Tiling semigroups

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Abstract

It has recently been shown how to construct an inverse semigroup from any tiling: a construction having applications in K-theoretical gap-labelling. In this paper, we provide the categorical basis for this construction in terms of an appropriate group acting partially and without fixed points on an inverse category associated with the tiling.

1 Introduction

In [6] and [7], the first author showed how to construct an inverse semigroup from any tiling of Euclidean space; we call such semigroups *tiling semigroups*. This work was motivated by questions in solid-state physics, particularly by those concerning quasi-crystals. Our motivation here is to understand the mathematical nature of tiling semigroups. We show that tiling semigroups can best be understood in terms of 'partial actions of groups'. Such partial group actions were introduced by Exel [2] and their theory further developed in [11]. The fact that the group acts only *partially* is accounted for by the fact that the tilings of most interest are those which are *non-periodic*.

We recall here the basic definitions concerning partial group actions. Let G be a group, and X a set. A *partial action* of G on X is a function $\theta \colon G \to I(X)$ to the (inverse) semigroup of all partial bijections on the set X satisfying the following two axioms:

(P1) $\theta(s^{-1}) = \theta(s)^{-1}$ for all $s \in G$.

(P2) $\theta(s)\theta(t) \subseteq \theta(st)$ for all $s, t \in G$.

We say that the partial action is *unital* if the following axiom also holds

(P3) $\theta(1) = 1_X$, the identity on X.

In Section 2, all partial group actions will be assumed unital.

If the inclusion in axiom (P2) is actually equality, then we say that the action is *global*; this is equivalent to an action of G on a subset of X, which is equal to X when the action is unital.

If $\theta\colon G \to I(X)$ is a partial action of the group G, and $\theta(g)(x)$ is defined, then we shall write $\theta(g)(x) = g \cdot x$. We write $\exists g \cdot x$ to mean that $g \cdot x$ is defined. Observe that the following two conditions hold:

(PA1) $\exists g \cdot x$ implies that $\exists g^{-1} \cdot (g \cdot x)$, and $g^{-1} \cdot (g \cdot x) = x$.

(PA2) $\exists g \cdot (h \cdot x)$ implies that $\exists (gh) \cdot x$ and $g \cdot (h \cdot x) = (gh) \cdot x$.

If the partial action is unital then the following equation also holds:

(PA3) $\exists 1 \cdot x$ for every $x \in X$, and $1 \cdot x = x$.

Conversely, a partial function from $G \times X$ to X denoted by $(g, x) \mapsto g \cdot x$ which satisfies conditions (PA1) and (PA2) defines a partial action of G on X, which is unital if condition (PA3) also holds.

Throughout this paper, we shall use basic results from inverse semigroup theory; see Howie [4], Lawson [9] or Petrich [12].

2 Inverse semigroups from tilings

We begin by describing the general construction in Section 2.1 and then applying it to tilings in Section 2.2. We conclude Section 2.2 with a brief discussion of 1-dimensional tiling semigroups.

2.1 Partial actions of groups on categories

We shall make use of the following notation concerning categories. If C is a category and $x \in C$ then $\mathbf{d}(x)$ denotes the right identity of x, and $\mathbf{r}(x)$ denotes the left identity. Our convention is that the product xy is defined in C if and only if $\mathbf{d}(x) = \mathbf{r}(y)$. We write $\exists xy$ to mean 'the product xy exists in the category'. The set of identities of C is denoted by C_o. A category C is said to be *inverse* if for each $x \in C$ there exists a unique element x^{-1} such that the products $xx^{-1}x$ and $x^{-1}xx^{-1}$ exist and satisfy $x = xx^{-1}x$ and $x^{-1} = x^{-1}xx^{-1}$. Standard properties of inverse monoids can be extended to inverse categories without difficulty. In particular, inverse categories are equipped with a natural partial order $\leq$.

An inverse category (or semigroup) S is said to be *combinatorial* if for all $s, t \in S$, whenever $s^{-1}s = t^{-1}t$ and $ss^{-1} = tt^{-1}$ then $s = t$. An inverse semigroup S with zero is said to be *0-E-unitary* if for every $s \in S$ such that $0 \neq e \leq s$ where e is an idempotent then s is an idempotent.

Let C be a category. A subcategory D of C is said to be *factorisation closed* if for all $x, y \in C$ such that $\exists xy$ and $xy \in D$ then $x, y \in D$.

