Periodic "weighted" operators

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Abstract. Define the periodic weighted operator $Ty = -\rho^{-2}(\rho^2 y')'$ in $L^2(\mathbb{R}, \rho(x)^2dx)$. Suppose a function $\rho \in W_1^2(\mathbb{R}/\mathbb{Z})$ is 1-periodic real positive and let $q = \rho'/\rho \in L^2(0,1)$. The spectrum of T consists of intervals $\sigma_n = [\lambda_{n-1}^-, \lambda_n^+]$ and $\lambda_0^+ = 0$, separated by gaps $\gamma_n = (\lambda_n^-, \lambda_n^+), n \geqslant 1$, with the lengths $|\gamma_n|$. Let h_n be a height of the corresponding slit in the quasimomentum domain and let $g_n, n \geqslant 1$, be the gap with the length $|g_n|$ of the operator $\sqrt{T} \geqslant 0$. Introduce the sequences $\gamma = \{|\gamma_n|\}, h = \{h_n\}, g = \{|g_n|\}$, and the norms $\|f\|_{(m)}^2 = \sum_{n\geqslant 1}(2\pi n)^{2m} f_n^2, m \in \mathbb{Z}$. The following results are obtained: i) the quasimomentum k for the weighted operator T is constructed and the needed properties of k have been studied, ii) double-side estimates of $\|\gamma\|_{(-1)}, \|h\|, \|g\|$ in terms of $\|q\|^2 = \int_0^1 q(x)^2dx$, iii) the asymptotics of gap length $|\gamma_n|$ as $n \to \infty$, iv) let the function $q \in C^1(0,1)$ and $q'(x) < 0, x \in (0,1)$. Then each gap of T is nondegenerate. The proofs are based on the analysis of the quasimomentum as the conformal mapping, the embedding theorems and the identities between the quasimomentum and the potential. In order to prove these results the asymptotics of the fundamental solutions and the Lyapunov function are found at high energy .

1 Introduction and main results

Consider problems arising when one-dimensional waves propagate in periodic homogeneous media. The mathematical model in this case is the wave equation $(\tau u_y')_y' = \nu u_{tt}$, where $\nu(y)$ is density, $u(y)$ is the shear displacement and $\tau(y)$ is the shear modulus. We have the periodic weighted operator $T_{\nu,\tau}$ which is defined by the following formula $T_{\nu,\tau} = -\frac{1}{\nu}\frac{d}{dx}\tau\frac{d}{dx}$ in $L^2(\mathbb{R}, \nu(x)dx)$, with the scalar product $(\cdot,\cdot)_\nu$, where a 1-periodic real functions $\nu, \tau, 1/\nu, 1/\tau \in L^\infty(0,1)$. Without loss generality, assume $\int_0^1 \sqrt{\nu(x)/\tau(x)}dx = 1$ and $\int_0^1 1/\tau(x)dx = 1$. It is well known [L], [Kr] that the spectrum of $T_{\nu,\tau}$ is absolutely continuous and consists of intervals $\sigma_n = [\lambda_{n-1}^+, \lambda_n^-]$, where $\lambda_{n-1}^+ \leqslant \lambda_n^- \leqslant \lambda_n^+, n \geqslant 1$. These intervals are separated by the gaps $\gamma_n = (\lambda_n^-, \lambda_n^+)$, with the length $|\gamma_n| \geqslant 0$. If a gap γ_n is degenerate, i.e. $|\gamma_n| = 0$, then the corresponding segments σ_n, σ_{n+1} merge.

Using some unitary transformations the operator $T_{\nu,\tau}$ can be rewritten in various forms. For example, the operator $T_{\nu,\tau}$ can be transformed, even with nonsmooth coefficients, by

$$x = x(y) = \int_0^y (1/\tau(y))dy, \qquad b(x) = (\nu(y(x))\tau(y(x)))^{1/2} > 0, \tag{1.1}$$

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where $y(x)$ is the inverse function for $x(y)$. Then we obtain the following periodic weighted operator

$$T_b f = -b(x)^{-2} f'', \quad in \quad L^2(\mathbb{R}, b(x)^2 dx),$$

where b is a 1-periodic real positive function, such that $b, 1/b \in L^\infty(0,1)$ and $\int_0^1 b(x)dx = 1$. Let us recall the result of Atkinson and Mingarelli [AM] that if $b \in L^1(0,1)$, then

$$\sqrt{\lambda_n^\pm} = (\pi n)(1 + o(1)), \qquad \text{as} \quad n \to \infty, \tag{1.2}$$

and these asymptotics give the rough estimates of gap lengths. Such operator arised in the following situations. First, let $c = 1/b$, where c is the velocity in the periodic media, then we have the following operator

$$T_c y = -c^2(x) y''(x), \qquad \text{in} \qquad L^2(\mathbb{R}, c^{-2}(x)dx).$$

Second, the operator T_b is also studied in the photonic crystal problem, where $\varepsilon(x) = b(x)^2$ is the dielectric coefficient, periodic in x (see [N]). Third, (see [K6]), consider a hydrogen atom in an external, homogeneous, time-periodic magnetic field $B(t) = z2b(t)(0,0,1) \in \mathbb{R}^3$ with a vector potential $A(t,r) = [B(t), r]/2$ where $r \in \mathbb{R}^3$ is the radius vector of the electron and z is the parameter. In this case the energy operator has the form

$$\mathcal{H}(t) = (-i\nabla - A(t,r))^2/2 - \frac{1}{|r|}, \quad \text{in} \quad L^2(\mathbb{R}^3).$$

For scattering in $L^2(\mathbb{R}^3)$ it is important the spectral properties of the weighted operator $T_b f = -b(t)^{-2} f''$ in $L^2(\mathbb{R}, b(t)^2 dt)$.

Introduce fundamental solutions $\widetilde{\varphi}(x,\lambda,b), \widetilde{\vartheta}(x,\lambda,b)$ of the following equation

$$-y'' = \lambda b^2 y, \quad \lambda = z^2, \; z \in \mathbb{C}, \tag{1.3}$$

with the conditions $\widetilde{\varphi}(0,z,b) = \widetilde{\vartheta}'(0,z,b) = 0, \;\; \widetilde{\varphi}'(0,z,b) = \widetilde{\vartheta}(0,z,b) = 1$. The Lyapunov function is defined by $\widetilde{\Delta}(\lambda,b) = \frac{1}{2}(\widetilde{\varphi}'(1,\lambda,b) + \widetilde{\vartheta}(1,\lambda,b))$. Krein proved the following results about the Lyapunov function $\widetilde{\Delta}(z,b)$ for $b \in L^2(0,1)$, (see [Kr]):

i) the zeroes of the entire function $\widetilde{\Delta}(\lambda,b)$ are real and simple,

ii) the zeroes of $\widetilde{\Delta}(\lambda,b) \pm 1$ are real,

iii) if $\widetilde{\Delta}'(\lambda_1,b) = 0$ for some λ_1 then $|\widetilde{\Delta}(\lambda_1,b)| \geqslant 1$ and $\widetilde{\Delta}(\lambda_1,b)\widetilde{\Delta}''(\lambda_1,b) < 0$.

The sequence $\lambda_0^+ < \lambda_1^- \leqslant \lambda_1^+ < \dots$ is the spectrum of Eq. (1.3) with the periodic boundary conditions of period 2, i.e. $f(x+2) = f(x), x \in \mathbb{R}$. Here the equality means that $\lambda_n^- = \lambda_n^+$ is the double eigenvalue. We note that $\widetilde{\Delta}(\lambda_n^\pm, b) = (-1)^n, \; n \geqslant 1$. The lowest eigenvalue $\lambda_0^+ = 0$ is simple, $\widetilde{\Delta}(0,b) = 1$, and the corresponding eigenfunction equals 1. The eigenfunctions corresponding to $\lambda_n^\pm$ have period 1 when n is even and they are antiperiodic, $f(x+1) = -f(x), \; x \in \mathbb{R}$, when n is odd. For each $n \geqslant 1$ there exists the unique point $\lambda_n \in [\lambda_n^-, \lambda_n^+]$ such that $\widetilde{\Delta}_\lambda(\lambda_n, b) = 0$. With the Lyapunov function $\widetilde{\Delta}(\lambda,b)$ we associate a quasimomentum domain $K = K(h) = \mathbb{C} \setminus \cup\Gamma_n$, where $\Gamma_n = (\pi n + ih_n, \pi n - ih_n \,)$ is an excised slit with the height $h_n = h_{-n} \geqslant 0, \; n \geqslant 1$, and $h_0 = 0$. The value h_n is defines by the equation $\cos h_n = (-1)^n \widetilde{\Delta}(\lambda_n, b)$ and let $h = \{h_n\}$. Below we need the spectral domain $\mathcal{Z} = \mathbb{C} \setminus \cup g_n$ where $g_n = (z_n^-, z_n^+) = -g_{-n}, z_n^\pm = \sqrt{\lambda_n^\pm} > 0, n \geqslant 1$, is a horizontal slit with the length $|g_n| \geqslant 0$ and $|g_0| = 0$. Define $z_n = \sqrt{\lambda_n} > 0$. Assume that there exists a conformal mapping $z : K \to \mathcal{Z}$ and let $\lambda(k) = z(k)^2$ and $k(z)$ is the inverse function. With an endpoint of the gap $\gamma_n \neq \emptyset$, we associate the effective mass $M_n^\pm = 1/\lambda''(\pi n \pm 0), n \geqslant 1$, and $M_0 = 1/\lambda''(0)$. In this case we have $\pm M_m^\pm > 0$ and $M_0 > 0$ (see [KK2]).

Using another unitary mapping the operator $T_{\nu,\tau}$ can be transformed, even with nonsmooth coefficients, by

$$x = \int_0^y (\nu(y)/\tau(y))^{1/2} dy, \qquad \rho(x) = (\nu(y(x))\tau(y(x)))^{1/4} > 0. \tag{1.4}$$

Then we obtain the following periodic weighted operator

$$T_\rho f = -\rho^{-2}(\rho^2 f')', \quad in \quad L^2(\mathbb{R}, \rho(x)^2 dx), \tag{1.5}$$

where ρ is a 1-periodic real positive function $\rho, 1/\rho \in L^\infty(0,1)$ and $\int_0^1 \rho^{-2}(x)dx = 1$.

For integer $m \geqslant 0$, $W_m^2(0,1)$ denotes the Sobolev space of all complex valued functions on $[0,1]$ which have m derivatives in $L^2(0,1)$ and so on. Assume that ρ belongs to the Sobolev space $W_1^2(\mathbb{R}/\mathbb{Z})$, then the operator T_ρ has the following form

$$T_\rho f = -f'' - 2qf', \quad in \quad L^2(\mathbb{R}, \rho(x)^2 dx), \quad \text{where} \quad q \equiv \frac{\rho'}{\rho} \in L^2(\mathbb{R}/\mathbb{Z}).$$

Below we use another assumption $\rho(0) = 1$. Introduce solutions $\varphi(x,z,q), \vartheta(x,z,q)$ of the following equation

$$-y'' - 2qy' = z^2 y, \quad z \in \mathbb{C}, \tag{1.6}$$

with the conditions $\varphi(0,z,q) = \vartheta'(0,z,q) = 0, \quad \varphi'(0,z,q) = \vartheta(0,z,q) = 1$. The Lyapunov function for T_ρ is defined by

$$\Delta(z,q) = \frac{1}{2}(\varphi'(1,z,q) + \vartheta(1,z,q)).$$

The goal of the present paper is to estimate the norm $\|q\|$ in term of various parameters and is motivated by the papers [CMK], [K2], [K5-6]. These estimates will be used to solve the inverse problem for T_ρ [K5]. Below, we find the asymptotics of gap lengths in terms of Fourier coefficients of the function q:

$$\hat{q}_n \equiv \int_0^1 q(t) e^{-i2\pi nt} dt, \ n \in \mathbb{Z}.$$

We introduce the real Hilbert spaces $\ell^p = \{f = \{f_n\}_1^\infty$, with the norm $\|f\|_p^p = \sum_{n>0} f_n^p < \infty\}, p \geqslant 1$, and the real Hilbert spaces H of functions with the norm $\|q\|^2 = \int_0^1 q(x)^2 dx$ and with the following condition $\int_0^1 q(x)dx = 0$. Define the ball $B(p,t) = \{q : \|q - p\|_C < t\} \subset H$, for some $t > 0, \ p \in H$, and introduce the constant ε_q by the formula $\varepsilon_q = 8^{-2} e^{-4\|q\|}$. We formulate the first main result.

