alpha$ of V which consists of the eigenvectors of t_1, we have

$$\chi_{\bar{\hat{\rho}}_+}(t_1) = \chi_{\hat{\rho}_1}(I) > 0.$$

Hence $\bar{\hat{\rho}}_+$ restricts to $\check{\rho}_+$. The other representation is obtained by Lemma 4.3.1.
(ii) Giving the opposite signature to the action of t, we modify $V \otimes V$ to be the virtual representation space of $\hat{\rho}_1 \times \sharp_{t_1}\hat{\rho}_1$. Then the statement is verified similarly. Q.E.D.

Lemma 3. *Suppose that ρ is an irreducible representation of $\Omega_{f_{\mathfrak{p}}}$ and is not t_1-invariant.*
(i)Suppose that ρ is r_1-invariant. Then irreducible representations of $O_{f_{\mathfrak{p}}}$ which restrict to $\mathrm{Ind}_{\Omega_{f_{\mathfrak{p}}}}^{SO_{f_{\mathfrak{p}}}} \rho \in I_2$ are given by

$$\mathrm{Res}_{SO_{f_{\mathfrak{p}}}}^{O_{f_{\mathfrak{p}}}} \mathrm{Ind}_{\Omega_{f_{\mathfrak{p}}} \rtimes \langle r_1 \rangle}^{O_{f_{\mathfrak{p}}}} \bar{\rho}_+ \text{ and } \mathrm{Res}_{SO_{f_{\mathfrak{p}}}}^{O_{f_{\mathfrak{p}}}} \mathrm{Ind}_{\Omega_{f_{\mathfrak{p}}} \rtimes \langle r_1 \rangle}^{O_{f_{\mathfrak{p}}}} \bar{\rho}_-$$

where $\bar{\rho}_\pm$ is the representations described in Lemma 1(i).
(ii)Suppose that ρ is $r_1 t_1$-invariant. Then irreducible representation of $O_{f_{\mathfrak{p}}}$ which restricts to $\mathrm{Ind}_{\Omega_{f_{\mathfrak{p}}}}^{SO_{f_{\mathfrak{p}}}} \rho \in I_3$ are given by

$$\mathrm{Res}_{SO_{f_{\mathfrak{p}}}}^{O_{f_{\mathfrak{p}}}} \mathrm{Ind}_{\Omega_{f_{\mathfrak{p}}} \rtimes \langle r_1 \rangle}^{O_{f_{\mathfrak{p}}}} \check{\rho}_+ \text{ and } \mathrm{Res}_{SO_{f_{\mathfrak{p}}}}^{O_{f_{\mathfrak{p}}}} \mathrm{Ind}_{\Omega_{f_{\mathfrak{p}}} \rtimes \langle r_1 \rangle}^{O_{f_{\mathfrak{p}}}} \check{\rho}_-$$

where $\check{\rho}_\pm$ is the representations described in Lemma 1(ii).

Proof. By the associativity of induced modules,

$$\langle \mathrm{Res}_{SO_{f_{\mathfrak{p}}}}^{O_{f_{\mathfrak{p}}}} \mathrm{Ind}_{\Omega_{f_{\mathfrak{p}}} \rtimes \langle r_1 \rangle}^{O_{f_{\mathfrak{p}}}} \bar{\rho}_+, \mathrm{Ind}_{\Omega_{f_{\mathfrak{p}}}}^{SO_{f_{\mathfrak{p}}}} \rho \rangle_{SO_{f_{\mathfrak{p}}}} \neq 0.$$

Hence $\mathrm{Res}_{SO_{f_{\mathfrak{p}}}}^{O_{f_{\mathfrak{p}}}} \mathrm{Ind}_{\Omega_{f_{\mathfrak{p}}} \rtimes \langle r_1 \rangle}^{O_{f_{\mathfrak{p}}}} \bar{\rho}_+$ contains $\mathrm{Ind}_{\Omega_{f_{\mathfrak{p}}}}^{SO_{f_{\mathfrak{p}}}} \rho$ as its irreducible component. Since these representations have the same virtual dimension $2\chi_\rho(\varepsilon)$, they coincide. This proves (i). (ii) is similarly proved. Q.E.D.

6.5 *Computation* For irreducible representation ρ of G set

$$\begin{aligned}
\mathfrak{N}_\rho :&= \Big| - \chi_{\bar{\rho}}(\theta_E^{-1} r_1) - \chi_{\bar{\rho}}(r_1) \\
&\quad + \frac{1}{4} \sum_{i=0}^{3} \left\{ \chi_{\bar{\rho}}(\theta_A^i \theta_B^{-1} r_1) + \chi_{\bar{\rho}}(\theta_D^i \theta_E^{-1} r_1) + \chi_{\bar{\rho}}(\theta_C^i r_1) + \chi_{\bar{\rho}}(\theta_F^i r_1) \right\} \Big| \\
\mathfrak{M}_\rho :&= \left| \chi_{\bar{\rho}}(r_2 r_3 r_1) + \chi_{\bar{\rho}}(\theta_A^{-1} \theta_F \theta_E r_2 r_3 r_1) \right|
\end{aligned}$$

where $\bar{\rho}$ is an irreducible representation of $G \rtimes \langle r_1 \rangle$ which restricts to ρ.

Proposition 1. *Suppose that $q \equiv \pm 1 \mod 8$ and ρ is a r_1-invariant irreducible representation of $G = \Omega_{f_{\mathfrak{p}}}$ with $p^* \rho = (\rho_1, \rho_1) \in \mathfrak{I}$. Then*

$$\begin{aligned}
\mathfrak{N}_\rho &= \left| -\chi_{\rho_1}(I) + \chi_{\rho_1}(-I) + 2\chi_{\rho_1}(\mathfrak{T}_4) \right|, \\
\mathfrak{M}_\rho &= 2\chi_{\rho_1}(I).
\end{aligned}$$

Proof. Applying Lemma 6.3.5 and Lemma 6.4.1(i), we obtain the first formula by the direct computation. Similarly the second formula follows from Lemma 6.3.7(i) and Lemma 6.4.1(i). Q.E.D.

TABLE 1. The character table of $SL_2(\mathbb{F}_q)$

	1_{SL_2}	St	H_n	D_m	$G_1^{\pm}$	$G_2^{\pm}$
I	1	q	$q+1$	$q-1$	$\frac{q+1}{2}$	$\frac{q-1}{2}$
$-I$	1	q	$(-1)^n(q+1)$	$(-1)^m(q-1)$	$(-1)^{\nu}\frac{q+1}{2}$	$(-1)^{\nu'}\frac{q-1}{2}$
$\mathfrak{U}_1$	1	0	1	-1	$\kappa^{\pm}$	$\lambda^{\pm}$
$\mathfrak{U}_d$	1	0	1	-1	$\kappa^{\mp}$	$\lambda^{\mp}$
$-\mathfrak{U}_1$	1	0	$(-1)^n$	$-(-1)^m$	$(-1)^{\nu}\kappa^{\pm}$	$(-1)^{\nu'}\lambda^{\pm}$
$-\mathfrak{U}_d$	1	0	$(-1)^n$	$-(-1)^m$	$(-1)^{\nu}\kappa^{\mp}$	$(-1)^{\nu'}\lambda^{\mp}$
$\mathfrak{T}_{\alpha^i}$	1	1	$e_{q-1}^{ni}+e_{q-1}^{-ni}$	0	$(-1)^i$	0
$\mathfrak{T}_{\beta^j}$	1	-1	0	$-(e_{q+1}^{mj}+e_{q+1}^{-mj})$	0	$-(-1)^j$

$i \neq 0 \in \mathbb{Z}/(q-1)\mathbb{Z}$, $j \neq 0 \in \mathbb{Z}/(q+1)\mathbb{Z}$, $e_k := \exp(\frac{2\pi\sqrt{-1}}{k})$, $\nu := \frac{q-1}{2}$, $\nu' := \frac{q+1}{2}$, $\kappa^{\pm} := \frac{1\pm\sqrt{(-1)^{\nu}q}}{2}$, $\lambda^{\pm} := \frac{-1\pm\sqrt{(-1)^{\nu}q}}{2}$.

TABLE 2. The value of $\mathfrak{N}_\rho$

ρ_1	$q\equiv 1 \mod 4$	$q \equiv -1 \mod 4$
St	2	2
H_n (n even)	4	0
(n odd)	2(q+1)	2(q+1)
D_m (m even)	0	4
(m odd)	2(q-1)	2(q-1)
$G_1^{\pm}$	2	q+1
$G_2^{\pm}$	q-1	2

Computation I In the character table in Table 1 α denotes a generator of $\mathbb{F}_q^*$ and β denotes a element of $\mathbb{F}_{q^2}^*$ which generates a cyclic group of order $q+1$ and verifies $\beta+\beta^{-1} \in \mathbb{F}_q$. If $q \equiv 1 \mod 4$, the conjugacy class of order four is rationally diagonalized, hence coincides with the conjugacy class represented by $T_{\alpha^{\frac{q-1}{4}}}$ in the notion of Table 1. If $q \equiv -1 \mod 4$, it is a nonsplit semisimple class, hence coincides with $T_{\beta^{\frac{q+1}{4}}}$. According to this dichotomy we can compute $\mathfrak{N}_\rho$ basing on Proposition 1, the result of which is summarized in Table 2.

Proposition 2. *Suppose that $q \equiv \pm 3 \mod 8$ and $\hat{\rho}$ is a r_1-invariant irreducible representation of $G = SO_{f_{\mathfrak{p}}}$.*
(i) If $\hat{\rho} \in I_1$ and $\hat{p}^*\hat{\rho}$ extends to $(\hat{\rho}_1, \hat{\rho}_1) \in \hat{\mathfrak{I}}$, then

$$\mathfrak{N}_{\hat{\rho}} = -\chi_{\rho_1}(I) + \chi_{\rho_1}(-I) + 2\chi_{\rho_1}(\mathfrak{T}_4).$$
$$\mathfrak{M}_{\hat{\rho}} = 2\chi_{\hat{\rho}}(I)$$

(ii) If $\hat{\rho} \in I_1$ *and* $\hat{p}^*\hat{\rho}$ *extends to* $(\hat{\rho}_1, \sharp_{t_1}\hat{\rho}_1) \in \hat{\mathfrak{I}}$*, then* $\mathfrak{N}_{\hat{\rho}} = \mathfrak{M}_{\hat{\rho}} = 0$.
(iii) If $\hat{\rho} = \mathrm{Ind}_{\Omega_{f_\mathfrak{p}}}^{SO_{f_\mathfrak{p}}} \rho \in I_2$ *for* $\rho \in \mathrm{Irr}(\Omega_{f_\mathfrak{p}})$ *with* $p^*\rho = (\rho_1, \rho_1) \in \mathfrak{I}$*, then*

$$\begin{aligned} \mathfrak{N}_{\hat{\rho}} &= -\chi_{\rho_1}(I) + \chi_{\rho_1}(-I) + 2\chi_{\rho_1}(\mathfrak{T}_4), \\ \mathfrak{M}_{\hat{\rho}} &= 2\chi_{\rho_1}(I). \end{aligned}$$

(iv) If $\hat{\rho} = \mathrm{Ind}_{\Omega_{f_\mathfrak{p}}}^{SO_{f_\mathfrak{p}}} \rho \in I_3$ *for* $\rho \in \mathrm{Irr}(\Omega_{f_\mathfrak{p}})$ *with* $p^*\rho = (\rho_1, {}^{t_1}\rho_1) \in \mathfrak{I}$*, then*

$$\begin{aligned} \mathfrak{N}_{\hat{\rho}} &= -\chi_{\rho_1}(I) + \chi_{\rho_1}(-I) + 2\chi_{\rho_1}(\mathfrak{T}_4), \\ \mathfrak{M}_{\hat{\rho}} &= 2\chi_{\rho_1}(I). \end{aligned}$$

Proof. (i) and (ii) are consequences of Lemma 6.3.6, Lemma 6.3.7 and Lemma 6.4.2. (iii) and (iv) follow from Lemma 6.3.6, Lemma6.3.7 and Lemma 6.4.3. Q.E.D.

Computation II The action of $*d:\ s \mapsto T_d\ {}^*sT_d^{-1}$ permutes the conjugacy classes of $\pm u_1$ and $\pm u_d$ and fixes all the semisimple classes.
Hence by Lemma 6.2.2, $\langle r_1, t_1\rangle$-invariant irreducible representations of $\Omega_{f_\mathfrak{p}}$ correspond to (ρ_1, ρ_1) with $\rho_1 = St$, H_n's and D_m's according to Table 1. These representation extend to the irreducible representations in I_1, (ρ_1, ρ_1) or $(\rho_1, \sharp_{t_1}\rho_1)$. For the former case, the result in Table 2 for the corresponding ρ_1 is carried over to this case by Proposition 2(i). For the latter case, the formula automatically gives zero.
It follows from Table 1 that the action of t_1 permutes G_i^+ and G_i^-. Hence the irreducible representations in I_2 are

$$\mathrm{Ind}_{\Omega_{f_\mathfrak{p}}}^{SO_{f_\mathfrak{p}}} p_*(G_1^+ \times G_1^+) \quad \text{and} \quad \mathrm{Ind}_{\Omega_{f_\mathfrak{p}}}^{SO_{f_\mathfrak{p}}} p_*(G_2^+ \times G_2^+).$$

Those in I_3 are

$$\mathrm{Ind}_{\Omega_{f_\mathfrak{p}}}^{SO_{f_\mathfrak{p}}} p_*(G_1^+ \times G_1^-) \quad \text{and} \quad \mathrm{Ind}_{\Omega_{f_\mathfrak{p}}}^{SO_{f_\mathfrak{p}}} p_*(G_2^+ \times G_2^-).$$

Again the formulas in Proposition 2(iii) and (iv) show that the result in Table 2 for the corresponding ρ_1 is carried over.

