Exploring Surfaces through Methods from the Theory of Integrable Systems. Lectures on the Bonnet Problem*

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Abstract

A generic surface in Euclidean 3-space is determined uniquely by its metric and curvature. Classification of all special surfaces where this is not the case, i.e. of surfaces possessing isometries which preserve the mean curvature, is known as the Bonnet problem. Regarding the Bonnet problem, we show how analytic methods of the theory of integrable systems – such as finite-gap integration, isomonodromic deformation, and loop group description – can be applied for studying global properties of special surfaces.

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1 Lecture. Quaternionic Description of Surfaces. Bonnet Problem

1.1 Differential equations of surfaces

Let $\mathcal{F}$ be a smooth orientable surface in 3-dimensional Euclidean space. The Euclidean metric induces a metric Ω on this surface, which in turn generates the complex structure of a Riemann surface $\mathcal{R}$. Under such a parametrization, which is called *conformal*, the surface $\mathcal{F}$ is given by an immersion

$$F = (F_1, F_2, F_3) : \mathcal{R} \to \mathbb{R}^3,$$

and the metric is conformal: $\Omega = e^u \, dz d\bar{z}$, where z is a local coordinate on $\mathcal{R}$.

One should keep in mind that a complex coordinate is defined up to holomorphic $z \to w(z)$ transformation. This freedom will be used to simplify the corresponding equations.

The conformal parametrization gives the following normalization of $F(z, \bar{z})$:

$$< F_z, F_z >=< F_{\bar{z}}, F_{\bar{z}} >= 0, \; < F_z, F_{\bar{z}} >= \frac{1}{2} e^u, \tag{1}$$

where the brackets denote the scalar product

$$< a, b >= a_1 b_1 + a_2 b_2 + a_3 b_3,$$

and F_z and $F_{\bar{z}}$ are the partial derivatives $\frac{\partial F}{\partial z}$ and $\frac{\partial F}{\partial \bar{z}}$, where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The vectors $F_z, F_{\bar{z}}$, as well as the normal N, with

$$< F_z, N >=< F_{\bar{z}}, N >= 0, \quad < N, N >= 1, \tag{2}$$

define a moving frame on the surface, which due to (1, 2) satisfies the following Gauss-Weingarten equations:

$$\sigma_z = \mathcal{U}\sigma, \; \sigma_{\bar{z}} = \mathcal{V}\sigma, \; \sigma = (F_z, F_{\bar{z}}, N)^T, \tag{3}$$

$$\mathcal{U} = \begin{pmatrix} u_z & 0 & Q \\ 0 & 0 & \frac{1}{2} H e^u \\ -H & -2e^{-u} Q & 0 \end{pmatrix}, \; \mathcal{V} = \begin{pmatrix} 0 & 0 & \frac{1}{2} H e^u \\ 0 & u_{\bar{z}} & \bar{Q} \\ -2e^{-u}\bar{Q} & -H & 0 \end{pmatrix}, \tag{4}$$

where

$$Q =< F_{zz}, N >, \quad < F_{z\bar{z}}, N >= \frac{1}{2} H e^u. \tag{5}$$

The quadratic differential Qdz^2 is called the *Hopf differential.* The first and the second fundamental forms

$$\begin{aligned} < dF, dF > &= < I\begin{pmatrix} dx \\ dy \end{pmatrix}, \begin{pmatrix} dx \\ dy \end{pmatrix} >, \; z = x + iy, \\ - < dF, dN > &= < II\begin{pmatrix} dx \\ dy \end{pmatrix}, \begin{pmatrix} dx \\ dy \end{pmatrix} > \end{aligned}$$

are given by the matrices

$$I = e^u \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; II = \begin{pmatrix} Q + \bar{Q} + He^u & i(Q - \bar{Q}) \\ i(Q - \bar{Q}) & -(Q + \bar{Q}) + He^u \end{pmatrix}. \tag{6}$$

The *principal curvatures* k_1 and k_2 are the eigenvalues of the matrix $II \cdot I^{-1}$. This gives the following expressions for the *mean* and the *Gaussian curvatures*:

$$\begin{gathered} H = \tfrac{1}{2}\,(k_1 + k_2) = \tfrac{1}{2}\ \mathrm{tr}\ (II \cdot I^{-1}), \\ K = k_1 k_2 = \det\ (II \cdot I^{-1}) = H^2 - 4Q\bar{Q}e^{-2u}. \end{gathered}$$

A point P of the surface $\mathcal{F}$ is called *umbilic* if the principal curvatures at this point coincide $k_1(P) = k_2(P)$. The Hopf differential vanishes $Q(P) = 0$ exactly at umbilic points of the surface.

Coordinates in which both fundamental forms are diagonal are called *curvature line coordinates* and the corresponding parametrization (not necessarily conformal) is called a *curvature line parametrization.* A curvature line parametrization always exists in a neighborhood of a non-umbilic point. Near umbilic points, curvature lines form more complicated patterns.

The Gauss–Codazzi equations, which are the compatibility conditions of equations (3, 4),

$$\mathcal{U}_{\bar{z}} - \mathcal{V}_z + [\mathcal{U}, \mathcal{V}] = 0,$$

have the following form:

$$\begin{array}{lrcl} \text{Gauss equation} & u_{z\bar{z}} + \frac{1}{2}\, H^2\, e^u - 2|Q|^2\, e^{-u} & = & 0, \\ \text{Codazzi equation} & Q_{\bar{z}} & = & \frac{1}{2}\, H_z\, e^u. \end{array} \tag{7}$$

These equations are necessary and sufficient for existence of the corresponding surface.

Theorem 1 (Bonnet theorem). *Given a metric $e^u\, dz d\bar{z}$, a quadratic differential $Q\, dz^2$, and a function H on $\mathcal{R}$ satisfying the Gauss–Codazzi equations, there exists an immersion*

$$F : \tilde{\mathcal{R}} \to \mathbb{R}^3$$

with the fundamental forms (6). Here $\tilde{\mathcal{R}}$ is the universal covering of $\mathcal{R}$. The immersion F is unique up to Euclidean motions in $\mathbb{R}^3$.

We finish this section with some basic facts about a special class of surfaces. A conformal curvature line parametrization is called *isothermic*. In this case the preimages of the curvature lines are the lines $x = \text{const}$ and $y = \text{const}$ on the parameter domain, where $z = x + iy$ is a conformal coordinate. A surface is called *isothermic* if it allows isothermic parametrization. Isothermic surfaces are divided by their curvature lines into infinitesimal squares. Written in terms of an isothermic coordinate z the Hopf differential $Q(z, \bar{z})dz^2$ of an isothermic surface is real, i.e. $Q(z, \bar{z}) \in \mathbb{R}$.

In terms of arbitrary conformal coordinates, isothermic surfaces can be characterized as follows.

Lemma 1 *Let $F : \mathcal{R} \to \mathbb{R}^3$ be a conformal immersion of an umbilic free surface in $\mathbb{R}^3$. The surface is isothermic if and only if there exists a holomorphic non-vanishing differential $f(z)dz^2$ on $\mathcal{R}$ and a function $q : \mathcal{R} \to \mathbb{R}_*$ such that the Hopf differential is of the form*

$$Q(z, \bar{z}) = f(z)\, q(z, \bar{z}). \tag{8}$$

It is easy to see that $w = \int \sqrt{f(z)}\, dz$ is an isothermic coordinate.

1.2 Quaternionic description of surfaces

We construct and investigate surfaces in $\mathbb{R}^3$ by analytic methods. For this purpose it is convenient to use the Lie algebra isomorphism $so(3) = su(2)$ and to rewrite the equations (3, 4) for the moving frame in terms of 2 by 2 matrices. This quaternionic description turns out to be useful for analytic studies of general curves and surfaces in 3- and 4-spaces as well as for investigation of special classes of surfaces [Bob1, KS2, DPW, Bob2, KPP, PP].

Let us denote the algebra of quaternions by $\mathbb{H}$, the multiplicative quaternion group by $\mathbb{H}_* = \mathbb{H} \setminus \{0\}$, and their standard basis by $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, where

$$\mathbf{ij} = \mathbf{k}, \ \ \mathbf{jk} = \mathbf{i}, \ \ \mathbf{ki} = \mathbf{j}. \tag{9}$$

This basis can be represented by the Pauli matrices σ_α as follows:

$$\begin{gathered} \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\ \mathbf{i}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\ \mathbf{j}, \\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\ \mathbf{k}, \quad \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{gathered} \tag{10}$$

We identify $\mathbb{H}$ with 4-dimensional Euclidean space

$$q = q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \longleftrightarrow q = (q_0, q_1, q_2, q_3) \in \mathbb{R}^4.$$

The sphere $S^3 \subset \mathbb{R}^4$ is then naturally identified with the group of unitary quaternions $\mathbb{H}_1 = SU(2)$. 3-dimensional Euclidean space is identified with the space of imaginary quaternions Im $\mathbb{H}$

$$X = -i \sum_{\alpha=1}^{3} X_\alpha \sigma_\alpha \in \mathrm{Im}\ \mathbb{H} \ \longleftrightarrow \ X = (X_1, X_2, X_3) \in \mathbb{R}^3. \tag{11}$$

The scalar product of vectors in terms of quaternions and matrices is then

$$< X,Y >= -\frac{1}{2}(XY+YX) = -\frac{1}{2}\mathrm{tr}\, XY. \tag{12}$$

We will also denote by F and N the matrices obtained in this way from the vectors F and N.

Let us take $\Phi \in \mathbb{H}_*$ which transforms the basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ into the frame F_x, F_y, N:

$$F_x = e^{u/2}\Phi^{-1}\mathbf{i}\Phi,\ F_y = e^{u/2}\Phi^{-1}\mathbf{j}\Phi,\ N = \Phi^{-1}\mathbf{k}\Phi. \tag{13}$$

Then

$$F_z = -ie^{u/2}\Phi^{-1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\Phi,\ \ F_{\bar z} = -ie^{u/2}\Phi^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\Phi, \tag{14}$$

and all the conditions (1) are automatically satisfied.

The quaternion Φ satisfies linear differential equations. To derive them we introduce matrices

$$U = \Phi_z\Phi^{-1}, \qquad V = \Phi_{\bar z}\Phi^{-1}. \tag{15}$$

The compatibility condition $F_{z\bar z} = F_{\bar z z}$ for (14) implies

$$V_{22} - V_{11} = \frac{u_{\bar z}}{2},\ \ U_{11} - U_{22} = \frac{u_z}{2},\ \ U_{21} = -V_{12},$$

where U_{kl} and V_{kl} are the matrix elements of U and V. In the same way one obtains from (14)

$$\begin{aligned} F_{z\bar z} = \frac{1}{2}He^uN \quad &\rightarrow \quad U_{21} = -V_{12} = \frac{1}{2}He^{u/2} \\ F_{zz} = u_zF_z + QN \quad &\rightarrow \quad U_{12} = -Qe^{-u/2} \\ F_{\bar z\bar z} = u_{\bar z}F_{\bar z} + \bar QN \quad &\rightarrow \quad V_{21} = \bar Qe^{-u/2}. \end{aligned}$$

Recall that Φ is defined up to multiplication by a scalar factor. We normalize this factor by the condition

$$\det\Phi = e^{\frac{u}{2}}, \tag{16}$$

for reasons which will be clarified in the next section. For the traces of U and V this implies

$$\mathrm{tr}\, U = \frac{u_z}{2}, \qquad \mathrm{tr}\, V = \frac{u_{\bar z}}{2}.$$

Finally one arrives at the following

Theorem 2 *Using the isomorphism (11), the moving frame $F_z, F_{\bar z}, N$ of a conformally parametrized surface (z is a conformal coordinate) is described by formulas (13),(14), where $\Phi \in \mathbb{H}_*$ satisfies the equations (15) with U, V of the form*

$$U = \begin{pmatrix} \frac{u_z}{2} & -Qe^{-u/2} \\ \frac{1}{2}He^{u/2} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & -\frac{1}{2}He^{u/2} \\ \bar Qe^{-u/2} & \frac{u_{\bar z}}{2} \end{pmatrix}. \tag{17}$$

Corollary 1 *The conformal frame Φ satisfies the Dirac equation*

$$e^{-u/2}\begin{pmatrix} 0 & \partial_z \\ -\partial_{\bar z} & 0 \end{pmatrix}\Phi = \frac{1}{2}H\Phi. \tag{18}$$

It turnes out that at this point the whole construction can be reversed. Namely, starting with a solution to the Dirac equation one can derive a Weierstrass type representation (see (21) below) for conformally parametrized surfaces. This idea was recently developed by Konopelchenko [Kon] and further in [Tai, PP, KS2], although in other forms the Weierstrass representation of surfaces was known already to Eisenhart [Eis] and Kenmotsu [Ken].

Theorem 3 *Let $D \subset \mathbb{C}$ be a simply connected domain and $(s_1, \bar s_2)^T : D \to \mathbb{C}^2$ be a solution to the Dirac equation with the potential $p \in C^\infty(D)$*

$$\begin{pmatrix} 0 & \partial_z \\ -\partial_{\bar z} & 0 \end{pmatrix}\begin{pmatrix} s_1 \\ \bar s_2 \end{pmatrix} = p\begin{pmatrix} s_1 \\ \bar s_2 \end{pmatrix}. \tag{19}$$

Then

$$\Phi = \begin{pmatrix} s_1 & -s_2 \\ \bar s_2 & \bar s_1 \end{pmatrix} : D \to \mathbb{H}_* \tag{20}$$

is a conformal frame (14) of the conformally immersed surface

$$\begin{aligned} F_1 + iF_2 &= \int s_1^2\, dz - \bar s_2^2\, d\bar z \\ F_3 &= \int s_1 s_2\, dz + \bar s_1 \bar s_2\, d\bar z. \end{aligned} \tag{21}$$

The metric and the mean curvature of the surface are given by

$$e^u\, dz d\bar z = (\mid s_1 \mid^2 + \mid s_2 \mid^2)^2\, dz d\bar z, \quad H = 2pe^{-u/2}. \tag{22}$$

Proof. Note that $(-s_2, \bar s_1)^T$ is also a solution to (19) due to the symmetry of the Dirac equation. At this point Φ given by (20) can be identified with the conformal frame Φ of Corollary 1. The formula for the metric (22) follows from (16). Substituting it into our previous formulas (14) for conformal frame one defines

$$F_z := -i\begin{pmatrix} s_1 s_2 & -s_2^2 \\ s_1^2 & -s_1 s_2 \end{pmatrix}, \quad F_{\bar z} := -i\begin{pmatrix} \bar s_2 \bar s_1 & \bar s_1^2 \\ -\bar s_2^2 & -\bar s_2 \bar s_1 \end{pmatrix}.$$

These formulas are automatically compatible. Integrating them one arrives at (21).