Lemma 1 *Let C be a category, and let A and B be two factorisation closed subcategories of C.*

(i) *$A \cap B$ is either empty or a factorisation closed subcategory of C.*

(ii) *Let $\alpha\colon A \to B$ be an isomorphism of categories. Let A' be a factorisation closed subcategory of C contained in A. Then $\alpha(A')$ is a factorisation closed subcategory of C contained in B.*

Proof (i) Suppose that $A \cap B$ is non-empty. Then $A \cap B$ is a subcategory of C. Let $xy \in A \cap B$. Then $xy \in A$ and $xy \in B$. But both A and B are factorisation closed, and so $x, y \in A$ and $x, y \in B$. Thus $x, y \in A \cap B$.

(ii) Let $x, y \in C$ such that $\exists xy$ and $xy \in \alpha(A') \subseteq B$. Since B is factorisation closed, we have that $x, y \in B$. Now $\alpha^{-1}(x), \alpha^{-1}(y) \in A$, and $\alpha(\alpha^{-1}(x)\alpha^{-1}(y)) = xy \in \alpha(A')$. Thus $\alpha^{-1}(x)\alpha^{-1}(y) \in A'$. But A' is factorisation closed, so that $\alpha^{-1}(x), \alpha^{-1}(y) \in A'$. Hence $x, y \in \alpha(A')$. ∎

Let $\mathcal{I}(C)$ be the set of all isomorphisms between factorisation closed subcategories of C, including the empty function. The proof of the following is a straightforward application of Lemma 1.

Proposition 2 *$\mathcal{I}(C)$ is an inverse monoid under composition of partial functions.* ∎

Let C be a category and G a group. We say that G *acts partially on the category C without fixed points* if there is a unital partial action $\theta\colon G \to \mathcal{I}(C)$ such that for every identity $e \in C_o$ whenever $\theta(g)(e) = e$ then $g = 1$.

Lemma 3 *Let G be a group acting partially and without fixed points on the category C.*

(i) *If $g \cdot x = x$ then $g = 1$.*

(ii) *If $g \cdot x = h \cdot x$ then $g = h$.*

Proof (i) We have $\mathbf{d}(g \cdot x) = \mathbf{d}(x)$. But $\theta(g)$ is a functor, and so $\mathbf{d}(g \cdot x) = g \cdot \mathbf{d}(x)$. Thus $g \cdot \mathbf{d}(x) = \mathbf{d}(x)$. Hence $g = 1$ by the fixed point condition.

(ii) Suppose that $g \cdot x = h \cdot x$. Then $h^{-1} \cdot (g \cdot x) = h^{-1} \cdot (h \cdot x) = x$ by property (PA1) of Section 1. But by property (PA2), we have that $h^{-1} \cdot (g \cdot x) = (h^{-1}g) \cdot x$. Thus $(h^{-1}g) \cdot x = x$, by property (PA3). Hence $h^{-1}g = 1$ by (i) above, and so $h = g$. ∎

It is straighforward to check that the definition of a group G acting partially and without fixed points on a category C is axiomatised by (PA1), (PA2) and (PA3) together with the following axioms:

(PA4) If $\exists g \cdot x$ then $\exists g \cdot \mathbf{d}(x)$, and $\mathbf{d}(g \cdot x) = g \cdot \mathbf{d}(x)$; if $\exists g \cdot x$ then $\exists g \cdot \mathbf{r}(x)$, and $\mathbf{r}(g \cdot x) = g \cdot \mathbf{r}(x)$.

(PA5) If the product xy is defined in C and $\exists g \cdot (xy)$ then $\exists g \cdot x$ and $\exists g \cdot y$, the product $(g \cdot x)(g \cdot y)$ exists in C and $g \cdot (xy) = (g \cdot x)(g \cdot y)$.

(PA6) If the product xy is defined in C and $\exists g \cdot x$ and $g \cdot y$ then $\exists g \cdot (xy)$ and $g \cdot (xy) = (g \cdot x)(g \cdot y)$.

(PA7) If $e \in C_o$ and $g \cdot e = e$ then $g = 1$.

Let G be a group which acts partially and without fixed points on the category C. Define a relation $\sim$ on C by $x \sim y$ if and only if there exists $g \in G$ such that $g \cdot x = y$. It is easy to check using (PA1), (PA2) and (PA3) that $\sim$ is an equivalence relation on the set C. Denote the $\sim$-equivalence class containing x by $[x]$, and the set of equivalence classes by C/G.