Theorem 1.1. *i) Let $b \in L^2(0,1)$. Then there exists a unique conformal mapping (the quasimomentum) $k : \mathcal{Z} \to K(h)$ such that $\cos k(z) = \widetilde{\Delta}(z^2, b)$ and $k(0) = 0, k(iy) = \alpha iy(1 + o(1))$ as $y \to \infty$ for some $\alpha > 0$.*
ii) Let, in addition, $q \in H$. Then $k(\cdot)$ is such that the following relations are fulfilled

$$k(z) = z - \frac{Q_0 + o(1)}{z}, \qquad \text{as} \qquad z \to i\infty. \tag{1.7}$$

$$Q_0 \equiv \frac{1}{\pi}\int v(t)dt = \frac{\|q\|^2}{2} = \frac{1}{2\pi}\int\int |z'(k) - 1|^2 dvdu, \qquad k = u + iv, \tag{1.8}$$

$$M_n^{\pm} = -(-1)^n \frac{\Delta'(z_n^{\pm}, q)}{2z_n^{\pm}} = \frac{\pm|\hat{q}_n| + \ell^d(n)}{2\pi n}, \tag{1.9}$$

$$z_n^{\pm} = \pi n \pm |\hat{q}_n| + \ell^d(n), \tag{1.10}$$

$$|\gamma_n| = (2\pi n + \ell^d(n))(2|\hat{q}_n| + \ell^d(n)), \tag{1.11}$$

$$h_n = |\hat{q}_n| + \ell^d(n), \tag{1.12}$$

$$|g_n| = 2|\hat{q}_n| + \ell^d(n), \tag{1.13}$$

for any fixed $d > 1$, as $n \to \infty$, uniformly on $B(p, \varepsilon_p)$ for each fixed $p \in H$.
Remark. i) In analogy to the notation $O(1/n)$ we use the notation $\ell^d(n), d \geqslant 1$. For example, (1.12) means that there exists a sequence $\beta = \{\beta_n(q)\} \in \ell^d$ such that $\|\beta\|_d \leqslant 1$ and $h_n = |\hat{q}_n| + \beta_n O(1), \quad n \to \infty$, uniformly on $B(p, \varepsilon_p)$ for each fixed $p \in H$.

For $q \in H$ we introduce the sequences $\gamma = \{|\gamma_n|\}_1^\infty$, $h = \{h_n\}_1^\infty, g = \{|g_n|\}_1^\infty$. It is well known that if $q \neq 0$ then the length of the spectral band $z_n^- - z_{n-1}^+ < \pi$, i.e. each spectral band shrinks (see e.g. [K2]). Define the band shrinkage $B_n = \pi - (z_n^- - z_{n-1}^+) > 0$ and the sequence $B = \{B_n\}$.. If a function $\rho = 1$ then the corresponding spectral band has the form $\sigma_n^0 = [\pi^2(n-1)^2, \pi^2 n^2], n > 0$, with the band length $|\sigma_n^0| = \pi^2(2n-1)$. It is well known that if $q \neq 0$ then the length of the spectral band $|\sigma_n^0| > |\sigma_n|$, i.e. each spectral band shrinks (see e.g. [K2]). Define the band shrinkage $\beta_n = |\sigma_n^0| - |\sigma_n| > 0$ and the sequence $\beta = \{\beta_n\}$. Introduce the real Hilbert spaces

$$\ell^2_{(m)} = \{f = \{f_n\}_1^\infty, \ \ \|f\|_{(m)} < \infty\}, \ \ \text{with the norm} \ \ \|f\|^2_{(m)} = \sum_{n>0}(2\pi n)^{2m} f_n^2, \ \ m \in \mathbb{Z},$$

and let $\ell_0^2 = \ell^2$. We formulate the second main results concerning estimates.
Theorem 1.2. *For each periodic function $q \in H$ the following estimates are fulfilled:*

$$\|\gamma\|_{(-1)} \leqslant \sqrt{2}\|q\|(1 + \|q\|), \qquad \|q\| \leqslant 8\pi\|\gamma\|_{(-1)}(1 + \|\gamma\|_{(-1)}), \tag{1.14}$$

$$\frac{1}{\sqrt{2}}\|g\| \leqslant \|q\| \leqslant 2\|g\|(1 + \frac{2}{\pi}\|g\|), \tag{1.15}$$

$$\frac{1}{2}\|g\| \leqslant \|h\| \leqslant \pi\|g\|(1 + \frac{4}{\pi^2}\|g\|^2), \tag{1.16}$$

$$\frac{\sqrt{\pi}}{2\sqrt{2}}\|q\| \leqslant \|h\| \leqslant \frac{\pi}{2}\|q\|(1+\frac{\|q\|}{\pi}), \tag{1.17}$$

$$\frac{1}{2}\|\beta\|_{(-1)} \leqslant \|B\| \leqslant 16\min\{\|q\|, \|h\|, \|g\|(1+\|g\|)\}. \tag{1.18}$$

Remark. i) Proving we use well known results from the spectral theory for the Hill operator and reformulate the problem for the differential operator as the problem for the conformal mapping theory. ii) The important point in the proof is the identities (1.8).

Let $\mu_n(c), n \geqslant 1$, be the Dirichlet spectrum of the following equation

$$-y'' = \frac{\lambda}{c(x)^2}y, \tag{1.19}$$

with the boundary condition $y(0) = y(1) = 0$. Let $\nu_n(c), n \geqslant 0$, be the Neumann spectrum of Eq. (1.19) with the boundary condition $y'(0) = y'(1) = 0$. It is well-known that $\mu_n(c), \nu_n(c) \in \gamma_n(c) = [\lambda_n^-, \lambda_n^+], n \geqslant 1$, where $\gamma_n(c)$ is the gap of T_c. Consider the Dirichlet and Neumann eigenvalues and gap lengths. We have the following result.

Theorem 1.3. *Let a 1-periodic piecewise continuous function $c \in C^2(0,1)$ be convex, $c''(x) > 0$ as $x \in (0,1)$. Then each $\mu_n(c) < \nu_n(c), n \geqslant 1$, and $|\gamma_n(c)| > 0$, i.e. any gap of T_c is open.*

The simple example: c is 1-periodic and $c = 1 + (x - x_0)^2, x \in (0,1)$, for some fixed $x_0 \in [0,1]$.

Remark. i) Proving we use well known results from the Sturm theory for Eq. (1.19).

Lyapunov was the first who consider the spectral properties of the weighted periodic operator [L]. He proved that the spectrum of T_b has the band structure and studied the needed properties of the Lyapunov function $\widetilde{\Delta}$. Later Krein [Kr] reproved these results for the more general case including 2×2 systems. Firsova [F] and Marchenko and Ostrovski [MO] introduced simultaneously the global quasimomentum into the spectral theory of the Hill operator. Double-sided estimates for various parameters of the Hill operator (the norm of a periodic potential, effective masses, gap lengths, height of slits $|h_n|$ and so on) were obtained in [KK1-2], [K2-4], [K7] and for the Dirac operator in [KK2], [K2], [K4]. The precise double-sided estimates for gap lengths was found in [K8]. Results of this paper is used to solve the inverse problem for the operator T_ρ, roughly speaking the inverse problem for the mapping $q \to h = \{h_n\}$ (see [K5]). There exists a big difference between the spectral properties of the Hill operator and the weighted periodic operator. The gap length of T_ρ goes to ∞ at high energy and the gap length of the Hill operator tends to 0 at high energy. Roughly speaking, it is clear since the gap length is the Fourier coefficient of potentials as $n \to \infty$. The potential belongs to $L^2(0,1)$ for the Hill operator and "potentials" for the weighted periodic operators (after the transformation , see (1.26)) have the form $q' + q^2$, where $q \in L^2(0,1)$ and then the corresponding Fourier coefficient is increasing, in general.

Below we need some results about the Hill operator $T_H = -\frac{d^2}{dt^2} + V(t)$ acting in the Hilbert space $L^2(\mathbb{R})$ and a 1-periodic real function $V \in L^2(0,1)$. It is well known [Ti] that the spectrum of T_H is absolutely continuous and consists of intervals $\sigma_n = [\lambda_{n-1}^+, \lambda_n^-]$, where $\lambda_{n-1}^+ \leqslant \lambda_n^- \leqslant \lambda_n^+, n \geqslant 1$, and let $\lambda_0^+ = 0$. These intervals are separated by the gaps

$\gamma_n = (\lambda_n^-, \lambda_n^+)$, with the length $|\gamma_n| \geqslant 0$. If a gap γ_n is degenerate, i.e. $|\gamma_n| = 0$, then the corresponding segments σ_n, σ_{n+1} merge. Introduce the fundamental solutions $y_1(x,z), y_2(x,z)$ of the following equation

$$-y'' + V(x)y = z^2 y, \quad z \in \mathbb{C}, \tag{1.20}$$

satisfying $y_2(0,z) = y_1'(0,z) = 0, \quad y_2'(0,z) = y_1(0,z) = 1$. We define the Lyapunov function $D(z,V) = (y_2'(1,z) + y_1(1,z))/2$ and the quasimomentum (see [F, MO]) for the Hill operator

$$k(z) = \arccos D(z,V), \ z \in \mathcal{Z} = \mathbb{C} \setminus \cup g_n, \ g_n = (z_n^-, z_n^+) = -g_n, \ z_n^{\pm} = \sqrt{\lambda_n^{\pm}} > 0, \ n \geqslant 1.$$

The function $k(z)$ is a conformal mapping from $\mathcal{Z}$ onto a quasimomentum domain $K = \mathbb{C} \setminus \cup \Gamma_n$, where $\Gamma_n = (\pi n + ih_n, \pi n - ih_n\ )$ is an excised slit with the height $h_n = h_{-n} \geqslant 0, \ n \in \mathbb{Z}$, and $h_0 = 0$. Note that $k(\cdot)$ maps the slit $g_{\pm n}$ onto the vertical cut $\Gamma_{\pm n}$. Let $z(k)$ be the inverse function for $k(z)$. The quasimomentum has the asymptotics

$$k(z) = z - \frac{Q_0}{z} - \ldots, \qquad \text{as} \quad z \to \infty, \quad \text{where} \quad Q_0 = \frac{1}{2}\int_0^1 V(t)dt > 0. \tag{1.21}$$

Assume that τ, ν have two derivatives. Define the unitary transform $U : L^2(\mathbb{R}, \nu dx) \to L^2(\mathbb{R})$ such that $y(t) = (Uf)(t) = r(x(t))f(x(t))$, where the new variable $t = \int_0^x (\nu/\tau)^{1/2} dx$ and $r = (\nu\tau)^{1/4}$. Then we obtain

$$(T_{\nu,\tau} f, f)_\nu = \int_0^1 \tau(x) f'(x)^2 dx = \int_0^1 [\dot{y}(t)^2 + V(t)y(t)^2]dt = (T_H y, y),$$

where the Hill operator $T_H = -\frac{d^2}{dt^2} + V(t)$ acts in the Hilbert space $L^2(\mathbb{R}, dt)$ and a 1-periodic real function $V(t)$ has the following form

$$V(t) = \xi(t)^2 + \dot{\xi}(t), \qquad \xi(t) = \frac{r(x(t))'\dot{x}(t)}{r(x(t))}.$$