1.3 Spinor description of surfaces

As shown in [Bob2], the quaternionic description of the previous section is actually a global one. Let $\cup_i D_i = \mathcal{R}$ be an open covering of $\mathcal{R}$ with local coordinates $z_i : D_i \to \mathbb{C}$. Conditions (14, 16) determine a quaternionic valued smooth $\Phi(z_i, \bar{z}_i)$ uniquely up to sign on each D_i. To establish the global nature of Φ recall that a holomorphic line bundle S is called a *spin bundle* if it satisfies $S \otimes S = K$, where K is the canonical bundle.

Denote the first column of Φ by

$$\begin{pmatrix} S_1 \\ \bar{S}_2 \end{pmatrix}.$$

Lemma 2 *S_1 and S_2 are smooth sections of the same holomorphic spin bundle S.*

Proof. Consider two intersecting $D_i \cap D_j \neq \emptyset$ with corresponding $\Phi_i(z_i, \bar{z}_i)$ and $\Phi_j(z_j, \bar{z}_j)$. Identifying the representations for the Gauss map in terms of Φ_i and Φ_j one obtains on $D_i \cap D_j$

$$\Phi_i = \begin{pmatrix} c_{ij} & 0 \\ 0 & \bar{c}_{ij} \end{pmatrix} \Phi_j.$$

with some $c_{ij} : D_i \cap D_j \to \mathbb{C}_*$. Further, identifying the tangent frames

$$F_{z_i} = F_{z_j} \frac{dz_j}{dz_i}$$

and using $\Phi \sigma_2 \Phi^T \sigma_2 = \det \Phi$ one obtains

$$\Phi_i^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Phi_i = \frac{dz_j}{dz_i} \Phi_j^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Phi_j,$$

which finally implies

$$c_{ij}^2 = \frac{dz_j}{dz_i}.$$

The transition functions $c_{ij} : D_i \cap D_j \to \mathbb{C}_*$ defined through Φ_i obviously satisfy the cocycle condition $c_{ij} c_{jk} = c_{ik}$ and thus define a line bundle S with $S \otimes S = K$.

In local coordinates S_n are written naturally as $S_n = s_n(z_i, \bar{z}_i)\sqrt{dz_i}$.

Using the equivalence of spinor representation of conformal frames of surfaces and solutions of the Dirac equation, proven in Corollary 1 and Theorem 3, we arrive at the following global reformulation [Tai, PP] of Theorem 3.

Theorem 4 *A half-density p (i.e. a smooth section of $K^{\frac{1}{2}} \otimes \bar{K}^{\frac{1}{2}}$) and two not simultaneously vanishing spinors S_1, S_2 (i.e. smooth sections of $S \cong K^{\frac{1}{2}}$ with $(S_1, S_2) \neq (0,0)\ \forall P \in \mathcal{R}$) satisfying the Dirac equation (19) determine through*

$$\begin{aligned} F_1 + iF_2 &= \int S_1^2 - \bar{S}_2^2 \\ F_3 &= \int S_1 S_2 + \bar{S}_1 \bar{S}_2 \end{aligned} \tag{23}$$

a conformal immersion $F : \tilde{\mathcal{R}} \to \mathbb{R}^3$, where $\tilde{\mathcal{R}}$ is the universal covering of $\mathcal{R}$. The metric and the mean curvature of the immersion are given by

$$e^u \, dz d\bar{z} = (\mid S_1 \mid^2 + \mid S_2 \mid^2)^2, \quad H = 2pe^{-u/2}.$$

Remark. In the case of minimal surfaces $H = 0$ the spinors S_1 and S_2 are holomorphic and the representation (23) is known as the spinor Weierstrass representation [Sul, Bob2, KS1].

On a Riemann surface of genus g there exist 2^{2g} non-isomorphic spin bundles which are distinguished by different *spin structures*. For a geometric interpretation of the spin structure of the spin bundle S in terms of the immersion (23) we refer to [Bob2]. Spin structures classify regular homotopies of immersions [Pin].

1.4 Alternative descriptions of surfaces and the Bonnet problem

Bonnet Theorem 1 characterizes surfaces via the coefficients e^u, H, Q of their fundamental forms. These coefficients are not independent and are subject to the Gauss–Codazzi equations (7). A natural question is whether some of these data are superfluous. The following natural candidates for more "economic" characterization of surfaces were studied.

(i) The most geometric setting of the problem is the oldest one and is due to Bonnet. He posed the question whether one can eliminate the Hopf differential from the description of surfaces, i.e. whether the *metric* and the *mean curvature function*

$$e^u, \qquad H$$

alone suffice to describe a surface completely. Generic surfaces are determined uniquely by the metric and the mean curvature function. Bonnet himself [Bon] made the initial progress in the investigation of the special surfaces where it is not the case, i.e. which possess non-congruent isometric "relatives" with the same mean curvature function. The rest of these lectures is devoted to this problem, which is fairly named the *Bonnet problem.*

(ii) The *conformal Hopf differential*

$$q := Qe^{-\frac{u}{2}}.$$

Note that whereas the Hopf differential is a quadratic differential, i.e. a section of the line bundle K^2, the conformal Hopf differential is more exotic - it is a section of $K^{\frac{3}{2}} \otimes \bar{K}^{-\frac{1}{2}}$. The reason for its introduction by U. Pinkall is that q is invariant with respect to the Möbius transformations of the ambient $\mathbb{R}^3$. A non-isothermic surface is uniquely determined by q up to Möbius transformations. Counting dimensions, one immediately observes that generic sections of $K^{\frac{3}{2}} \otimes \bar{K}^{-\frac{1}{2}}$ do not correspond to surfaces in $\mathbb{R}^3$. A proper equation for q of surfaces in $\mathbb{R}^3$ is still unknown.

(iii) The *Dirac potential* or *mean-curvature half-density*

$$p = \frac{1}{2} H e^{-\frac{u}{2}}.$$

As one can see from its definition, this potential is a half-density, i.e. a section of the line bundle $K^{\frac{1}{2}} \otimes \bar{K}^{\frac{1}{2}}$. Recently, description of surfaces through Dirac spinors attracted much attention [KS2, Tai, PP]. Unfortunately, one has neither existence nor uniqueness in this description. A generic Dirac operator (with generic potential) has trivial kernel, thus generic half-densities do not yield surfaces. On the other hand, there may exist many immersions with the same potential p, for example all special surfaces appearing in the Bonnet problem.

Returning to the Bonnet problem, note that already Bonnet indicated all special surfaces which possess non-congruent isometric "relatives" with the same mean curvature function. There are three cases when this happens.

1. Constant mean curvature surfaces. Let $\mathcal{F}$ be a surface with constant mean curvature H. The Gauss–Codazzi equations (7) are obviously invariant with respect to the transformation

$$Q \to Q_t = e^{it} Q, \qquad t \in \mathbb{R}.$$

Applying the Bonnet theorem one obtains the one parameter family $\mathcal{F}_t$, $\mathcal{F} = \mathcal{F}_0$ of isometric surfaces with the same constant mean curvature H. In the last ten years there was much interest in studying global properties of surfaces with constant mean curvature and now they are rather well investigated by various methods (see for example [Wen, PS, Kap, Bob1, GKS]) including methods of the theory of integrable systems.

2. Bonnet pairs are exactly two non-congruent isometric surfaces $\mathcal{F}'$ and $\mathcal{F}''$ with the same mean curvature function. The theory of Bonnet pairs is very closely related [Bia2, KPP] to the theory of isothermic surfaces and as such belongs also to geometry described by integrable systems. Up to now, global theory of Bonnet pairs is not well developed, in particular it is unknown whether there exist compact Bonnet pairs, a question first posed in [LT].

3. Bonnet families. In [Bon], Bonnet himself was able to show that besides the surfaces with constant mean curvature there exists a class of surfaces, depending on finitely many parameters which possess one-parameter family of isometries preserving the mean curvature. These surfaces were studied by many authors [Haz, Gra, Car, Che, BE1, Rou] and recently global classification [BE2] of them was obtained using methods from the theory of integrable systems.

The remaining three lectures are devoted to consideration of these three cases.

2 Lecture. Constant mean curvature surfaces

The content of this lecture is based on results obtained in [Bob1].

2.1 Associated family

If the mean curvature of $\mathcal{F}$ is constant, then the Gauss-Codazzi equations

$$u_{z\bar{z}} + \frac{1}{2}H^2 e^u - 2Q\bar{Q}e^{-u} = 0, \qquad Q_{\bar{z}} = 0,$$

are invariant with respect to the transformation

$$Q \to Q^t = \Lambda Q, \quad |\Lambda| = 1, \tag{24}$$

Integrating the equations for the moving frame with the coefficient Q replaced by $Q^t = \Lambda Q$ we obtain a one-parameter family $\mathcal{F}^t$ of surfaces. All the surfaces $\mathcal{F}^t$ are isometric and have the same constant mean curvature. Treating t as a deformation parameter we obtain the first family of special surfaces indicated by Bonnet (see Section 1.4).

Theorem 5 *Every constant mean curvature surface has a one-parameter family of isometric deformations preserving the mean curvature. The deformation is described by the transformations (24).*

Without loss of generality we normalize

$$H = 1.$$

The quaternion $\Phi(z, \bar{z}, \Lambda)$ solving the system (15, 17) with $Q^t = \Lambda Q$ describes the moving frame $F_z, F_{\bar{z}}, N$ (13, 14) of the corresponding surface. Knowing the family $\Phi(z, \bar{z}, \Lambda)$ in a neighbourhood of $\Lambda = e^{2it}$ allows us to derive an immersion formula [Bob1] without integration the frame with respect to $z, \bar{z}$, but just by differentiation by t. Before presenting this important formula we pass to a gauge equivalent frame function

$$\Phi_0 = e^{-u/4} \begin{pmatrix} \frac{1}{\sqrt{i\lambda}} & 0 \\ 0 & \sqrt{i\lambda} \end{pmatrix} \Phi, \qquad \Lambda = \lambda^2, \tag{25}$$

normalized by

$$\Phi_0(z, \bar{z}, \lambda = e^{it}) \in SU(2), \qquad t \in \mathbb{R}. \tag{26}$$

This lecture deals essentially with the theory of CMC tori. Since the canonical bundle in this case is trivial, introducing a global complex coordinate, one can describe spinors in terms of doubly-periodic functions (see Section 2.3).

Theorem 6 *Let $\Phi_0(z,\bar z,\lambda=e^{it})$ be a solution of the system*

$$\Phi_{0\,z}=U_0(\lambda)\Phi_0, \qquad \Phi_{0\,\bar z}=V_0(\lambda)\Phi_0, \tag{27}$$

$$U_0(\lambda)=\left(\begin{array}{cc} \frac{u_z}{4} & i\lambda Q e^{-u/2} \\ \lambda\frac{i}{2}e^{u/2} & -\frac{u_z}{4}\end{array}\right),\ V_0(\lambda)=\left(\begin{array}{cc} -\frac{u_{\bar z}}{4} & \frac{i}{2\lambda}e^{u/2} \\ \frac{i}{\lambda}\bar Q e^{-u/2} & \frac{u_{\bar z}}{4}\end{array}\right) \tag{28}$$

normalized by (26). Then F and N, defined by the formulas

$$F=-\Phi_0^{-1}\frac{\partial}{\partial t}\Phi_0+\frac{i}{2}\Phi_0^{-1}\sigma_3\Phi_0, \qquad N=-i\Phi_0^{-1}\sigma_3\Phi_0, \tag{29}$$

describe a CMC surface and its Gauss map, with metric e^u, mean curvature $H=1$, and Hopf differential $Q^t=e^{2it}Q$.

Conversely, let F be a conformal parametrization of a CMC surface with metric e^u, mean curvature $H=1$, and Hopf differential Q^t. Then F is given by formula (29) where Φ_0 is a solution of (27, 28) as above.

Proof. First we note that both F and N are imaginary quaternions and therefore can be identified with vectors in $\mathbb{R}^3$. By identification (25) the system (28) coincides with the quaternionic representation (17) for the equations for the moving frame with the Hopf differential λQ. Differentiating (29) we get

$$\begin{aligned} F_z &= -\Phi_0^{-1}\frac{\partial U_0(\lambda)}{\partial t}\Phi_0+\frac{i}{2}\Phi_0^{-1}[\sigma_3,U_0(\lambda)]\Phi_0=-ie^{u/2}\Phi^{-1}\left(\begin{array}{cc}0&0\\1&0\end{array}\right)\Phi,\\ F_{\bar z} &= -ie^{u/2}\Phi^{-1}\left(\begin{array}{cc}0&1\\0&0\end{array}\right)\Phi, \end{aligned}$$

which coincides with (14).

Remark. In a neighborhood of a non-umbilic point $Q\neq 0$ by a conformal change of coordinate $z\to w(z)$ one can always normalize

$$Q=\frac{1}{2}.$$

Thus, umbilic free CMC surfaces are isothermic. In this parametrization the Gauss equation becomes the elliptic sinh-Gordon equation

$$u_{z\bar z}+\sinh u=0. \tag{30}$$

2.2 Loop group formulation

The matrices

$$A=U_0+V_0, \qquad B=i(U_0-V_0)$$

corresponding to real vector fields $\partial_x = \partial_z + \partial_{\bar z}$ and $\partial_y = i(\partial_z - \partial_{\bar z})$ belong to the loop algebra

$$g_H[\lambda] = \{\xi : S^1 \to su(2) : \xi(-\lambda) = \sigma_3 \xi(\lambda)\sigma_3\},$$

and Φ_0 in (27) lies in the corresponding loop group

$$G_H[\lambda] = \{\phi : S^1 \to SU(2) : \phi(-\lambda) = \sigma_3 \phi(\lambda)\sigma_3\}. \tag{31}$$

Here S^1 is the set $|\lambda| = 1$. When defined for general complex λ, elements of $g_H[\lambda]$ and $G_H[\lambda]$ satisfy the real reduction

$$\xi(\lambda) = \sigma_2 \overline{\xi(\frac{1}{\bar\lambda})}\sigma_2, \qquad \phi(\lambda) = \sigma_2 \overline{\phi(\frac{1}{\bar\lambda})}\sigma_2.$$

For applying analytic methods of the theory of integrable systems it is crucial that CMC surfaces can be characterized in terms of this loop group completely without referring to the special geometric nature of the coefficients of A and B. It is not difficult to prove the following

Theorem 7 *Let $\phi : D \to G_H[\lambda]$ be a smooth map on $D \subset \mathbb{C}$ satisfying*

$$\phi_z \phi^{-1} = A\lambda + B$$

with $A : D \to GL(2, \mathbb{C})$. Then the gauge equivalent

$$\Phi_0 = \exp(\frac{i}{2} \arg A_{21}\, \sigma_3)\phi$$

satisfies (27) with U_0, V_0 of the form (28) and describes the conformal frame of the immersion

$$F = -\phi^{-1}\frac{\partial}{\partial t}\phi + \frac{i}{2}\phi^{-1}\sigma_3\phi, \qquad \lambda = e^{it}$$

of D in $\mathbb{R}^3$ with the mean curvature $H = 1$.