Theorem 1. *Suppose that* $\mathfrak{p} \notin P_{B_{4,4,4}}$*. Nontrivial* r_1*-invariant irreducible representation* ρ *of* G *appears as an irreducible component of* $H_1(\Gamma_\mathfrak{p}\backslash\mathbb{H}^3, \mathbb{C})$*, unless* $q \equiv \pm 3 \mod 8$ *and* $\rho \in I_1$ *corresponds to* $(\hat{\rho}_1, \sharp_{t_1}\hat{\rho}_1) \in \hat{\mathfrak{I}}$*.*

Proof. The result follows from Theorem 4.4.2, Proposition 1 and Proposition 2. Q.E.D.

Remark. For the proof of Theorem 1 $\mathfrak{N}_\rho$ plays no role. We have computed $\mathfrak{N}_\rho$ only to see experimentally of the difference between the actions of r_1 and $r_1r_2r_3$.(see also Remark following Theorem 7.5.1 and Problem 6(iii) in 9.2.)

7. Anisotropic case

In this section we assume that $-a$ *is a non-square element of* $F_\mathfrak{p}$*.*

7.1 *Structure of the non-split 4-orthogonal group* Let $f_{q,1}$ be the anisotropic quadratic form on $W := \oplus_{i=1}^4 \mathbb{F}_q v_i$ defined by

$$f_{q,1} := v_1 v_3 + \frac{1}{2}(v_2^2 - dv_4^2)$$

where d is a non-square element in $\mathbb{F}_q^*$. Extending the base field to $\mathbb{F}_{q^2}/\mathbb{F}_q$, we take the new bases $\{u_i\}_{i=1}^4$ by

$$u_1 = v_1, \ u_2 = v_2 + \frac{1}{\sqrt{d}} v_4, \ u_3 = v_3, \ u_4 = v_2 - \frac{1}{\sqrt{d}} v_4 \tag{7.1}$$

so that the base field extension of $f_{q,1}$ is represented with respect to the new basis as

$$f_{q,1} = u_1 u_3 + u_2 u_4,$$

which is of the same form as $f_{q^2,0}$ in the notion of 6.1.

Lemma 1. *Let F_0 be the Frobenius morphism of $W \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2}$ over $V = \oplus_i \mathbb{F}_q u_i$.*
(i) Frobenius morphism F_1 of $W \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2}$ over W acts on $\mathrm{End}(W \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2})$ by

$$\alpha \mapsto {}^{F_0 r_0}\alpha$$

where r_0 is the reflection defined in 6.1.
(ii) Let S_1 and S_2 be the subgroups of $\Omega_{f_{q^2,0}}$ defined in 6.1. Then

$$\Omega_{f_{q,1}} = \left\{ g \in \Omega_{f_{q^2,0}}; g = s\ {}^{F_1}s \text{ with } s \in S_1 \right\}$$

In particular,

$$i: \ SL_2(\mathbb{F}_{q^2}) \ni s \mapsto i_1(s)^{F_1} i_1(s) \in \Omega_{f_{q,1}}$$

induces the isomorphism of $PSL_2(\mathbb{F}_{q^2})$ onto $\Omega_{f_{q,1}}$.
(iii) Let I_4 be the identity element of $\mathrm{End}(W \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2})$. Then

$$SO_{f_{q,1}} = \Omega_{f_{q,1}} \times \langle \pm I_4 \rangle.$$

(iv) The action of $r_0 \in O_{f_{q,1}} \setminus SO_{f_{q,1}}$ on $\Omega_{f_{q,1}}$ induces the action of canonical Frobenius morphism F of $\mathbb{F}_{q^2}/\mathbb{F}_q$ on $PSL_2(\mathbb{F}_{q^2})$ through isomorphism i in (ii).

Proof. It follows from the choice of u_i's in (7.1) that

$${}^{F_1}u_1 = u_1, \ {}^{F_1}u_2 = u_4, \ {}^{F_1}u_3 = u_3, \ {}^{F_1}u_4 = u_2.$$

Hence $F_1 = F_0 \cdot r_0$. This proves (i). By definition

$$\Omega_{f_{q,1}} = \Omega_{f_{q^2,0}}^{F_1} \cap \mathrm{Ker}\, Sp_{f_{q,1}}$$

where upperscript F_1 denote the fixed point set of the action of F_1. By definition S_1 and S_2 are stabilized by F_0. Suppose that $s_1 \in S_1$, $s_2 \in S_2$ and $s = s_1 s_2 \in \Omega_{f_{q^2,0}}$ is fixed by F_1. Then it follows from (i) that $s_2 = \pm^{r_0 F_0} s_1$. Hence we have

$$\Omega_{f_{q^2,0}}^{F_1} = \{ g \in \Omega_{f_{q^2,0}}; \ g = \delta s_1 {}^{F_1} s_1 \\ \text{where } s_1 \in S_1 \text{ and } \delta \text{ is a central element of } \Omega_{f_{q^2,0}}. \}.$$

Set
$$H := \left\{ g \in \Omega_{f_{q^2,0}};\ g = {s_1}^{F_1} s_1 \text{ with } s_1 \in S_1. \right\}.$$

Then H is isomorphic to $PSL_2(\mathbb{F}_{q^2})$ and $\Omega^{F_1}_{f_{q^2,0}} = H \times \langle \pm I_4 \rangle$. Since H is a simple group and $Sp_{f_{q,1}}(-I_4)$ is nontrivial, $\operatorname{Ker} Sp_{f_{q,1}} \cap \Omega^{F_1}_{f_{q^2,0}} = H$. This proves (ii) and (iii). (iv) follows immediately from our description of $\Omega_{f_{q,1}}$. Q.E.D.

It follows from Lemma 5.2.3 that we may identify $L_{\mathfrak{p}}$ with W by an isometry as in 6.2. Fix any such isometry and denote it by ν. ν induces the isomorphisms of the orthogonal groups.

Lemma 2. *The action of r_1 on $\Omega_{f_{\mathfrak{p}}}$ is conjugate by an element of $\Omega_{f_{\mathfrak{p}}}$ to the action induced by r_0. In particular,there is an element of $g \in \Omega_{f_{q,1}}$ such that $j := g \circ \nu \circ i$ induces an isomorphism of $PSL_2(\mathbb{F}_{q^2}) \rtimes \langle F \rangle$ to $\Omega_{f_{\mathfrak{p}}} \rtimes \langle r_1 \rangle$ taking F to r_1.*

Proof. Since both actions are induced by reflections, Witt's theorem shows that they are conjugate by an element of $\Omega_{f_{\mathfrak{p}}}$ if they have the same spinorial norm. In fact, we have
$$Sp_{f_{\mathfrak{p}}}(r_1) = a^{-1},\ Sp_{f_{q,1}}(r_0) = -d.$$
Since we are assuming $-a$ is a non-square, this proves the lemma. The last statements follows immediately from Lemma 1. Q.E.D.

Lemma 3. *(i) Suppose that $q \equiv \pm 1 \mod 8$. Then $\operatorname{Irr}(G)$ is identified with*
$$\mathfrak{J} := \left\{ \rho \in \operatorname{Irr}(SL_2(\mathbb{F}_{q^2}));\ \rho(-I) = id. \right\}.$$

(ii) Suppose that $q \equiv \pm 3 \mod 8$. Then $\operatorname{Irr}(G)$ is identified with
$$\hat{\mathfrak{J}} := \left\{ \hat{\rho} \in \operatorname{Irr}(SL_2(\mathbb{F}_{q^2}) \times \mathbb{Z}_2);\ \hat{\rho}(-I,0) = id. \right\}.$$

(iii) r_1-invariant irreducible representations of G are identified with F-invariant representations of $\mathfrak{J}$ or $\hat{\mathfrak{J}}$.

Proof. (i) and (ii) follow immediately from Lemma 1(ii) and (iii). (iii) follows from Lemma 1(iii) and Lemma 2. Q.E.D.

7.2 *General formula for irreducible characters of $\mathbb{Z}_2$-extensions* In this subsection K denotes a general finite group and r an automorphism of order two of K. Suppose that ρ is a r-invariant irreducible representation of K. By Lemma 4.3.1 we have two irreducible representations of $K \rtimes \langle r \rangle$ which restrict to ρ and the sum of these two representations is zero on $K \rtimes \langle r \rangle \setminus K$. Let V be a virtual representation space of ρ. For $l \in \operatorname{End}_{\mathbb{C}}(V)$ set
$$f(l) = \frac{1}{|K|} \sum_{g \in K} \rho(g) \cdot l \cdot \rho({}^r g^{-1}) \in \operatorname{End}_{\mathbb{C}}(V).$$

Lemma 1. *Suppose $\bar{\rho}$ is an irreducible representation of $K \rtimes \langle r \rangle$ which restricts to ρ. Then for any $l \in \mathrm{End}_{\mathbb{C}}(V)$, $f(l) \in \mathbb{C}\bar{\rho}(r)$.*

Proof. It follows from Schur's lemma and r-invariance of ρ that subspace J of $\mathrm{End}(V)$ defined by

$$J := \{l \in \mathrm{End}_{\mathbb{C}}(V); \rho(g) \cdot l = l \cdot \rho({}^r g) \text{ for any } g \in K\}.$$

is one dimensional. Clearly $\bar{\rho}(r)$ is a nonzero element of J. Hence $J = \mathbb{C}\bar{\rho}(r)$. Since

$$\begin{aligned} f(l) \cdot \rho(h) =& \frac{1}{|K|} \sum_{g \in K} \rho(g) \cdot l \cdot \rho({}^r g^{-1} h) \\ =& \rho({}^r h) \left(\frac{1}{|K|} \sum_{g \in K} \rho({}^r h^{-1} g) \cdot l \cdot \rho({}^r g^{-1} h) \right) = \rho({}^r h) \cdot f(l), \end{aligned}$$

we have $f(l) \in \mathbb{C}\bar{\rho}(r)$. Q.E.D.

Lemma 2. *Suppose that* $\mathrm{Trace}(f(id_V)) \neq 0$. *Then*

$$\bar{\rho}(r) = \sqrt{\frac{\chi_\rho(\varepsilon)}{\mathrm{Trace}(f(id_V))}}\, f(id_V)$$

where $\bar{\rho}$ is a suitably chosen irreducible representation of $K \rtimes \langle r \rangle$ restricting to ρ according to the choice of the square root.

Proof. By Lemma 1, we have

$$c\bar{\rho}(r) = f(id_V) = \frac{1}{|K|} \sum_{g \in K} \rho(g\, {}^r g^{-1}) \tag{7.2}$$

for some $c \in \mathbb{C}$. Since r is of order two, we have

$$\begin{aligned} c^2 id_V =& \frac{1}{|K|^2} \sum_{g,\, h \in K} \rho(g\, {}^r g^{-1}) \rho(h\, {}^r h^{-1}) \\ =& \frac{1}{|K|^2} \sum_{g,\, h \in K} \rho({}^{{}^r h}({}^r h^{-1} g\, {}^r({}^r h^{-1} g)^{-1})). \end{aligned} \tag{7.3}$$

Taking the trace of equation (7.3), we have

$$c^2 \chi_\rho(\varepsilon) = \mathrm{Trace}(f(id_V)). \tag{7.4}$$

It follows from equation (7.4) that

$$c = \pm \sqrt{\frac{\mathrm{Trace}(f(id_V))}{\chi_\rho(\varepsilon)}}. \tag{7.5}$$

The signature depends on the choice of two irreducible representations restricting to ρ. Then the result follows from (7.2) and (7.5). Q.E.D.

Lemma 3. *Suppose that* Trace $f(id_V) \neq 0$ *and* $\bar{\rho}$ *is as in Lemma 2. Then for any* $\theta \in K$ *there holds*

$$\chi_{\bar{\rho}}(\theta r) = \sqrt{\frac{\chi_\rho(\varepsilon)}{\mathrm{Trace}(f(id_V))}}\,\mathrm{Trace}(f(\rho(\theta))).$$

Proof. By definition

$$f(\rho(\theta)) = \frac{1}{|K|}\sum_{g \in K} {}^{\rho(g)}\left(\rho(\theta\, {}^r g^{-1} g)\right).$$

Hence by Lemma 2, we have

$$\begin{aligned}\mathrm{Trace}(f(\rho(\theta))) =& \frac{1}{|K|}\sum_{g\in K} \chi_\rho(\theta\, {}^r g^{-1} g)\\ &= \mathrm{Trace}(\rho(\theta)\cdot f(id_V)) = \sqrt{\frac{\mathrm{Trace}(f(id_V))}{\chi_\rho(\varepsilon)}}\chi_{\bar{\rho}}(\theta r).\end{aligned}$$

This proves the lemma. Q.E.D.