We consider two cases. First, let $\tau = \nu = \rho^2$, then $t = x, \rho = \xi$ and we obtain the Hill operator $T_H = UT_\rho U^*$ with a potential

$$V(x) = \frac{\rho''(x)}{\rho(x)} = q'(x) + q^2(x), \qquad x = t.$$

Second, let $\tau = 1, \nu = b^2 = 1/c^2, r = \sqrt{b}$, then we obtain the Hill operator $T_H = UT_c U^*$ with a potential

$$V(t) = \frac{c'(x)^2}{4} - \frac{c(x)c''(x)}{2}, \qquad t = \int_0^x b(x)dx = \int_0^x \frac{dx}{c(x)}. \tag{1.22}$$

We write the exact formulas for q and b, c. We have

$$\rho(t)^2 = b(x(t)) = \frac{1}{c(x(t))}, \qquad t = \int_0^x b(x)dx = \int_0^x \frac{dx}{c(x)}, \tag{1.23}$$

hence

$$q(t) = \frac{b'(x(t))}{2b^2(x(t))} = -\frac{c'(x(t))}{2}. \tag{1.24}$$

Using last identities we write the Fourier coefficients and Q_0 in terms of the functions c, b. Identity (1.11) implies

$$\hat{q}_n = -\int_0^1 \frac{c'(x)}{2c(x)} \exp\Big(-i2\pi n \int_0^x \frac{ds}{c(s)}\Big) dx = \int_0^1 \frac{b'(x)}{2b(x)} \exp\Big(-i2\pi n \int_0^x b(y)dy\Big) dx, \tag{1.25}$$

$$2Q_0 = \|q\|^2 = \frac{1}{4}\int_0^1 \frac{c'(x)^2}{c(x)} dx = \frac{1}{4}\int_0^1 \frac{b'(x)^2}{b^3(x)} dx. \tag{1.26}$$

2 Preliminaries

In this section we find the asymptotics estimates which need to construct the asymptotics estimates of the fundamental solutions and the Lyapunov function at large z. For the Wronskian $w(x,q) = \vartheta(x,z,q)\varphi'(x,z,q) - \vartheta'(x,z,q)\varphi(x,z,q)$ of Eq. (1.6) we have the well known formula

$$w(x,q) = \frac{\rho(0)^2}{\rho(x)^2}, \qquad x \in \mathbb{R}, \tag{2.1}$$

and since $w(0,q) = 1$ then we get

$$w(0) = w(1) = 1.$$

Define the function $\Delta_-(z,q) = (1/2)(\varphi'(1,z,q) - \vartheta(1,z,q))$, hence (2.1) yields

$$\Delta_-^2 - (\Delta^2 - 1) = (\Delta_- - \Delta)(\Delta_- + \Delta) + 1 = 1 - \vartheta\varphi' = -\vartheta'\varphi.$$

and hence

$$\Delta_-^2(z,q) - (\Delta^2(z,q) - 1) = -\vartheta'(1,z,q)\varphi(1,z,q). \tag{2.2}$$

Let H_C be the complexification of the real Hilbert space H with the norm $\|\cdot\|_C$ Define the ball $B(p,t) = \{q : \|q-p\|_C < t\} \subset H_C$, for some $t > 0$, $p \in H_C$, Let $C(0,1)$ be the space of continuous functions equipped with norm $\|f\|_c = \sup|f(x)|, x \in [0,1]$. For each function $q \in H_C$ and any $z \in \mathbb{C}$ define the linear operators $a_\pm(z,q), \omega_\pm(z,q)$ acting in $C([0,1])$ by the formulas

$$\omega_\pm(z,q) = \frac{1}{\rho} a_\pm(z,q)\rho, \qquad (a_\pm(z,q)f)(x) = e^{\pm izx}\int_0^x e^{\mp izt} q(t) f(t) dt, \qquad f \in C(0,1).$$

In [PT] the following estimates were proved

$$|(a_\pm(z,q))^n f)(x)| \leqslant \frac{1}{n!}\|q\|_C^n e^{|\operatorname{Im} z|x} \max_{0 \leqslant t \leqslant x} |f(t)|, \quad n \geqslant 1, \tag{2.3}$$

$$|(a_\pm(z,q)a_\mp(z,q))^{(n/2)} f)(x)| \leqslant \frac{1}{n!}\|q\|_C^n e^{n|\operatorname{Im} z|x} \max_{0 \leqslant t \leqslant x} |f(t)|, \quad n \in 2\mathbb{N}, \tag{2.4}$$

where $x \in [0,1]$, $z \in \mathbb{C}$, $q \in H_C$, $f \in C(0,1)$. Using (2.3) we deduce that for each $z \in \mathbb{C}$ and $q \in H_C$ there exists the inverse operator

$$r_\pm(z,q) = (1 + a_\pm(z,q))^{-1} = 1 - a_\pm(z,q) + a_\pm(z,q)^2 - a_\pm(z,q)^3 + \dots,$$

and this series converge uniformly on bounded subsets of $\mathbb{C} \times H_C$. Define the functions $e_\pm = e_\pm(x,z) = \exp(\pm izx)$. We now consider the solutions $f_\pm(x,z,q)$ of the following equation

$$f_\pm(z,q) = e_\pm(z) - (a_+(z,q) + a_-(z,q))f_\pm(z,q), \quad f \in C([0,1]). \tag{2.5}$$

We need

Lemma 2.1. *i) Let $f, f' \in C([0,1])$. Then for each $q \in H_C$ and any $z \in \mathbb{C}$ the following identities are fulfilled*

$$(r_\pm(z,q)e_\pm(z)f)(x) = \frac{e^{\pm izx}}{\rho(x)}(f(0) + \int_0^x \rho(t)f'(t)dt), \tag{2.6}$$

$$r_\pm e_\pm = \frac{e_\pm}{\rho}, \tag{2.7}$$

$$r_\pm a_\pm = \omega_\pm = \rho^{-1} a_\pm \rho, \tag{2.8}$$

$$r_\pm = \rho^{-1}(1 - a_\pm)\rho, \tag{2.9}$$

$$\|(r_\pm(z,q) - 1)f\|_c \leqslant \|f\|_c \|q\|_C e^{|\operatorname{Im} z| + \|q\|_C}, \qquad x \in [0,1], \quad z \in \mathbb{C}. \tag{2.10}$$

ii) Let $f = f_0 - a_\pm f + g$, and $f = f_1 + r_\pm f_0$ for some $f, f_0, g \in C(0,1)$. Then for each $z \in \mathbb{C}, q \in H_C$ the following identity are fulfilled:

$$f_1 = -a_\pm f_1 + g. \tag{2.11}$$

Proof. i) We consider the case " + " , it is the same for " − ". Let $v = e_- r_+(e_+ f)$ then we have

$$r_+(e_+f) = e_+f - a_+(z,q)r_+(e_+f) = e^{izx}(f - \int_0^x q(t)v(t)dt)$$

and

$$v = f(x) - \int_0^x q(t)v(t)dt.$$

Differentiating last identity we obtain $v' = f' - qv$ which implies (2.6), since $v(0) = f(0)$.

Using (2.6) with $f \equiv 1$ we obtain (2.7).

Taking $f = e_\mp a_\pm g$, $g \in C([0,1])$, and using (2.6) we obtain

$$(r_\pm a_\pm g)(x) = \frac{e_\pm}{\rho} \int_0^x e_\mp(t)q(t)g(t)dt,$$

which yields (2.8-9).

Using (2.3) it is easy to get (2.10). Note that, in [PT] there is an analysis of the equation very closed to (2.5). In fact, we repeat the corresponding proof in [PT].

ii) Substituting $f=f_1+r_\pm f_0$ into the equation $f=f_0-a_\pm f+g$ we obtain (2.11) since $r_\mp=1-a_\mp r_\mp$. $\Box$

Define the following mapping $\Omega_\pm$ in the space $C([0,1])$:

$$\Omega_\pm=\rho^{-1}a_\pm a_\mp\rho, \qquad (\Omega_\pm(z,q)f)(x)=\frac{e^{\pm izx}}{\rho(x)}\int_0^x e^{\mp i2zt}q(t)dt\int_0^t e^{\pm izs}\rho(s)q(s)f(s)ds.$$

Using (2.4) we deduce that for each $z\in\mathbb{C}$ and $q\in H_C$ there exists the inverse operator

$$R_\pm(z,q)=(I-\Omega_\pm(z,q))^{-1}=I+\Omega_\pm(z,q)+\Omega_\pm(z,q)^2+\Omega_\pm(z,q)^3+\dots, \tag{2.12}$$

and this series converge uniformly on bounded subsets of $\mathbb{C}\times H_C$.

The main goal of this Section is to find the asymptotics of $f_\pm$ as $|z|\to\infty$.

Lemma 2.2. *For each $z\in\mathbb{C}, q\in H_C$, the following estimates and identities are fulfilled*

$$\|(R_\pm(z,q)-I)f\|_c\leqslant\|q\|_C\|f\|_c e^{|\operatorname{Im} z|+2\|q\|_C}, \quad f\in C(0,1), \tag{2.13}$$

$$r_\mp R_\pm=\frac{1}{\rho}(I-a_-+a_+a_--a_-a_+a_-...)\rho, \tag{2.14}$$

$$\|(r_\mp(z,q)R_\pm(z,q)-I)f\|_c\leqslant\|q\|_C\|f\|_c e^{|\operatorname{Im} z|+2\|q\|_C}, \quad f\in C(0,1), \tag{2.15}$$

$$(I+a_-+a_+)r_-R_+r_+=I. \tag{2.16}$$

and series (2.14) converge uniformly on bounded subsets of $\mathbb{C}\times H_C$.

Proof. Using (2.4) it is easy to get (2.13). Using (2.9), (2.12) we deduce that

$$r_\mp R_\pm=\rho^{-1}(I-a_-)\rho(1+(\omega_+\omega_-)+(\omega_+\omega_-)^2...)=\rho^{-1}(I-a_-)(1+(a_+a_-)+(a_+a_-)^2...)\rho,$$

and multiplication gives (2.14). Moreover, by (2.4) these series converge uniformly on bounded subsets of $\mathbb{C}\times H_C$.

Using (2.14) and (2.3-4) we obtain (2.15).

In order to show (2.16), we rewrite the left side (2.16) in the form

$$(I+a_-+a_+)r_-R_+r_+=(I+a_+r_-)R_+r_+$$

and

$$(I+a_+r_-)r_+=r_++a_+r_-r_+=r_++a_+r_+-a_+a_-r_-r_+=I-w, \quad w=a_+a_-r_-r_+, \tag{2.17}$$

Moreover,

$$(I+a_+)(I-\Omega_+)=(I+a_+)-(I+a_+)\Omega_+=(I+a_+)-a_+r_-a_-=(I-w)(I+a_+), \tag{2.18}$$

Hence (2.18) implies

$$R_+r_+=(I-\Omega_+)^{-1}(I+a_+)^{-1}=[(I+a_+)(I-\Omega_+)]^{-1}=r_+(I-w)^{-1}. \tag{2.19}$$

Then (2.17), (2.19) yield

$$(I+a_-+a_+)r_-R_+r_+=(I+a_-+a_+)r_-r_+(I-w)^{-1}=(I-w)(I-w)^{-1}=I.\square$$

Below we need the following identities

$$\omega_-(z,-q)f=-\rho e_-\int_0^x e_+\frac{q}{\rho}f\,dx=-\rho a_-(f/\rho) \tag{2.20}$$

$$\omega_+(z,-q)f=-\rho e_+\int_0^x e_-\frac{q}{\rho}f\,dx=-\rho a_+(f/\rho) \tag{2.21}$$

$$\Omega_+(z,-q)f=\rho e_+\int_0^x e_-^2 q\,dx\int_0^x e_+\frac{q}{\rho}f\,dx \tag{2.22}$$

Now we will prove the basic results about the functions $f_\pm$.