2.3 CMC Tori. Analytic formulation

Methods of Section 2.2 can be used not only in local but also in global studies of CMC surfaces. It is a classical result of Hopf [Hop] that the only CMC surface of genus zero is a round sphere. Indeed the holomorphic quadratic differential $Q\,dz^2$ on a sphere must vanish identically. Then (4) implies in particular $N + F = C = \text{const}$, which yields $< F - C, F - C >= 1$.

Classification of CMC tori is not as simple as of spheres but analytic tools enable us to achieve success in this case also. The reason for a simplification in the case $g = 1$ is the fact that, unlike the case of Riemann surfaces of genus $g \geq 2$, on a torus it is possible to introduce a global complex coordinate.

Any Riemann surface of genus 1 is conformally equivalent to the factor of the complex plane by a lattice $\mathbb{C}/\mathcal{L}$. The corresponding conformal parametrization of a torus is given by a doubly-periodic mapping $F : \mathbb{C}/\mathcal{L} \to \mathbb{R}^3$. The metric and the Hopf differential in this parametrization are described by doubly-periodic functions $u(z, \bar z),\ Q(z, \bar z)$. Note that $H = \text{const}$ implies $Q_{\bar z} = 0$ and

$Q(z)$ is a bounded elliptic function, thus a constant. This constant is not zero, otherwise, as follows from the consideration above the surface is a sphere. Thus CMC tori have no umbilic points. As before we normalize the Gauss equation to (30) by $Q = \frac{1}{2}$.

Denoting the generators of $\mathcal{L}$ by

$$Z_1 = X_1 + iY_1, \qquad Z_2 = X_2 + iY_2$$

one obtains the following

Proposition 1 *Any torus with mean curvature $H = 1$ can be conformally parametrized by a doubly-periodic immersion $F : \mathbb{C} \to \mathbb{R}^3$*

$$F(z + Z_i, \bar{z} + \bar{Z}_i) = F(z, \bar{z}), \qquad i = 1, 2$$

with the Hopf differential $Q = \frac{1}{2}$. In this parametrization the metric $u(z, \bar{z})$ is a doubly-periodic solution to the elliptic sinh-Gordon equation (30).

Note that due to the ellipticity of equation (30) all CMC tori are real analytic.

To describe all CMC tori one should solve the following problems.

1. Describe all doubly-periodic solutions $u(z, \bar{z})$ of the elliptic sinh-Gordon equation (30).

2. Integrate linear system (27) with

$$U_0(\lambda) = \frac{1}{2} \begin{pmatrix} \dfrac{u_z}{2} & i\lambda e^{-u/2} \\ i\lambda e^{u/2} & -\dfrac{u_z}{2} \end{pmatrix}, \; V_0(\lambda) = \frac{1}{2} \begin{pmatrix} -\dfrac{u_{\bar{z}}}{2} & \dfrac{i}{\lambda} e^{u/2} \\ \dfrac{i}{\lambda} e^{-u/2} & \dfrac{u_{\bar{z}}}{2} \end{pmatrix} \tag{32}$$

   to find $\Phi_0(z, \bar{z}, \lambda = e^{it})$.

3. Formula (29) for F describes the corresponding CMC immersion. In general, this immersion is not doubly-periodic. One should specify parameters of the solution $u(z, \bar{z})$, which yield doubly-periodic $F(z, \bar{z})$.

These three problems can be solved simultaneously using methods of the finite-gap integration theory. In the rest of the lecture we give an idea of how this solution is found.

2.4 Higher flows and the fundamental theorem

Let $u(z, \bar{z})$ be a solution of the sinh-Gordon equation. The perturbation $u_\epsilon(z, \bar{z}) = u(z, \bar{z}) + \epsilon v(z, \bar{z})$ of $u(z, \bar{z})$ satisfies (30) up to the terms of order $O(\epsilon^2)$ if and only if $v(z, \bar{z})$ is a solution of the linearized elliptic sinh-Gordon equation

$$(\partial_{z\bar{z}} + \cosh u(z, \bar{z}))v(z, \bar{z}) = 0. \tag{33}$$

The elliptic sinh-Gordon equation is one of the possible real versions of the sine-Gordon equation, which is one of the basic models of the theory of integrable systems. Integrable systems possess infinitely many conservation laws, which induce infinitely many commuting flows of the corresponding dynamical system. In particular, applying standard algebraic tools of the theory to the sine-Gordon equation one can prove that there exists $v(u_z,\dots,u_z^{(k)})$, which solves (33) and is a polynomial of all its arguments. Such a polynomial can be treated as a tangential vector field to the space of solutions of the elliptic sinh-Gordon equation. These vector fields induce flows on the phase space of the dynamical system (30), which in the theory of solitons are called *higher flows.*

There exists a regular algebraic description of these commuting flows through *formal Killing field* (see [FPPS]), which is in our case a symmetric $K_0(-\lambda)=\sigma_3 K_0(\lambda)\sigma_3$ formal power series solution

$$K_0(\lambda)=\sum_{m=1}^{\infty} K_m\lambda^{-m} \tag{34}$$

of

$$K_0(\lambda)_z=\big[U_0(\lambda),K_0(\lambda)\big],\qquad K_0(\lambda)_{\bar z}=\big[V_0(\lambda),K_0(\lambda)\big].$$

Coefficients K_m can be computed recursively.

Lemma 3 *The diagonal terms*

$$K_{2n}=v_n\sigma_3,\qquad n=1,\dots$$

of the formal Killing field (34) define tangential vector fields v_n

$$(\partial_{z\bar z}+\cosh u)v_n=0. \tag{35}$$

v_n are polynomials of $u_z,\dots,u_z^{(2n-1)}$.

Any complex vector field v_n generates two real tangential vector fields

$$w_{2n-1}=v_n+\bar v_n,\qquad w_{2n}=i(v_n-\bar v_n),\quad n=1,\dots$$

Lemma 4 *Let $u(z,\bar z)$ be a doubly periodic solution*

$$u(z+Z_i,\bar z+\bar Z_i)=u(z,\bar z),\quad i=1,2\quad \mathrm{Im}\ Z_1/Z_2\neq 0$$

of the elliptic sinh-Gordon equation and $w_n, n=1,\dots$ be the corresponding tangential real vector fields. Only finitely many tangential vectors w_n are linearly independent.

Proof. All w_n are also doubly-periodic. Equation (35) determines an elliptic operator L on the torus T :

$$Lw_n=(\partial_z\partial_{\bar z}+\cosh u)w_n=0.$$

It is well known that the spectra of this operator is discrete, which implies in particular that all the eigenspaces are finite dimensional. All tangential vectors w_n belong to the kernel of L. This observation proves the lemma

$$\dim \operatorname{span}\{w_n\}_{n=1,\dots} < \infty.$$

This lemma is the reason for the existence of a *polynomial Killing field.*

Theorem 8 *Let $u(z,\bar{z})$ be a doubly-periodic solution of the elliptic sinh-Gordon equation (30), and U_0, V_0 are given by (32). Then in the loop algebra $g_H[\lambda]$ there exists a polynomial Killing field*

$$W_0(\lambda) = \sum_{n=-(2N-1)}^{2N-1} W_n \lambda^n \tag{36}$$

which satisfies

$$\begin{aligned} W_0(\lambda)_z &= \big[U_0(\lambda), W_0(\lambda)\big], \\ W_0(\lambda)_{\bar{z}} &= \big[V_0(\lambda), W_0(\lambda)\big]. \end{aligned} \tag{37}$$

This fundamental theorem in different forms appeared first in [PS, Hit].

The coefficients of $W_n, n > 0$ are polynomials of $u_z, \dots, u_z^{(2N-1-n)}$ and $e^{\pm u/2}$, the coefficients of $W_n, n < 0$ are polynomials of $u_{\bar{z}}, \dots, u_{\bar{z}}^{(2N-1+n)}$ and $e^{\pm u/2}$, W_0 is a polynominal of $u_z, u_{\bar{z}}, \dots, u_z^{(2N-1)}, u_{\bar{z}}^{(2N-1)}$. The leading coefficient is of the form

$$W_{2N-1} = \alpha \begin{pmatrix} 0 & e^{-u/2} \\ e^{u/2} & 0 \end{pmatrix}, \qquad 0 \neq \alpha \in \mathbb{C}.$$

Solutions possessing polynomial Killing fields are called *solutions of finite type* or *finite-gap solutions.* The theory of finite-gap solutions is a well established branch [DKN, BBEIM] of the theory of integrable equations. Due to Theorem 8, all doubly-periodic solutions of the elliptic sinh-Gordon equation are finite-gap.

2.5 The spectral curve and Baker-Akhiezer function

Let $u(z,\bar{z})$ be a solution of the elliptic sinh-Gordon equation with the polynomial Killing field $W_0(\lambda)$. The curve

$$\det(W_0(\lambda) - \mu I) = 0 \tag{38}$$

is called the *spectral curve* of the solution $u(z,\bar{z})$. The spectral curve is independent of $z, \bar{z}$.

Compactified at $\mu = \infty$ the hyperelliptic curve (38) determines a compact Riemann surface $\hat{C}$ of genus $\hat{g}$. Due to symmetries of the loop algebra $g_H[\lambda]$, besides the hyperelliptic involution $(\mu,\lambda) \to (-\mu,\lambda)$ it possesses two more involutions: a holomorphic

$$\pi : (\mu,\lambda) \to (\mu,-\lambda) \tag{39}$$

and an anti-holomorphic $\hat{\tau} : (\mu, \lambda) \to (\bar{\mu}, \frac{1}{\bar{\lambda}})$.

The factor Riemann surface $C = \hat{C}/\pi$ plays central role for the explicit construction of section 2.6. The covering $\hat{C} \to C$, $(\mu, \lambda) \mapsto (\mu, \Lambda)$, $\Lambda := \lambda^2$ is unramified and C is a Riemann surface of genus g, where $\hat{g} = 2g - 1$. The anti-holomorphic involution

$$\tau : (\mu, \Lambda) \to (\bar{\mu}, \frac{1}{\bar{\Lambda}}). \tag{40}$$

acts on C.

Due to (37) the system

$$\phi_z = U_0 \phi, \qquad \phi_{\bar{z}} = V_0 \phi, \qquad W_0 \phi = \mu \phi$$

has a common vector valued solution $\phi(P, z, \bar{z})$, which is called the *Baker-Akhiezer function.* Here $P = (\mu, \lambda)$ is a point on $\hat{C}$. In the finite-gap integration theory of the sine-Gordon equation, usually a gauge equivalent function

$$\psi = \begin{pmatrix} e^{u/4} & 0 \\ 0 & e^{-u/4} \end{pmatrix} \phi$$

is used. Immersion formula (29) is obviously invariant with respect to this transformation.

Suitably normalized

$$\psi(P, 0, 0) = \begin{pmatrix} 1 \\ * \end{pmatrix}$$

this function satisfies

$$\psi(\pi P, z, \bar{z}) = \sigma_3 \psi(P, z, \bar{z}). \tag{41}$$

The Baker-Akhiezer function ψ has essential singularities at the points $\infty^\pm, 0^\pm \in \hat{C}$ defined by $\lambda(\infty^\pm) = \infty, \lambda(0^\pm) = 0$. The involution π interchanges these points $\pi(\infty^+) = \infty^-, \pi(0^+) = 0^-$. Denote their projection on C by ∞ and 0 respectively. Due to the symmetry (41) the pole divisor of ψ on $\hat{C} \setminus \{\infty^\pm, 0^\pm\}$ is the lift of a divisor $\mathcal{D}$ on $C \setminus \{\infty, 0\}$.

Finally after some computations one can prove the following analytic properties of ψ.

Theorem 9 *The Baker-Akhiezer function ψ possesses the following analytic properties:*

1. *ψ is transformed by (41) under the action of the involution π,*
2. *ψ is meromorphic on $\hat{C} \setminus \{\infty^\pm, 0^\pm\}$. The pole divisor $\mathcal{D}$ of ψ on $C \setminus \{\infty, 0\}$ is independent of $z, \bar{z}$, and is a non-special divisor of degree g. The Abel map $\mathcal{A}(\mathcal{D})$ of $\mathcal{D}$ on C satisfies*

$$\mathcal{A}(\mathcal{D} - \tau\mathcal{D}) = \mathcal{A}(0 - \infty), \tag{42}$$

3. *ψ has essential singularities at the points $\infty^\pm, 0^\pm$ of the form*

$$\begin{aligned} \psi(P, z, \bar{z}) &= \left(\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} + o(1) \right) \exp(\pm \frac{i\lambda z}{2}), \qquad P \to \infty^\pm. \\ \psi(P, z, \bar{z}) &= O(1) \exp(\mp \frac{i\bar{z}}{2\lambda}), \qquad P \to 0^\pm. \end{aligned}$$

2.6 Baker-Akhiezer function. Formulas

Due to the symmetry (41) the Baker-Akhiezer function ψ can be described in terms of the data $\{C, \mathcal{D}\}$. Here C is a hyperelliptic Riemann surface of genus g with the anti-holomorphic involution (40) and branch points at $\lambda = 0, \infty$ and $\mathcal{D}$ is a non-special divisor of degree g on C satisfying (42). We call these data admissible. It is crucial that the construction of Section 2.5 can be reversed and a result similar to Theorem 7 holds.

Theorem 10 *Let $\{C, \mathcal{D}\}$ be admissible data. There exists a Baker-Akhiezer function ψ with these data and ψ is uniquely characterized by the analytic properties listed in Theorem 9.*

Admissible $\{C, \mathcal{D}\}$ generate a finite-gap solution of the elliptic sinh-Gordon equation and thus a surface with constant mean curvature $H = 1$, which we call a CMC surface of *finite type*. It follows from Sections 2.4, 2.5 that all CMC tori are CMC surfaces of finite type.