We define the action L of K on itself by

$$L(g):\ K \ni \theta \mapsto g\theta\, {}^r g^{-1} \in K$$

and call it *r-conjugation*. We denote the r-conjugate orbit of $\theta \in K$ by L_θ and call it *r-conjugacy class* of θ. $\mathfrak{L}_r$ denotes the set of r-conjugacy classes in K and $D_K(\theta)$ the isotropy group of $\theta \in K$ by the r-conjugation.

Lemma 4. *Let* $\mathcal{C}_K$ *be the set of conjugacy classes of* K. *Then*

$$\mathrm{Trace}(f(\rho(\theta))) := \frac{1}{|D_K(\theta)|}\sum_{c \in \mathcal{C}_K} |c \cap L_\theta|\ \rho(c).$$

Proof. Immediate from definitions of $f(\rho(\theta))$, L_θ and $D_K(\theta)$. Q.E.D.

Lemma 5. *Suppose* $\theta \in K$. *Then subset* $L_\theta r = \{lr;\ l \in L_\theta\}$ *of* $K \rtimes \langle r\rangle$ *is the* K*-orbit of* θr *by the conjugation. The map* $\mathfrak{L}_r \ni L_\theta \mapsto L_\theta r$ *identifies* $\mathfrak{L}_r$ *with the set of all the* K*-orbits in* $K \rtimes \langle r\rangle \setminus K$. *Moreover, it induces the one-to-one corresponding of* $\mathfrak{L}_r/\langle r\rangle$ *with the set of conjugacy classes of* $K \rtimes \langle r\rangle$ *in* $K \rtimes \langle r\rangle \setminus K$.

Proof. Immediate from the definitions. Q.E.D.

Lemma 6. *Suppose that* $\bar{\rho}$ *is as in Lemma 2.*

$$\sum_{L_\theta(r) \in \mathfrak{L}_r} \frac{|\chi_{\bar{\rho}}(\theta r)|^2}{|D_K(\theta)|} = 1$$

Proof. By orthogonal relations of irreducible characters, we have

$$
\begin{aligned}
1 =& \langle \bar{\rho}, \bar{\rho} \rangle_{K \rtimes \langle r \rangle} = \frac{1}{2|K|} \sum_{x \in K \rtimes \langle r \rangle} |\chi_{\bar{\rho}}(x)|^2 \\
=& \frac{1}{2} \frac{1}{|K|} \sum_{g \in K} |\chi_{\rho}(g)|^2 + \frac{1}{2|K|} \sum_{g \in K} |\chi_{\bar{\rho}}(gr)|^2 \\
=& \frac{1}{2} \langle \rho, \rho \rangle_K + \frac{1}{2|K|} \sum_{g \in K} |\chi_{\bar{\rho}}(gr)|^2.
\end{aligned}
$$

Hence by Lemma 5,

$$
1 = \frac{1}{|K|} \sum_{g \in K} |\chi_{\bar{\rho}}(gr)|^2 = \frac{1}{|K|} \sum_{L_\theta \in \mathfrak{L}_r} |\chi_{\bar{\rho}}(\theta r)|^2 \frac{|K|}{|D_K(\theta)|} = \sum_{L_\theta \in \mathfrak{L}_r} \frac{|\chi_{\bar{\rho}}(\theta r)|^2}{|D_K(\theta)|}.
$$

This proves the lemma. Q.E.D.

7.3 *F-conjugacy class in SL_2* We shall specialize the argument in 7.2 to $K = SL_2(\mathbb{F}_{q^2})$ and $r = F$, the canonical Frobenius morphism. We denote by S the geometric point $SL_2(\overline{\mathbb{F}}_q)$ of algebraic group SL_2 over $\mathbb{F}_q$. In particular $K = S^{F^2}$. Set

$$
\begin{aligned}
T :=& \left\{ t \in S;\ t = \begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix},\ \alpha \in \overline{\mathbb{F}}_q^*. \right\}, \\
V :=& \left\{ v \in S;\ v = \pm \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix},\ \beta \in \overline{\mathbb{F}}_q. \right\}, \\
B :=& \left\{ b \in S;\ b = \pm \begin{pmatrix} \alpha & \beta \\ & \alpha^{-1} \end{pmatrix},\ \alpha \in \overline{\mathbb{F}}_q^*,\ \beta \in \overline{\mathbb{F}}_q. \right\}, \\
J :=& \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
N :=& \langle T, J \rangle.
\end{aligned}
$$

Lemma 1. *(i) For each non-central element s of S there exists the unique maximal abelian subgroup of S containing s and it coincides with $C_S(s)$.*
(ii) If s is semisimple and not central, $C_S(s)$ is conjugate to T in S.
(iii) If s is not semisimple, $C_S(s)$ is conjugate to V in S.
(iv) $N_S(T) = N$, $N_S(V) = B$.

Proof. All the statement can be verified directly. Q.E.D.

Define the automorphism of variety S by $n:\ S \ni \theta \mapsto \theta\ {}^F\theta \in S$. It is readily verified that n induces a map of set $\mathfrak{L}_F$ of F-conjugacy classes to set $\mathcal{C}_K$ of conjugacy classes.

Lemma 2. *Let $z \in \{\pm I\}$ be a central element of S. Then $n^{-1}(z)$ coincides with a F-conjugacy class of K. In particular, $n^{-1}(I) = L_I$ and $n^{-1}(-I) = L_{\theta_4}$ for any F-rational element θ_4 of order four.*

Proof. It is readily verified that $n^{-1}(z)$ is stable under the F-conjugation by elements of $K = S^{F^2}$. Suppose that $x, y \in n^{-1}(z)$. By Lang-Steinberg theorem, we can find $\zeta \in S$ such that $y = \zeta x\ {}^F\zeta^{-1}$. Hence

$$
z = y\ {}^F y = \zeta x\ {}^F x\ {}^{F^2}\zeta^{-1} = \zeta z\ {}^{F^2}\zeta^{-1}.
$$

Since z is central, this shows that $\zeta \in K$. This proves the lemma. Q.E.D.

Lemma 3. *Let z be as in Lemma 2. For any element x of $n^{-1}(z)$, $C_K(x)$ is F-stable.*

Proof. Since $x\,{}^F x = z$, $C_K(x) = C_K({}^F x) = {}^F C_K(x)$. Q.E.D.

Lemma 4. *Suppose ${}^gT$ is F-stable for some $g \in S$. Then one of the following holds.*
(i) $g^{-1}\,{}^F g \in T$ and ${}^gT$ is isomorphic over $\mathbb{F}_q$ to $\overline{\mathbb{F}}_q^*$ with canonical Frobenius morphism F. Any such tori are conjugate each other by an element of S^F.
(ii) $g^{-1}\,{}^F g \in JT$ and ${}^gT$ is isomorphic over $\mathbb{F}_q$ to $\overline{\mathbb{F}}_q^*$ with Frobenius morphism F' defined by $\alpha \mapsto {}^{F_0}\alpha^{-1}$. Any such tori are conjugate each other by an element of S^F.

Proof. Since ${}^gT$ is F-stable, ${}^gT = {}^F({}^gT) = {}^{{}^F g}T$. Hence $g^{-1}\,{}^F g \in N$ by Lemma 1(iv). If $g^{-1}{}^F g \in C_K(T) = T$, we can find $t \in T$ such that $t^{-1}\,{}^F t = g^{-1}\,{}^F g$ applying the Lang-Steinberg theorem to torus T. Hence we may assume $g \in S^F$ and the conjugation by g induces the morphism over $\mathbb{F}_q$ of $(T, F) \simeq (\overline{\mathbb{F}}_q^*)$ to $({}^gT, F)$. This proves all the statement of (i). (ii) is similarly proved, observing J acts on T by $t \mapsto t^{-1}$. Q.E.D.

Lemma 5. *Any F-stable conjugate of V is conjugate each other by an element of S^F and are isomorphic to $(\overline{\mathbb{F}}_q, F) \times \mathbb{Z}_2$ over $\mathbb{F}_q$.*

Proof. Suppose that ${}^gV$ is F-stable. Then $g^{-1}\,{}^F g \in N_S(V) = B$. Applying the Lang-Steinberg theorem to connected algebraic group B, we can find element b of B with $g^{-1}\,{}^F g = b^{-1}\,{}^F b$. Hence $gb^{-1} \in S^F$ and ${}^{gb^{-1}}V = {}^gV$. This proves the first statement. The second statement follows immediately. Q.E.D.

Lemma 6. *Let $\mathfrak{T}_\alpha$ and $\pm\mathfrak{U}_*$ be the conjugacy classes of $K = SL_2(\mathbb{F}_{q^2})$ described in 6.3 (but replacing the base field with $\mathbb{F}_{q^2}$). Suppose that $\alpha^2 \neq 1$. Then*

$$|L_I \cap \mathfrak{T}_\alpha| = \begin{cases} q(q+1) & \text{if } \alpha\,{}^{F_0}\alpha = 1, \\ q(q-1) & \text{if } \alpha\,{}^{F_0}\alpha^{-1} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$|L_I \cap \pm\mathfrak{U}_\beta| = \begin{cases} q^2 - 1 & \text{if 4 divides } q+1 \text{ and } \beta = 1 \\ & \text{or if 4 divides } q-1 \text{ and } \beta = d, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose x is an element of $L_I \cap \mathfrak{T}_\alpha$. Then Lemma 1(ii) and Lemma 3 show that $T_1 := C_S(x)$ is an F-stable torus. Then by Lemma 2 and Lemma 4, either T_1 is isomorphic to $(\overline{\mathbb{F}}_q,\ F)$ and $\alpha\,{}^F\alpha = 1$ or T_1 is isomorphic to $(\overline{\mathbb{F}}_q,\ F')$ and $\alpha\,{}^F\alpha^{-1} = 1$. First consider the former case. Since each torus contains two elements of the conjugacy class and each noncentral semisimple element is contained in the unique torus by Lemma 1 (ii), Lemma 4(i) imply

$$|L_I \cap \mathfrak{T}_\alpha| = 2\frac{|S^F|}{|N_{S^F}(T_1)|} = 2\frac{q(q^2-1)}{2(q-1)} = q(q+1).$$

For the latter case we can similarly verify that

$$|L_I \cap \mathfrak{T}_\alpha| = 2\frac{|S^F|}{|N_{S^F}(T_1)|} = 2\frac{q(q^2-1)}{2(q+1)} = q(q-1).$$

This proves the first formula. For the second formula we can similarly reduce the problem to counting the number of elements in question in (V, F). The equation $v\ {}^F v = I$ in V is equivalent to $\beta + {}^F\beta = 0$ with

$$v = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \text{ or } -\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}.$$

The non-zero roots of the equation are $\{\zeta^k\}$(k:odd) with primitive $2(q-1)$-th root ζ of unity. All these roots are simultaneously squares in $\mathbb{F}_{q^2}$ or not according to whether $(q+1)/2$ is even or odd. Hence by Lemma 5 we have

$$|L_I \cap \mathfrak{U}_\beta| = (q-1)\frac{|S^F|}{|N_{S^F}(V)|} = (q-1)\frac{q(q^2-1)}{q(q-1)} = q^2 - 1$$

provided $(q+1)/2$ is even and $\beta = 1$ or $(q+1)/2$ is odd and $\beta = d$. This proves the second formula. Q.E.D.

Lemma 7. *Suppose that $\theta_4 \in K$ is an F-rational element of order four. Then*

$$|L_{\theta_4} \cap \mathfrak{T}_\alpha| = \begin{cases} q(q+1) & \text{if } \alpha\ {}^{F_0}\alpha = -1, \\ q(q-1) & \text{if } \alpha\ {}^{F_0}\alpha^{-1} = -1, \\ 0 & \text{otherwise.} \end{cases}$$

$$|L_{\theta_4} \cap \pm\mathfrak{U}_\beta| = 0.$$

Proof. The first formula follows from Lemma 2 and the same argument as Lemma 6. For the second formula we just have to note that equation $v\ {}^F v = -I$ can not hold true for $v \in V$ since F preserves the unipotent elements of SL_2. Q.E.D.

Lemma 8. *(i) Suppose that $x \in K$ and $y \in S$ with $y\ {}^F y^{-1} = x$. Then $C_K(n(x))$ is yFy^{-1}-stable and $D_K(x) = {}^y(S^F) = C_K(n(x))^{yFy^{-1}}$. In particular*

$$|D_K(x)| = q(q^2-1)$$

for $x \in L_I \cup L_{\theta_4}$.
(ii) If $x \notin L_I \cup L_{\theta_4}$ is semisimple, $|D_K(x)| \le q+1$.