Let $S = (C/G)^0$ be the set C/G with the new symbol 0 adjoined. Let $[x], [y] \in C/G$. If there exist $g, h \in G$ such that the product $(g \cdot x)(h \cdot y)$ exists in the category C then put $[x] \otimes [y] = [(g \cdot x)(h \cdot y)]$, otherwise put $[x] \otimes [y] = 0$, and define $0 \otimes 0 = 0$ and $[x] \otimes 0 = 0 = 0 \otimes [x]$.

Theorem 4 *Let G be a group acting partially and without fixed points on a category C. Then $S = (C/G)^0$ is a semigroup with zero. If C is an inverse category, then S is an inverse semigroup.*

Proof We begin with the general case where C is a category. We show first that $\otimes$ is well-defined. Suppose that

$$[x] \otimes [y] = [(a \cdot x)(b \cdot y)]$$

where $a, b \in G$ and $(a \cdot x)(b \cdot y)$ is defined in C, and also

$$[x] \otimes [y] = [(c \cdot x)(d \cdot y)]$$

where $c, d \in G$ and $(c \cdot x)(d \cdot y)$ is defined in C. We shall show that

$$(a \cdot x)(b \cdot y) \sim (c \cdot x)(d \cdot y).$$

By assumption, we have that

$$\mathbf{d}(a \cdot x) = \mathbf{r}(b \cdot y) \text{ and } \mathbf{d}(c \cdot x) = \mathbf{r}(d \cdot y).$$

By axiom (PA4), we have that

$$a \cdot \mathbf{d}(x) = b \cdot \mathbf{r}(y) \text{ and } c \cdot \mathbf{d}(x) = d \cdot \mathbf{r}(y).$$

By axioms (PA1), (PA2) and (PA3) we therefore have that

$$(b^{-1}a) \cdot \mathbf{d}(x) = \mathbf{r}(y) \text{ and } (d^{-1}c) \cdot \mathbf{d}(x) = \mathbf{r}(y).$$

Hence $b^{-1}a = d^{-1}c$ by Lemma 3. Put $g = ca^{-1} = db^{-1}$. Now $g \cdot (a \cdot x) = c \cdot x$ and $g \cdot (b \cdot y) = d \cdot y$. By assumption, the product $(a \cdot x)(b \cdot y)$ exists in C. Thus by axiom (PA6) it follows that $g \cdot ((a \cdot x)(b \cdot y))$ exists and

$$g \cdot ((a \cdot x)(b \cdot y)) = (c \cdot x)(d \cdot y).$$

Thus $[(a \cdot x)(b \cdot y)] = [(c \cdot x)(d \cdot y)]$, and so the product is well-defined.

We now show that S is a semigroup. Suppose that $([x] \otimes [y]) \otimes [z] \neq 0$. Then $[x] \otimes [y] = [(a \cdot x)(b \cdot y)]$ for some $a, b \in G$, and $([x] \otimes [y]) \otimes [z] = [c \cdot ((a \cdot x)(b \cdot y))(d \cdot z)]$ for some $c, d \in G$. By axiom (PA5), and axioms (PA1),(PA2) and (PA3) we obtain

$$([x] \otimes [y]) \otimes [z] = [((ca) \cdot x)((cb) \cdot y)(d \cdot z)].$$

It is evident that this is equal to $[x] \otimes ([y] \otimes [z])$. Likewise, $[x] \otimes ([y] \otimes [z]) \neq 0$ implies that $[x] \otimes ([y] \otimes [z] = ([x] \otimes [y]) \otimes [z]$. Consequently, S has an associative binary operation.

Suppose now that C is an inverse category. To show that S is an inverse semigroup we need only prove that its idempotents commute and that it is regular. The latter property is straightforward to prove: C is an inverse category and so for each $x \in C$ we have that $x = xx^{-1}x$. Thus $[x] = [x] \otimes [x^{-1}] \otimes [x]$, and so S is a regular semigroup.