The Baker-Akhiezer functions and hence CMC surfaces of finite type can be described explicitly. Let

$$M^2 = \Lambda \prod_{i=1}^{g} (\Lambda - \Lambda_i)(\Lambda - \frac{1}{\bar{\Lambda}_i}), \qquad |\Lambda_i| < 1 \quad \forall i$$

be a non-singular hyperelliptic curve C with an anti-holomorphic involution $\tau : \Lambda \to \frac{1}{\bar{\Lambda}}$. Choose a canonical homology basis $a_1, b_1, \ldots, a_g, b_g$ with a_i-cycles surrounding the cuts $[\Lambda_i, \frac{1}{\bar{\Lambda}_i}]$, i.e. $\tau a_i = -a_i$. Let $\omega_1, \ldots, \omega_g$ be the dual basis

$$\int_{a_n} \omega_m = 2\pi i \delta_{nm}$$

of holomorphic differentials. The period matrix

$$B_{nm} = \int_{b_n} \omega_m$$

determines the Riemann theta function

$$\theta(u) = \sum_{k \in \mathbb{Z}^g} \exp(\frac{1}{2}(Bk, k) + (u, k)), \qquad u \in \mathbb{C}^g.$$

We need also the Abelian differentials of the second kind Ω_∞, Ω_0 normalized by the condition

$$\int_{a_n} \Omega_\infty = \int_{a_n} \Omega_0 = 0, \qquad n = 1, \ldots, g$$

and the following asymptotics at the singularities

$$\begin{aligned} \Omega_\infty &\to d\sqrt{\Lambda}, \qquad && \Lambda \to \infty \\ \Omega_0 &\to d\frac{1}{\sqrt{\Lambda}}, \qquad && \Lambda \to 0. \end{aligned}$$

Denote the vector of b-periods of Ω_∞ by

$$U = (U_1, \dots, U_g), \qquad U_n = \int_{b_n} \Omega_\infty.$$

Finally note that in explicit description one can replace the divisor $\mathcal{D}$ of admissible data by its Abel map $D \in Jac(C)$. One can show that in the chosen normalizations the reality condition (42) is equivalent to the condition $D \in i\mathbb{R}^g$.

Theorem 11 *The Baker-Akhiezer function with the data $\{C, D\}$ is given by the formulas*

$$\begin{aligned}
\psi_1(P, z, \bar{z}) &= \frac{\theta(\int_\infty^P \omega + W)\theta(D)}{\theta(\int_\infty^P \omega + D)\theta(W)} \exp(\frac{i}{2}\int_\infty^P (z\Omega_\infty + \bar{z}\Omega_0)) \\
\psi_2(P, z, \bar{z}) &= \frac{\theta(\int_\infty^P \omega + W + \Delta)\theta(D)}{\theta(\int_\infty^P \omega + D)\theta(W + \Delta)} \exp(\frac{i}{2}\int_\infty^P (z\Omega_\infty + \bar{z}\Omega_0)).
\end{aligned}$$

Here $\Delta = \pi i(1, \dots, 1)$, the vector $D \in i\mathbb{R}^g$ is arbitrary,

$$W = i\mathrm{Re}\,(Uz) + D$$

and the integration paths in all the integrals are identical. The corresponding solution to the sinh-Gordon equation is given by

$$u(z, \bar{z}) = 2\log\frac{\theta(W + \Delta)}{\theta(W)}. \tag{43}$$

Applying now Theorem 6 to $\psi(P, z, \bar{z})$ with $P = P_0 = (M_0, \Lambda_0)$, $\mid \Lambda_0 \mid= 1$ one arrives at the following final formulas for CMC immersions of finite type [Bob1].

Theorem 12 *The quaternion valued solution $\Phi(z, \bar{z}, \lambda_0)$ of the linear system (27, 32) with the finite-gap coefficient (43) is given by*

$$\Phi = \frac{i}{\sqrt{\theta(W)\theta(W+\Delta)}} \begin{pmatrix} \theta(W + l) & \theta(W - l) \\ \theta(W + \Delta + l) & -\theta(W + \Delta - l) \end{pmatrix} \exp(i\sigma_3 \mathrm{Re}\,(zL)),$$

where $l = \int_\infty^{P_0} \omega$ is the Abel map of $P_0 = (M_0, \Lambda_0)$ chosen on the unit circle $\Lambda_0 = \lambda^2 = e^{2it}$ and $L = \int_\infty^{P_0} \Omega_\infty$. The matrix Φ is normalized by

$$\det \Phi = 2\frac{\theta(l)\theta(l + \Delta)}{\theta(0)\theta(\Delta)}.$$

The corresponding CMC immersion is given by (29). This immersion is doubly-periodic granted a lattice $\mathcal{L}$ with the basic vectors Z_1, Z_2 exists such that

$$\mathrm{Re}\,(Z_k U) \in 2\pi\mathbb{Z}^g, \quad \mathrm{Re}\,(2Z_k \int_\infty^{P_0} \Omega_\infty) \in 2\pi\mathbb{Z}, \qquad k = 1, 2, \tag{44}$$

and the differential Ω_∞ vanishes at the point P_0

$$\Omega_\infty(\lambda_0) = 0. \tag{45}$$

CMC tori are singled out from general quasiperiodic immersions of finite type by the periodicity conditions (44, 45), which are in fact conditions on the corresponding hyperelliptic curve C of genus g only. One can show [Jag, Bob1] that there are no CMC tori with $g = 1$ and that for $g > 1$ there exists a discrete set of spectral curves C generating CMC tori. The parameter $D \in i\mathbb{R}^g$ remains arbitrary. So the CMC tori with $g > 2$ (changes of D in the plane $\mathrm{span}\{\mathrm{Re}\, U, \mathrm{Im}\, U\}$ are equivalent to reparametrization of the torus) possess commuting deformation flows. These deformations are area preserving [Bob1].

2.7 Examples of CMC Tori

All finite-gap solutions of the sinh-Gordon equation of genus $g = 1$ and $g = 2$ are doubly-periodic. There are no CMC tori with $g = 1$. The simplest CMC tori were found by Wente [Wen] and analytically studied by Abresch [Abr] and Walter [Wal]. These tori presented in Figures 1, 2 possess a family of plane curvature lines. This implies the additional symmetry $\Lambda \to 1/\Lambda$ of the corresponding spectral curve of genus $g = 2$. The Wente torus in Figure 1 is comprised of three congruent fundamental domains shown in Figure 2.

Spectral curves of genus $g = 2$ without additional symmetries also generate CMC tori. An example is presented in Figure 3.

Taking spectral curves with $g = 3$ and the symmetry $\Lambda \to 1/\Lambda$ one obtains all CMC tori with spherical curvature lines. The fundamental domain of such an example is shown in Figure 4.

Figure 5 visualizes a CMC torus corresponding to a curve of genus $g = 5$. This torus possesses a 3-parameter family of area preserving deformations. Finally Figure 6 presents classical surfaces of Delaunay which correspond to spectral curves of genus $g = 1$ and are CMC surfaces of revolution.

The figures of this section are produced by Matthias Heil using formulas presented in this lecture and the software for calculations on hyperelliptic Riemann surfaces developed by him for Sfb288 in Berlin. Further examples can be found in [Hei].

3 Lecture. Bonnet Pairs

In this lecture we present some preliminary results on local and global geometry of Bonnet pairs. This is a work in progress.

3.1 Basic Facts about Bonnet Pairs

Let $\mathcal{F}_1, \mathcal{F}_2 \subset \mathbb{R}^3$ be a smooth Bonnet pair (Bonnet mates), i.e. two isometric non-congruent surfaces with coinciding mean curvatures at the corresponding points. As conformal immersions of the same Riemann surface

$$F_1 : \mathcal{R} \to \mathbb{R}^3, \qquad F_2 : \mathcal{R} \to \mathbb{R}^3$$

they are described by the corresponding Hopf differentials Q_1, Q_2, the common metric $e^u\, dz d\bar{z}$ and the mean curvature function H. Since the surfaces are non-congruent the Hopf differentials differ $Q_1 \not\equiv Q_2$.

The Gauss-Codazzi equations immediately imply

Proposition 2 *Let Q_1 and Q_2 be the Hopf differentials of a Bonnet pair $F_{1,2} \to \mathbb{R}^3$. Then*

$$h = Q_2 - Q_1 \tag{46}$$

is a holomorphic quadratic differential $h\, dz^2$ on $\mathcal{R}$ and

$$\mid Q_1 \mid = \mid Q_2 \mid . \tag{47}$$

Due to the second statement of Proposition 2 the umbilic points of $\mathcal{F}_1$ and $\mathcal{F}_2$ correspond. Denote by

$$\mathcal{U} = \{P \in \mathcal{R} :\ Q_k(P) = 0\}$$

the corresponding set of umbilic points on $\mathcal{R}$.

Proposition 3 *Let Q_1 and Q_2 be the Hopf differentials of a Bonnet pair $F_{1,2} \to \mathbb{R}^3$. Then there exist a holomorhic quadratic differential h on $\mathcal{R}$ and a smooth real valued function $\alpha : \mathcal{R} \to \mathbb{R}$ such that*

$$Q_1 = \frac{1}{2} h(i\alpha - 1), \qquad Q_2 = \frac{1}{2} h(i\alpha + 1). \tag{48}$$

Proof. Define a smooth quadratic differential $g\, dz^2$ by

$$g = Q_1 + Q_2.$$

Identity (47) implies

$$h\bar{g} + g\bar{h} = 0.$$

Thus the quotient

$$\alpha = -i\frac{g}{h}$$

is a real valued smooth function $\alpha : \mathcal{R} \setminus \mathcal{U}_h \to \mathbb{R}$, where

$$\mathcal{U}_h = \{P \in \mathcal{R} : \ h(P) = 0\}$$

is the zero set of h. Let us show that α can be smoothly extended to the whole of $\mathcal{R}$. At any $z_0 \in \mathcal{U}_h$ the holomorphic differential h is of the form

$$h(z) = (z - z_0)^J h_0(z), \quad h_0(z_0) \neq 0, \quad J \in \mathbb{N}.$$

Real-valuedness of α near z_0 implies

$$g(z) = (z - z_0)^J g_0(z)$$

with g_0 smooth, which in its turn implies smoothness of α at z_0.

Corollary 2 *Umbilic points of a Bonnet pair are isolated. The umbilic set coincides with the zero set of h*

$$\mathcal{U} = \mathcal{U}_h.$$

The number $-J$ where J is defined above is called the *index* of the umbilic point. We call the zero divisor

$$D_u = (h)$$

of h the *umbilic divisor* of a Bonnet pair.

For compact Riemann surfaces $\mathcal{R}$, Propositions 2, 3 imply the following

Theorem 13
(i) There exist no Bonnet pairs of genus $g = 0$.
(ii) Bonnet pairs of genus $g = 1$ have no umbilic points.
(iii) The umbilic divisor D_u of a Bonnet pair of genus $g \geq 1$ is of degree $4g - 4$ and its class is $D_u \equiv 2K$, where K is the canonical divisor.

Proof. A holomorphic quadratic differential on a sphere vanishes identically $h \equiv 0$, which means $Q_1 = Q_2$, and the surfaces are congruent. A holomorphic quadratic differential on a torus does not have zeros, thus $\mathcal{U} = \emptyset$ for tori.

The point (i) of Theorem 13 was proven in [LT].

Taking into account the similarity of the analytic description of Bonnet surfaces and CMC surfaces and the progress in the investigation of CMC surfaces achieved by methods from the theory of integrable systems (see Section 2), the most promising open problem to attack by these methods seems to be the problem of existence and description of Bonnet tori mates.

For tori one has $\mathcal{R} = \mathbb{C}/\mathcal{L}$. Scaling the lattice $\mathcal{L}$ appropriately one can always normalize $h = -i$, i.e.

$$Q_1 = \frac{1}{2}(\alpha + i), \qquad Q_2 = \frac{1}{2}(\alpha - i). \tag{49}$$

The corresponding Gauss-Codazzi equations of Bonnet mates become

$$\begin{aligned} 2u_{z\bar{z}} + H^2 e^u - (1+\alpha^2)e^{-u} &= 0, \\ \alpha_{\bar{z}} - e^u H_z &= 0, \\ \alpha_z - e^u H_{\bar{z}} &= 0. \end{aligned} \tag{50}$$

Note that the Gauss-Codazzi equations of isothermic surfaces $Q = \alpha/2 \in \mathbb{R}$ differ only slightly

$$\begin{aligned} 2u_{z\bar{z}} + H^2 e^u - \alpha^2 e^{-u} &= 0, \\ \alpha_{\bar{z}} - e^u H_z &= 0, \\ \alpha_z - e^u H_{\bar{z}} &= 0. \end{aligned} \tag{51}$$

3.2 Lax Representation and Connection to Isothermic Surfaces

Looking for a Lax representation for Bonnet pairs, it is natural to try to merge the frame equations of two Bonnet mates. For tori, cylinders, or simply connected domains it is enough to consider the case of $\mathcal{R}$ being a domain D in $\mathbb{C}$. Cylinders and tori are distinguished by the corresponding periodicity lattices $\mathcal{L}$. Since our main interest lies in the investigation of tori let us restrict ourselves to the case of umbilic free Bonnet pairs. As in Section 2, introducing a global complex variable z on D we normalize the corresponding frame matrices traceless and the Hopf differentials as in (49). The following theorem can be checked directly.

Theorem 14 *Normalized by (49), conformal frames $\Phi_1, \Phi_2 : D \to SU(2)$ of a Bonnet pair*

$$\Phi_{k\,z} = U_k \Phi_k, \qquad \Phi_{k\,\bar{z}} = V_k \Phi_k$$
$$U_k = \begin{pmatrix} \frac{u_z}{4} & -Q_k e^{-u/2} \\ \frac{H}{2} e^{u/2} & -\frac{u_z}{4} \end{pmatrix}, \qquad V_k = \begin{pmatrix} -\frac{u_{\bar{z}}}{4} & -\frac{H}{2} e^{u/2} \\ \bar{Q}_k e^{-u/2} & \frac{u_{\bar{z}}}{4} \end{pmatrix}$$

can be extended

$$\Phi(z, \bar{z}, \lambda = 0) = \begin{pmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{pmatrix} (z, \bar{z}) \tag{52}$$

to $\Phi(z, \bar{z}, \lambda)$ satisfying

$$\Phi_z = U\Phi, \qquad \Phi_{\bar{z}} = V\Phi \tag{53}$$

with

$$\begin{aligned} U &= \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{U_1} & 0 & 0 \\ & & -i\lambda e^{u/2} & 0 \\ \hline 0 & -i\lambda e^{-u/2} & \multicolumn{2}{c}{U_2} \\ 0 & 0 & & \end{array} \right), \\ V &= \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{V_1} & 0 & -i\lambda e^{u/2} \\ & & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{V_2} \\ -i\lambda e^{-u/2} & 0 & & \end{array} \right). \end{aligned} \tag{54}$$

Conversely, the linear system (54) with $Q_1 = Q \neq Q_2 = \bar{Q}$ is compatible if and only if the metric e^u, the mean curvature function H, and the Hopf differentials Q_1, Q_2 on D satisfy the Gauss-Codazzi equations of Bonnet pairs. Conformal frames of the Bonnet mates are determined through (52) by a suitably normalized common solution $\Phi(z, \bar{z}, \lambda)$ of (53) evaluated at $\lambda = 0$.

Remark. In the general case of an arbitrary Riemann surface $\mathcal{R}$ and holomorphic quadratic differential h, a spinor form of the Lax representation for Bonnet pairs similar to (14) can be easily made by merging the spinor frames (see Section 1) of the corresponding surfaces.