Proof. Suppose that $g \in D_K(x)$. Then by definition, $gx\ {}^F g^{-1} = x$. Applying n on the both sides, we have $gn(x)g^{-1} = n(x)$. Hence $g \in C_K(n(x))$. It follows from the assumption that $gy\ {}^F(gy)^{-1} = y\ {}^F y^{-1}$ for any $g \in D_K(x)$. Hence $g \in {}^y(S^F)$. This proves $D_K(x) \subset C_K(n(x)) \cap {}^y(S^F)$. Since

$${}^{yFy^{-1}}n(x) = {}^y({}^F y^{-1} n({}^F x)\ {}^F y) = xn({}^F x)x^{-1} = n(x),$$

$C_K(n(x))$ is yFy^{-1}-stable. ${}^y(S^F) \subset D_K(x)$ follows from that

$$hx\ {}^F h^{-1} = hy\ {}^F(hy)^{-1} = x$$

for $h \in {}^y S^F$. This proves (i). (ii) follows from that any $\mathbb{F}_q$-rational structure on T has at most $q+1$ rational points. Q.E.D.

7.4 *Conjugacy classes*

Lemma 1. *Let θ_4 be as in Lemma 7.3.2. Suppose that $g \in \Omega_{f_\mathfrak{p}}$ verifies $g\,{}^{r_1}g = I_4$. Then $gr_1 \in \Omega_{f_\mathfrak{p}} \rtimes \langle r_1 \rangle$ is conjugate to r_1 in $\Omega_{f_\mathfrak{p}} \rtimes \langle r_1 \rangle$ if $Tr_{\mathbb{F}_q^2}(gr_1) = 2$ and to $i(\theta_4)r_1$ if $Tr_{\mathbb{F}_q^2}(gr_1) = -2$.*

Proof. Let $g = j(s)$ for $s \in SL_2(\mathbb{F}_{q^2})$ and epimorphism j defined in Lemma1.2. Then it follows from the assumption that $s\,{}^F s = \pm I$. Lemma 7.2.5 and Lemma 7.3.2 show that gr_1 is conjugate to r_1 if $s\,{}^F s = I$ and to $j(\theta_4)r_1$ if $s\,{}^F s = -I$. It is directly verified that the dichotomy is distinguished by $Tr_{\mathbb{F}_{q^2}}$. This proves the Lemma. Q.E.D.

Lemma 2. *Any element of order four in $\Omega_{f_\mathfrak{p}}^{r_1}$ is conjugate to each other in $\Omega_{f_\mathfrak{p}}^{r_1}$ where upperscript r_1 denotes the fixed point set by the action.*

Proof. Suppose that $g = j(s)$ is an element of order four in $\Omega_{f_\mathfrak{p}}^{r_1}$. Then ${}^F s = \pm s$ and $s \in SL_2(\mathbb{F}_{q^2})$ is of order eight. Hence $T_1 := C_K(s)$ is an F-stable torus. The elementary observation of congruence modulo 8 shows that T_1 is of type (i) in terms of Lemma 7.3.4 if $q \equiv 1$ or $-3 \mod 8$ and of type (ii) otherwise. Since any elements of order eight in a F-stable torus are conjugate by an element of $SL_2(\mathbb{F}_{q^2})^F$ modulo the center, the statement follows from Lemma 7.3.4. Q.E.D.

Lemma 3. *Suppose that $q \equiv \pm 1 \mod 8$ and θ_8 is an element of order eight in $SL_2(\mathbb{F}_{q^2})$ with ${}^F\theta_8 = \pm\theta_8$.*
(i) $\theta_E^{-1} r_1$ is conjugate to r_1 in $\Omega_{f_\mathfrak{p}} \rtimes \langle r_1 \rangle$.
(ii) $\theta_C^2 r_1$, $\theta_F^2 r_1$, $\theta_A^2\theta_B^{-1} r_1$ and $\theta_D^2\theta_E^{-1} r_1$ are conjugate to $j(\theta_4)r_1$ in $\Omega_{f_\mathfrak{p}} \rtimes \langle r_1 \rangle$.
(iii) $r_2r_3r_1$ and $\theta_A^{-1}\theta_F\theta_E r_2r_3r_1$ are conjugate to $j(\theta_4)r_1$ in $\Omega_{f_\mathfrak{p}} \rtimes \langle r_1 \rangle$.
(iv) $\theta_C^{\pm 1} r_1$, $\theta_F^{\pm 1} r_1$, $\theta_A^{\pm 1}\theta_B^{-1}$ and $\theta_D^{\pm 1}\theta_E^{-1} r_1$ are conjugate to $j(\theta_8)r_1$ in $\Omega_{f_\mathfrak{p}} \rtimes \langle r_1 \rangle$.

Proof. For element gr_1 in the statements (i), (ii) and (iii), we can verify that $g\,{}^{r_1}g = I$ by relations in Lemma 2.2.2 and Lemma 2.2.3. By Lemma 6.3.5 we can compute the traces of these elements, extending the base field to $\mathbb{F}_{q^2}$. Then (i),(ii) and (iii) follow from Lemma 1. Since θ_C and θ_F are of order four by Lemma 6.3.3 and they are fixed by r_1, they are conjugate to $j(\theta_8)$ in $\Omega_{f_\mathfrak{p}}$ by Lemma 2, hence the statement for $\theta_C r_1$ and $\theta_F r_1$ follows. Similarly the statement for $\theta_D\theta_E^{-1} r_1$ follows from (i) and the commutativity of θ_D and $\theta_E^{-1} r_1$. The argument also applies to $\theta_D\theta_B^{-1} r_1$. Q.E.D.

Lemma 4. *Suppose that $q \equiv \pm 3 \mod 8$.*
(i) $\theta_E^{-1} r_1$ is conjugate to $-I_4 \cdot j(\theta_4)r_1$ in $O_{f_\mathfrak{p}}$.
(ii) $\theta_C^2 r_1$ and $\theta_F^2 r_1$ are conjugate to $j(\theta_4)r_1$ in $O_{f_\mathfrak{p}}$.
(iii) $\theta_A^2\theta_B^{-1} r_1$ and $\theta_D^2\theta_E^{-1} r_1$ are conjugate to $-I_4 \cdot r_1$.
(iv) $r_2r_3r_1$ is conjugate to $j(\theta_4)r_1$ in $O_{f_\mathfrak{p}}$.
(v) $\theta_A^{-1}\theta_F\theta_E r_1r_2r_3$ is conjugate to $-I_4 \cdot r_1$.
(vi) $\theta_C^{\pm 1} r_1$ and $\theta_F^{\pm 1} r_1$ are conjugate to $-I_4 \cdot j(\theta_8)r_1$.
(vii) $\theta_A^{\pm 1}\theta_B^{-1} r_1$ and $\theta_D^{\pm 1}\theta_E r_1$ are conjugate to $j(\theta_8)r_1$.

Proof. All the statement follow from Lemma 7.1.1(iii), Lemma 1, Lemma 2 and the corresponding argument in the proof of Lemma 3.

7.5 *Computation*

Lemma 1. *Let the notions be as in Table 1. F-invariant irreducible representations of $SL_2(\mathbb{F}_{q^2})$ are St, $G_1^{\pm}$ and H_n ($n \in (q \pm 1)\mathbb{Z}$).*

Proof. F permutes the semisimple conjugacy class $\mathfrak{T}_{\alpha^i} \mapsto \mathfrak{T}_{\alpha^{qi}}$ and fixes all the other conjugacy classes. Hence it is readily verified that St and $G_1^{\pm}$ are F-stable in $\mathfrak{J}$ and $G_2^{\pm}$ is not an element of $\mathfrak{J}$ since $\nu' = (q^2+1)/2$ is odd. It is directly verified that Harish-Chandra character H_n is F-invariant if and only if

$$\frac{nq}{q^2-1} \pm \frac{n}{q^2-1} \in \mathbb{Z},$$

which is equivalent to $n \in (q \pm 1)\mathbb{Z}$. We can similarly verify the F-invariance of Deligne-Lustzig character D_m to obtain that none of them is F-invariant. Q.E.D.

Lemma 2. *Suppose that $q \equiv \pm 1 \bmod 8$. Let ρ be an r_1-invariant irreducible representation of G and $\sigma \in \mathfrak{J}$ the F-invariant irreducible representation of $SL_2(\mathbb{F}_{q^2})$ corresponding to ρ in terms of Lemma 7.1.3. Then for the irreducible representation $\bar{\sigma}$ of $SL_2(\mathbb{F}_{q^2}) \rtimes \langle F \rangle$ restricting to σ, we have*

$$\begin{aligned}\mathfrak{N}_\rho &= |-\chi_{\bar{\sigma}}(F) + \chi_{\bar{\sigma}}(\theta_4 F) + 2\chi_{\bar{\sigma}}(\theta_8 F)|,\\ \mathfrak{M}_\rho &= 2|\chi_{\bar{\sigma}}(\theta_4 F)|.\end{aligned}$$

Proof. The formulas follow immediately from Lemma 7.4.3 and the definitions in 6.5 of $\mathfrak{N}_\rho$ and $\mathfrak{M}_\rho$. Q.E.D.

Lemma 3. *Suppose that $q \equiv \pm 3 \bmod 8$. Let ρ be an r_1-invariant irreducible representation of G, $\sigma \times \tau$ the F-invariant irreducible representation of $SL_2(\mathbb{F}_{q^2}) \times \mathbb{Z}_2$ corresponding to ρ in terms of Lemma 7.1.3 and $\bar{\sigma} \times \tau$ an irreducible representation of $(SL_2(\mathbb{F}_{q^2}) \rtimes \langle F \rangle) \times \mathbb{Z}_2$ which restricts to $\sigma \times \tau$.*
(i) If $\tau = 1_{\mathbb{Z}_2}$,

$$\begin{aligned}\mathfrak{N}_\rho &= 2|\chi_{\bar{\sigma}}(\theta_8 F)|\\ \mathfrak{M}_\rho &= |\chi_{\bar{\sigma}}(F) + \chi_{\bar{\sigma}}(\theta_4 F)|.\end{aligned}$$

(ii) If τ is the nontrivial irreducible representation of $\mathbb{Z}_2$,

$$\mathfrak{N}_\rho = \mathfrak{M}_\rho = |\chi_{\bar{\sigma}}(\theta_4 F) - \chi_{\bar{\sigma}}(F)|.$$

Proof. The formulas follow immediately from Lemma 7.4.4 and the definitions in 6.5 of $\mathfrak{M}_\rho$ and $\mathfrak{N}_\rho$. Q.E.D.

Computations By Lemma 7.3.6 and Lemma 7.3.8 we can compute the formula of $\mathrm{Trace}(f(I))$ in Lemma 7.2.4. In fact we have $\mathrm{Trace}(f(I)) \neq 0$ for any F-invariant representations of $SL_2(\mathbb{F}_{q^2})$ in Lemma 1. Hence according to the formula in Lemma 7.3.3 and Table 1 we can compute values $\bar{\sigma}(F)$ and $\bar{\sigma}(\theta_4 F)$, the result of which is found in Table 3.

Theorem 1. *Suppose that $-a$ is a non-square in $F_{\mathfrak{p}}$. Let ρ, σ, $\bar{\sigma}$ and τ as in Lemma 2 or Lemma 3. Then ρ appears as an irreducible component of $H_1(\Gamma\backslash\mathbb{H}^3, \mathbb{C})$ unless $q \equiv \pm 3 \bmod 8$ and $\sigma \times \tau$ is one of $H_{(q-1)l} \times 1_{\mathbb{Z}_2}$ (k: odd), $St \times -1_{\mathbb{Z}_2}$ and*

TABLE 3. The value of $\bar{\sigma}(F)$ and $\bar{\sigma}(\theta_4 F)$

σ	$\bar{\sigma}(F)$	$\bar{\sigma}(\theta_4 F)$
St	q	q
$H_{(q+1)k}$ ($k \neq 0$ even)	$q+1$	$q+1$
(k odd)	$q+1$	$-q$
$H_{(q-1)l}$ ($l \neq 0$ even)	$q-1$	$q-1$
(l odd)	$q-1$	$-(q-1)$
if $q+1 \equiv 0 \mod 4$		
G_1^+	q	-1
G_1^-	1	$-q$
if $q-1 \equiv 0 \mod 4$		
G_1^+	1	q
G_1^-	q	1

$H_{(q\pm1)k} \times -1_{\mathbb{Z}_2}$ (k: even) where $-1_{\mathbb{Z}_2}$ is the unique nontrivial irreducible representation of $\mathbb{Z}_2$.

Proof. It follows from the formulas for $\mathfrak{M}_\rho$ in Lemma 2 and Lemma 3 and the result of computations in Table 3 that $\mathfrak{M}_\rho \neq 0$ except for the irreducible representations in the last part of the statement. Hence the theorem follows from Theorem 4.4.2. Q.E.D.