Lemma 2.3. *i) For each $q\in H_C$ and $z\in\mathbb{C}$ there exists a unique solution $f_\pm$ of Eq. (2.5) which has the form*

$$f_\pm=r_\mp R_\pm r_\pm e_\pm=r_\mp R_\pm\frac{e_\pm}{\rho}=\frac{1}{\rho}(I-a_\mp+a_\pm a_\mp-a_\mp a_\pm a_\mp+....)e_\pm, \tag{2.23}$$

and series converge uniformly on bounded subsets of $[0,1]\times\mathbb{C}\times H_C$.
ii) For each $x\in[0,1]$ the functions $f_\pm(x,z,q)$ are entire on $\mathbb{C}\times H_C$. Moreover, $f_\pm$ are analytic as a map from $\mathbb{C}\times H_C$ into $W^2_{1C}(0,1)$ and the following estimates are fulfilled

$$|f_\pm(x,z,q)|\leqslant e^{|\operatorname{Im} z|x+2\|q\|_C}. \tag{2.24}$$

If the sequence q^ν converges weakly to q in H_C, as $\nu\to\infty$, then $f_\pm(x,z,q^\nu)\to f_\pm(x,z,q)$ uniformly on bounded subsets of $[0,1]\times\mathbb{C}$.
iii) Moreover, for any fixed $d>1/2$ the following asymptotic estimates are fulfilled

$$f_\pm=\frac{1}{\rho}(I-a_\mp+a_\pm a_\mp-a_\mp a_\pm a_\mp)e_\pm+[\ell^d(n)]^4, \tag{2.25}$$

where

$$(e_\mp(z)a_\mp(z)e_\pm(z))(x)=\int_0^x e^{\pm i2zt}q(t)dt=\ell^d(n), \tag{2.26}$$

$$(e_\mp(z)a_\pm(z)a_\mp(z)e_\pm(z))(x)=\int_0^x e^{\mp i2zt}q(t)dt\int_0^t e^{\pm i2zt_1}q(t_1)dt_1=(\ell^d(n))^2, \tag{2.27}$$

$$(e_\mp(z)a_\mp(z)a_\pm(z)a_\mp(z)e_\pm(z))(x)=\int_0^x dt\int_0^t dt_1\int_0^{t_1} e^{\pm i2z(t-t_1+t_2)}q(t)q(t_1)q(t_2)dt_2=(\ell^d(n))^3, \tag{2.28}$$

as $n\to\infty$, uniformly on $[0,1]\times\{|z-\pi n|\leqslant\pi\}\times H_C$.
Remark that $[\ell^d(n)]^2$ in (2.27) means that there exist two sequences $\beta=\{\beta_n(q)\},\alpha=\{\alpha_n(q)\}\in\ell^2$ such that $\|\beta\|_d\leqslant 1,\|\alpha\|_d\leqslant 1$, and

$$\int_0^x e^{\mp i2zt}q(t)dt\int_0^t e^{\pm i2zt_1}q(t_1)dt_1=[\ell^d(n)]^2=\alpha_n(q)\beta_n(q)O(1),\qquad n\to\infty,$$

uniformly on bounded subsets of $[0,1] \times \{|z-\pi n| \leqslant \pi\} \times H_C$.
Proof. i-ii) In [PT] there is an analysis of the equation very closed to (2.5). Really repeating the corresponding proof in [PT], we have the statements of i-ii).
iii) (2.26-28) are found in Lemmas 7.1-2. Using (2.23) we obtain

$$f_\pm = \frac{1}{\rho}(I - a_\mp + a_\pm a_\mp - a_\mp a_\pm a_\mp)e_\pm + r_-(z)R_+(z)f_\pm^0(z), \quad f_\pm^0 = [a_\pm a_\mp)]^2 e_\pm$$

and the estimate (2.15) yields

$$\|r_-(z)R_+(z)f_\pm^0(z)\|_c \leqslant e^{|\operatorname{Im} z|+2\|q\|_C}\|q\|_C\|f_\pm^0(z)\|_c.$$

Then last estimates together with Lemmas 7.1-2 gives (2.25). $\square$

3 Fundamental solutions

In this section some useful results concerning the fundamental solutions and the Lyapunov function for Eq. (1.3) will be presented. First, we describe the needed properties of the fundamental solutions for Eq.(1.6) which satisfy the following equations

$$\varphi(x,z,q) = \frac{\sin zx}{z} - \int_0^x \frac{\sin z(x-t)}{z} 2q(t)\varphi'(t,z,q)dt, \tag{3.1}$$

$$\vartheta(x,z,q) = \cos zx - \int_0^x \frac{\sin z(x-t)}{z} 2q(t)\vartheta'(t,z,q)dt \tag{3.2}$$

where $x \in [0,1], z \in \mathbb{C}, q \in H_C$, and then the equations for $\varphi'(x,z,q), \vartheta'(x,z,q)$:

$$\varphi'(x,z,q) = \cos zx - \int_0^x \cos z(x-t) 2q(t)\varphi'(t,z,q)dt, \tag{3.3}$$

$$\vartheta'(x,z,q) = -z\sin zx - \int_0^x \cos z(x-t) 2q(t)\vartheta'(t,z,q)dt. \tag{3.4}$$

There are the identities (see [CM])

$$\varphi'(x,z,-q) = \rho(x)^2\vartheta(x,z,q), \qquad \varphi(x,z,-q) = -\frac{\rho(x)^2}{z^2}\vartheta'(x,z,q), \tag{3.5}$$

and hence in order to study φ, ϑ we investigate φ', ϑ'. Define the functions

$$f_+ = \varphi' - i\frac{\vartheta'}{z}, \qquad f_- = \varphi' + i\frac{\vartheta'}{z},$$

and then

$$\varphi' = \frac{1}{2}(f_+ + f_-), \qquad \vartheta' = \frac{z}{2i}(f_- - f_+). \tag{3.6}$$

Using (3.3-4) we have the following equations

$$f_\pm(x,z,q) = e^{\pm izx} - \int_0^x \cos z(x-t) 2q(t) f_\pm(t,z,q)dt \tag{3.7}$$

and we rewrite them in the short form (2.5). This equation was studied in Section 2, and by Lemma 2.3, Eq. (3.7) has the solutions in the form (2.23). Now we formulate some results about properties of the fundamental solutions.

Lemma 3.1. *For each $x \in [0,1]$ the functions $\varphi(x,z,q), \vartheta(x,z,q)$ are entire on $\mathbb{C} \times H_C$. Moreover, φ and ϑ are analytic as a map from $\mathbb{C} \times H_C$ into $W^2_{2C}(0,1)$ and the following estimates are fulfilled*

$$\max\{|z\varphi(x,z,q)|, |\varphi'(x,z,q)|, |\vartheta(x,z,q)|, |z^{-1}\vartheta'(x,z,q)|\} \leqslant e^{|\operatorname{Im} z|x + 2\|q\|_C}. \tag{3.8}$$

If the sequence q_n converges weakly to q in H_C, then $\psi(x,z,q_n) \to \psi(x,z,q)$ uniformly on bounded subsets of $[0,1] \times \mathbb{C}$ where ψ is one from the functions $\varphi, \varphi', \vartheta, \vartheta'$. Moreover, for any fixed $d > 1$

$$\varphi(x,z,q) = \frac{1}{z\rho(x)}(\sin zx + \int_0^x \sin z(2t-x)q(t)dt + \ell^d(n)), \tag{3.9}$$

$$\varphi'(x,z,q) = \frac{1}{\rho(x)}(\cos zx - \int_0^x \cos z(2t-x)q(t)dt + \ell^d(n)), \tag{3.10}$$

as $n \to \infty$, uniformly on bounded subsets of $[0,1] \times \{|z - \pi n| \leqslant \pi\} \times H_C$.

Proof. The proof repeats the corresponding arguments for the functions $f_\pm$ and follows from Lemma 2.3. □

Now we consider some properties of the fundamental solutions $\widetilde{\varphi}, \widetilde{\vartheta}$ of Eq. (1.3). Rewrite Eq. (1.3) in the following form

$$-y'' - (zp)^2 y = z^2 r y, \qquad p = \|b\|_C > 0, \qquad r = b^2 - p^2. \tag{3.11}$$

We have the following equations for $\widetilde{\varphi}, \widetilde{\vartheta}$

$$\widetilde{\varphi}(x,\lambda,b) = \frac{\sin zpx}{zp} - z\int_0^x \frac{\sin zp(x-t)}{p} r(t)\widetilde{\varphi}'(t,\lambda,b)dt, \tag{3.12}$$

$$\widetilde{\vartheta}(x,\lambda,b) = \cos zpx - z\int_0^x \frac{\sin zp(x-t)}{p} r(t)\widetilde{\vartheta}'(t,\lambda,b)dt. \tag{3.13}$$

Define the function

$$A(x,z,b) = e^{|\operatorname{Im} zp|x + \frac{|z|}{p}\int_0^1 |b(t)^2 - p^2|dt}.$$

Now we prove some properties of the fundamental solutions.

Lemma 3.2 *i) For each $x \in [0,1]$ the functions $\widetilde{\varphi}(x,z,b), \widetilde{\vartheta}(x,z,b), \widetilde{\varphi}'(x,z,b), \widetilde{\vartheta}'(x,z,b)$, are entire on $\mathbb{C} \times L^2_C(0,1)$. They are real valued on $\mathbb{R} \times L^2_R(0,1)$. Moreover, the solutions $\widetilde{\varphi}(\cdot,z,b), \widetilde{\vartheta}(\cdot,z,b)$, are analytic as maps from $\mathbb{C} \times L^2_C(0,1)$ into $W^2_{2,C}(0,1)$. If the sequence b_n converges weakly to b in $L^2(0,1)$, then $\psi(x,z,b_n) \to \psi(x,z,q)$ uniformly on bounded subsets of $[0,1] \times \mathbb{C}$ where ψ is one from the functions $\widetilde{\varphi}, \widetilde{\varphi}', \widetilde{\vartheta}, \widetilde{\vartheta}'$.*

ii) For each $x \in [0,1], x \in \mathbb{C}, b \in L^2_C(0,1)$ the following estimates are fulfilled:

$$|zp\widetilde{\varphi}(x,z,b)|, |\widetilde{\varphi}'(x,z,b)|, |\widetilde{\vartheta}(x,z,b)|, |\widetilde{\vartheta}(x,z,b)/z| \leqslant e^{|\operatorname{Im} z|px + 2|z|p}, \tag{3.14}$$

$$|zp\widetilde{\varphi}(x,z,b)|, |\widetilde{\varphi}'(x,z,b)|, |\widetilde{\vartheta}(x,z,b)|, |\widetilde{\vartheta}(x,z,b)/z| \leqslant e^{|\operatorname{Im} z|px+2|z|p}, \tag{3.15}$$

Proof. i) Solutions of Eq. (1.3) have the following form

$$y = C_1 + C_2 x + \lambda \int_0^x (x-t) b^2(t) y(t) dt,$$

where C_1, C_2 are some constants. Using this equation and repeating the arguments for the Hill operator from [PT] we have the statements of i).

i) We show estimates (3.14-15). The proof is given for $\widetilde{\vartheta}$. It is the same for $\widetilde{\varphi}$. Recall the simple estimates

$$|\cos z| \leqslant e^{|\operatorname{Im} z|}, \qquad |\frac{\sin zx}{z}| \leqslant e^{|\operatorname{Im} z|}, \qquad x \in [0,1]. \tag{3.16}$$

Repeating the arguments from [PT] for Eq. (3.13) we obtain the series

$$\widetilde{\vartheta}(x,\lambda,b) = \cos zpx - z \int_0^x \frac{\sin zp(x-t)}{p} r(t) \cos zpt dt$$

$$+z^2 \int_0^x \frac{\sin zp(x-t)}{p} r(t) dt \int_0^t \frac{\sin zp(t-t_1)}{p} r(t_1) \cos zpt_1 dt_1 + \ldots$$

and the estimate

$$|\widetilde{\vartheta}(x,\lambda,b)| \leqslant e^{|\operatorname{Im} z|px} (1 + \frac{|z|}{p} \int_0^x |r(t)| dt + \frac{|z^2}{p^2} \int_0^x |r(t)| dt \int_0^t |r(t_1)| dt_1 + \ldots) \leqslant$$

$$e^{|\operatorname{Im} z|px + \frac{|z|}{p} \int_0^x |r(t)|dt} \leqslant e^{|\operatorname{Im} z|px+2|z|p}$$

since $\int_0^x |r(t)| dt \leqslant 2p^2$. $\square$

We need the following results about the Lyapunov function.