Remark. In the case $Q_1 = Q_2 = Q = \bar{Q}$ system (53, 54) becomes a Lax representation for isothermic surfaces in $\mathbb{R}^3$ in isothermic coordinates.

The matrices $U + V$ and $i(U - V)$ corresponding to real vector fields ∂_x and ∂_y possess the symmetries

$$A(-\lambda) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} A(\lambda) \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \tag{55}$$

$$\overline{A(\bar{\lambda})} = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} A(\lambda) \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}. \tag{56}$$

Denote by

$$g_B[\lambda] = \{A : \mathbb{R} \to gl(2, \mathbb{H}) \mid A(-\lambda) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} A(\lambda) \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}\}$$

the corresponding loop algebra, and by

$$G_B[\lambda] = \{\phi : \mathbb{R} \to GL(2, \mathbb{H}) \mid \phi(-\lambda) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \phi(\lambda) \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}\}$$

the corresponding loop group.

By the normalization (52) the solution $\Phi(z, \bar{z}, \lambda)$ is determined uniquely up to right multiplication by a matrix depending only on λ

$$\Phi(z, \bar{z}, \lambda) \to \Phi(z, \bar{z}, \lambda) G(\lambda), \qquad G(0) = 1. \tag{57}$$

$\Phi(z, \bar{z}, \lambda)$ can be chosen to lie in $G_B[\lambda]$. Then the matrix

$$\begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix} := \Phi^{-1} \Phi_\lambda \mid_{\lambda=0} \tag{58}$$

is off-diagonal and its coefficients are quaternion valued functions of z and $\bar{z}$.

For a description of the geometry of S and T, the notion of isothermic surfaces in $\mathbb{R}^4$ and of the dual isothermic surface is required. An immersion $f : D \to \mathbb{R}^4$ is called *isothermic* if it is conformal and the vector f_{xy} lies in the tangent plane $f_{xy} \in \operatorname{span}\{f_x, f_y\}$. It is convenient to describe isothermic surfaces in $\mathbb{R}^4 = \mathbb{H}$ in quaternionic form, i.e. as mappings $f : D \to \mathbb{H}$ with the coordinates

$$f = f_0\mathbf{1} + f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}.$$

Its differential is $df = f_x\,dx + f_y\,dy$. An important property of an isothermic immersion $f : D \to \mathbb{H}$ is the closedness of the form

$$df^* := -f_x^{-1}dx + f_y^{-1}dy \tag{59}$$

The corresponding immersion determined up to translation by this form and denoted by $f^* : D \to \mathbb{H}$ is also isothermic and is called the *dual isothermic surface*. Note that the dual isothermic surface is defined through one-forms and therefore the periodicity properties of f are not respected. The relation (59) is an involution. Moreover one can check that

Lemma 5 *The transformation (59) is characteristic for isothermic surfaces.*

Proposition 4
(i) Let Q_1 and Q_2, normalized by (49), be Hopf differentials of a Bonnet pair $F_{1,2} : D \to \operatorname{Im}\mathbb{H}$. Then $T : D \to \mathbb{H}$ defined by (58) is an isothermic surface in the three-dimensional sphere $S^3 \subset \mathbb{R}^4 = \mathbb{H}$ and $S : D \to \mathbb{H}$ in (58) is its dual $S = T^*$. The isothermic surfaces S and T are related to the Bonnet pair by

$$\begin{aligned} dF_1 &= dS\,T = dT^*\,T \\ dF_2 &= T\,dS = T\,dT^*. \end{aligned} \tag{60}$$

(ii) Let $Q_1 = Q_2 = Q$ in (54) be real. Then $S : D \to \operatorname{Im}\mathbb{H}$ given by (58) is the isothermic surface determined by the fundamental forms with the coefficients e^u, H, Q, and $T : D \to \operatorname{Im}\mathbb{H}$ is its dual $T = S^*$.

Proof. Let us prove the first statement. Formula (58) implies

$$d\begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix} = \Phi^{-1}\left(U_\lambda dz + V_\lambda d\bar{z}\right)\Phi\,|_{\lambda=0}$$

or equivalently

$$dS = e^{u/2}\Phi_1^{-1}(\mathbf{i}dx + \mathbf{j}dy)\Phi_2, dT = e^{-u/2}\Phi_2^{-1}(\mathbf{i}dx - \mathbf{j}dy)\Phi_1. \tag{61}$$

These frames are obviously conformal and are related by (59). Lemma 5 implies that T is isothermic. Moreover dT can be integrated explicitly

$$T = \Phi_2^{-1}\Phi_1. \tag{62}$$

Indeed, differentiating the last expression one obtains

$$dT = \Phi_2^{-1}(d\Phi_1\Phi_1^{-1} - d\Phi_2\Phi_2^{-1})\Phi_1 =$$
$$e^{-u/2}\Phi_2^{-1}\left((Q_2 - Q_1)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}dz + (\bar{Q}_1 - \bar{Q}_2)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}d\bar{z}\right)\Phi_1$$

which coincides with the previous expression for dT. Integrating, one obtains $T = \Phi_2^{-1}\Phi_1 + \text{const}$. The constant can be normalized to zero by transformation (57) with an appropriate $G(\lambda) \in G_B[\lambda]$. Obviously the surface given by (62) lies in the three sphere $S^3 = \mathbb{H}_1$. Using (62) we obtain

$$\begin{aligned} dS\,T &= -ie^{-u/2}\Phi_1^{-1}\begin{pmatrix} 0 & d\bar{z} \\ dz & 0 \end{pmatrix}\Phi_1 \\ TdS &= -ie^{-u/2}\Phi_2^{-1}\begin{pmatrix} 0 & d\bar{z} \\ dz & 0 \end{pmatrix}\Phi_2 \end{aligned}$$

which coincides with (60).

The proof of the second claim is even simpler (see [BP]).

The next theorem is essentially due to Bianchi [Bia2]. A modern version of it in terms of quaternions is derived in [KPP].

Theorem 15 *$F_{1,2} : D \to \operatorname{Im}\mathbb{H} = \mathbb{R}^3$ build a Bonnet pair if and only if there exists an isothermic surface $T : D \to \mathbb{H}_1 = S^3 \subset \mathbb{R}^4$ (or equivalently an isothermic surface $R : D \to \operatorname{Im}\mathbb{H} = \mathbb{R}^3$) such that*

$$\begin{aligned} dF_1 &= dT^*T = \frac{1}{2}(1-R)dR^*(1+R) \\ dF_2 &= TdT^* = \frac{1}{2}(1+R)dR^*(1-R). \end{aligned} \tag{63}$$

Proof. Let us show first the equivalence of the representations in terms of T and R. The class of isothermic surfaces is invariant under Möbius transformations. In particular, isothermic surfaces in S^3 and in $\mathbb{R}^3$ are related by stereographic projection, which in quaternionic form can be represented by

$$T = \frac{1+R}{1-R}$$

with $R \in \operatorname{Im}\mathbb{H} = \mathbb{R}^3, T \in \mathbb{H}_1 = S^3$. For the frames this implies

$$\begin{aligned} dT &= 2(1-R)^{-1}dR(1-R)^{-1} \\ dT^* &= \frac{1}{2}(1-R)dR^*(1-R), \end{aligned}$$

which proves the equivalence of the two representations of the theorem. The passing from Bonnet pairs to isothermic surfaces is proven in Proposition 4. Conversely, given R, the representation (63) shows that $dF_{1,2}$ are conformal and lie in $\operatorname{Im}\mathbb{H}$. Due to $dF_2 = TdF_1T^{-1}$ the immersions are isometric.

Similarly one can show (see [KPP]) that the forms dF_1, dF_2 defined by (63) are closed and that the mean curvature functions of the corresponding surfaces coincide.

To study a global version of Theorem 15 in the case of Bonnet tori mates let $F_{1,2} : \mathbb{C} \to \mathbb{R}^3$ be a Bonnet pair with doubly periodic frames $dF_{1,2}$ with the same period lattice $\mathcal{L}$. The frames $\Phi_{1,2}$ are periodic up to a sign

$$\Phi_k(z + Z_i) = (-1)^{p_{ik}}\Phi_k(z),$$

where Z_1, Z_2 are generators of $\mathcal{L}$ and $p_{ik} \in \mathbb{Z}_2$ characterize the spin structures of the immersions. The isothermic surface given by (62) is a torus in S^3 if the spin structures of F_1 and F_2 coincide ($p_{i1} = p_{i2}, i = 1, 2$) and is a torus in three dimensional real projective space $\mathbb{R}P^3 = S^3/\{-1\}$ if the spin structures differ. Conversely, an isothermic immersion $T : \mathbb{C}/\mathcal{L} \to S^3/\{-1\}$ generates by (63) a Bonnet pair with frames $dF_{1,2}$ defined on $\mathbb{C}/\mathcal{L}$. The spin structures of these two surfaces are the same iff T is an immersion to S^3 with the lattice $\mathcal{L}$.

Corollary 3 *Bonnet mates with doubly periodic frames $dF_{1,2}$ are in one to one correspondence with isothermic tori in $\mathbb{R}P^3$. Bonnet mates with doubly periodic frames $dF_{1,2}$ and coinciding spin structure are in one to one correspondence with isothermic tori in S^3. The corresponding relations are given by formulas (63).*

Formula (63) allows us to control the periodicity of the frame of a Bonnet pair. To be able to control the periodicity of the immersion one needs an analog of formula (29) describing the corresponding immersion without integration. We call a solution

$$\begin{aligned} \Phi : D \times \mathbb{R} &\to G_B[\lambda] \\ (z, \lambda) &\mapsto \Phi(z, \bar{z}, \lambda) \end{aligned}$$

of (53, 54) *normalized* (see proof of Proposition 4) if the coefficient T of its decomposition (58) at $\lambda = 0$ is a unitary quaternion $T : D \to \mathbb{H}_1$, i.e. (62) holds. Obviously this solution can be extended to all $\lambda \in \mathbb{C}$.

Theorem 16 *Let $\Phi(z, \bar{z}, \lambda)$ a normalized solution of (54) with Q of the form (49). Then the corresponding Bonnet pair $F_{1,2}(z, \bar{z})$ is restored by the following coefficients of quaternionic 2 by 2 matrices*

$$\begin{pmatrix} F_1 & 0 \\ 0 & * \end{pmatrix} = \frac{1}{2}\Phi^{-1}\Phi_{\lambda\lambda} \,|_{\lambda=0}, \tag{64}$$

$$\begin{pmatrix} * & 0 \\ 0 & F_2 \end{pmatrix} = \frac{1}{2}\Phi^{-1}\Phi_{\lambda\lambda} - \left(\Phi^{-1}\Phi_\lambda\right)_\lambda |_{\lambda=0}. \tag{65}$$

Proof. To prove the formula for dF_1 let us differentiate it by z and $\bar{z}$

$$d(\frac{1}{2}\Phi^{-1}\Phi_{\lambda\lambda}) = \Phi^{-1}(U_\lambda dz + V_\lambda d\bar{z})\Phi_\lambda,$$

where we used $U_{\lambda\lambda} = V_{\lambda\lambda} = 0$. Evaluating this expression at $\lambda = 0$ using (52), (62) and (61), one finally obtains

$$d(\frac{1}{2}\Phi^{-1}\Phi_{\lambda\lambda})_{|\lambda=0} = \begin{pmatrix} dS\,T & 0 \\ 0 & dT\,S \end{pmatrix}.$$

Now the formula for dF_1 follows from (60). The formula for dF_2 is proven by an analogous computation

$$d(\frac{1}{2}\Phi^{-1}\Phi_{\lambda\lambda} - \left(\Phi^{-1}\Phi_\lambda\right)_\lambda)_{|\lambda=0} = \begin{pmatrix} S\,dT & 0 \\ 0 & T\,dS \end{pmatrix}.$$

3.3 Loop Group Description

As we have seen already in Proposition 4 the theory developed in Section 3.2 includes two different cases: the case of Bonnet pairs when the Hopf differentials $Q_1 = Q$ and $Q_2 = \bar{Q}$ are different and thus generate two non-congruent surfaces, and the case $Q = \bar{Q}$ of isothermic surfaces in $\mathbb{R}^3$. The loop group $G_B[\lambda]$ and the loop algebra $g_B[\lambda]$ of Bonnet pairs are described in Section 3.2. In the case of isothermic surfaces $Q = \bar{Q}$, the corresponding loop group and algebra are specialized further as follows:

$$
\begin{aligned}
G_I[\lambda] &= \{\phi \in G_B[\lambda] \mid \phi^T(\lambda) \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \phi(\lambda) = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\} \\
g_I[\lambda] &= \{A \in g_B[\lambda] \mid A^T(\lambda) = -\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} A(\lambda) \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\}.
\end{aligned}
$$

As in the previous sections the main strategy for applying analytic methods of the theory of integrable systems consists of two steps: first, to characterize the frame equations through analytic properties of Φ as a function of λ without referring to the special geometric nature of the coefficients of the frame equations and, second, to construct those $\Phi(\lambda)$ explicitly. For this purpose it is more convenient to pass to a gauge equivalent

$$\Phi_0 = \begin{pmatrix} e^{-u/4}\mathbf{1} & 0 \\ 0 & e^{u/4}\mathbf{1} \end{pmatrix} \Phi \tag{66}$$

to (53), linear problem

$$\Phi_{0\,z} = U\Phi_0, \qquad \Phi_{0\,\bar{z}} = V\Phi_0 \tag{67}$$

$$
\begin{aligned}
U_0 &= \begin{pmatrix} 0 & -Qe^{-u/2} & 0 & 0 \\ \frac{H}{2}e^{u/2} & -\frac{u_z}{2} & -i\lambda & 0 \\ 0 & -i\lambda & \frac{u_z}{2} & -\bar{Q}e^{-u/2} \\ 0 & 0 & \frac{H}{2}e^{u/2} & 0 \end{pmatrix}, \\
V_0 &= \begin{pmatrix} -\frac{u_{\bar{z}}}{2} & -\frac{H}{2}e^{u/2} & 0 & -i\lambda \\ \bar{Q}e^{-u/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{H}{2}e^{u/2} \\ -i\lambda & 0 & Qe^{-u/2} & \frac{u_{\bar{z}}}{2} \end{pmatrix},
\end{aligned}
\tag{68}
$$

normalizing the solution at $\lambda = \infty$. Note that all immersion formulas (64, 58) are preserved under this gauge transformation.