Remark. (1) The reader may suspect that the consideration of general r-conjugacy in 7.2 and the argument of F-conjugacy in 7.3 are unnecessarily general to compute $\bar{\sigma}(*F)$ in this particular situation. In fact it follows from the construction of the Harish-Chandra representation that we can rigorously construct the extended representation $\bar{H}_{(q+1)k}$ and compute all the value of these characters on $SL_2(\mathbb{F}_{q^2}) \rtimes \langle F \rangle$. However for $\bar{H}_{(q-1)l}$ such a construction or its modification would be impossible since they are constructed from F^2-split, but F-nonsplit, tori and its corresponding unipotent subgroups. Hence even for this particular situation the arguments in 7.2 , 7.3 and more are required to determine completely which irreducible characters satisfy the criterion in Theorem 4.4.1 or Theorem 4.4.2. In fact I could not determine whether $\sigma = \bar{H}_{(q-1)l}$ verifies the criterion of Theorem 4.4.1 because of the lack of the technique to compute the value at $\theta_8 F$, which is the only case left to be verified in our setting.
(2) Though we cannot compute the explicit value of $\chi_{\bar{\sigma}}(\theta_8 r_1)$ we can verify $\mathfrak{M}_\rho \neq 0$ if $q \equiv \pm 1 \mod 8$ and $\sigma = H_{(q\pm1)k}(k \text{ odd})$ or $G_1^{\pm}$. In fact if $\mathfrak{M}_\rho = 0$ and these assumptions hold, $\chi_{\bar{\sigma}}(\theta_8 r_1) = l(q)$ for some nonconstant linear function l. Then Lemma 7.2.6 and Lemma 7.3.8 show that

$$\frac{l(q)^2}{q+1} + \frac{|\bar{\sigma}(F)|^2}{q(q^2-1)} + \frac{|\bar{\sigma}(\theta_4 F)|^2}{q(q^2-1)} \leq 1.$$

This cannot be true at least for sufficiently large q and actually for any q.

8. A conclusion from group theory

Let $T,\ N,\ B$ and F be as in 7.3. We denote by T' a F-stable conjugate of T which is not isomorphic to T over $\mathbb{F}_q$ (cf. Lemma 7.3.4) and by N' the normalizer of T'. π denotes the projection of $SL_2(\mathbb{F}_q)$ to $PSL_2(\mathbb{F}_q)$. A_n and Σ_n denote the alternating group and the symmetric group of degree n, respectively.

Theorem (Dickson). *Let q be a power of odd rational prime p. Then any subgroup of $SL_2(\mathbb{F}_q)$ verifies one of the followings*
(i) Conjugate to a subgroup of B^F, N^F or ${N'}^F$.
(ii) Isomorphic to $SL_2(\mathbb{F}_{q'})$ for power q' of p with ${q'}^m = q$. Such subgroups are all conjugate if m is odd. They form two conjugacy classes if m is even.
(iii) Conjugate to $\langle SL_2(\mathbb{F}_{q'}), T_{\sqrt{d}}\rangle$ for a generator d of multiplicative group $\mathbb{F}^*$ provided m is even in the notion of (ii).
(iv) Isomorphic to $\pi^{-1}(\Sigma)$ where Σ is a subgroup of $PSL_2(\mathbb{F}_q)$ which is isomorphic to $A_4,\ \Sigma_4$ or A_5.
If $\Sigma \simeq A_4$, such subgroups are all conjugate if $q \equiv \pm 3 \mod 8$. They form two conjugacy classes if $q \equiv \pm 1 \mod 8$.
If $\Sigma \simeq \Sigma_4$, such a subgroup actually exists if and only if $q \equiv \pm 1 \mod 8$ and is conjugate to each other.
If $\Sigma \simeq A_5$, such a subgroup actually exists if and only if $q \equiv \pm 1 \mod 5$ and is conjugate to each other.

Proof. See [Di] for the proof or [Sz] for a modern description.

Lemma 1. *Let K be a finite group and L_i $(i = 1, 2)$ a subgroup of K. Then*

$$\langle \mathrm{Ind}_{L_1}^K 1_{L_1},\ \mathrm{Ind}_{L_2}^K 1_{L_2}\rangle_K = |L_1\backslash K/L_2|. \tag{8.1}$$

In particular, the following three statements are equivalent.
(i) $|L_1||L_2| \neq |K||L_1 \cap L_2|$,
(ii) $\mathrm{Ind}_{L_1}^K 1_{L_1}$ and $\mathrm{Ind}_{L_2}^K 1_{L_2}$ have a common nontrivial irreducible component as K-module.
(iii) $\langle \mathrm{Ind}_{L_1}^K 1_{L_1},\ \mathrm{Ind}_{L_2}^K 1_{L_2}\rangle \neq 1$

Proof. Formula (8.1) is an immediate consequence of Mackey's formula for operator $\mathrm{Res}_{L_2}^K \circ \mathrm{Ind}_{L_1}^K$. The equivalence of (i) and (iii) follows immediately from formula (8.1) and the equivalence of $|L_1\backslash K/L_2| = 1$ and $|K| = |L_1L_2|$. Since $\mathrm{Ind}_{L_i}^K 1_{L_i}$ is represented by the transitive permutation of K on $\mathbb{C}[K/L_i]$, $\mathrm{Ind}_{L_i}^K 1_{L_i}$ has exactly one trivial irreducible component. Hence the equivalence of (ii) and (iii) is obvious.
Q.E.D

Lemma 2. *Let $\bar{H}$ be a proper subgroup of $SL_2(\mathbb{F}_q)$ and St as in 6.5. Then*

$$\langle \mathrm{Res}_{\bar{H}}^{SL_2(\mathbb{F}_q)} St,\ 1_{\bar{H}}\rangle_{\bar{H}} = 0$$

if and only if one of the followings holds
(i) $q \equiv -1 \mod 4$ and $\bar{H}$ is conjugate to N'.
(ii) $q = 3,\ 5,\ or\ 11$ and $\bar{H} = \pi^{-1}(\Sigma)$ with $\Sigma \simeq A_4$.
(iii) $q = 5,\ 7,\ or\ 23$ and $\bar{H} = \pi^{-1}(\Sigma)$ with $\Sigma \simeq \Sigma_4$.
(iv) $q = 5$ and $\bar{H} = \pi^{-1}(\Sigma)$ with $\Sigma \simeq \Sigma_5$.

Proof. Since $St = \mathrm{Ind}_{B^F}^{SL_2(\mathbb{F}_q)} 1_{B^F} - 1_{SL_2(\mathbb{F}_q)}$,

$$\langle \mathrm{Res}_{\bar{H}}^{SL_2(\mathbb{F}_q)} St,\ 1_{\bar{H}}\rangle_{\bar{H}} = \langle Ind_{B^F}^{SL_2(\mathbb{F}_q)} 1_{B^F},\ \mathrm{Ind}_{\bar{H}}^{SL_2(\mathbb{F}_q)} 1_{\bar{H}}\rangle_{\bar{H}} - 1. \tag{8.2}$$

By Dickson's Theorem we can directly verify that

$$|B^F||\bar{H}| = |\bar{H} \cap B^F||SL_2(\mathbb{F}_q)|$$

if and only if $\bar{H}$ is conjugate to N' provided q is sufficiently large. The exceptional cases are verified by hand. Then the lemma follows from Lemma 1 and equation (8.2). Q.E.D.

Proposition 1. *Suppose that $-a = -(1+\sqrt{5})/2$ is a non-square in $F_{\mathfrak{p}}$. Then if subgroup H of $G = B_{4,4,4}/\Gamma_{\mathfrak{p}}$ does not contain a noncentral normal subgroup, H fixes $H_1(\Gamma\backslash\mathbb{H}^3,\ \mathbb{C})$ nontrivially.*

Proof. In this case $G \simeq PSL_2(\mathbb{F}_{q^2})$ or $PSL_2(\mathbb{F}_{q^2}) \times \mathbb{Z}_2$. Since any subgroup of $SL_2(\mathbb{F}_{q^2})$ can verify none of the statements (i)-(iv) in Lemma 2, $\bar{H} := \pi^{-1}(H)$ always have nontrivial fixed point set in St or $St \times 1_{\mathbb{F}_{q^2}}$. Thus the proposition follows from Theorem 7.5.1. Q.E.D

Lemma 3. *Let K, L_1 and L_2 be as in Lemma 1. Then subgroup $L_1 \times L_2$ of $K \times K$ has nontrivial fixed point in $\sigma := \sum_{1 \neq \rho \in \mathrm{Irr}(K)} \rho \times \rho$ if and only if one of the equivalent statements (i),(ii) and (iii) in Lemma 1.*

Proof. Since

$$\begin{aligned}
\langle \mathrm{Res}^{K\times K}_{L_1 \times L_2} \sigma,\ 1_{L_1\times L_2}\rangle_{L_1\times L_2} &= \sum_{1\neq\rho\in\mathrm{Irr}(K)} \langle \mathrm{Res}^K_{L_1} \rho,\ 1_{L_1}\rangle_{L_1} \langle \mathrm{Res}^K_{L_2} \rho,\ 1_{L_2}\rangle_{L_2} \\
&= \sum_{1\neq\rho\in\mathrm{Irr}(K)} \langle \rho,\ \mathrm{Ind}^K_{L_1} 1_{L_1}\rangle_{L_1} \langle \rho,\ \mathrm{Ind}^K_{L_2} 1_{L_2}\rangle_{L_2},
\end{aligned}$$

$L_1 \times L_2$ has nontrivial fixed point in σ if and only if statement (ii) in Lemma 1 holds true. Q.E.D.

Lemma 4. *Suppose that $\mathfrak{p} \notin P_{B_{4,4,4}}$ is a prime ideal of $k := \mathbb{Q}(\sqrt{5})$ which devides rational prime p. Let $K := SL_2(F_{\mathfrak{p}})$ and L_i a proper subgroup of K. Then pair $(L_1,\ L_2)$ verifies statements (i), (ii) and (iii) in Lemma 1 unless one of the followings holds true*
(i) $p \equiv -1 \mod 4$, $p \equiv \pm 1 \mod 5$ and $(L_1,\ L_2)$ is conjugate to $(B^F,\ N'^F)$ or $(M,\ N'^F)$ where M is the unique subgroup of index 2 in B^F.
(ii) $p = 11$ and $(L_1,\ L_2)$ is conjugate to $(B^F,\ \pi^{-1}(\Sigma))$ with $\Sigma \simeq A_4$.

Proof. First observe that the classical reciprocity for quadratic fields shows that $q := |F_{\mathfrak{p}}|$ coincides with rational prime p if $p \equiv \pm 1 \mod 5$ and $q = p^2$ otherwise. Then we can verify directly that statement (i) in Lemma 1 for each pair of proper subgroups, which are classified by Dickson's Theorem. Q.E.D.

Lemma 5. *Let K be as in Lemma 1, H a subgroup of $K \times K$ and π_i $(i = 1, 2)$ the projection of $K \times K$ onto the i-th component. Then*
(i) $L = \pi_2(\mathrm{Ker}\,\pi_1)$ is an normal subgroup of $L_2 := \pi_2(H)$.
(ii) There is an epimorphism, say φ, of $L_1 := \pi_1(H)$ onto L_2/L such that

$$H = \bigcup_{x\in L_1} (x,\ \varphi(x)L).$$

Proof. By definition we have $\pi_1^{-1}(x) = (x,\ \overline{x}L)$ for $x \in L_1$ and $\overline{x} \in \pi_2(\pi_1^{-1}(x))$. Hence choosing such $\overline{x}$ for each $x \in L_1$ we have $H = \bigcup_{x\in X}(x,\ \overline{x}L)$. Since H is a group, we have

$$\overline{x}_1 L \overline{x}_2 L \subset \overline{x_1 x_2} L$$

for any $x_1,\ x_2 \in L_1$. Hence ${}^{\overline{x}_2^{-1}}L = L$ and $\overline{x}_1\overline{x}_2 L = \overline{x_1x_2}L$. This proves the statements. Q.E.D.