It therefore remains to show that the idempotents commute. The first step is to locate the non-zero idempotents of S: we prove that $[x]$ is an idempotent if and only if x is an idempotent. (Note that this result does not use the fact that C is inverse). Suppose that $[x] \otimes [x] = [x]$. Then there exist $a, b \in G$ such that $[x] = [(a \cdot x)(b \cdot x)]$. Thus there exists $c \in G$ such that $x = c \cdot ((a \cdot x)(b \cdot x))$. Hence $x = ((ca) \cdot x)((cb) \cdot x)$. But

$$\mathbf{d}(x) = \mathbf{d}((cb) \cdot x) = (cb) \cdot \mathbf{d}(x).$$

Thus $cb = 1$ by Lemma 3. Similarly,

$$\mathbf{r}(x) = \mathbf{r}((ca) \cdot x) = (ca) \cdot \mathbf{r}(x)$$

and so $ca = 1$. Hence $x = x^2$. The converse is immediate.

Let $[x]$ and $[y]$ be non-zero idempotents of S. Suppose that $[x] \otimes [y] \neq 0$. Then $[x] \otimes [y] = [(a \cdot x)(b \cdot y)]$ for some $a, b \in G$. Now x and y are both idempotents, and so $a \cdot x$ and $b \cdot y$ are both idempotents using axiom (PA5). Since their product is defined, they belong to the same endomorphism monoid of C. By assumption, C is inverse, and so $(a \cdot x)(b \cdot y) = (b \cdot y)(a \cdot x)$. Thus $[x] \otimes [y] = [y] \otimes [x]$. If $[x] \otimes [y] = 0$ then the above argument implies that $[y] \otimes [x] = 0$ also. Thus in all cases idempotents commute. But a regular semigroup in which idempotents commute is inverse. Hence S is inverse. ■

The case where G acts globally on a category is more well-behaved, as might be expected. We shall need the following definition. A semigroup S is said to be *categorical at zero* if for all $a, b, c \in S$, whenever $ab \neq 0$ and $bc \neq 0$ then $abc \neq 0$.

Theorem 5 *Let G be a group acting globally and without fixed points on the category C. Then C/G is a category and $S = (C/G)^0$ is categorical at zero.*

Proof Since G acts globally on C, the map $\theta\colon G \to \mathcal{I}(C)$ is a monoid homomorphism. Thus for each $g \in G$, the domain of $\theta(g)$ is C, and so $g \cdot x$ is defined for every $x \in C$. Define the partial product $\star$ on the set C/G by $[x] \star [y] = [x] \otimes [y]$ if this is non-zero, and undefined otherwise. We prove that $(C/G, \star)$ is a category. Observe first that if $e \in C_o$ then $[e]$ consists entirely of identities of C because every $g \in G$ acts as a functor which necessarily maps identities to identities. We may therefore denote by C_o/G the set of $\sim$-equivalence classes of identities. Clearly, $C_o/G \subseteq C/G$. For each $[x] \in C/G$ define $\mathbf{d}[x] = [\mathbf{d}(x)]$ and $\mathbf{r}[x] = [\mathbf{r}(x)]$. Both these maps are well-defined. Observe now that $[x] \star [y]$ exists if and only if $\mathbf{d}[x] = \mathbf{r}[y]$: to see this, suppose first that $[x] \star [y]$ exists. Then there exist $g, h \in G$ such that $(g \cdot x)(h \cdot y)$ exists in the category C. Thus $g \cdot \mathbf{d}(x) = h \cdot \mathbf{r}(y)$ and so $(h^{-1}g) \cdot \mathbf{d}(x) = \mathbf{r}(y)$. Hence $\mathbf{d}[x] = \mathbf{r}[y]$; this result uses only properties of partial actions. Suppose now that $\mathbf{d}[x] = \mathbf{r}[y]$. Then there exists $g \in G$ such that $g \cdot \mathbf{d}(x) = \mathbf{r}(y)$. In the case of partial actions, we cannot infer from the existence of $g \cdot \mathbf{d}(x)$ the existence of $g \cdot x$. However, in the case we are now considering, $g \cdot x$ is defined. Thus $\mathbf{d}(g \cdot x) = g \cdot \mathbf{d}(x) = \mathbf{r}(y)$. Hence the product $(g \cdot x)y$ is defined in C and so $[x] \star [y]$ is defined. It is now straightforward to check that $(C/G, \star)$ is a category with set of identities C_o/G. Finally, the semigroup obtained by adjoining a zero to a category is always categorical at zero. ■

2.2 Tiling semigroups

In this section, we shall apply the construction used in Theorem 4 to construct inverse semigroups from tilings.