Theorem 17 *Let*

$$
\begin{aligned}
\Phi_0 : D \times \mathbb{R} &\rightarrow G_B[\lambda] \\
z, \lambda &\mapsto \Phi_0(z, \bar{z}, \lambda)
\end{aligned}
$$

be a smooth mapping satisfying

$$\Phi_{0\,z}\Phi_0^{-1} = A(z,\bar{z})\lambda + B(z,\bar{z})$$

with some $A(z,\bar{z})$ *and* $B(z,\bar{z})$*, having the asymptotics*

$$\Phi_0(z,\bar{z},\lambda) = (L + M(z,\bar{z})\lambda^{-1} + o(\lambda^{-1}))\exp(-i\lambda(J_1 z + J_2\bar{z}))C(\lambda) \tag{69}$$

at $\lambda \to \infty$ *with*

$$L = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \; J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; J_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and some $M(z,\bar{z})$ *and* $C(\lambda)$*. If, for the coefficients of* M*, it holds that*

$$M_{31}(z,\bar{z}) = -M_{42}(z,\bar{z}), \tag{70}$$

then

$$U_0 := \Phi_{0\,z}\Phi_0^{-1}, \qquad V_0 := \Phi_{0\,\bar{z}}\Phi_0^{-1}$$

can be parametrized as in (68) with some real valued $u(z,\bar{z}), H(z,\bar{z})$*. If in addition*

$$M_{21}(z,\bar{z}) = -M_{12}(z,\bar{z}), \tag{71}$$

then $Q(z,\bar{z})$ *in (68) is also real valued* $Q = \bar{Q}$ *and* Φ *related to* Φ_0 *by (66) can be chosen in* $G_I[\lambda]$*.*

Proof. Due to the assumption of the theorem, U_0 is of the form $U_0 = A\lambda + B$. Substituting the asymptotics (69) one obtains for the coefficients

$$A = -iLJ_1L^{-1}, \qquad B = [ML^{-1}, A].$$

Similarly the symmetry (55) and the asymptotics (69) imply $V_0 = C\lambda + D$ with

$$C = -iLJ_2L^{-1}, \qquad D = [ML^{-1}, C].$$

The matrices A and C are of the required form. For the matrices B and D a simple computation gives

$$\begin{aligned} B &= -i\begin{pmatrix} 0 & M_{12} & 0 & 0 \\ -M_{31} & M_{22} - M_{32} & 0 & 0 \\ 0 & 0 & M_{32} - M_{22} & -M_{21} \\ 0 & 0 & M_{42} & 0 \end{pmatrix} \\ D &= -i\begin{pmatrix} M_{11} - M_{41} & -M_{42} & 0 & 0 \\ M_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{31} \\ 0 & 0 & -M_{12} & M_{41} - M_{11} \end{pmatrix}, \end{aligned}$$

where we have used the symmetry of M

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} M \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = -M.$$

The anti-holomorphic involution $\lambda \to \bar{\lambda}$ of the loop group implies

$$M_{12} = \bar{M}_{21},\ M_{42} = \bar{M}_{31},\ M_{11} = -\bar{M}_{22},\ M_{32} = -\bar{M}_{41}.$$

Finally, comparing the coefficients and using the compatibility conditions one shows that U_0, V_0 are of the form (68).

3.4 Surfaces of Finite Type

We have shown that isothermic surfaces and Bonnet pairs can be studied in frames of the theory of integrable systems. Applying standard methods one can define for these surfaces higher flows, the Bäcklund-Darboux transformations[1], finite-gap solutions etc.

In contrast to the case of CMC surfaces, the linearizations of the Gauss-Codazzi equations of isothermic surfaces (51) and of Bonnet pairs (50) are not elliptic. This fact prevents us from applying the arguments of Section 2 and claiming that all corresponding immersions with doubly periodic fundamental forms are generated by finite-gap solutions. We call surfaces corresponding to the finite-gap solutions of the Gauss-Codazzi equations or equivalently surfaces with a polynomial Killing field (see Section 2) surfaces of *finite type*. This class of surfaces is worth studying, in particular, since it may contain Bonnet tori mates.

A polynomial Killing field $W(\lambda)$ of an isothermic surface or a Bonnet pair of finite type is an element of the loop algebra $g_B[\lambda]$. The coresponding spectral curve $\hat{C}$

$$\det(W(\lambda) - \mu I) = 0$$

is a four-sheeted covering $\hat{C} \to \bar{\mathbb{C}} \ni \mu$. Due to symmetries of the loop algebra it possesses a holomorphic involution $\pi : (\mu, \lambda) \to (\mu, -\lambda)$ and an anti-holomorphic involution $\hat{\tau} : (\mu, \lambda) \to (\bar{\mu}, \bar{\lambda})$. The Riemann surface $C = \hat{C}/\pi$ is an algebraic curve of μ and $\Lambda = \lambda^2$. It also possesses an anti-holomorphic involution $\tau : (\mu, \Lambda) \to (\bar{\mu}, \bar{\Lambda})$. In the case of isothermic surfaces in $\mathbb{R}^3$ the Killing field lies in the subalgebra $g_I[\lambda]$, which implies the additional symmetry $\sigma : (\mu, \Lambda) \to (-\mu, \Lambda)$ of C. The factor curve $C_\sigma := C/\sigma$ is quadratic in μ^2 and thus a hyperelliptic one. This fact simplifies the theory and leads to a more effective description of the corresponding surfaces.

Proposition 5 *The spectral curve C of an isothermic surface is a double covering of a hyperelliptic curve.*

[1]The Bäcklund-Darboux transformations of isothermic surfaces are known already in local differential geometry [Dar, Bia3] and in theory of solitons [Cie]

Let us indicate the steps of construction of the finite-gap solutions in this case. Although the considerations are similar to those of Section 2, technically they are more involved. By more or less standard technique one describes explicitly finite-gap solutions of the complexified system (51), i.e. of the system corresponding to the loop algebra without the real reduction (56). The spectral curve in this case remains to be a double covering of a hyperelliptic curve, but does not necessarily possess the real involution τ. The differentials analogous to the differentials Ω_1 and Ω_2 turn out to be Abelian differentials of the second kind with singularities at the branch points of the covering $C \to C_\sigma$. They are odd with respect to the involution σ. Their vectors of b-periods describing the velocity of the linear dynamics on the Jacobian lie in the odd part of the Jacobian with respect to the involution σ. The dynamic of the corresponding nonlinear system lies on the Prym variety $Prym_\sigma(C)$ of the covering $C \to C_\sigma$.

The real reduction (56) leads to constraints on parameters of the finite-gap solution. It is a technical but rather involved problem to classify all possible cases leading to real finite-gap solutions of (51) and thus to isothermic surfaces of finite type. This is not yet done. Even more important is to realize this program for the system (50). Note that due to explicit formulas (58, 64) for the corresponding immersions, the isolation of tori from general surfaces of finite type is then straightforward.

4 Lecture. Bonnet Families

Here we classify Bonnet families. The content of this lecture is based on results obtained jointly with Ulrich Eitner [BE1, BE2].

4.1 Definition of Bonnet Surfaces and Simplest Properties

Let $\mathcal{F}$ be a smooth surface in $\mathbb{R}^3$ with non-constant mean curvature function. $\mathcal{F}$ is called a *Bonnet surface* if it possesses a one-parameter family

$$\mathcal{F}_\tau, \qquad \tau \in (-\epsilon, \epsilon),\ \epsilon > 0,\ \mathcal{F}_0 = \mathcal{F}$$

of non-trivial[2] isometric deformations preserving the mean curvature function. The family $(\mathcal{F}_\tau)_{\tau\in(-\epsilon,\epsilon)}$ is called a *Bonnet family.* A Bonnet family can be described as a conformal mapping

$$F: \begin{array}{ccc} (-\epsilon,\epsilon)\times\mathcal{R} & \longrightarrow & \mathbb{R}^3 \\ (\tau, z) & \mapsto & F(\tau, z, \bar z) \end{array} \qquad \epsilon > 0, \tag{72}$$

where z is a local coordinate z on the Riemann surface $\mathcal{R}$.

The set $\mathcal{U} \subset \mathcal{R}$ of preimages of the umbilic points on $\mathcal{F}_\tau$ is independent of τ (see Section 3). Obviously, the set

$$\mathcal{V} = \{P \in \mathcal{R}\ :\ dH(P) = 0\}$$

of preimages of critical points of the mean curvature function on $\mathcal{F}_\tau$ is also τ-independent.

Similarly to Proposition 3 one can prove

Proposition 6 *The holomorphic quadratic differential*

$$\varphi(z,\tau)\, dz^2 := \frac{\partial}{\partial\tau} Q(z,\bar z,\tau)\, dz^2. \tag{73}$$

vanishes exactly at the umbilic points.

In a neighbourhood of a non-umbilic point $Q(P) \neq 0$ there exist smooth real-valued functions $\psi(z,\bar z,\tau)$, $q(z,\bar z)$ such that $Q(z,\bar z,\tau) = e^{\mathrm{i}\,\psi(z,\bar z,\tau)}\, q(z,\bar z)$. Differentiating we obtain

$$Q(z,\bar z,\tau) = -\mathrm{i}\,\frac{\varphi(z,\tau)}{\psi_\tau(z,\bar z,\tau)} \tag{74}$$

This representation is a special case of (8), which implies

Corollary 4 *Bonnet surfaces are isothermic away from umbilic points.*

[2] We call an isometry of a surface non-trivial if it is not induced by an isometry of the ambient space.

Representation (74) implies that ψ_τ is a harmonic function on $\mathcal{R} \setminus \mathcal{U}$. Moreover, from a more detailed analysis of the local behaviour at an umbilic point one can deduce [BE2] that the function ψ_τ can be extended to a nowhere vanishing harmonic function on $\mathcal{R}$

$$\psi_\tau : \mathcal{R} \to \mathbb{R}_*.$$

With so defined ψ_τ, the identity

$$\varphi = \mathrm{i}\, \psi_\tau\, Q \tag{75}$$

holds on all $\mathcal{R}$.

Theorem 18 *Let $\mathcal{F}$ be a Bonnet surface. Then*

(i.) Umbilic points are critical points of the mean curvature function $\mathcal{U} \subset \mathcal{V}$.

(ii.) The set $\mathcal{V}$ of critical points of the mean curvature function is discrete in $\mathcal{R}$.

Proof: From the Codazzi equations it follows that $dH = 0$ if and only if $Q_{\bar{z}} = 0$. Differentiating (75) with respect to $\bar{z}$ one obtains

$$\mathrm{i}\, \psi_{\tau \bar{z}}\, Q + \mathrm{i}\, Q_{\bar{z}} \equiv 0.$$

Thus $Q = 0$ implies $Q_{\bar{z}} = 0$. To show that $\mathcal{V}$ is a discrete subset of $\mathcal{R}$, we use

$$Q_{\bar{z}} = 0 \Longleftrightarrow \mathrm{i}\, \psi_{\tau \bar{z}}\, Q = 0 \Longleftrightarrow \mathrm{i}\, \psi_{\tau z}\, Q = 0 \Longleftrightarrow \frac{\psi_{\tau z}\, Q_\tau}{\psi_\tau} = 0$$

where we use that ψ_τ is a never vanishing harmonic function on $\mathcal{R}$. Since $\psi_{\tau z}\, Q_\tau\, dz^3$ is a cubic holomorphic differential, its zeros (which comprise the set $\mathcal{V}$) are discrete.

4.2 Local Theory away from Critical Points

In this section we develop local theory of Bonnet surfaces in $\mathbb{R}^3$ away from possible critical points of the mean curvature function $F : (-\epsilon, \epsilon) \times \mathcal{R} \setminus \mathcal{V} \to \mathbb{R}^3$. The preimages of holomorphic local charts $z : U \subset \mathcal{R} \setminus \mathcal{V} \to \mathbb{C}$ are always assumed to be simply connected.

The following "stationary" characterisation of Bonnet surfaces is classical [Gra].

Theorem 19 *An umbilic free surface $\mathcal{F}$ is a Bonnet surface if and only if*
(i.) $\mathcal{F}$ is isothermic.
(ii.) $1/Q$ is harmonic, i.e.

$$\left(\frac{1}{Q(z, \bar{z})} \right)_{z\bar{z}} = 0, \tag{76}$$

where z is an isothermic coordinate and $Q\, dz^2$ is the Hopf differential.

Proof: "$\Rightarrow$": follows from Corollary 4, (74), and harmonicity of ψ_T.
"$\Leftarrow$": Let z be an isothermic coordinate and $e^{u(z,\bar{z})}, H(z,\bar{z}), Q(z,\bar{z})$ be a solution to the Gauss-Codazzi equations (7) with Q satisfying (76). Locally there exists a holomorphic function $h(z)$ such that $Q(z,\bar{z}) = 1/(h(z) + \bar{h}(\bar{z}))$. Define $Q(z,\bar{z},\tau)$ via

$$Q(z,\overline{z},\tau) = \left(\frac{1 - \mathrm{i}\,T\,\overline{h(z)}}{1 + \mathrm{i}\,T\,h(z)}\right) \frac{1}{h(z) + \overline{h(z)}}, \tag{77}$$

where T is a deformation parameter equivalent to τ. One can easily check that $|Q(z,\bar{z},\tau)| = |Q(z,\bar{z})|$ and $Q_{\bar{z}}(z,\bar{z},\tau) = Q_{\bar{z}}(z,\bar{z})$ holds. Thus $e^{u(z,\bar{z})}$, $H(z,\bar{z})$, $Q(z,\bar{z},\tau)$ is also a solution of (7) for all T. The surfaces corresponding to different T are isometric and have the same mean curvature. They form a Bonnet family.

It is easy to see that the Codazzi equations imply

$$h'(z)\,H_z(z,\bar{z}) = \overline{h'(z)}\,H_{\bar{z}}(z,\bar{z}).$$

Introducing locally the new conformal coordinate

$$w = \int \frac{1}{h'(z)}\,dz \tag{78}$$

one finds that the mean curvature function depends on

$$t = w + \bar{w}$$

only. This finally leads to the fact that the Gauss-Codazzi equations can be reduced to an ordinary differential equation, which is derived below.

One can directly check that Q satisfies

$$Q_{\bar{w}}(w,\bar{w},T) = \bar{Q}_w(w,\bar{w},T) = -|Q(w,\bar{w},T)|^2.$$

Inserting

$$e^u = \frac{2\,Q_{\bar{w}}}{H_w} = -\frac{2\,|Q|^2}{H'} \tag{79}$$

into the Gauss equation one obtains

$$\left(\frac{H''(t)}{H'(t)}\right)' - H'(t) = |Q|^2\left(2 - \frac{H^2(t)}{H'(t)}\right), \quad ' = \frac{d}{d\,t}. \tag{80}$$

For a general holomorphic function $h(w)$, equation (80) does not possess a solution depending on t only.