Proposition 2. *Let $\mathfrak{p} \notin P_{B_{4,4,4}}$ be a prime ideal of $\mathbb{Q}(\sqrt{5})$ and p denotes the projection of $SL_2(\mathbb{F}_q) \times SL_2(\mathbb{F}_q)$ to $G := B_{4,4,4}/\Gamma_\mathfrak{p}$ defined in 6.2. Suppose that $-a$ is a square in $F_\mathfrak{p}$ and $q := |F_\mathfrak{p}| \equiv \pm 1 \mod 8$. Then subgroup H of G fixes $H_1(\Gamma_\mathfrak{p}\backslash\mathbb{H}^3,\ \mathbb{C})$ nontrivially unless H contains a noncentral normal subgroup or $(\pi_1(p^{-1}(H)),\ \pi_2(p^{-1}(H)))$ verifies statement (i) in Lemma 4.*

Proof. It follows from Theorem 6.5.1 that $\sigma := \sum_{1\neq\rho\in\mathrm{Irr}(SL_2(\mathbb{F}_q))} \rho\times\rho$ projects to a component of $H_1(\Gamma_\mathfrak{p}\backslash\mathbb{H}^3,\mathbb{C})$ through the pull-back by p. We shall prove that σ is nontrivially fixed by subgroup $\bar{H} := p^{-1}(H)$ of $SL_2(\mathbb{F}_q)\times SL_2(\mathbb{F}_q)$ unless H is a subgroup described in statement (i) in Lemma 4.
Let the notions be as in Lemma 4 with $K := SL_2(\mathbb{F}_q),\ L_1 := \pi_1(\bar{H})$ and $L_2 := \pi_2(\bar{H})$. Suppose first that one of L_1 and L_2, say L_1, coincides with $SL_2(\mathbb{F}_q)$. Then the almost simpleness of $SL_2(\mathbb{F}_q)$ and Lemma 5 show that $\bar{H} = SL_2(\mathbb{F}_q)\times L_2$ or

$$H = \{(x,\ \varphi(x)z);\ x \in SL_2(\mathbb{F}_q),\ z = \pm I\}$$

for some epimorphism of $SL_2(\mathbb{F}_q)$ onto $PSL_2(\mathbb{F}_q)$. In the former case H contains the non-central normal subgroup $SL_2(\mathbb{F}_q)\times I$. The latter group fixes $St\times St$ nontrivially since $St\times St$ projects to the irreducible representation of $PSL_2(\mathbb{F}_q)\times PSL_2(\mathbb{F}_q)$ and is invariant with respect to any automorphism of $PSL_2(\mathbb{F}_q)$.
Now suppose that both L_1 and L_2 are proper. Since $\bar{H} \subset L_1\times L_2$, it follows from Lemma 3 and Lemma 4 that $\bar{H}$ fixes σ nontrivially unless $(L_1,\ L_2)$ verifies (i) or (ii) in the statement of Lemma 4. This proves the proposition. Q.E.D.

Remark. It follows from Lemma 5 that there are two non-conjugate subgroups H_1 and H_2 which are properly contained in $B^F\times {N'}^F$ and $(\pi_1(H_i),\ \pi_2(H_i)) = (B^F,\ {N'}^F)$ provided $q \equiv -1 \mod 4$. For these groups one can expect that they fix σ nontrivially. But one can verify directly that it is not the case.

Now we shall apply the preceding Lemmas for $K = \widehat{SL_2}(\mathbb{F}_q)$, which was defined in 6.2. However for the notational convinience we set $\hat{K} := \widehat{SL_2}(\mathbb{F}_q)$, $K := SL_2(\mathbb{F}_q)$ and

$$\widehat{K\times K} := (K\times K)\rtimes\langle t_1\rangle = \left\{(g\delta,\ h\delta)\in\hat{K}\times\hat{K};\ g,\ h\in K,\ \delta = t_1 \text{ or } I\right\}.$$

As before π_1 and π_2 denote the projections of $\hat{K}\times\hat{K}$ to the first and second component, respectively. For subgroups $\hat{L}_1$ and $\hat{L}_2$ of $\hat{K}$ set

$$\widehat{L_1\times L_2} := \widehat{K\times K}\cap(\hat{L}_1\times\hat{L}_2).$$

Recall that r_1 acts on $\widehat{K\times K}$ by $r_1 :\ (g\delta,\ h\delta) \mapsto\ (h\delta,\ g\delta)$. It follows from Lemma 6.2.5 that r_1-invariant representation τ of $\widehat{K\times K}$ verifies exactly one of the

followings
(I_1^+) $\tau = \mathrm{Res}^{\hat{K}\times\hat{K}}_{\widehat{K\times K}}\,\hat{\rho}\times\hat{\rho}$ with $\hat{\rho}\in\mathrm{Irr}(K)$ and $\mathrm{Res}^{\hat{K}}_{K}\,\hat{\rho}\in\mathrm{Irr}(K)$,
(I_1^-) $\tau = \mathrm{Res}^{\hat{K}\times\hat{K}}_{\widehat{K\times K}}\,\hat{\rho}\times\sharp_{t_1}\hat{\rho}$ with $\hat{\rho}\in\mathrm{Irr}(K)$ and $\mathrm{Res}^{\hat{K}}_{K}\,\hat{\rho}\in\mathrm{Irr}(K)$,
(I_2) $\tau = \mathrm{Ind}^{\hat{K}\times\hat{K}}_{\widehat{K\times K}}\,\rho$ with $\rho\neq {}^{t_1}\rho$,
(I_3) $\tau = \mathrm{Ind}^{\hat{K}\times\hat{K}}_{\widehat{K\times K}}\,\rho\times{}^{t_1}\rho$ with $\rho\neq {}^{t_1}\rho$.
We define a representation σ of $\widehat{K\times K}$ by $\sigma := \sum_\tau \tau$ where τ runs over all the nontrivial r_1-invariant irreducible representations which does not verfy (I_1^-).

Lemma 6. *Let* $\hat{\sigma} := \sum_{1\neq\hat{\rho}\in\mathrm{Irr}(\hat{K})}\hat{\rho}\times\hat{\rho}$. *Then* $\mathrm{Res}^{\hat{K}\times\hat{K}}_{\widehat{K\times K}}\,\hat{\sigma} = \sigma + 1_{\widehat{K\times K}}$.

Proof. Mackey's criterion and Lemma 4.3.1 show that irreducible representation $\bar{\rho}$ of $\hat{K}$ verifies exactly one of the followings
(a) $\mathrm{Res}^{\hat{K}}_{K}\,\hat{\rho}$ is irreducible,
(b) $\hat{\rho} = \mathrm{Ind}^{\hat{K}}_{K}\,\rho$ for $\rho\in\mathrm{Irr}(K)$ with $\rho\neq{}^{t_1}\rho$.
Suppose first that $\hat{\rho}$ is of type (b). Then we have

$$\mathrm{Res}^{\hat{K}\times\hat{K}}_{\widehat{K\times K}}(\hat{\rho}\times\hat{\rho}) = \mathrm{Ind}^{\widehat{K\times K}}_{K\times K}(\rho\times\rho) + \mathrm{Ind}^{\widehat{K\times K}}_{K\times K}(\rho\times{}^{t_1}\rho)$$

since the characters of the representations on the both sides coincide on $K\times K$ and vanish on $\widehat{K\times K}\setminus K\times K$. If $\bar{\rho}$ is of type (a), $\hat{\rho}\times\hat{\rho}$ restricts to an irreducible representation of type (I_1^+). Thus unless $\hat{\rho}$ is the unique nontrivial extension of 1_K, $\mathrm{Res}^{\hat{K}\times\hat{K}}_{\widehat{K\times K}}(\hat{\rho}\times\hat{\rho})$ is a sum of non-trivial r_1-invariant irreducible representations which do not verify (I_1^-) and any such irreducible representations are covered exactly once by unique $\hat{\rho}\times\hat{\rho}$. Thus the lemma follows. Q.E.D.

Lemma 7. *Suppose that* $\hat{L}_1$ *and* $\hat{L}_2$ *are subgroups of* $\hat{K}$. *Let* $L_i := \hat{L}_i\cap K$ *for* $i=1,2$.
(a) Suppose that $\hat{L}_1 = L_1$ *and* $\hat{L}_2 = L_2$. *Then* $\widehat{L_1\times L_2} = L_1\times L_2$ *fixes* σ *trivially if and only if the following two equations hold*

$$|L_1||L_2| = |L_1\cap L_2||K|,\ \ |L_1||L_2| = |{}^{t_1}L_1\cap L_2||K|.$$

(b) Suppose that $\hat{L}_1\neq L_1$ *and* $\hat{L}_2\neq L_2$. *Then* $\widehat{L_1\times L_2}$ *fixes* σ *trivially if and only if one of the followings holds*
(i) $|L_1||L_2| = |L_1\cap L_2||K|$,
(ii) $2|L_1||L_2| = |L_1\cap L_2||K|$ *and* $\hat{L}_1\cap\hat{L}_2 = L_1\cap L_2$.

Proof. Lemma 6 shows that

$$(8.3)\ \ \langle \mathrm{Res}^{\widehat{K\times K}}_{\widehat{L_1\times L_2}}\,\sigma,\ 1_{\widehat{L_1\times L_2}}\rangle_{\widehat{L_1\times L_2}} = \sum_{1\neq\hat{\rho}\in\mathrm{Irr}(\hat{K})}\langle \mathrm{Res}^{\hat{K}\times\hat{K}}_{\widehat{L_1\times L_2}}\,\hat{\rho}\times\hat{\rho},\ 1_{\widehat{L_1\times L_2}}\rangle_{\widehat{L_1\times L_2}} - 1.$$

Suppose first that $\hat{L}_1 = L_1$ and $\hat{L}_2 = L_2$. Then it follows from Lemma 1, Lemma 3 and equation (8.3) that

$$\langle \mathrm{Res}^{\widehat{K\times K}}_{\widehat{L_1\times L_2}}\,\sigma,\ 1_{\widehat{L_1\times L_2}}\rangle_{\widehat{L_1\times L_2}} = |L_1\backslash\hat{K}/L_2| - 2.$$

Since $K \lhd \hat{K}$ and $|\hat{K}/K| = 2$, $|L_1\backslash \hat{K}/L_2|$ has at least two elements , $L_1L_2 \subset K$ and $L_1 t_1 L_2 = t_1{}^{t_1}L_1L_2 \subset \hat{K}\setminus K$. Hence $|L_1\backslash\hat{K}/L_2| = 2$ if and only if L_1L_2 and ${}^{t_1}L_1L_2$ exhaust K. This proves (a). Now suppose that $\hat{L}_1 \neq L_1$ and $\hat{L}_2 \neq L_2$. Then writing down equation (8.3) by terms, we have

$$\langle \mathrm{Res}^{\widehat{K\times K}}_{\widehat{L_1\times L_2}}\,\sigma,\ 1_{\widehat{L_1\times L_2}}\rangle_{\widehat{L_1\times L_2}} = \frac{1}{2|L_1||L_2|}\sum_{\hat\rho\neq 1}\sum_{l_i\in L_i}\left(\hat\rho(l_1)\hat\rho(l_2)+\hat\rho(l_1u)\hat\rho(l_2u')\right)-1$$

$$=\frac{1}{|L_2|}\sum_{\hat\rho\neq 1}\sum_{l_2\in L_2}\big\{\langle \mathrm{Res}^{\hat K}_{L_1}\,\hat\rho,\ 1_{L_1}\rangle_{L_1}\hat\rho(l_2)$$

$$\big(\langle \mathrm{Res}^{\hat K}_{\hat L_1}\,\hat\rho,\ 1_{\hat L_1}\rangle_{\hat L_1}-\frac{1}{2}\langle \mathrm{Res}^{\hat K}_{L_1}\,\hat\rho,\ 1_{L_1}\rangle_{L_1}\big)\rho(l_2u')\big\}-1$$

$$=-1+\sum_{\hat\rho\neq 1}\big\{\frac{1}{2}\langle \mathrm{Res}^{\hat K}_{L_1}\,\hat\rho,\ 1_{L_1}\rangle_{L_1}\langle \mathrm{Res}^{\hat K}_{L_2}\,\hat\rho,\ 1_{L_1}\rangle_{L_2}$$

$$+\big(\langle \mathrm{Res}^{\hat K}_{\hat L_1}\,\hat\rho,\ 1_{\hat L_1}\rangle_{\hat L_1}-\frac{1}{2}\langle \mathrm{Res}^{\hat K}_{L_1}\,\hat\rho,\ 1_{L_1}\rangle_{L_1}\big)\big(2\langle \mathrm{Res}^{\hat K}_{\hat L_2}\,\hat\rho,\ 1_{\hat L_2}\rangle_{\hat L_2}-\langle \mathrm{Res}^{\hat K}_{L_2}\,\hat\rho,\ 1_{L_2}\rangle_{L_2}\big)\big\}$$

$$=\langle \mathrm{Res}^{\hat K}_{L_1}\,\bar\rho,\ 1_{L_1}\rangle_{L_1}\langle \mathrm{Res}^{\hat K}_{L_2}\,\bar\rho,\ 1_{L_1}\rangle_{L_2}-\langle \mathrm{Res}^{\hat K}_{\hat L_1}\,\bar\rho,\ 1_{\hat L_1}\rangle_{\hat L_1}\langle \mathrm{Res}^{\hat K}_{L_2}\,\bar\rho,\ 1_{L_2}\rangle_{L_2}$$

$$-\langle \mathrm{Res}^{\hat K}_{L_1}\,\bar\rho,\ 1_{L_1}\rangle_{L_1}\langle \mathrm{Res}^{\hat K}_{\hat L_2}\,\bar\rho,\ 1_{\hat L_2}\rangle_{\hat L_2}+2\langle \mathrm{Res}^{\hat K}_{\hat L_1}\,\bar\rho,\ 1_{\hat L_1}\rangle_{\hat L_1}\langle \mathrm{Res}^{\hat K}_{\hat L_2}\,\bar\rho,\ 1_{\hat L_2}\rangle_{\hat L_2}-1$$

where u and u' are elements of $\hat{L}_1\setminus L_1$ and $\hat{L}_2\setminus L_2$, respectively. Then Lemma 1 shows that

$$\langle \mathrm{Res}^{\widehat{K\times K}}_{\widehat{L_1\times L_2}}\,\sigma,\ 1_{\widehat{L_1\times L_2}}\rangle_{\widehat{L_1\times L_2}} = 2\left(|\hat L_1\backslash\hat K/\hat L_2|-1\right) + |L_1\backslash\hat K/L_2| - |\hat L_1\backslash\hat K/L_2| - |L_1\backslash\hat K/\hat L_2|. \tag{8.4}$$