By a *tile* in $\mathbb{R}^n$, we mean a connected, bounded subset of $\mathbb{R}^n$ which is the closure of its interior. An *n-dimensional tiling* is an infinite set of tiles which cover $\mathbb{R}^n$ overlapping, at most, at their boundaries. Tilings are interesting not only from a mathematical point of view, but also in view of their applications. The point of departure for the work of this section was an application of tilings to solid-state physics. Solids can be modelled by tilings, with each tile of the tiling representing a particular atom or molecule. In classical crystallography, such tilings were assumed to be periodic, but with the discovery of quasi-crystals, this assumption was shown to be untenable. As a consequence, mathematicians and physisists have become increasingly interested in non-periodic tilings, such as those devised by Roger Penrose. It was against this background that one of us began an investigation into K-theoretical gap-labelling [5] in order to obtain a deeper understanding of how the long-range order of a solid, as modelled by a tiling, influences the structure of Schrödinger operators describing the motion of particles in the tiling. As a consequence of this work, it was shown that an inverse semigroup, which we call a *tiling semigroup*, could be associated with every tiling. The aim of this section is to show that tiling semigroups arise naturally as an application of our work on partial group actions on categories.

A finite subset of a tiling which is a union of tiles is called a *pattern*. It is important to bear the following point in mind: we have not introduced any equivalence relation on the set of tiles so that congruent tiles located in different places are different. If the subset of $\mathbb{R}^n$ covered by a pattern is connected then we say the pattern is *connected*. All patterns will be assumed to be connected.

Every pattern is a union of tiles, and any such tile is said to *belong to* that pattern. ¿From now on, we shall assume that a tiling $\mathcal{T}$ in $\mathbb{R}^n$ has been chosen, and make all our definitions relative to this tiling.

Given a tiling, we can construct an inverse category in the following way. Let $C = C(\mathcal{T})$ consist of all triples of the form (p_2, P, p_1) where P is a pattern and p_1 and p_2 are distinguished tiles belonging to P. Define $\mathbf{d}(p_2, P, p_1) = (p_1, p_1, p_1)$ and $\mathbf{r}(p_2, P, p_1) = (p_2, p_2, p_2)$. We define a partial product on C as follows:

$$(p_2, P, p_1)(q_2, Q, q_1) = \begin{cases} (p_2, P \cup Q, q_1) & \text{if } p_1 = q_2 \\ \text{undefined} & \text{else.} \end{cases}$$

It is evident that this partial product is well-defined.

Proposition 6 *Let $\mathcal{T}$ be a tiling. Then $C = C(\mathcal{T})$ with the partial product defined above is an inverse category. Furthermore, the endomorphism monoids of C are semilattices and C is combinatorial.*

Proof It is straightforward to check that C is a category. The identities of C are the triples of the form (p, p, p) and so are in one-to-one correspondence with the tiles of $\mathcal{T}$; the endomorphism monoid of C at the identity (p, p, p) is a semilattice whose elements can be identified with all patterns to which p belongs: in particular, the idempotents of C commute. If we define $(p_2, P, p_1)^{-1} = (p_1, P, p_2)$, then it is easy to check that

$$(p_2, P, p_1) = (p_2, P, p_1)(p_2, P, p_1)^{-1}(p_2, P, p_1).$$

The commutativity of the idempotents now implies that C is an inverse category. A straightfoward calculation shows that C is combinatorial. ■

Let G be the group of all translations of $\mathbb{R}^n$. We shall now define a partial action of G on C. Let $g \in G$ and (p_2, P, p_1). If $g \cdot P$ is a pattern of the tiling $\mathcal{T}$, then we define

$$g \cdot (p_2, P, p_1) = (g \cdot p_2, g \cdot P, g \cdot p_1),$$

otherwise $g \cdot (p_2, P, p_1)$ is undefined. We write $\exists g \cdot (p_2, P, p_1)$ to mean that it is defined. The proof of the following is straightforward.

Proposition 7 *Let $\mathcal{T}$ be a tiling and $C = C(\mathcal{T})$ its associated inverse category. The above definition defines a partial action of the group G on the inverse category C which satisfies the axioms (PA4), (PA5), (PA6) and (PA7). Thus G acts partially and without fixed points on the category C.* ■

The inverse semigroup $S(\mathcal{T}) = (C/G)^0$, constructed according to Theorem 4, is called the *tiling semigroup* of $\mathcal{T}$.