The identity $|Q|^2_w = |Q|^2_{\bar{w}}$ implies

$$\left(\frac{h''(w)}{h'(w)} - \overline{\frac{h''(w)}{h'(w)}}\right)\left(h(w) + \overline{h(w)}\right) = 2\,(h'(w) - \overline{h'(w)}).$$

This equation can be solved explicitly. Up to appropriate normalizations $h(w)$ is one of the following five forms

$$\begin{array}{rclrclrcl} h_1(w) & = & w, & h_2(w) & = & e^{4\,\mathrm{i}\,w}, & h_3(w) & = & -\dfrac{1}{w}, \\ h_4(w) & = & \tanh(2\,w), & h_5(w) & = & \tan(2\,w). & & & \end{array}$$

Finally one arrives at the following classical result of E. Cartan [Car] (see [BE1, BE2] for detail).

Theorem 20 *Away from umbilic points there are 3 types of Bonnet families, which are characterized by the modulus of the Hopf differential:*

Type A : $|Q^A(w,\bar{w},T)|^2 = \dfrac{4}{\sin^2(2t)}.$

Type B : $|Q^B(w,\bar{w},T)|^2 = \dfrac{4}{\sinh^2(2t)}.$

Type C : $|Q^C(w,\bar{w},T)|^2 = \dfrac{1}{t^2}$.

The families are given by the surfaces with the fundamental forms presented in Table 1.

Type	$Q(w,\bar{w})$	H solves	e^u
A_1	$-2\left(\dfrac{\sin(2\bar{w})}{\sin(2w)}\right)\dfrac{1}{\sin(2\,t)}$	$\left(\left(\dfrac{H''}{H'}\right)' - H'\right)\dfrac{\sin^2(2t)}{4} = 2 - \dfrac{H^2}{H'}$	$-\dfrac{8}{\sin^2(2t)H'}$
A_2	$2\left(\dfrac{\cos(2\bar{w})}{\cos(2w)}\right)\dfrac{1}{\sin(2\,t)}$		
B	$-2\left(\dfrac{\sinh(2\bar{w})}{\sinh(2w)}\right)\dfrac{1}{\sinh(2\,t)}$	$\left(\left(\dfrac{H''}{H'}\right)' - H'\right)\dfrac{\sinh^2(2t)}{4} = 2 - \dfrac{H^2}{H'}$	$-\dfrac{8}{\sinh^2(2t)H'}$
C	$-\dfrac{\bar{w}}{w}\dfrac{1}{t}$	$\left(\left(\dfrac{H''}{H'}\right)' - H'\right)t^2 = 2 - \dfrac{H^2}{H'}$	$-\dfrac{2}{t^2H'}$

Table 1: Table of fundamental functions

Here $H(t)$ is any smooth solution with $H' < 0$ of the corresponding ordinary differential equation in Table 1.
The corresponding one-parameter families of isometries are intrinsic isometries of the surface described by imaginary translations of the coordinate w

$$w \to w + \mathrm{i}\,\rho(T). \tag{81}$$

The surfaces of type A_1 and A_2 are isometric with the same mean curvature function.

Corollary 5 *Bonnet surfaces are real analytic.*

The equations for the mean curvature function in Table 1 were first derived by N.Hazzidakis in [Haz]. We call them Hazzidakis equations.

A Bonnet surface in $\mathbb{R}^3$ is said to be *of type* **A**, **B**, or **C**, respectively, if away from critical points of the mean curvature function it is of the corresponding type. One can show that a Bonnet surface is exactly of one of the types **A**, **B**, or **C**.

4.3 Local Theory at Critical Points

In this section we derive a differential equation describing Bonnet surfaces near (isolated) critical points.

The identity (75) at an umbilic point implies the following local representation of the Hopf differential.

Proposition 7 *Let $P \in \mathcal{U} \subset \mathcal{R}$ be an umbilic point of a Bonnet surface. Then there exists a neighbourhood U of P, a local conformal chart $z : U \to \mathbb{C}$ with $z(P) = 0$, a holomorphic non-vanishing function $\varphi : U \to \mathbb{C}_*$, and an integer $J > 0$ such that the Hopf differential on U is*

$$Q(z, \bar{z}, \tau) = -\mathrm{i}\, z^J \left(\frac{\varphi(z, \tau)}{\psi_\tau(z, \bar{z}, \tau)} \right). \tag{82}$$

The function $\psi_{\tau\, z}$ is holomorphic on U and therefore can be represented as $\psi_{\tau z} = z^M\, \theta(z)$, $\theta(0) \neq 0$ with some $M \geq 0$. Analysing the Codazzi equations

$$\bar{Q}_z = \bar{z}^J\, z^M\, \theta\, \overline{\left(\frac{\mathrm{i}\,\varphi}{\psi_\tau^2} \right)} = \frac{1}{2}\, H_{\bar{z}}\, e^u, \quad Q_{\bar{z}} = z^J\, \bar{z}^M\, \bar{\theta} \left(\frac{\mathrm{i}\,\varphi}{\psi_\tau^2} \right) = \frac{1}{2}\, H_z\, e^u. \tag{83}$$

one obtains

$$\psi_{\tau z} = z^{J+1}\, \theta(z), \qquad \theta(0) \neq 0. \tag{84}$$

Non-umbilic critical points correspond to $J = 0$. We call $-J$ with J given by (82, 84) the *index of the critical point.*

The derivation of an ordinary differential equation at a critical point is similar to the one of section 4.2. The Codazzi equations (83) imply

$$\left(\frac{\theta}{\mathrm{i}\,\varphi} \right) z\, H_z = \overline{\left(\frac{\theta}{\mathrm{i}\,\varphi} \right)}\, \bar{z}\, H_{\bar{z}} \in \mathbb{R}. \tag{85}$$

Introducing a new conformal parameter $w(z)$ by

$$\frac{z}{f}\, \frac{\partial}{\partial z} = \frac{w}{f(0)}\, \frac{\partial}{\partial w}, \tag{86}$$

with $w(0) = 0$ and $f(z) = -i\, \varphi(z)/\theta(z)$, one arrives at

$$w\, H_w = \bar{w}\, H_{\bar{w}}.$$

The last identity shows that the mean curvature function is a function of $s = |w|^2$ only. Proceeding further as in Section 4.2, after apropriate normalizations (see [BE2] for detail) one obtains the following

Theorem 21 *Let $\mathcal{F}$ be a Bonnet surface in $\mathbb{R}^3$ with an (isolated) critical point of index $-J$. Then there exists a local conformal chart w at the critical point $w = 0$ such that the coefficients of its fundamental forms are given by*

$$\begin{aligned} Q(w,\bar{w})\,dw^2 &= (J+2)\left(\frac{1-\bar{w}^{J+2}}{1-w^{J+2}}\right)\frac{w^J}{1-s^{J+2}}\,dw^2, \\ e^{u(w,\bar{w})}\,|dw|^2 &= -2\,(J+2)^2\,\frac{s^J}{(1-s^{J+2})^2\,H'(s)}\,|dw|^2, \end{aligned} \qquad s = |w|^2\,, \tag{87}$$

and $H(w,\bar{w}) = H(|w|^2)$ is a solution of

$$\left(\frac{sH''(s)}{H'(s)}\right)' - H'(s) = \frac{(J+2)^2\,s^{J+1}}{(1-s^{J+2})^2}\left(2 - \frac{H^2(s)}{s\,H'(s)}\right), \quad ' = \frac{d}{d\,s}, \tag{88}$$

with the asymptotics

$$H(s) = H(0) + s^{J+1}\,B(s) \tag{89}$$

at $s = 0$, where $B(s)$ is a non-vanishing, smooth function.
Conversely, any solution of (88) with the asymptotics (89) at $s = 0$ via (87) determines a Bonnet surface with critical point of index $-J$. The Bonnet family is given by the intrinisic isometry

$$w \to e^{\mathrm{i}\,\alpha}\,w, \quad \alpha \in \mathbb{R}. \tag{90}$$

The existence of Bonnet surfaces in $\mathbb{R}^3$ with critical points of the mean curvature function will be proven in Section 4.5.

Finally note that Bonnet surfaces with critical points are necessarily of type $\mathbf{B}$. The relation between the corresponding coordinates w of this section and $w_{\mathbf{B}} = w$ of Section 4.2 is given by

$$w = e^{-\frac{4}{J+2}\,w_{\mathbf{B}}}. \tag{91}$$

We call a Bonnet surface with an isolated critical point a Bonnet surface of type $\mathbf{B}_{\mathcal{V}}$.

4.4 Bonnet Surfaces via Painlevé Transcendents

In this section we give an explicit description of Bonnet surfaces in terms of solutions of Painlevé equations, which are certain ordinary differential equations of the second order

$$y'' = R(y', y, x). \tag{92}$$

Solutions of the Painlevé equations – Painlevé transcendents – are treated nowadays as non-linear special functions. The theory of these special functions is rather well developed (see for example [IN, IKSY, Its]). In Section 4.5 we use elements of this theory for global classification of Bonnet surfaces. Of special importance for us is the *Painlevé property* of these equations. Recall that a differential equation (92) is said to possess the Painlevé property if it is free of movable branch and essential singular points, i.e. the only singularities of the solutions which change their position if one varies the initial data are poles.

Regarding examples of the Bonnet surfaces of type **B**, let us expain how the quaternionic representation of Bonnet surfaces naturally leads to the above mentioned remarkable connection. Note that the Hazzidakis equations of types **A** and **B** are analytically equivalent $H_{\mathbf{A}}(it) = -iH_{\mathbf{B}}(t)$.

Since the mean curvature and the metric of a Bonnet surfaces of type **B** depend on $t = w + \bar{w}$ only, it is natural to change the parametrization once more using t (or a function of t) as one of the variables. After some computations one arrives to the idea of considering

$$x = e^{-4\,(w+\bar{w})}, \qquad \lambda = e^{-4\,w} \tag{93}$$

as new variables. In these coordinates the Hazzidakis equation is

$$4\left(x\,\frac{\mathcal{H}''(x)}{\mathcal{H}'(x)}\right)' + \mathcal{H}'(x) = \frac{4}{(x-1)^2}\left(2 + \frac{\mathcal{H}^2(x)}{4\,x\,\mathcal{H}'(x)}\right), \tag{94}$$

where $\mathcal{H}(x) \equiv H(t)$. The frame equations (15, 17) in parametrization (93) are now

$$\frac{\partial \Phi}{\partial \lambda}\,\Phi^{-1} = \frac{B_0(x)}{\lambda} + \frac{B_1(x)}{\lambda - 1} + \frac{B_x(x)}{\lambda - x}, \tag{95}$$

$$\frac{\partial \Phi}{\partial x}\,\Phi^{-1} = -\frac{B_x(x)}{\lambda - x} + C(x), \tag{96}$$

where all the coefficients of the matrices are given by some explicit formulas through $\mathcal{H}(x), \mathcal{H}'(x)$ and $\mathcal{H}''(x)$. Ignore for the moment the complicated formulas for the coefficients in the matrices (95). What is more important is the special dependence of (95) on λ. Equation (95) is a 2 by 2 matrix dimensional Fuchsian system with four regular singularities (at $\lambda = 0, 1, x, \infty$). Equation (96) describes deformations preserving the monodromy group of the system. It is well known (see for example [JM]) that such isomonodromic deformations are the issue of the Painlevé VI equation. In particular, all the coefficients of the matrices can be expressed in terms of solutions of this equation – Painlevé transcendents. A rather involved calculation (see [BE1]) of the corresponding gauge transformation identifying the corresponding descriptions yields finally the following

Theorem 22 *Equation (94) possesses the first integral*

$$x^2\left(\frac{\mathcal{H}''(x)}{\mathcal{H}'(x)} + \frac{2}{x-1}\right)^2 + \frac{x\,\mathcal{H}'(x)}{2} + \frac{\mathcal{H}^2(x)}{2\,(x-1)^2\,\mathcal{H}'(x)} + \frac{\mathcal{H}(x)\,(x+1)}{2\,(x-1)} = \theta^2. \tag{97}$$

Let $\mathcal{H}(x)$ be a solution of (94) with $\mathcal{H}'(x) \neq 0$ and θ be a fixed root of (97). Then the function $y(x)$ defined by

$$y(x) \equiv -\frac{2}{\mathcal{H}'(x)}\left(\frac{x\,(x-1)\,\mathcal{H}''(x) + (\theta - x\,(\theta - 2))\,\mathcal{H}'(x)}{\mathcal{H}(x) + (x-1)\,\mathcal{H}'(x)}\right)^2 \tag{98}$$

solves the Painlevé VI equation of the following form

$$\begin{aligned}\frac{d^2 y}{d x^2} &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right) y'^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right) y'+\\ &\quad \frac{y\,(y-1)\,(y-x)}{2\,x^2\,(x-1)^2}\left(\theta^2\,\frac{(x-1)}{(y-1)^2}-\theta\,(\theta+2)\,\frac{x\,(x-1)}{(y-x)^2}\right).\end{aligned} \tag{99}$$

Conversely, for any solution of the Painlevé VI (99) which is not of the form $y(x)=\mathrm{const}\cdot x^{-\theta}$, *the function*

$$\mathcal{H}(x) \equiv -2\,\frac{(x-1)\,(\theta^2\,y(x)^2-x^2 y'^2(x))}{y(x)\,(y(x)-1)(y(x)-x)} \tag{100}$$

is a solution of (94) with first integral $\pm\theta$ *(97).*

The mappings (98) and (100) with the same θ *are inverse one to another.*

In a similar way (see [BE1] for the corresponding formulas), Bonnet surfaces of type **C** are described through solutions of the Painlevé V equation.

Corollary 6 *The Hazzidakis equations for Bonnet surfaces of all types possess the Painlevé property.*

4.5 Global Classification of Bonnet Surfaces

In this section we classify all *maximal* or *global* Bonnet immersions $F:\mathcal{R}\to\mathbb{R}^3$. These surfaces are characterized by the following natural "maximality" property: given an immersed Bonnet surface $\mathcal{F}\subset\mathbb{R}^3$ there exists a global Bonnet immersion containing $\mathcal{F}$, i.e.

$$\mathcal{F}=F(U),\quad U\subset\mathcal{R}.$$

Let us first prove the existence of critical points.

Theorem 23 *For arbitrary* $H(0)\in\mathbb{R}$, $H_0<0$ *there exists a real analytic Bonnet surface of type* $\mathbf{B}_{\mathcal{V}}$ *with a critical point of index* $-J$ *and with the mean curvature function and the metric at the critical point given by*

$$H(0)\ \textit{and}\ -2\,\frac{(J+2)^2}{(J+1)\,H_0}\,dw\,d\bar{w}.$$

Proof. Substituting the ansatz

$$H(s)=H(0)+s^{J+1}\sum_{i=0}^{\infty}H_i\,s^i,\quad H_0\neq 0$$

into equation (88) and using the corresponding recurrence formulas for H_i, one can prove that the series converges absolutely in a neighbourhood of $s = 0$. Thus, for arbitrary $H(0) \in \mathbb{R}$, $H_0 \in \mathbb{R}_*$ there exists a real analytic solution of equation (88) at $s = 0$ with the asymptotic

$$H(s) = H(0) + s^{J+1} H_0 + O(s^{J+2}). \tag{101}$$

The claim follows now from Theorem 21.