Since $K \lhd \hat{K}$ and $|\hat{K}/K| = 2$,

$$L_1\backslash\hat K/L_2 \ni L_1xL_2 \mapsto \hat L_1 x L_2 \in \hat L_1\backslash \hat K/L_2$$

is exactly two-to-one and similarly for $L_1\backslash\hat{K}/\hat{L}_2$. Hence

$$\frac{1}{2}|L_1\backslash\hat K/L_2| = |\hat L_1\backslash\hat K/L_2| = |L_1\backslash\hat K/\hat L_2|.$$

Thus it follows from equation (8.4)

$$\langle \mathrm{Res}^{\widehat{K\times K}}_{\widehat{L_1\times L_2}}\,\sigma,\ 1_{\widehat{L_1\times L_2}}\rangle_{\widehat{L_1\times L_2}} = 2\left(|\hat L_1\backslash\hat K/\hat L_2|-1\right)$$

and again the equivalence of the triviality of the fixed point set and $|\hat{L}_1||\hat{L}_2| = |\hat{K}||\hat{L}_1\cap\hat{L}_2|$. Then statement (b) follows from the assumptions that $|\hat{L}_1/L_1| = |\hat{L}_2/L_2| = |\hat{K}/K| = 2$ and $|\hat{L}_1\cap\hat{L}_2/L_1\cap L_2| = 1$ or 2. Q.E.D.

Lemma 8. *Let $\hat{L}_i$ and L_i $(i = 1, 2)$ be as in Lemma 7 with*

$$2|L_1||L_2| = |L_1\cap L_2||K|. \tag{8.5}$$

Suppose that $N_{\hat{K}}(L_1) \cap N_{\hat{K}}(L_2)$ is not contained in K and

$$2|N_K(L_1)||N_K(L_2)| = |N_K(L_1) \cap N_K(L_2)||K|. \tag{8.6}$$

Then $\hat{L}_1 \cap \hat{L}_2$ is not contained in K.

Proof. By assumptiom $\hat{L}_i = \langle L_i,\ u_i \rangle$ for some $u_i \in N_{\hat{K}}(L_i) \setminus K$. Since $N_{\hat{K}}(L_1) \cap N_{\hat{K}}(L_2) \cap (\hat{K} \setminus K) \neq \emptyset$, $u_i = n_i \alpha$ for some $n_i \in N_K(L_i)$ and $\alpha \in N_{\hat{K}}(L_1) \cap N_{\hat{K}}(L_2) \cap (\hat{K} \setminus K)$. Then $\hat{L}_i \cap (\hat{K} \setminus K) = L_i u_i = L_i n_i \alpha$. Thus $\hat{L}_1 \cap \hat{L}_2$ contains an element of $\hat{K} \setminus K$ if and only if $n_1 n_2^{-1} \in L_1 L_2$. By our assumption this is always the case since $L_1 L_2 \subset N_K(L_1) N_K(L_2)$ and both double cosets have a common cardinality by equations (8.5) and (8.6). This proves the Lemma. Q.E.D.

Lemma 9. *Suppose that $q \equiv \pm 3 \mod 8$. Let ϕ be an automorphism of $P\hat{K} := \hat{K}/\langle \pm I \rangle$. Then*

$$\frac{1}{|P\hat{K}|} \sum_{\hat{\rho} \in \mathrm{Irr}(P\hat{K})} \sum_{g \in P\hat{K}} \hat{\rho}(g) \hat{\rho}({}^{\phi}g) \geq 3.$$

Proof. It follows from the first and the second orthogonal relation of irreducible characters that

$$\frac{1}{|P\hat{K}|} \sum_{\hat{\rho} \in \mathrm{Irr}(P\hat{K})} \sum_{g \in P\hat{K}} \hat{\rho}(g) \hat{\rho}({}^{\phi}g) = \sharp \left\{ c \in \mathcal{C}_{P\hat{K}};\ c^{-1} = {}^{\phi}c \right\} \tag{8.7}$$

where $\mathcal{C}_{P\hat{K}}$ is the set of conjugacy classes of $P\hat{K}$. Since the conjugacy classes of $\pm I$ and $\pm\theta_4$ (see 7.3 for the notion) are the unique classes of order one and two contained in the characteristic subgroup $PSL_2(\mathbb{F}_q)$ of $P\hat{K}$, they verifies $c^{-1} = {}^{\phi}c$ for any automorphism ϕ. Hence the right hand side of equation (8.7) is larger than or equal to 3, if we can prove that there is only one conjugacy class of order two contained $P\hat{K} \setminus PSL_2(\mathbb{F}_q)$.

Since $q \equiv \pm 3 \mod 8$, the order of Sylow 2-groups of $PSL_2(\mathbb{F}_q)$ is four and that of $P\hat{K}$ is eight. Hence 2-Sylow groups of $PSL_2(\mathbb{F}_q)$ are conjugate to $N_{PSL_2(\mathbb{F}_q)}(\theta_4)$, which is isomorphic to the normal subgroup of alternative group A_4 spanned by all element of type $(2,\ 2)$. Let S_2 be a 2-Sylow group stabilized by t_1. Lemma 6.1.5 and the statement of Dickson's Theorem for A_4 and Σ_4 show that t_1 acts on S_2 nontrivially, i.e. permutes a pair of elements of order two and fixes a non-trivial element. Thus we have a rigorous description of Sylow 2-group $\hat{S}_2 = S_2 \rtimes \langle t_1 \rangle$ of $P\hat{K}$. It is readily verified that $\hat{S}_2 \setminus S_2$ has two elements of order two and they are conjugate in $\hat{S}_2$. Thus it follows from Sylow's theorem that all elements of order 2 in $P\hat{K} \setminus PSL_2(\mathbb{F}_q)$ are conjugate. This proves the lemma. Q.E.D.

Proposition 3. *Let $\mathfrak{p}$ and $-a$ be as in Proposition 2 and $\hat{p}$ denotes the projection of $(SL_2(\mathbb{F}_q) \times SL_2(\mathbb{F}_q)) \rtimes \langle t_1 \rangle$ to $G := B_{4,4,4}/\Gamma_{\mathfrak{p}}$ defined in 6.2. Suppose that $q := |F_{\mathfrak{p}}| \equiv \pm 3 \mod 8$. Subgroup H of G fixes $H_1(\Gamma_{\mathfrak{p}} \backslash \mathbb{H}^3,\ \mathbb{C})$ nontrivially unless H contains a noncentral normal subgroup of G or $(\pi_1(\hat{p}^{-1}(H)) \cap SL_2(\mathbb{F}_q),\ \pi_2(\hat{p}^{-1}(H)) \cap SL_2(\mathbb{F}_q))$ verifies one of the statements in Lemma 4.*

Proof. It follows from Theorem 6.5.1 that representation σ, the sum of all the r_1-invariant representations which do not verify (I_1^-), projects to a component of

$H_1(\Gamma_{\mathfrak{p}}\backslash\mathbb{H}^3,\ \mathbb{C})$ through $\hat{p}$. We shall prove that σ is nontrivially fixed by $\bar{H} = \hat{p}^{-1}(H)$ verifies the assumption of the statement.
For $i = 1,2$ set $\hat{L}_i := \pi_i(\bar{H})$ and $L_i := \hat{L}_i \cap SL_2(\mathbb{F}_q)$. By definition of $(SL_2(\mathbb{F}_q) \times SL_2(\mathbb{F}_q)) \rtimes \langle t_1 \rangle$ both $\hat{L}_1 = L_1$ and $\hat{L}_2 = L_2$ hold, if $\bar{H}$ is contained in $SL_2(\mathbb{F}_q) \times SL_2(\mathbb{F}_q)$, and neither of these equations holds otherwise.

Suppose first that $\hat{L}_1 = L_1$ and $\hat{L}_2 = L_2$. Since the conjugacy class of B^F in $K = SL_2(\mathbb{F}_q)$ and Steinberg character St are invariant by the action of any automorphism of K, the statement of the theorem follows from Lemma 1, Lemma 4, Lemma 7 (a) and the argument of Theorem 2.

Suppose then that one of $\hat{L}_1$ and $\hat{L}_2$, say $\hat{L}_1$, coincides with $\hat{K} = SL_2(\mathbb{F}_q) \rtimes \langle t_1 \rangle$. Then considering $\bar{H}$ as a subgroup of $\hat{K}\times\hat{K}$, we can apply Lemma 5 to the situation. In the notion of Lemma 5, $\mathrm{Ker}\,\varphi = \{I\},\ \{\pm I\},\ K$ or $\hat{K}$. If $\mathrm{Ker}\,\varphi = K$ or $\hat{K}$, $\bar{H}$ contains a normal subgroup $K \times I$ of $\widehat{K \times K}$. If $\mathrm{Ker}\,\varphi = \{I\}$ or $\pm\{I\}$, $\bar{H}$ projects to subgroup H' of $P\hat{K} \times P\hat{K} := \hat{K}/\langle \pm I \rangle \times \hat{K}/\langle \pm I \rangle$ defined by

$$H' := \left\{(x,\ \psi(x));\ x \in P\hat{K}\right\}$$

for some $\psi \in \mathrm{Aut}(P\hat{K})$. Lemma 9 show that there is at least one irreducible representation, denoted by $\hat{\rho}$, of $P\hat{K}$ which restricts to a nontrivial representation of $PSL_2(\mathbb{F}_q)$ such that $\hat{\rho}\times\hat{\rho}$ is fixed by H' nontrivially. Then it follows from Lemma 6 that σ is fixed by $\bar{H}$ nontrivially.

Finally assume that both $\hat{L}_1$ and $\hat{L}_2$ are proper but not contained in $SL_2(\mathbb{F}_q)$. Lemma 7(b) shows that $\bar{H}$ fixes σ nontrivially unless L_1 and L_2 verify either of statements (i) and (ii) in the Lemma. If the statement (i) in Lemma 7(b) holds then H verifies one of the statement of Lemma 4. Suppose that $2|L_1||L_2| = |L_1 \cap L_2||K|$. Then it is directly verified by Dickson's theorem that the equation holds only if $q \equiv 1 \bmod 4$ and $(L_1,\ L_2)$ is conjugate to a subgroup of $(B^F,\ N'^F)$ with $N_K(L_1) = B^F$ and $N_K(L_2) = N'^F$. Since the direct computation shows that $J(*d)$ normalizes some conjugates of B^F and N'^F with $*d$ being the automorphism defined in Lemma 6.2.2, $\hat{L}_1 \cap \hat{L}_2 \neq L_1 \cap L_2$ by Lemma 8. Hence the statement (ii) in Lemma 7(b) cannot hold true for any $(L_1,\ L_2)$. Q.E.D.

Theorem 1. *Let Γ be a subgroup of finite index in $B_{4,4,4}$ and T_Γ the subgroup of Γ defined in 1.1. If there exists prime ideal $\mathfrak{p} \notin P_{B_{4,4,4}}$ of $\mathbb{Q}(\sqrt{5})$ such that the image of T_Γ in $G := B_{4,4,4}/\Gamma_{\mathfrak{p}}$ does not contain any noncentral normal subgroup of G, $\Gamma\backslash\mathbb{H}^3$ has a finite-sheeted cover of positive first Betti number.*

Proof. The theorem follows from Proposition in 1.1 if we can prove that the image of T_Γ in G has a nontrivial fixed point in $H_1(\Gamma_{\mathfrak{p}}\backslash\mathbb{H}^3,\ \mathbb{C})$. By Proposition 1 we may assume that $-a$ is a square in local field $F_{\mathfrak{p}}$. Since any elliptic element of $B_{4,4,4}$ is of order two or four, the image of T_Γ in G is generated by elements of order two or four. Hence $p^{-1}(T_\Gamma)$ or $\hat{p}^{-1}(T_\Gamma) \cap (SL_2(\mathbb{F}_q) \times SL_2(\mathbb{F}_q))$ is generated by elements of order two, four and eight. (Here the notions in 6.2 are carried over and we abbreviate the image of T_Γ again to T_Γ.) Then it is readily verified that $p^{-1}(T_\Gamma)$ or $\hat{p}^{-1}(T_\Gamma) \cap (SL_2(\mathbb{F}_q) \times SL_2(\mathbb{F}_q))$ cannot verify that the statements in Lemma 4. Then the theorem follows from Proposition 2 and Proposition 3. Q.E.D.