Theorem 8 *Tiling semigroups are 0-E-unitary, combinatorial and have finite $\mathcal{D}$-classes.*

Proof Let $S = S(\mathcal{T})$ be a tiling semigroup. To show that S is 0-E-unitary, let $[x] \in S(\mathcal{T})$ such that $[y] \leq [x]$ where $[y]$ is a non-zero idempotent. Then $[y] = [x] \otimes [y]$. Thus there exist elements $g, h \in G$ such that $(g \cdot x)(h \cdot y)$ is defined in $C(\mathcal{T})$. It follows that there exists $k \in G$ such that $k \cdot y = (g \cdot x)(h \cdot y)$. Now $[y]$ an idempotent implies y is an idempotent in $C(\mathcal{T})$ by the proof of Theorem 4. Hence $k \cdot y$ is an idempotent. Also $h \cdot y$ is an idempotent. Thus $k \cdot y \leq g \cdot x$ in the inverse category $C(\mathcal{T})$ and so $g \cdot x$ belongs to an endomorphism monoid of $C(\mathcal{T})$. But by Proposition 6, these endomorphism monoids are semilattices and so $g \cdot x$ is an idempotent. Hence x is an idempotent, and consequently, $[x]$ is an idempotent. Thus S is 0-E-unitary.

To show that S is combinatorial, let $[x], [y] \in C(\mathcal{T})$ such that $[x]^{-1} \otimes [x] = [y]^{-1} \otimes [y]$ and $[x] \otimes [x]^{-1} = [y] \otimes [y]^{-1}$. Then $[x^{-1}x] = [y^{-1}y]$ and $[xx^{-1}] = [yy^{-1}]$. Thus there exists $g, h \in G$ such that $g \cdot (x^{-1}x) = y^{-1}y$ and $h \cdot (xx^{-1}) = yy^{-1}$. If we let $x = (p_2, P, p_1)$ and $y = (q_2, Q, q_1)$. Then $g \cdot (p_1, P, p_1) = (q_1, Q, q_1)$ and $h \cdot (p_2, P, p_2) = (q_2, Q, q_2)$. Thus $g = h$, and so $g \cdot (p_2, P, p_1) = (q_2, Q, q_1)$. Hence $[x] = [y]$.

Finally, we describe the $\mathcal{D}$-classes of S. Let $[x], [y] \in S$ such that $[x] \, \mathcal{D} \, [y]$. Then there exists $[z] \in S$ such that $[x]^{-1}[x] = [z]^{-1}[z]$ and $[z][z]^{-1} = [y]^{-1}[y]$. Thus there exist $g, h \in G$ such that $g \cdot x^{-1}x = z^{-1}z$ and $h \cdot y^{-1}y = zz^{-1}$. Let $x = (p_2, P, p_1)$, $y = (q_2, Q, q_1)$ and $z = (r_2, R, r_1)$. Then $g \cdot (p_1, P, p_1) = (r_1, R, r_1)$ and $h \cdot (q_1, Q, q_1) = (r_2, R, r_2)$. Hence $(h^{-1}g) \cdot P = Q$. Conversely, if there is an element of G which maps P to Q then $[x] \, \mathcal{D} \, [y]$: this is because given two elements (x_2, X, x_1) and $(x_2', X, x_{1'})$ of $C = C(\mathcal{T})$ then the triple (x_1, X, x_1') can be used to show that they are $\mathcal{D}$-related in C. Thus the non-zero $\mathcal{D}$-classes are in bijective correspondence with the equivalence classes of patterns of the tiling under the (partial) action of the group G. If we choose such a class, then the elements of the corresponding $\mathcal{D}$-class are obtained by choosing a representative and then picking all possible pairs of tiles from this representative; since the representative consists of only finitely many tiles this can be done in only finitely many ways. Thus every $\mathcal{D}$-class is finite. ■

This approach to defining tiling semigroups is equivalent to the one originally adopted in the papers [6] and [7], and outlined in [9]. There are, however, at least two advantages to the categorical approach: it provides us with built-in flexibility in the choice of groups; and it opens up the possibility of studying tilings in other spaces, such as hyperbolic spaces.

We conclude with some remarks on the structure of 1-dimensional tiling semigroups by way of illustration.