Let the local coordinate w be defined by (78) for the Bonnet surfaces of type **A**, **B**, and **C** and by (86) for the Bonnet surfaces of type $\mathbf{B}_{\mathcal{V}}$. We denote by $\mathbf{U}$ the largest connected domain in the w-plane, for which Q (see Table **1** and (87) respectively) is bounded. Furthermore, let $\mathcal{D} = \{t = w + \bar{w} \,|w \in \mathbf{U}\}$ for the types **A**, **B**, and **C**, and $\mathcal{D} = \{s = |w|^2 \,|w \in \mathbf{U}\}$ for the type $\mathbf{B}_{\mathcal{V}}$. In the following Table **2**, all the cases $\mathbf{U}$, $\mathcal{D}$, and the harmonic function ψ_τ, parametrized by w, are listed:

Type	$\mathbf{U}$	$\mathcal{D}$	ψ_τ
$\mathbf{A}_1$, $\mathbf{A}_2$	$\{w \in \mathbb{C} \mid 0 < \mathrm{Re}(w) < \frac{\pi}{4}\}$	$(0, \frac{\pi}{2})$	$2\,\mathrm{Re}(\tan(2w))$
$\mathbf{B}$	$\{w \in \mathbb{C} \mid \mathrm{Re}(w) > 0\}$	$(0,\infty)$	$2\,\mathrm{Re}(\tanh(2w))$
$\mathbf{B}_{\mathcal{V}}$	$\{w \in \mathbb{C} \mid \ \lvert w\rvert < 1\}$	$[0,1)$	$2\,\mathrm{Re}\left(\frac{1+w^{J+2}}{1-w^{J+2}}\right)$
$\mathbf{C}$	$\{w \in \mathbb{C} \mid \mathrm{Re}(w) > 0\}$	$(0,\infty)$	$2\,\mathrm{Re}(w)$

Table 2: Global description of Bonnet surfaces

Proposition 8 *Let H be a solution of one of the Hazzidakis equations of types* **A**, **B**, **C**, *or* $\mathbf{B}_{\mathcal{V}}$ *(see Table* **1** *and (88)) with $H'(t) < 0$ at some point $t \in \mathcal{D}$ (the corresponding domains $\mathcal{D}$ are listed in Table* **2***). Then H is real analytic on $\mathcal{D}$.*

The proof of this proposition is based on using the Painlevé property (see Corollary 6) together with the following

Lemma 6 *Let $\mathcal{D} \ni t$ be an open interval with smooth positive-valued functions f, g, $|Q|^2 : \mathcal{D} \to \mathbb{R}_+$ on it. Let $H = H(t)$ be a real-valued solution of the generalized Hazzidakis equation*

$$\left(f(t)\, \frac{H''(t)}{H'(t)} \right)' - H'(t) = |Q|^2 \left(2 - \frac{H^2(t)}{g(t)\, H'(t)} \right), \tag{102}$$

smooth on $\mathcal{D} \setminus \mathcal{P}$, where $\mathcal{P}$ is a discrete set of poles of $H(t)$.
If $H'(t_0) < 0$ at some $t_0 \in \mathcal{D}$, then H is smooth everywhere on $\mathcal{D}$, i.e. $\mathcal{P} = \emptyset$, with $H'(t) < 0$ for all $t \in \mathcal{D}$.

To prove smoothness of $H(t)$ one shows that all poles of $H(t)$ are necessarily simple with negative residues and that in addition $H'(t)$ never changes its sign. These two properties contradict one another.

It is not difficult to show that immersions $F : \mathbf{U} \to \mathbb{R}^3$ of Bonnet surfaces of type $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, or $\mathbf{B}_{\mathcal{V}}$ given in Tables 1 and 2 are maximal. The function ψ_τ defined in Table 2 is a non-vanishing function on $\mathbf{U}$. The continuity of ψ_τ yields that this function can not be extended beyond $\mathbf{U}$.

Finally using the arguments of Section 4.2 it is easy to show that the classified global Bonnet surfaces are all different, i.e. for two global Bonnet immersions $F_i : \mathcal{R}_i \to \mathbb{R}^3$, $i = 1, 2$ there exist no open sets $U_i \subset \mathcal{R}_i$, $i = 1, 2$ on which the surfaces coincide.

Theorem 24 *Any Bonnet surface in $\mathbb{R}^3$ can be conformally parametrized $F : \mathcal{R} \to \mathbb{R}^3$ by a corresponding global Bonnet immersion $F : \mathbf{U} \to \mathbb{R}^3$, $\mathcal{R} \subset \mathbf{U}$. The latter are of one of the types $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{B}_{\mathcal{V}}$. The corresponding domains $\mathbf{U}$ are listed in Table 2.*
Given $t_0 \in \mathcal{D}$ (see Table 2) and arbitrary $H(t_0)$, $H'(t_0) < 0$, $H''(t_0)$ there exists a unique solution $H(t)$ of the Hazzidakis equation of type $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ (see Table 1), real analytic on D. This function determines the fundamental forms (Table 1) of the corresponding global Bonnet immersions of the type $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$. Given $H(0)$, $H_0 < 0$ there exists a unique solution $H(s)$ of the Hazzidakis equation (88), real analytic on $\mathcal{D}$, with the asymptotics (101). It determines by (87) the fundamental forms of the corresponding global Bonnet surface of type $\mathbf{B}_{\mathcal{V}}$.

4.6 Examples of Bonnet Surfaces

Let us present some figures of Bonnet surfaces of types $\mathbf{B}$ and $\mathbf{B}_{\mathcal{V}}$. Tubes correspond to parameter lines $t =$ const, i.e. to the trajectories of the isometric flow preserving the mean curvature function. Both the mean curvature function and the metric are preserved along these lines. The last fact can be clearly observed - the strips bounded by two consequent parameter lines $t = t_1$ and $t = t_2$ are of constant width. The isometry is intrinsic, i.e. is not induced by a Euclidean motion of the ambient $\mathbb{R}^3$. The immersion domain $\mathbf{U}$ of Bonnet surfaces of type $\mathbf{B}$ is naturally split into fundamental domains

$$\mathbf{U}_n = \{w \in \mathbb{C} \,|\, (n-1)\frac{\pi}{2} < \mathrm{Im}(w) < n\,\frac{\pi}{2}\}.$$

Indeed the fundamental forms (see Table **1**) are invariant with respect to the shift

$$w \to w + \mathrm{i}\,\frac{\pi}{2},$$

and thus immersed $\mathbf{U}_n$ with different n's are congruent in $\mathbb{R}^3$. A Bonnet surface comprised of three fundamental domains is shown in Figure 7. For an appropriate choice of parameters, several copies of the fundamental domain can close up and thus comprise a closed surface with a critical point. Figure 8 shows such a case with three fundamental domains. It is worth mentioning that it was this figure which lead us to suggest the existence of Bonnet surfaces with critical points.

Figures 9 and 10 present another example of type $\mathbf{B}_{\mathcal{V}}$. The surface in Figure 9 is an immersed disk. The index of the critical point is $J = 6$. A more detailed view of one of the cusps of this surface is shown in Figure 10. As before, the tubed curves are integral curves of the isometry field. The surface is probably embedded.

The figures of this section are produced by Ulrich Eitner. Further examples can be found in [BE1, BE2].

References

[Abr] Abresch, U.: Constant mean curvature tori in terms of elliptic functions. J. reine angew. Math. **394** (1987) 169-192

[BBEIM] Belokolos, E.D., Bobenko, A.I., Enolskii, V.Z., Its, A.R., Matveev, V.B.: Algebro-Geometric Approach to Nonlinear Integrable Equations, Springer, Berlin (1994)

[Bia1] Bianchi,L.: Vorlesung über Differentialgeometrie, Leipzig, Berlin, p. 454 (1910)

[Bia2] Bianchi, L.: Sulle superficie a linee di curvature isotherme. Rend. Acc. Naz. dei Lincei, (5) **12** (1903), 511-520

[Bia3] Bianchi, L.: Ricerche sulle superficie isotherme and sulla deformatione delle quadriche. Ammali di Math. Ser. III **11** (1905) 93-157

[Bob1] Bobenko, A.I.: Constant mean curvature surfaces and integrable equations. Russ. Math. Surveys **46** (4) (1991), 1 - 45

[Bob2] Bobenko, A.I.: Surfaces in Terms of 2 by 2 Matrices. Old and New Integrable Cases. In: Fordy, A.P., Wood, J.(eds.) Harmonic Maps and Integrable Systems, Braunschweig Wiesbaden: Vieweg (1994) 83-127

[BE1] Bobenko, A.I., Eitner, U.: Bonnet surfaces and Painlevé equations. J. reine angew. Math. **499** (1998), 47-79

[BE2] Bobenko, A.I., Eitner, U.: Painlevé equations in differential geometry of surfaces (in preparation).

[BP] Bobenko, A., Pinkall, U.: Discrete isothermic surfaces. J. reine angew. Math. **475** (1996) 187-208

[Bon] Bonnet, O.: Mémoire sur la théorie des surfaces applicables, J. Ec. Polyt. **42** (1867), 72-92

[Car] Cartan, E.: Sur les couples de surfaces applicables avec conversation des courbieres principles, Bull. Sc. Math **66**, 1-30 (1942)

[Che] Chern, S.S., Deformation of Surfaces Preserving Principal Curvatures. In: Chavel, I., Farkas, H., (eds.), Differential Geometry and Complex Analysis , Springer 1985, 155-163

[Cie] Cieslinski, J.: The Darboux-Bianchi transformation for isothermic surfaces. Classical results versus the soliton approach. Diff. Geom. and its Appl. **7** (1997) 1-28

[Dar] Darboux, G.: Sur les surfaces isothermiques. Comptes Rendus **122** (1899) 1299-1305, 1483-1487, 1538

[DKN] Dubrovin, B.A., Krichever, I.M., Novikov, S.P.: Integrable systems I. In: Contemprorary problems of mathematics, Fundamental Directions, Itogi nauki i Tekhniki, VINITI AN SSSR, Moscow, vol. 4 (1985) 210-315

[DPW] Dorfmeister, J., Pedit, F., Wu, H.: Weierstrass type representation of harmonic maps into symmetric spaces. GANG preprint III. **25** (1994) (to appear in Comm. in Anal. Geom.)

[Eis] Eisenhart, L.P.: A Treatize on the Differential Geometry of Curves and Surfaces. Allyn and Bacon, Boston (1909)

[FPPS] Ferus, D., Pedit, F., Pinkall, U., Sterling, I.: Minimal tori in S^4. J. reine angew. Math. **429** (1992) 1-47

[Gra] Graustein, W.C.: Applicability with Preservation of Both Curvatures. Bull. Amer. Math. Soc. **30** (1924), 19-23

[GKS] Grosse-Braukmann, K., Kusner, R., Sullivan, J.: Classification of embedded constant mean curvature surfaces with genus zero and three ends. Preprint (1997)

[Haz] Hazzidakis, J.N.: Biegung mit Erhaltung der Hauptkrümmungsradien. Journal für Reine und Angewandte Mathematik **117** (1887), 42-56

[Hei] Heil, M.: Numerical tools for the study of finite gap solutions of integrable systems. PhD Thesis, TU Berlin (1995)

[Hit] Hitchin, F.S.: Harmonic maps from a 2-torus to the 3-sphere. J. Diff. Geom. **31** (1991) 627-710

[Hop] Hopf, H.: Differential geometry in the large. Lecture Notes in Mathematics **1000** Springer, Berlin (1983)

[Its] Its, A.R.: The Painlevé Transcendents as Nonlinear Special Functions. In: Levi D., Winternitz P. (eds.) The Painlevé Transcendents, Their Asymptotics and Physical Applications. (pp. 40-60), New York: Plenum 1992

[IN] Its, A.R., Novokshenov, V.Y.: The Isomonodromic Deformation Method in the Theory of Painlevé Equations. Lectures Notes in Mathematics No. 1191, Berlin, 1986

[IKSY] Iwasaki, K., Kimura, H., Shimomura, S., Yoshida, M.: From Gauss to Painlevé. A Modern Theory of Special Functions, Braunschweig, Vieweg, 1991

[Jag] Jaggy, C.: On the classification of constant mean curvature tori in $\mathbb{R}^3$. Comment. Math. Helvetici **69** (1994) 640-658

[JM] Jimbo, M.T., Miwa, T.: Monodromy Perserving Deformation of Linear Ordinary Differential Equations with Rational Coefficients, II. Physica **2 D**, 407-448 (1981)

[KPP] Kamberov, G., Pedit, F., Pinkall, U.: Bonnet pairs and isothermic surfaces, Duke Math. J. **92** (3), (1998) 637-644

[Kap] Kapouleas, N.: Compact constant mean curvature surfaces in Euclidean three-space. J. Diff. Geom. **33** (1991) 683-715

[Ken] Kenmotsu, K.: Weierstrass formula for surfaces of prescribed mean curvature. Math. Ann. **245** (1979) 89-99

[Kon] Konopelchenko, B.G.: Induced surfaces and their integrable dynamics. Stud. Appl. Math. **96** (1996) 9-51

[KS1] Kusner, R., Schmidt, N.: The spinor representation of minimal surfaces in space, GANG Preprint III.27 (1993)

[KS2] Kusner, R., Schmidt, N.: The spinor representation of surfaces in space, GANG Preprint IV.18 (1996)

[LT] Lawson, H.B: Tribuzy, R. de, On the Mean Curvature Function for Compact Surfaces, J. Diff. Geom., 16, pp 179- 183 (1981)

[PP] Pedit, F., Pinkall, U.: Quaternionic analysis on Riemann surfaces and differential geometry, Doc. Math., Extra Volume ICM 1998, 389-400

[Pin] Pinkall, U.: Regular homotopy classes of immersed surfaces. Topology **24** (1985) 421-434

[PS] Pinkall, U., Sterling, I.: On the classification of constant mean curvature tori, Ann. Math. **130**, 407 - 451 (1989)

[Rou] Roussos, I.M.: Global Results on Bonnet surfaces, J. Geom., to appear.

[Sul] Sullivan, D.: The spinor representation of minimal surfaces in space. Notes (1989)

[Tai] Taimanov, I.: The Weierstrass representation of closed surfaces in $\mathbb{R}^3$. Sfb 288 Preprint **291** (1997)

[Wal] Walter, R.: Explicit examples to the H-problem of heinz Hopf. Geom. Dedicata **23** (1987) 187-213

[Wen] Wente, H.C.: Counterexample to a conjecture of H. Hopf. Pac. J. Math. **121** (1986) 193-243