9. Remarks and problems for further development

9.1 *Technical remarks* We shall summerize some remarks and the associated problems which is not relevent directly to the motivation in 1.1.
(1) *Exceptional primes* The finiteness of set P_Γ of the exceptional primes was shown in Lemma 5.3.1 by the general argument. We shall see that more precise information on $P_{B_{4,4,4}}$ is obtained with a little more work. Let q be the norm of prime ideal $\mathfrak{p}$ as usual. First observe that Dickson's theorem, which was quoted in Section 8, shows that the minimal index of proper subgroups of $\Omega_{f_\mathfrak{p}}$ is $q+1$ provided $q > 11$. Then equation (5.1) gives the following bound on q.

$$|\Gamma'_{v_0}/B_{4,4,4} \cap \Gamma'_{v_0}| \geq q+1 \tag{9.1}$$

for $\mathfrak{p} \in P_{B_{4,4,4}}$ and $q > 11$. Moreover it follows immediately from the argument on Lemma 5.3.1 that $|\Gamma'_{v_0}/B_{4,4,4} \cap \Gamma'_{v_0}|$ and $\Omega_{f_\mathfrak{p}}$ have a common nontrivial prime factor. We also have the information on the left hand side of (9.1) due to Borel [Bo]. He shows in [Bo] that in any (arithmetic) commensurable class of hyperbolic 3-orbifold there is a volume minimizing orbifold and he determined almost explicitely the value of the minimal volume. Though we have been considering $B_{4,4,4}$ as an arithmetic larrice in $SO(3,1)$, he considered arithmetic lattices in PSL_2, which are proved to be more general than the one in SO_3 by Reid [Re]. Hence the minimal volume is given in terms of $\zeta_K(2)$ and the ramification of quaternion algebra D over K where ζ_K the Dedekind zeta function of base field K of the arithmetic commensurable class of lattices of PSL_2 and D is the associated quaternion algebra. In [HLM1] base field K is determined to be $\mathbb{Q}(a^{\frac{3}{2}}\sqrt{-1})$ and the ramification of D is intensively studied. Hence computing the volume of our dodecahedron in terms of $\zeta_K(2)$ (or possibly with a help of the computer work), we can compute (or at least give an upper bound of) the left hand side of (9.1). Thus working out with these information it would not be hard to re-state our theorems without the ambiguity of exceptional primes.
(2) *The growth of the first Betti number* To show the existence of congruent coverings of arbitrary large first Betti number, Millson gives in [Mi] a rough lower bound for the first Betti number of congruent subgroup $\Gamma_\mathfrak{m}$ in terms of the number of distinct primes in the decomposition of ideal $\mathfrak{m}$. For our particular case, $B_{4,4,4}$, the results in Section 6 and Section 7 immediately gives a lower bound in terms of the norm of the prime ideals. In particular we have

Proposition 1. *Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Q}(\sqrt{5})$ and $\Gamma_\mathfrak{p}$ the congruent subgroup of $B_{4,4,4}$ associated to $\mathfrak{p}$. Then*

$$\liminf_{q\to\infty} \frac{b_1(\Gamma_\mathfrak{p}\backslash\mathbb{H}^3)}{q^4} > 0, \quad \limsup_{q\to\infty} \frac{b_1(\Gamma_\mathfrak{p}\backslash\mathbb{H}^3)}{q^6} < \infty$$

where q is the norm of $\mathfrak{p}$ and $b_1(\cdot)$ denotes the first Betti number of the topological space.

Proof. We shall carry over the notions in Section 4. The second inequality is almost obvious since the dimension of our 1-chain module is of order $O(q^6)$. To prove the first inequality recall that by definition of $\mathfrak{N}_\rho$ in 6.5

$$|\bar{\mathcal{H}}_{1,\rho}(gr_1) - \bar{\mathcal{H}}_{2,\rho}(gr_1)| = |\bar{\rho}(gr_1)|\mathfrak{N}_\rho.$$

Suppose that $\bar{\mathcal{H}}_{1,\rho} = n_1\bar{\rho} + n_2\sharp_{r_1}\bar{\rho}$ and $\bar{\mathcal{H}}_{1,\rho} = n'_1\bar{\rho} + n'_2\sharp_{r_1}\bar{\rho}$. Then $\mathfrak{N}_\rho = |(n_1 - n_2) - (n'_1 - n'_2)|$. Since $\mathcal{H}_{1,\rho} = \mathcal{H}_{2,\rho}$ by Lemma 4.1.2, $|\mathfrak{N}_\rho|\rho(\varepsilon) \le 2\mathcal{H}_{1,\rho}(\varepsilon)$. Thus we have

$$b_1(\Gamma_{\mathfrak{p}}\backslash\mathbb{H}^3) \ge \sum_\rho |\mathfrak{N}_\rho|\rho(\varepsilon).$$

Computing the right hand side by the results in 6.5 and 7.5, we have the first inequality in the statement. Q.E.D.

This estimates lead the following problems.

Problem 1. *Can one find the exponent α such that $b_1(\Gamma_{\mathfrak{p}}\backslash\mathbb{H}^3) \sim q^\alpha$? If such an exponent exists for all arithmetic Kleinian group, is it constant for all arithmetic Kleinian groups or for all lattices in the commensurable class? Does the value,*

$$\lim_{q\to\infty} \frac{b_1(\Gamma_{\mathfrak{p}}\backslash\mathbb{H}^3)}{q^\alpha}$$

(possibly lim sup *or* lim inf*) play some role in geometry? (e.g. verify some estimate in terms of the volume?)*

Problem 2. *Define a natural zeta function or other arithmetic functions associated to $b_1(\Gamma_{\mathfrak{p}}\backslash\mathbb{H}^3)$ (for general arithmetic Kleinian group) with a help of the growth estimate to give a good invariant of the lattice in the commensurable class.*

(3) *Computation of $\chi_{\bar{\rho}}(\theta_8 r_1)$* The significance of the value and the difficulty of its computation was mentioned in the remark following Theorem 7.5.1. This leads the following purely algebraic problem.

Problem 3. *Suppose that F is the canonical Frobenius morphism of $G := SL_2(\mathbb{F}_{q^2})$ and ρ is an F-invariant irreducible representation of G. Let $\bar{\rho}$ be a irreducible representation of $G \rtimes \langle F\rangle$ which restricts to ρ. Find an effective formula to compute $\chi_{\bar{\rho}}(gF)$ when $g\ {}^F g$ is not central.*

9.2 *Local problems*

Lemma 1. *Let Γ be a cocompact Kleinian group, r an automorphism of Γ of order two and Γ_0 a maximal r-normal subgroup of finite index in Γ. Then $G := \Gamma/\Gamma_0$ verifies one of the followings*
(a) G is a simple group.
(b) G is the direct product of two copies of a simple group and r acts on G by the permutation of the factors of the direct product.

Proof. Suppose that G is not simple. Let N_1 be a maximal normal subgroup of G. Set $N_2 := {}^r N_1$. Then $N_1 \cap N_2 = \{\varepsilon\}$ and $G = N_1N_2$ since Γ_0 is maximal r-normal and N_1 is maximal normal. Hence the following exact sequence splits.

$$1 \longrightarrow N_1 \longrightarrow G \longrightarrow N_2 \longrightarrow 1.$$

Since any normal subgroup N of N_2 lifts to normal subgroup NN_1 of G and N_1 is maximal, $N_2 \simeq N_1$ is a simple group. Then validity of Schreier's conjecture (see [Gor] Theorem 1.46) and the Eilenberg-MacLane theory of the extension of groups (see e.g. [Mac] Theorem 8.8 in Chapter IX) show that $G \simeq N_1 \times N_1$ provided N_1 is non-abelian. If $N_1 \simeq \mathbb{Z}_p$ for rational prime p, it is obvious that $N_2 \to \mathrm{Aut}(N_1) \simeq \mathbb{Z}_{p-1}$ is trivial and the lemma follows directly. Q.E.D.

In view of Lemma 1 our computations in Section 6 and Section 7 turn out to be a treatment of typical cases of two types of maximal r-normal subgroup of $B_{4,4,4}$ up to the central extension and $\mathbb{Z}_2$-extention by the spinorial norm. In fact the method in Section 6 readily deduces the following general statement for $r_1r_2r_3$-simple quotient of type (b) in Lemma 1.

Theorem 1. *Let Γ_0 be a maximal $r_1r_2r_3$-normal, but not maximal normal subgroup of finite index in $B_{4,4,4}$ and $G := B_{4,4,4}/\Gamma_0$. Then all the nontrivial $r_1r_2r_3$-invariant irreducible representation are irreducible components of G-module $H_1(\Gamma_0\backslash\mathbb{H}^3,\ \mathbb{C})$.*

Proof. By assunption G is of type (b) in Lemma 1 with $r = r_1r_2r_3$. We may assume that $G = K \times K$ and r acts on G by $G \ni (k_1,\ k_2) \mapsto (k_2,\ k_1) \in G$. Then we can trace the arguments in Section 6 to obtain the followings,
(1) It is immediate that any r-invariant irreducible representation of G is of form $\rho_1 \times \rho_1$ for some $\rho_1 \in \mathrm{Irr}(K)$.
(2) If $gr \in G \rtimes \langle r\rangle$ is of order two, g is of form $(k,\ k^{-1}) \in K \times K = G$. Hence in particular such gr's are all conjugate.
(3) The computations in 6.4 shows that $\bar{\rho}(r) = \bar{\rho}_1(\varepsilon)$ if $\rho = \rho_1 \times \rho_1$.
Then the theorem follows from (1), (2), (3) and Theorem 4.4.2. Q.E.D.

We use the term "local" for the argument of (r-)simple quotient by analogy to the format of adelic field theory. (more precisely, the investigation of r-simple quotient may be comparable to the local theory of classical quadratic extension.) Here we shall summerize the basic problems of the local theory of quotients of $B_{4,4,4}$. The first two problems are purely group theoretic and the last one would bind these abstract investigation to our object.

Problem 4. *According to the table of simple groups find all pair $(K,\ L_1,\ L_2)$ of simple group K and its subgroups L_1 and L_2 with*

$$|K||L_1 \cap L_2| = |L_1||L_2|.$$

By Theorem 1 the double coset argument in Section 8 is applicable to all r-simple quotients of type (b) of $B_{4,4,4}$. Hence the list of $(K,\ L_1,\ L_2)$ of such a property and Proposition in 1.1 would give us a quite precise informations of the first Betti number of orbifolds covered by the regular covering.

Problem 5. *(i) For a given pair $(G,\ r)$ of simple group G and its automorphism r of order 2 develop an effective method to determine irreducible representation of $G \rtimes \langle r\rangle$ which verifies the criterion in Theorem 4.4.1 or Theorem 4.4.2.*
(ii) The Steinberg character in the proof of Proposition 1 plays a special role. In which case such an irreducible representation exists?

Problem 6. *Let $r = r_1$ or $r_1r_2r_3$.*
(i) For which simple group K does there exist a r-simple quotient of type (b) with $G = K \times K$?
(ii) For which pair (G, r) of simple group G and its automorphism r of order two does there exist a r-simple quotient of type (a)?
(iii) For the proof of Theorem 6.5.1, Theorem 7.5.1 and Theorem 1 the action of

$r_1r_2r_3$ plays the main role and the action of r_1 usually gives a less information. Is it always the case for general quotients or what makes this difference?

9.3 Global theory It is appropriate to adopt the term "global" for arguments of general quotients of Kleinian groups, if we use the term "local" for simple quotients. However it is almost obvious that we are far from the position to give a substantial comments on the global aspects. Therefore we shall only illustrates the relationship to our motivation, i.e. the Thurston's problem quoted in 1.1.
For our motivation the global argument is necessary in two aspects. The first one is the investigation of the image of T_Γ in the quotient. In view of Proposition in 1.1 our motivation requires to find a quotient so that T_Γ projects to a "small" subgroup in the quotient. This is the same kind of the problem as problem 15 of the problem list in the last section of [Th]. There are still a lot to be done since almost nothing is known about the property of T_Γ, in particular the one which can be deduced from the existence of aspherical cocompact quotient of noncompact (possibly quite wild) hyperbolic orbifold $T_\Gamma\backslash\mathbb{H}^3$. (or from other homotopical properties which is required for the existence of the covering of positive first Betti number.)
The second aspect is the investigation of irreducible components of $H_1(\Gamma_0\backslash\mathbb{H}^3,\ \mathbb{C})$. In view of Propositon in 1.1 the requirement in this aspect is to obtain "sufficiently many" distinct irreducible components as we pass to the deeper quotients. In the current paper we developed a method to find only r-invariant irreducible component. I have no idea if it is enough for the purpose or not. Possibly more topological informations (e.g. intersection of the homology) or rationality of the homology would give us more irreducible component other than r-invariant ones.
In the whole the roles played by these two aspects are complemetary for our purpose, i.e. if we can prove a strong (resp. only week) result in one aspect, the requirement for the other decreases (resp. increases).

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