Lemma 3.3. *i) The function $\widetilde{\Delta}(\cdot,\cdot)$ is entire on $\mathbb{C} \times L^2_C(0.1)$. If the sequence $b^\nu, \nu \geqslant 1$, converges weakly to b in $L^2_C(0.1)$ then $\widetilde{\Delta}(\lambda, b^\nu) \to \widetilde{\Delta}(\lambda, b)$ uniformly on bounded subsets of $\mathbb{C}$.*

ii) For each $x \in [0,1], z \in \mathbb{C}, b \in L^2_C(0,1)$ the following estimates are fulfilled:

$$|\widetilde{\Delta}(\lambda,b)| \leqslant e^{|\operatorname{Im} z|px+2|z|p}. \tag{3.17}$$

iii) For each $b \in L^2_R(0,1)$ the zeroes of the equation $\Delta(z,b)^2 - 1 = 0$ are real and they satisfies the asymptotics estimates (1.2).

iv) For each $b \in L^2_R(0,1)$ the zeroes of the entire function $\widetilde{\Delta}(\lambda,b)$ are real and simple,

v) If $\widetilde{\Delta}'(\lambda_0,b) = 0$ for some λ_0 then $|\widetilde{\Delta}(\lambda_0,b)| \geqslant 1$ and $\widetilde{\Delta}(\lambda_0,b)\widetilde{\Delta}''(\lambda_0,b) < 0$.

Proof. Using Lemma 3.2 we obtain the statements of i) and ii).

Other results follow from [Kr] (see Sect. 1) and [AM] (see (1.2)). $\square$

Let $\widetilde{\mu}_n(b), n \geqslant 1$, be the spectrum of (1.3) with boundary condition $y(0) = y(1) = 0$. Let $\widetilde{\nu}_n(b), n \geqslant 1$, be the Neumann spectrum of b, that is the spectrum of (1.3) with boundary condition $y'(0) = y'(1) = 0$. Now we prove some properties of $\widetilde{\mu}_n(b), \widetilde{\nu}_n(b)$ for the problem (1.3).

Lemma 3.4 *i) Let $b \in L^2_R(0,1)$. Then the following identities are fulfilled:*

$$\|\widetilde{\varphi}(\cdot, \mu_n(b), b)\|_b^2 = \widetilde{\varphi}_\lambda(1, \mu_n(b), b)\widetilde{\varphi}'(1, \mu_n(b), b) > 0, \tag{3.18}$$

$$\|\widetilde{\vartheta}(\cdot, \nu_n(b), b)\|_b^2 = \widetilde{\varphi}_\lambda(1, \nu_n(b), b)\widetilde{\varphi}'(1, \nu_n(b), b) > 0, \tag{3.19}$$

$$\nu_0(b) = 0, \qquad \mu_n(b), \nu_n(b) \in [\lambda_n^-, \lambda_n^+], n \geqslant 1. \tag{3.20}$$

Thus, all roots of $\widetilde{\varphi}(1, z, b)$, $\widetilde{\vartheta}(1, z, b)$ are simple.

Proof. The proof is given for $\widetilde{\nu}_n(b)$. It is the same for $\widetilde{\mu}_n(b)$. Let $u = \widetilde{\vartheta}(x, \lambda, b)$. Differentiating Eq. (1.3) with respect z we obtain

$$-u''_\lambda = b^2(u + \lambda u_\lambda).$$

Multiplying last equation by u, Eq. (1.3) by u_λ and taking the difference, we have the identity

$$\lambda 2b^2u^2 = u''u_\lambda - u''_\lambda u = [u_\lambda, u]',$$

which yields

$$\|u\|_b^2 = [u_\lambda, u]|_0^1 = -\widetilde{\vartheta}'_\lambda(1, \lambda, b)\widetilde{\vartheta}(1, \lambda, b),$$

and since $\widetilde{\vartheta}'(1, \lambda, b) = 0$, at $\lambda = \widetilde{\nu}_n(b)$.

Define the function $\widetilde{\Delta}_-(\lambda, b) = (1/2)(\widetilde{\varphi}'(1, \lambda, b) - \widetilde{\vartheta}(1, \lambda, b))$, then the Wronskian equality yields

$$\widetilde{\Delta}_-^2 - (\widetilde{\Delta}^2 - 1) = (\widetilde{\Delta}_- - \widetilde{\Delta})(\widetilde{\Delta}_- + \widetilde{\Delta}) + 1 = 1 - \widetilde{\vartheta}\widetilde{\varphi}' = -\widetilde{\vartheta}'\widetilde{\varphi}.$$

and hence

$$\widetilde{\Delta}_-^2(\lambda, b) - (\widetilde{\Delta}^2(\lambda, b) - 1) = -\widetilde{\vartheta}'(1, \lambda, b)\widetilde{\varphi}(1, \lambda, b). \tag{3.21}$$

Using last identity we deduce that $\widetilde{\vartheta}'(1, \lambda, b)\widetilde{\varphi}(1, \lambda, b) < 0$ inside the spectrum, and zeroes of the functions $\widetilde{\vartheta}'(1, \lambda, b), \widetilde{\varphi}(1, \lambda, b)$ belong to the "closed" gaps $[\lambda_n^-, \lambda_n^+], n \geqslant 1$, or equal 0.

It is clear that $\widetilde{\nu}_0(b) = 0$ only and $\widetilde{\vartheta}(x, 0, b) = 1$.

Consider the gaps $\gamma_n, n \geqslant 1$. First, let $\gamma_n = \emptyset$. Then $\lambda = \lambda_n^- = \lambda_n^+$ is the periodic (anti-periodic) eigenvalue. Moreover, $\widetilde{\vartheta}'(x, \lambda, b), \widetilde{\varphi}(x, \lambda, b)$ are the periodic (anti-periodic) functions. Hence $\lambda_n^\pm$ is the Dirichlet eigenvalue and the Neumann eigenvalue. By (3.19), $\lambda_n^\pm(b)$ is the simple Dirichlet eigenvalue.

Second, consider the case of a gap $\gamma = (\lambda^-, \lambda^+) \neq \emptyset$. Let some Dirichlet eigenvalue μ belong to this gap γ. Assume that $\widetilde{\Delta}(\lambda, b) > 1, \lambda \in \gamma$. Then by the Wronskian identity $\widetilde{\vartheta}\widetilde{\varphi}' - \widetilde{\vartheta}'\widetilde{\varphi} = 1$, $\widetilde{\varphi}'(1, \mu, b) > 0$ and $\widetilde{\vartheta}(1, \mu, b) > 0$. Hence (3.19) yields $\widetilde{\varphi}(1, \mu, b) > 0$ and then there exists only one Dirichlet eigenvalue μ which belongs to this gap γ and no others.

We show now that in this "closed" gap $\bar{\gamma} = [\lambda^-, \lambda^+]$ there exists exactly one Dirichlet eigenvalue μ. We use the proof by contradiction. Assume that the function $\widetilde{\varphi}(1, \lambda, b)$ has not zero on the closed gap $[\lambda^-, \lambda^+]$. Define the interval $\gamma_\pm = [\lambda_- \mp t, \lambda_+ \pm t]$ for some $t > 0$. We take small $t > 0$ such that $0 < \widetilde{\Delta}(\lambda_\pm \pm t, b) < 1 - a, \widetilde{\Delta}(\lambda, b) > 1 + a, \lambda \in \gamma_-$, and $|\widetilde{\varphi}(1, \lambda)| > 2a, \lambda \in \gamma_+$ for some $a > 0$.

Let $b_m \to b \geqslant 0$ strongly in $L^2(0,1)$ as $m \to \infty$, where $b_m \in W_2^2(\mathbb{R}/\mathbb{Z})$ and $b_m(x) > 0, x \in [0,1]$ for any $m \geqslant 1$. Then by Lemma 3.2 and 3.3, $\psi(1, \lambda, b_n) \to \psi(1, \lambda, b)$ uniformly on bounded subsets of $\mathbb{C}$ where ψ is one from the functions $\widetilde{\varphi}, \widetilde{\varphi}', \widetilde{\vartheta}, \widetilde{\vartheta}', \widetilde{\Delta}$. Hence, there exists M such that $|\widetilde{\varphi}(1,\lambda,b_m)| > a, \lambda \in \gamma_+$ for any $m > M$. On the other hand there exists a gap $\gamma(b_m) \subset \gamma_+$. Using the unitary transformation (1.23) we obtain the Hill operator with a periodic potential $V_m \in L^2(0,1)$. In this case, it is well known, the function $y_2(1,\lambda,V_m)$ has a zero in the gap $\gamma(b_m)$, and then the function $\widetilde{\varphi}(1,\lambda,b_m)$ has the same zero in this gap $\gamma(b_m)$. This implies contradiction, and function $\widetilde{\varphi}(1,\lambda,b)$ has one zero in the gap γ. $\square$

4 Quasimomentum and asymptotics

We need some results about the Lyapunov function for Eq. (1.6). Define the functions $\Delta_1(z,q), \Delta_2(z,q)$ by the formulas

$$\Delta(z,q) = \cos z + \Delta_1(z,q) + \Delta_2(z,q), \quad z \in \mathbb{C}, q \in H_C \tag{4.1}$$

where

$$\Delta_1(z,q) = \int_0^1 q(t)dt \int_0^t q(s)\cos z(1-2t+2s)ds. \tag{4.2}$$

We need the following results about the Lyapunov function for Eq. (1.6).
Lemma 4.1. *i) If the sequence $q^\nu, \nu \geqslant 1$, converges weakly to q in H, then $\Delta(z,q^\nu) \to \Delta(z,q)$ uniformly on bounded subsets of $\mathbb{C}$. The function $\Delta(\cdot,\cdot)$ is entire on $\mathbb{C} \times H_C$.*
ii) For fixed $z \in \mathbb{C}$ the function $\Delta(z,q)$ is even with respect to $q \in H_C$ and

$$\Delta(z,-q) = \Delta(z,\rho) = \frac{1}{2}[\varphi'(1,z,q,) + \varphi'(1,z,-q)], \qquad q \in H_C, \tag{4.3}$$

iii) We have

$$\Delta_1(\pi n,q) = \frac{(-1)^n}{2} q_n^2, \qquad q \in H_C, \tag{4.4}$$

$$\Delta_2(z,q) = R_+(z,q)\Omega_+^2(z,q)e_+(z)/(2\rho) + R_-(z,q)\Omega_-^2(z,q)e_-(z)/(2\rho), \qquad q \in H_C. \tag{4.5}$$

iv) Moreover, for any fixed $d > 1$ the following asymptotic estimates are fulfilled:

$$\Delta_1(z,q) = \ell^d(n), \quad \Delta_2(z,q) = (\ell^d(n))^2, \text{ as } |z| \to \infty, \ |z - \pi n| \leqslant \pi/2. \tag{4.6}$$

These estimates are infinitely differentiable with respect to z and are satisfied uniformly on bounded subsets of $[0,1] \times H_C$.
Proof. Statements of i) and ii) were proved in [K5].
iii) Identities (4.1) and (3.5) imply

$$\Delta(z,\rho) = \frac{1}{4}(f_+(1,z,q) + f_-(1,z,q) + f_+(1,z,-q) + f_-(1,z,-q)). \tag{4.7}$$