Remarks

1. There is a convenient language-theoretic way of handling 1-dimensional tilings. Since a tile in $\mathbb{R}$ is a closed interval, a 1-dimensional tiling $\mathcal{T}$ is nothing other than an infinite set of closed intervals which cover $\mathbb{R}$ overlapping, at most, at their boundaries. Two tiles of $\mathcal{T}$ are said to be *congruent* if there is a translation of $\mathbb{R}$ which maps one to the other. Clearly, two tiles are congruent if and only if they have the same length. Let Σ be the set of congruence classes

of tiles; we call Σ an *alphabet* and its elements *letters*. It is evident that $\mathcal{T}$ can be identified with a bi-infinite string over Σ.

2. Identifying a 1-dimensional tiling $\mathcal{T}$ with a bi-infinite string over an alphabet Σ, we define the *language* $\mathcal{L} = \mathcal{L}(\mathcal{T})$ of the tiling $\mathcal{T}$ to consist of all finite strings which occur in $\mathcal{T}$. For each $x = x_1 \ldots x_m \in \mathcal{L}$ a *doubly-pointed marking of* x is obtained in the following way: pick a pair of letters x_i and x_j of x ($i = j$ is a possibility); x_i is called the *in-letter* and marked with an acute accent $\acute{x_i}$; x_j is called the *out-letter* and marked with a grave accent $\grave{x_j}$; if $i = j$ then x_i is both an in- and out-letter and marked with a check $\check{x_i}$. The string x with a doubly-pointed marking is called a *doubly-pointed string* of the tiling. The non-zero elements of the tiling semigroup $\mathcal{S} = \mathcal{S}(\mathcal{T})$ can be identified with all possible doubly-pointed strings of the tiling [6].

3. The multiplication in the tiling semigroup in terms of doubly-pointed strings is described in [9]. In summary: if Σ has cardinality n, then in Theorem 9.5.6 of [9] it is proved that the tiling semigroup $\mathcal{S}$ is a Rees quotient of a semigroup K_n, called a *Kachel semigroup on n letters.*[1] The semigroup K_n may be embedded in the McAlister semigroup M_n, introduced in [8].

4. The simplest 1-dimensional tiling is the bi-infinite string consisting of the repetition of a single letter ('a' say). Its tiling semigroup is the free inverse monoid on one generator FIM_1 with a zero adjoined (see Theorem 9.5.4 of [9]).

5. All tiling semigroups are 0-E-unitary, as we proved in Theorem 8. In the 1-dimensional case we can say a little more. To explain precisely what, we need the following definitions. A function $\theta\colon S \to T$ between inverse semigroups with zero is said to be *0-restricted* if $\theta^{-1}(0) = \{0\}$; it is said to be a *0-morphism* if, in addition, $\theta(ab) = \theta(a)\theta(b)$ for all $a, b \in S$ such that $ab \neq 0$. It is said to be *idempotent pure* if $\theta(a)$ an idempotent implies a is an idempotent. The class of 0-morphisms is studied in [1], [3] and [7].

Let $\mathcal{S} = \mathcal{S}(\mathcal{T})$ be the tiling semigroup of a 1-dimensional tiling $\mathcal{T}$ over the alphabet $\Sigma = \{a_1, \ldots, a_n\}$. Let FIM_1^0 be the free inverse monoid with a zero adjoined, regarded as the tiling semigroup of the bi-infinite string composed of the repetition of the letter 'a' (see point 4 above). Define a function θ from $\mathcal{S}$ to FIM_1^0 as follows: if $x \in \mathcal{S}$ is a doubly-pointed string then replace each letter of x by the letter 'a' preserving any accents; zero is mapped to zero. It is easy to check that θ is a surjective idempotent pure 0-morphism. The semigroup FIM_1 is E-unitary and admits a surjective idempotent pure homomorphism to the additive group $\mathbb{Z}$. Consequently, we can construct a surjective idempotent pure 0-morphism from any 1-dimensional tiling semigroup to the additive group $\mathbb{Z}$ with a zero adjoined.[2] This implies that 1-dimensional tiling semigroups

[1] *Kachel* is a German word for 'tile'.

[2] This map can be constructed directly: it is simply one more than the number of tiles between the accents counted with a sign depending on which accent is to the right, and where idempotents are mapped to the additive identity in $\mathbb{Z}$.

are *strongly 0-E-unitary* [1] and so are amenable to a structural description generalising the classical theory of E-unitary inverse semigroups (See [1] and [3] for details).

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