The substitution of (2.23) into (4.7) yields

$$\Delta(z,q) = (1-\Omega_+)^{-1}(e_+/2\rho) + (1-\Omega_-)^{-1}(e_-/2\rho), \tag{4.8}$$

and then we have (4.1), since

$$\Delta_1 = \omega_+\omega_-\frac{e_+}{2\rho} + \omega_-\omega_+\frac{e_-}{2\rho} = \frac{e_+}{2}\int_0^1 e_-^2 q dx \int_0^x e_+^2 q dx + \frac{e_-}{2}\int_0^1 e_+^2 q dx \int_0^x e_-^2 q dx \tag{4.9}$$

and

$$\Delta_2 = (1-\Omega_+)^{-1}\Omega_+^2(e_+/2\rho) + (1-\Omega_-)^{-1}\Omega_-^2(e_-/2\rho), \tag{4.10}$$

and this gives (4.5). We show (4.4). By (4.2),

$$\Delta_1(\pi n, q) = (-1)^n \int_0^1 q(t)dt \int_0^t q(x)[\cos 2\pi nt \cos 2\pi nx + \sin 2\pi nx \sin 2\pi nt]dx,$$

and

$$\Delta_1(\pi n, q) = (-1)^n \int_0^1 [f_s(t)f_s'(t) + f_c(t)f_c'(t)]dt,$$

where

$$f_c(t) = \int_0^t q(t)\cos 2\pi nt dt, \qquad f_s(t) = \int_0^t q(t)\sin 2\pi nt dt,$$

and the simple integration yields (4.4).
iv) The identity (4.9) and Lemma 7.1 imply the first asymptotics estimates in (4.6). We have

$$\Omega_\pm^2(e_\pm/\rho) = \rho^{-1}e_\pm \int_0^1 e_\mp^2 q dx \int_0^x e_\pm^2 q dx \int_0^x e_\mp^2 q dx \int_0^x e_\pm^2 q dx. \tag{4.11}$$

The asymptotics estimates of Lemma 7.2 yield

$$\Omega_\pm^2(e_\pm/\rho) = (\ell^d(n))^4, \quad n\to\infty, \tag{4.12}$$

and using (2.13) we deduce that

$$R_\pm\Omega_\pm^2(e_\pm/\rho) = (\ell^d(n))^4, \quad n\to\infty, \tag{4.13}$$

which together with (4.5) and the Cauchy formula $f(z) = (2\pi)^{-1}\int_{|t-\pi n|=\pi/2}(t-z)^{-1}f(t)dt$ implies (4.6) uniformly on bounded subsets of $[0,1]\times H_C$.□

We need some results about the convergence of the quasimomentums.

Lemma 4.2. *Let $z(\cdot,h): K(h)\to \mathcal{Z}(g)$ be a conformal mapping for some $h\in\ell^2$. Assume that the sequence of conformal mappings $z(\cdot,h^\nu): K(h^\nu)\to\mathcal{Z}(g^\nu)$ converges to $z(\cdot,h)$ uniformly on compact domains in $\cup K(h^\nu)$, where $h^\nu\to h$ strongly in ℓ^2, as $\nu\to\infty$. Then $Q_0(h^\nu)\to Q_0(h)$, as $\nu\to\infty$.*

Proof. For any $\varepsilon>0$ there exists an integer $N\geqslant 1$ such that $\|h-h^\nu\|\leqslant\varepsilon$ for $\nu\geqslant N$ and $\sum_{n\geqslant N} h_n^2\leqslant\varepsilon^2$. Let η be one from the vectors $h, h^\nu, \nu\geqslant N$. Then we have the identity (see [KK2]) $Q_0(\eta) = I_N(\eta) + R_N(\eta)$, where

$$I_N(\eta) = \frac{1}{2\pi i}\int_{|k|=N(\pi+\frac{1}{2})} z(k,\eta)dk, \quad R_N(\eta) = \frac{1}{\pi i}\sum_{n>N}\int_{|k-\pi n|=2\varepsilon} z(k,\eta)dk.$$

Using the estimate $0 \leqslant z(\pi n + iv + 0, \eta) - z(\pi n + iv - 0, \eta) \leqslant \eta_n$ from [KK2] we obtain

$$0 \leqslant \frac{1}{i} \int_{|k-\pi n|=2\varepsilon} z(k,\eta)dk = 2\int_0^{\eta_n} [z(\pi n + iv + 0, \eta) - z(\pi n + iv - 0, \eta)]dv \leqslant 2\eta_n^2,$$

which yields $R_N(h) \leqslant \varepsilon^2$, $R_N(h^\nu) \leqslant 4\varepsilon^2, \nu > N$, and together with $I_N(h^\nu) \to I_N(h)$ as $\nu \to \infty$ we derive $Q_0(h^\nu) \to Q_0(h)$. $\square$

We now prove the main first result about the weighted periodic operator.

Proof of Theorem 1.1. Using the asymptotic estimates

$$z_n = \pi n + \ell^d(n), \quad n \to \infty, \tag{4.14}$$

from [K5], and (4.6) we have

$$z_n^{\pm} = \pi n + o(1), \quad n \to \infty. \tag{4.15}$$

We improve last result. Let $r_n = z_n^{\pm} - \pi n$. Then using (4.6) we deduce that

$$(-1)^n = (-1)^n \cos r_n + \ell^d(n), \quad n \to \infty,$$

and

$$2\sin^2(r_n/2) = \ell^d(n), \quad n \to \infty,$$

which yields

$$r_n = \ell^{d/2}(n), \quad n \to \infty, \tag{4.16}$$

Using again (4.6), (4.16), (1.9) we get the following asymptotics estimates

$$2\sin^2(r_n/2) = (-1)^n \Delta_1(\pi n + r_n, q) + (\ell^d(n))^2, \quad n \to \infty,$$

$$(-1)^n \Delta_2(\pi n + r_n, q) = (1/2)|\hat{q}_n|^2 + \Delta_2'(\pi n, q) r_n + (\ell^d(n))^2, \quad n \to \infty, \tag{4.17}$$

and then last relations and (4.16) yield

$$r_n^2 = |\hat{q}_n|^2 + 2x_n r_n + (\ell^d(n))^2, \quad n \to \infty,$$

where $x_n = (-1)^n \Delta_2'(\pi n)$, which implies

$$r_n = x_n \pm \sqrt{|\hat{q}_n|^2 + (\ell^d(n))^2} = \pm|\hat{q}_n| + \ell^d(n), \quad n \to \infty.$$

Using last estimates we have (1.10) and (1.13). Moreover, we have (1.11):

$$|\gamma_n| = (z_n^+ + z_n^-)(z_n^+ - z_n^-) = (2\pi n + \ell_n^d)(2|\hat{q}_n| + \ell_n^d), \quad n \to \infty.$$

We show (1.12). Asymptotics estimates (4.6) implies $h_n \to 0$. We improve last result. Identity (4.1) and (4.6) yield

$$(-1)^n \cosh h_n = \Delta(z_n, q) = \cos z_n + \Delta_1(z_n, q) + (\ell^d(n))^2, \quad n \to \infty.$$

By Lemma 4.1, the following asymptotics estimates hold

$$\Delta_1(z_n, q) = (-1)^n(1/2)|\hat{q}_n|^2 + \Delta_1'(\pi n, q)s_n + (\ell^d(n))^2, \quad n \to \infty, \quad s_n = z_n - \pi n, \quad (4.18)$$

Hence we obtain

$$h_n^2 = |\hat{q}_n|^2 + 2x_n s_n + (\ell^d(n))^2, \quad n \to \infty,$$

which yields (1.12).

We show that there exists the quasimomentum with needed properties. We use the following result from [MO]. Let a function $F(z), z \in \mathbb{C}$, satisfies

Condition A a) F is even entire and $F(0) = 1$,

b) each zero of the equation $F^2(z) = 1$ is real,

c) the number of zeroes of $F^2(z) = 1$ is equal to infinity.

Then there exists exists a unique conformal mapping $k : \mathbb{C}_+ \to K_+(h)$ such that $\cos k(z) = F(z)$ and $k(0) = 0, k(iy) = \alpha iy(1 + (o))$ as $y \to \infty$, for some $\alpha > 0$, where the domain $K_+(h) = \mathbb{C}_+ \setminus \cup G_n$ and the slit $G_n = [\pi n, \pi n + ih_n], n \in \mathbb{Z}$, for some $h = \{h_n\}$. Note that the function $k(\cdot)$ has analytic extension on $\mathbb{C}_-$ by the formula $k(z) = \bar{k}(\bar{z}), z \in \mathbb{C}_-$.

By Krein's results (see Sect.1 and [Kr]), $\widetilde{\Delta}(z^2, b)$ satisfies a) and b) of Condition A. Using (1.2) we deduce that the number of zeroes of $\widetilde{\Delta}^2(z^2, b) = 1$ is equal to infinity. Then by results of Marchenko and Ostrovski [MO], there exists a unique conformal mapping $k : \mathcal{Z} \to K$ such that $\widetilde{\Delta}(z^2, b) = \cos k(z),\ z \in \mathbb{C}$, and $k(0) = 0,\ k(iy) = \alpha iy(1 + o(1))$ as $y \to \infty$ for some $\alpha > 0$.

Assume that $q \in H$. Asymptotic estimate (1.12) yields $h \in \ell^2$. In the paper [KK1] the following results were proved: assume that the quasimomentum $k : \mathcal{Z} \to K$ such that $h \in \ell^2$ then (1.7) is fulfilled and

$$Q_0(h) = \frac{1}{\pi}\int v(t)dt = \frac{1}{2\pi}\int\int |z'(k) - 1|^2 dv du < \infty, \qquad k = u + iv.$$

Hence we have to show $2Q_0(h) = \|q\|^2$. Let $q \in H$ and $\rho(x) = \exp(\int_0^x q(t)dt)$. Then there exists a sequence of the smooth functions $q^\nu \in W_1^2(\mathbb{R}/\mathbb{Z})$ such that $q^\nu \to q$ strongly in H as $\nu \to \infty$. Then for each $\nu > 1$ there exists the smooth potential $V^\nu = (q^\nu)' + (q^\nu)^2$ and $D(z, V^\nu) = \Delta(z, q^\nu)$. Moreover, there exists the quasimomentum $k^\nu : \mathcal{Z} \to K(h^\nu)$ such that $2Q_0(h^\nu) = \|q^\nu\|^2$ and $\|q^\nu\| \to \|q\|$. We need to show $Q_0(h^\nu) \to Q_0(h)$ as $\nu \to \infty$. Indeed, we have $q^\nu \to q$ strongly in H as $\nu \to \infty$, and the author [K5] proved $h^\nu \to h$ strongly in ℓ^2 as $\nu \to \infty$. Moreover, assume that the sequence of conformal mappings $z(\cdot, h^\nu) : K(h^\nu) \to \mathcal{Z}(g^\nu)$ converges to $z(\cdot, h)$ uniformly on compact domains in $\cap K(h^\nu)$, where $h^\nu \to h$ strongly in ℓ^2, as $n \to \infty$. Then by Lemma 4.2, $Q_0(h^\nu) \to Q_0(h)$, as $\nu \to \infty$. We have to show the convergence of the conformal mappings $z(\cdot, h^\nu) : K(h^\nu) \to \mathcal{Z}(g^\nu)$. Indeed, by Lemma 4.1, $\Delta(z, q^\nu) \to \Delta(z, q)$ uniformly on bounded subsets of $\mathbb{C}$. Hence by the Inverse Function Theorem we obtain the convergence of the mappings $z(\cdot, h^n) : K(h^\nu) \to \mathcal{Z}(g^\nu)$ to $z(\cdot, h)$ uniformly on compact domains in $\cap K(h^\nu)$, as $\nu \to \infty$.

We show (1.9). By $\cos k = \Delta(z, q)$ we have $2z \sin k = \lambda'(k)\Delta'(z, q)$; the identity $z'(\pi n) = 0$ implies $-2z_n^\pm \cos \pi n = \lambda''(\pi n \pm 0)\Delta(z_n^\pm, q)$ and $M_n^\pm = -(-1)^n\Delta'(z_n^\pm, q)/2z_n^\pm$. Using (4.6) and ((1.10), we obtain $\Delta'(z_n^\pm, q) = -\sin z_n^\pm + \ell^d(n) = \mp(-1)^n|\hat{q}_n| + \ell^d(n)$ and we get (1.9). $\square$.

5 Estimates

We have proved that the mapping $k : \mathcal{Z} \to K$ is conformal and we now study some properties of this function. Define the value $M = 1 + (2h_+/\pi)$ where $h_+ = \max_{n\geqslant 1} h_n$. In [KK3] (see also [K3]) the following estimate was proved $h_+^2 < I_0$ and then identity (1.8) implies

$$h_+ < \|q\|, \tag{5.1}$$

which yields

$$M < 1 + \frac{2\|q\|}{\pi}. \tag{5.2}$$

We prove the second main result about the estimates.
Proof of Theorem 1.2. In [K2] the author studied the metric properties of the conformal mapping $k : \mathcal{Z} \to K$ with the norming asymptotics (1.7). Our mapping is such conformal mapping from $\mathcal{Z}$ onto K and in [K2] the following estimates were proved:

$$\frac{\pi}{4} I_D \leqslant 2\|h\|^2 \leqslant \frac{\pi^2}{2}\left(1 + \frac{I^{1/2}}{\pi}\right) I_D, \qquad I_D = \frac{1}{\pi}\int\int_{\mathbb{C}} |z'(k) - 1|^2 dv du. \tag{5.3}$$

$$\frac{1}{2}\|g\| \leqslant \|h\| \leqslant \pi\left(1 + \frac{2}{\pi^2}\|g\|^2\right)\|g\|, \tag{5.4}$$

$$\frac{1}{2}\|g\|^2 \leqslant I_D \leqslant 4\|g\|^2\left(1 + \frac{4}{\pi^2}\|g\|^2\right), \tag{5.5}$$

$$\|B\| \leqslant 16 \min\{\sqrt{I_D/2},\ \|h\|,\ \|g\|(1 + \frac{2}{\pi}\|g\|)\}. \tag{5.6}$$

Then inequalities (5.3-6) together with identity (1.8) yield (1.15-17).

We prove(1.14). The relation $|\gamma_n| = (z_n^- + z_n^+)|g_n|$ and the first estimate in (5.5) imply

$$\sum_{n\geqslant 1} \frac{|\gamma_n|^2}{(z_n^+ + z_n^-)^2} \leqslant 2I_D = \|q\|^2, \tag{5.7}$$

and using the following inequality from [K4]

$$z_n^+ \leqslant \pi n M, \quad n \geqslant 1, \tag{5.8}$$

we deduce that

$$\|\gamma\|_{-1}^2 = \sum_{n\geqslant 1} \frac{|\gamma_n|^2}{(2\pi n)^2} \leqslant 2M^2\|q\|^2, \tag{5.9}$$

which together with (5.2) implies the first estimate in (1.14). In order to show the second one we need the following inequality from [K3]

$$I_D \leqslant 4\|h\|\|\gamma\|_{-1}. \tag{5.10}$$

Substituting (1.8) and (5.3) into (5.10) we obtain

$$I_D^2 = \|q\|^4 \leqslant 16\|h\|^2\|\gamma\|_{(-1)}^2 \leqslant 4\pi^2\|q\|^2(1+\frac{\|q\|}{\pi})\|\gamma\|_{(-1)}^2,$$

and then

$$\|q\|^2 \leqslant 4\pi^2(1+\frac{\|q\|}{\pi})\|\gamma\|_{(-1)}^2,$$

which gives (1.14).

We prove the first estimate in (1.18). Due to the definitions of β_n and the fact $\beta_n > 0$ (see [K3]) we deduce that

$$0 < \beta_n = \pi^2(2n-1) - [(z_n^-)^2 - (z_{n-1}^+)^2] = \pi^2(2n-1) - |\sigma_n|(z_n^- + z_{n-1}^+)$$

$$= B_n\pi(2n-1) + |\sigma_n|\omega_n,$$

where $\omega_n \equiv \pi(2n-1) - (z_n^- + z_{n-1}^+)$. Hence using the estimate $|\sigma_n| \leqslant \pi$ from [K2] we obtain

$$\beta_n \leqslant B_n\pi(2n-1) + \pi\omega_n. \tag{5.11}$$

The first term in (5.11) is "good" and we must estimate the second one. Using the identities

$$z_n^- = z_{n-1}^+ + |\sigma_n|, \qquad z_{n-1}^+ = \sum_1^{n-1}(|\sigma_m| + |g_m|),$$

we deduce that

$$\omega_n = B_n + 2\sum_1^{n-1}(B_m - |g_m|). \tag{5.12}$$

Put $y_n = \sum_1^n B_m$. Substituting (5.12) into (5.11) we obtain

$$\beta_n < \pi y_n. \tag{5.13}$$

The Hardy inequality (see [HLP]) yields

$$\|y\|_{(-1)} \leqslant \frac{1}{\pi}\|B\|. \tag{5.14}$$

Then estimates (5.13-14) imply

$$\|\beta\|_{(-1)} \leqslant 2\pi\|y\|_{(-1)} \leqslant 2\|B\|,$$

and the first estimate in (1.18) has been proved. Using (5.6) we have others estimates in (1.18). $\square$

We now prove the result about open gaps and the Neumann and Dirichlet eigenvalues (see [GS] for the Hill operator).

Proof of Theorem 1.3 . Let $\lambda = \nu_n$ and ϑ_n be the Neumann eigenfunction for some $n > 0$. Then the function $u = \vartheta_n'$ satisfies the following equation

$$-c^2u'' - 2cc'u' = \lambda u$$

Defining the new function $v = cu$ we rewrite the last relation in the form

$$-c(cu'' + 2c'u' + c''u) + cc''u = \lambda u$$

and then

$$-v'' = (\frac{\lambda}{c^2} - \frac{c''}{c})v \tag{5.15}$$

The function ϑ_n has n zeroes on the interval $(0,1)$. Then the function ϑ_n' has $m \geqslant n-1$ zeroes on the interval $(0,1)$ and $M \geqslant n+1$ zeroes on the segment $[0,1]$. Let ϕ be the solution of the problem

$$-c^2\phi'' = \lambda\phi, \quad \phi(0) = 0, \ \phi'(1) = 1.$$

Then by the Sturm theorem ϕ has $N \geqslant n$ zeroes on the interval $(0,1)$ and $N+1$ zeroes on the segment $[0,1]$. If λ decreases then the zeroes of ϕ move to the right and when $(n+1)-$ th zero hits the point 1 we have the Dirichlet eigenvalue and hence $\mu_n < \nu_n$. $\square$

6 Examples

Example 1. Consider the velocity $c(x)$ as a 1-periodic function of $x \in \mathbb{R}$ and let $c(x) = c_0(1+x^2), x \in [-1/2, 1/2]$, then the new variable t has the form

$$t = \int_0^x \frac{dx}{c_0(1+x^2)} = \frac{1}{c_0} \arctan x$$

and c_0 is such that

$$\int_{-1/2}^{1/2} \frac{dx}{c(x)} = \int_{-1/2}^{1/2} \frac{dx}{c_0(1+x^2)} = 1,$$

hence $c_0 = 2\arctan(1/2)$ and $x = \tan c_0 t$. We have

$$\hat{q}_n = \int_{-1/2}^{1/2} q(t) e^{i2\pi nt} dt = -\frac{c_0}{2} \int_{-1/2}^{1/2} 2x e^{i2\pi nt} dt$$

$$= -c_0 \int_{-1/2}^{1/2} \tan c_0 t e^{i2\pi nt} dt = -\frac{c_0}{2\pi n} \int_{-1/2}^{1/2} \tan c_0 t e^{i2\pi nt} dt$$

$$= -\frac{c_0}{i2\pi n} \tan c_0 t e^{i2\pi nt}|_{-1/2}^{1/2} = -\frac{c_0}{i2\pi n} \tan \frac{c_0}{2} + ...$$

Example 2. We consider the equation

$$-y'' = z^2 b^2(x) y, \qquad x \in \mathbb{R}, \tag{6.1}$$

where b is 1-periodic function and let $b = 1, x \in A = [0, 1/2]$ and $b = a, x \in B = [1/2, 1]$. First, we find the fundamental solution φ:

$$\varphi(x,z) = \frac{\sin zx}{z}, \qquad x \in A, \tag{6.2}$$

$$\varphi(x,z) = C_1 \frac{\sin zax}{za} + C_2 \frac{\cos zax}{za}, \qquad x \in B, \tag{6.3}$$

and let $r = za/2$. There are the conditions:

$$\varphi(1/2, z) = \frac{\sin z/2}{z} = C_1 \frac{\sin r}{za} + C_2 \frac{\cos r}{za}, \tag{6.4}$$

$$\varphi'(1/2, z) = \cos z/2 = C_1 \cos r - C_2 \sin r. \tag{6.5}$$

These equations have the following solutions

$$C_1 = a \sin r \sin(z/2) + \cos r \cos(z/2), \qquad C_2 = a \cos r \sin(z/2) - \sin r \cos(z/2) \tag{6.6}$$

Using the same way we find the function ϑ:

$$\vartheta(x,z) = \cos zx, \qquad x \in A, \tag{6.7}$$

$$\vartheta(x,z) = M_1 \frac{\sin zax}{za} + M_2 \frac{\cos zax}{za}, \qquad x \in B, \tag{6.8}$$

and the coefficients M_1, M_2 have the form

$$M_1 = z(a \sin r \cos(z/2) - \cos r \sin(z/2)), \qquad M_2 = z[a \cos r \cos(z/2) + \sin r \sin(z/2)] \tag{6.9}$$

The Lyapunov function has the form

$$\Delta = \frac{1}{2}(\varphi'(1,z) + \vartheta(1,z)) = \frac{1}{2}[S_1 \cos za + S_2 \sin za] \tag{6.10}$$

where the functions S_1, S_2 are defined by the following formulas

$$S_1 = (a + \frac{1}{a}) \sin r \sin(z/2) + 2 \cos r \cos(z/2),$$

$$S_2 = -(a + \frac{1}{a}) \cos r \sin(z/2) + 2 \sin r \cos(z/2).$$

We study the function Δ. If parameter a is integer then the function $\Delta(z)$ is periodic with respect to z. Let $a = 2$. Then we obtain

$$\Delta(z) = \frac{9}{8} \cos \frac{3z}{2} - \frac{1}{8} \cos \frac{z}{2}$$

and Δ is $4\pi-$periodic. Note that if we take the step-function b with rational parameters then we obtain the Lyapunov function Δ periodic in z. In this case the asymptotics of the Lyapunov function is absent and we have the exact periodic picture.

7 Appendix

In this section we find the asymptotic estimates of the following integral

$$J_1^\pm(x,z)=\int_0^x e^{\pm izt}q(t)dt,\qquad J_2^\pm(x,z)=\int_0^x e^{\mp izt}q(t)J_1^\pm(t,z)dt,$$

$$J_3^\pm(x,z)=\int_0^x e^{\pm izt}q(t)J_2^\pm(t,z)dt,\qquad J_4^\pm(x,z)=\int_0^x e^{\mp izt}q(t)J_3^\pm(t,z)dt,$$

where $x\in[0,2\pi], |z-n|\leqslant 1/2, z\in\mathbb{C}, n\in\mathbb{Z}$, and $q\in L^2(0,2\pi)$. We have the Fourier series

$$q(x)=\sum q_m e^{imx}. \tag{7.1}$$

Define the constants

$$b_1^\pm(z)=\sum_m \frac{q_m}{i(m\pm z)},\quad b_{20}^\pm(z)=\sum_m \frac{q_m q_{-m}}{i(m\pm z)},\quad b_{21}^\pm(z)=\sum_{m+p\neq 0}\frac{q_m q_p}{(m\pm z)(m+p)}$$

and sequences $u_\pm=\{u_{\pm n}\}, v_\pm=\{v_{\pm n}\}, w_\pm=\{w_{\pm n}\}$, where positive components are defined by

$$u_{\pm n}=\sum\frac{|q_m|}{|m\pm n|_1},\quad v_{\pm n}^2=\sum\frac{|q_m q_{-m}|}{|m\pm n|_1},\quad w_{\pm n}^2=\sum\frac{|q_m q_p|}{|m\pm n|_1|m+p|_1},\quad |x|_1=|x+i|,\quad n\in\mathbb{Z},$$

and note that $v_{\pm n}\leqslant 2w_{\pm n}$. Below we need the simple estimate

$$|m-\zeta|\geqslant(1/2)|m|_1,\quad if\quad \zeta\in\mathbb{C},\ |\zeta|=1/2,\ m\in\mathbb{Z}. \tag{7.2}$$

We prove the needed estimates

Lemma 7.1. *Let $|z-n|\leqslant 1/2$ and $x\in[0,2\pi]$. Then*

$$|J_1^\pm(x,z)|\leqslant C_0u_{\pm n},\quad |J_2^\pm(x,z)|\leqslant C_0(u_{\pm m}^2+w_{\pm n}^2),\quad C_0=4e^\pi,\quad |z-n|\leqslant 1/2, \tag{7.3}$$

$$J_1^\pm(x,z)=-b_1^\pm(z)+e^{\pm ixz}\sum_m\frac{q_m e^{imx}}{i(m\pm z)},\quad |z-n|=1/2, \tag{7.4}$$

$$J_2^\pm(x,z)=xb_{20}^\pm(z)-b_{21}^\pm(z)-b_1^\pm(z)J_1^\pm(x,z)-\sum_{m+p\neq 0}\frac{q_m q_p e^{ix(m+p)}}{(m\pm z)(m+p)},\quad |z-n|=1/2, \tag{7.5}$$

$$u_\pm,\ w_\pm\in\ell^d,\quad d>1/2. \tag{7.6}$$

Proof. Enough to show (7.3) for the case $|z-n|=1/2$. The substitution of (7.1) and the simple integration give

$$J_1^\pm(x,z)=\int_0^x\sum_m q_m e^{imt\pm izt}=\sum_m q_m\frac{e^{ix(m\pm z)}-1}{i(m\pm z)}, \tag{7.7}$$

which yields (7.4). Moreover, using (7.2) we have

$$|J_1^{\pm}(x,z)| \leqslant |\sum_m q_m \frac{e^{ix(m-z)}-1}{i(m\pm z)}| \leqslant \sum_m \frac{|q_m|2e^{\pi}}{|m\pm z|} \leqslant C_0 \sum_m \frac{|q_m|}{|m\pm n|_1}.$$

We prove (7.4). The substitution of (7.4), (7.1) and the integration imply

$$J_2^{\pm}(x,z) = \int_0^x e^{\mp izt} q(t) J_1^{\pm}(x,z)dt = \int_0^x e^{\mp izt} q(t)[-b_1^{\pm}(z) + e^{\pm itz}\sum_m \frac{q_m e^{imt}}{i(m\pm z)}]dt$$

$$= -b_1^{\pm}(z)J_1^{\pm}(x,z) + \int_0^x \sum_{m,p} \frac{q_m q_p e^{it(m+p)}}{i(m\pm z)}dt$$

where the second term has the form

$$\int_0^x \sum_{m,p} \frac{q_m q_p e^{it(m+p)}}{i(m\pm z)}dt = x b_{20}^{\pm}(z) - \sum_{m+p\neq 0} \frac{q_m q_p (e^{ix(m+p)}-1)}{(m\pm z)(m+p)},$$

and summing we get (7.5). Using (7.5) we obtain the second inequality (7.3). Estimates (7.6) are well known , see [RS]. □

Introduce the constants

$$A_{mpr}^{\pm}(z) = \frac{q_m q_p q_r}{i(m\pm z)(m+p)(m+p+r\pm z)}, \quad b_3^{\pm}(z) = \sum_{m+p\neq 0} A_{mpr}^{\pm}(z),$$

$$b_4^{\pm}(z) = \sum_{m+p\neq 0} A_{mpr}^{\pm}(z) q_{-m-p-r}, \quad B_{mpr}^{\pm}(n) = \frac{|q_m q_p q_r|}{|m\pm n|_1 |m+p|_1 |m+p+r\pm n|_1}, \ n\in\mathbb{Z},$$

and sequences $U_\pm = \{U_{\pm n}\}, V_\pm = \{V_{\pm n}\}, W_\pm = \{W_{\pm n}\}$, where positive components are defined by

$$U_{\pm n}^3 = \sum B_{mpr}^{\pm}(n), \quad V_{\pm n}^4 = \sum B_{mpr}^{\pm}(n)|q_{-m-p-r}|, \quad W_{\pm n}^4 = \sum \frac{B_{mpr}^{\pm}(n)|q_s|}{|m+p+r+s|_1}$$

and note that $V_{\pm n} \leqslant 2W_{\pm n}$. Introduce the functions $\widetilde{J}_1^{\pm}(x,z) = \int_0^x e^{\pm izt} t q(t)dt$, $\widetilde{J}_2^{\pm}(x,z) = \int_0^x e^{\mp izt} q(t)dt$. which have the same estimates as $J_1^{\pm}$, $J_2^{\pm}$. We now consider $J_3^{\pm}, J_4^{\pm}$.

Lemma 7.2. *Let $|z-n| \leqslant 1/2$ and $x\in[0,2\pi]$. Then*

$$|J_3^{\pm}(x,z)| \leqslant w_{\pm n}^2(|\widetilde{J}_1^{\pm}(x,z)| + |J_1^{\pm}(x,z)|) + u_{\pm n}|J_2^{\mp}(x,z)| + U_{\pm n}^3, \tag{7.8}$$

$$|J_4^{\pm}(x,z)| \leqslant C_0[w_{\pm n}^2(|\widetilde{J}_2^{\pm}(x,z)| + |J_2^{\pm}(x,z)|) + u_{\pm n}|J_3^{\mp}(x,z)| + U_{\pm n}^3|J_1^{\mp}(x,z)| + W_{\pm n}^4], \tag{7.9}$$

$$U_\pm \quad W_\pm \in \ell^d, \quad d > 1/2, \tag{7.10}$$

and the identities for the case $|z-n| = 1/2$ and $x\in[0,2\pi]$

$$J_3^{\pm}(x,z) = b_{20}^{\pm}(z)\widetilde{J}_1^{\pm}(x,z) - b_{21}^{\pm}(z)J_1^{\pm}(x,z) - b_1^{\pm}(z)J_2^{\mp}(x,z) - \sum_{m+p\neq 0} A_{mpr}^{\pm}(z)(e^{ix(m+p+r\pm iz)}-1), \tag{7.11}$$

$$J_4^{\pm}(x,z) = b_{20}^{\pm}(z)\widetilde{J}_2^{\pm}(x,z) - b_{21}^{\pm}(z)J_2^{\pm}(x,z) - b_1^{\pm}(z)J_3^{\mp}(x,z) + b_3^{\pm}(z)J_1^{\mp}(x,z) \tag{7.12}$$
$$-xb_{40}^{\pm}(z) - \sum_{m+p\neq 0, m+p+r+s\neq 0} \frac{A_{mpr}^{\pm}(z)q_s(e^{ix(m+p+r+s)}-1)}{i(m+p+r+s)}.$$

Proof. Enough to show (7.8-9) for the case $|z-n| = 1/2$. The substitution of (7.5), (7.1) implies

$$J_3^{\pm}(x,z) = \int_0^x e^{\pm izt}q(t)J_2^{\pm}(x,z)dt$$
$$= \int_0^x e^{\pm izt}q(t)[tb_{20}^{\pm}(z) - b_{21}^{\pm}(z) - b_1^{\pm}(z)J_1^{\mp}(t,z) - \sum_{m+p\neq 0}\frac{q_m q_p e^{it(m+p)}}{(m\pm z)(m+p)}]dt$$

and then the simple integration yields

$$J_3^{\pm}(x,z) = b_{20}^{\pm}(z)\widetilde{J}_1^{\pm}(x,z) - b_{21}^{\pm}(z)J_1^{\pm}(x,z) - b_1^{\pm}(z)J_2^{\mp}(x,z) - \int_0^x\sum_{m+p\neq 0}\frac{q_m q_p q_r e^{it(m+p+r\pm iz)}}{(m\pm z)(m+p)}dt$$

$$= b_{20}^{\pm}(z)\widetilde{J}_1^{\pm}(x,z) - b_{21}^{\pm}(z)J_1^{\pm}(x,z) - b_1^{\pm}(z)J_2^{\mp}(x,z) - \sum_{m+p\neq 0}\frac{q_m q_p q_r(e^{ix(m+p+r\pm iz)}-1)}{(m\pm z)(m+p)(m+p+r\pm z)}$$

which gives (7.11). Using (7.11) we have (7.8). We now find $J_4^{\pm}(x,z)$. The substitution of (7.11) implies

$$J_4^{\pm}(x,z) = \int_0^x e^{\mp izt}q(t)J_3^{\pm}(x,z)dt$$
$$= \int_0^x e^{\mp izt}q(t)[b_{20}^{\pm}(z)\widetilde{J}_1^{\pm}(x,z) - b_{21}^{\pm}(z)J_1^{\pm}(x,z) - b_1^{\pm}(z)J_2^{\mp}(x,z) + b_3^{\pm}(z)$$
$$- \sum_{m+p\neq 0} A_{mpr}^{\pm}(z)e^{ix(m+p+r\pm iz)}]dt$$

Then

$$J_4^{\pm}(x,z) = b_{20}^{\pm}(z)\widetilde{J}_2^{\pm}(x,z) - b_{21}^{\pm}(z)J_2^{\pm}(x,z) - b_1^{\pm}(z)J_3^{\mp}(x,z) + b_3^{\pm}(z)J_1^{\mp}(x,z) - I_4^{\pm}(x,z)$$

where the last term has the form

$$I_4^{\mp}(x,z) = \int_0^x q(t)\sum_{m+p\neq 0} A_{mpr}^{\pm}(z)e^{ix(m+p+r)}dt$$

$$= \int_0^x \sum_{m+p\neq 0} A_{mpr}^{\pm}(z)q_s e^{ix(m+p+r+s)}dt = xb_4^{\pm}(z) + \sum_{m+p\neq 0, m+p+r+s\neq 0}\frac{A_{mpr}^{\pm}(z)q_s(e^{ix(m+p+r+s)}-1)}{i(m+p+r+s)},$$

which imply (7.12). Using (7.12) we obtain (7.9).

We estimate $U_{\pm n}, W_{\pm n}$ for real q and "+", the proof for complex q and "-" is similar. Let $y_p^2 = \sum u_m^2|m-p|_1^{-1}$. Using the Schwartz inequality we obtain

$$U_n^3 = \sum_m \frac{|q_m|}{|m+n|_1}\sum_p \frac{|q_p||u_{n+m+p}}{|m+p|_1} \leqslant \sum_m \frac{|q_m|}{|m+n|_1}y_n v_m = y_n\sum_m \frac{|q_m|v_m}{|m+n|_1}$$

which yields $U_+ \in \ell^d, d > 1/2$. Consider now the more complicated case $W_\pm$. Let $Y_p^2 = \sum v_m^2 |m-p|_1^{-1}$. Using twice the Schwartz inequality we deduce that

$$W_{-n} = \sum_m \frac{|q_m q_p|}{|m+n|_1 |m+p|_1} \sum_r \frac{|q_r||u_{m+p+r}}{|m+p+r-n|_1} \leqslant \sum_m \frac{|q_m q_p| v_{n+m+p} y_n}{|m+n|_1 |m+p|_1}$$

$$= y_n \sum_m \frac{|q_m|}{|m+n|_1} \sum_p \frac{|q_p| v_{m+p+n}}{|m+p|_1} \leqslant y_n \sum_m \frac{|q_m| v_m Y_n}{|m+n|_1} = Y_n y_n \sum_m \frac{|q_m| v_m}{|m+n|_1}$$

which yields $W_+ \in \ell^d, d > 1/2$.□

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