Partial actions of groups their globalisations and E-unitary inverse semigroups

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Abstract

A group G is said to act partially on a set Y if there is a map $\theta: G \to I(Y)$ into the semigroup of partial bijections on X such that $\theta(g)\theta(h) \subseteq \theta(gh)$ and $\theta(g)^{-1} = \theta(g^{-1})$ for all $g, h \in G$. We prove that each partial group action is the restriction of a universal global group action, and show that this result provides the key to understanding Munn's proof of the P-theorem.

1 Introduction

This is the first of two papers in which we develop the theory and applications of partial group actions; in this paper we concentrate on the purely algebraic part of the theory, whereas in [6] we deal with the topological part.

Partial group actions arise by weakening the usual definition of a group action, and form a natural subclass of the local transformation groups of differential geometry [13]. To define what they are, let G be a group and Y a set. A partial function from $G \times Y$ to Y is denoted by $(g, x) \mapsto g \cdot x$, and we write $\exists g \cdot x$ to mean that $g \cdot x$ is defined. We say that $(g, x) \mapsto g \cdot x$ defines a *partial action of G on Y* if the following two conditions hold:

(PA1) $\exists g \cdot x$ implies that $\exists g^{-1} \cdot (g \cdot x)$, and $g^{-1} \cdot (g \cdot x) = x$.

(PA2) $\exists g \cdot (h \cdot x)$ implies that $\exists (gh) \cdot x$ and $g \cdot (h \cdot x) = (gh) \cdot x$.

We say that the partial action is *unital* if the following condition also holds:

(PA3) $\exists 1 \cdot x$ for every $x \in X$, and $1 \cdot x = x$.[1]

[1] In this paper, all partial actions will be unital.

The essential difference between partial group actions and the usual 'global' group actions lies in condition (PA2): from the existence of $g \cdot (h \cdot x)$ we can infer the existence of $(gh) \cdot x$, but not conversely.

Partial group actions were first explicitly formulated by Exel [1], although, as the paper [9] shows, examples of partial group actions abound in mathematics. The authors use them to describe tiling semigroups in [5]. We describe two simple examples of general interest below.

Examples

1. The theory of Möbius transformations can be interpreted in terms of partial actions of groups. The Möbius group $PGL(2, \mathbb{C})$ acts partially on the complex plane in the following way. The matrix
$$A = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right)$$
of $PGL(2, \mathbb{C})$ induces the Möbius transformation $\mu(A)$ of the complex plane having the form $\mu(A)(z) = \frac{az+b}{cz+d}$. If B is another element of $PGL(2, \mathbb{C})$ then $\mu(A)\mu(B) \subseteq \mu(AB)$; this is because the composition of two Möbius transformations is not in general a Möbius transformation, but is contained in a unique Möbius transformation.

2. Let G be a group which acts globally on the set X. Let Y be any subset of X. Define the partial function $G \times Y \to Y$ by $\exists g \cdot x$ if and only if $g \cdot x \in Y$. The restriction of the action of the element g on the whole of X to the subset Y induces a partial function on Y with domain $g^{-1} \cdot Y \cap Y$ and image $g \cdot Y \cap Y$ (which could well be empty). It is easy to check that this defines a partial action of G on Y. We say that this partial action arises by *restricting* the global action.

The process described in Example 2 is often employed in group theory: for example, in the theory of groups acting on $\mathbb{R}$-trees, one restricts to finite subtrees and considers the corresponding partial action [2]; and in the theory of certain Fuchsian groups a similar procedure is adopted, which is formalised in [10] in terms which tacitly use partial group actions.

The reader will know that the partial action defined in Example 1 also arises by restriction of a global action: the complex plane may be embedded in the extended complex plane (i.e. the Riemann sphere) in such a way that the partial action of the Möbius group is the restriction of a global action. The main result of this paper shows that this situation is typical: in Section 2, we prove that every partial action of a group can be obtained by restricting a group action as in Example 2. Furthermore, we prove that every partial group action admits a universal globalisation. In Section 3, we apply our globalisation theorem to obtain an interpretation of Munn's proof of the P-theorem in inverse semigroup theory [12].

Throughout the paper, we shall use basic results from inverse semigroup theory; see Howie [3], Lawson [8] or Petrich [15].

2 Globalisation

It will be convenient to reformulate the definition of partial group actions in terms of special kinds of mappings. A function $\theta\colon S \to T$ between inverse semigroups is said to be a *dual prehomomorphism* if the following two axioms hold:

(DPH1) $\theta(s^{-1}) = \theta(s)^{-1}$ for all $s \in S$.

(DPH2) $\theta(s)\theta(t) \leq \theta(st)$ for all $s, t \in S$.

If S and T are both monoids then the dual prehomomorphism is said to be *unital* if the following axiom also holds:

(DPH3) $\theta(1) = 1$.

It is evident (or consult [9]) that (unital) partial actions of a group G on a set Y are equivalent to (unital) dual prehomomorphisms from G to the inverse semigroup $I(Y)$; a special case of this is the familiar result that global actions of G on the set Y are equivalent to unital homomorphisms.

If $\theta\colon G \to I(Y)$ is a unital dual prehomomorphism and $\theta(g)(x)$ is defined, then we write $\theta(g)(x) = g \cdot x$.

Let $\theta\colon G \to I(Y)$ be a unital partial action of the group G on the set Y. A *globalisation of* θ is a pair (ι, ϕ) consisting of an injection $\iota\colon Y \to X$ and a monoid homomorphism $\phi\colon G \to I(X)$ such that $\theta(g) = \iota^{-1}\phi(g)\iota$ for every $g \in G$; (note that we regard ι as a partial bijection from Y to X). We shall prove that every unital group action can be globalised in a universal way.

2.1 The basic construction

Let $\theta\colon G \to I(Y)$ be a unital partial action of the group G. Define the relation $\sim$ on the set $G \times Y$ by $(g, x) \sim (g', x')$ if and only if there exist $h \in G$ such that $g = g'h$ and $h \cdot x = x'$.

Lemma 1 *The relation $\sim$ is an equivalence relation on $G \times Y$.*

Proof For each $x \in Y$, $\exists 1 \cdot x$ and $1 \cdot x = x$ by condition (PA3). Consequently, $(g, x) \sim (g, x)$, and $\sim$ is reflexive. To prove that $\sim$ is symmetric, suppose that $(g, x) \sim (g', x')$. Then there exist $h \in G$ such that $g = g'h$ and $h \cdot x = x'$. By condition (PA1), $h^{-1} \cdot (h \cdot x)$ is defined and equals x. Thus $h^{-1} \cdot x' = x$. It follows that

$$(g', x') = (gh^{-1}, x') \sim (g, h^{-1} \cdot x') = (g, x),$$

and so $\sim$ is symmetric. Finally, we prove that $\sim$ is transitive. Suppose that $(g, x) \sim (g', x')$ and $(g', x') \sim (g'', x'')$. By definition, there exist $h, k \in G$ such that

$$g = g'h \text{ and } h \cdot x = x'$$

and

$$g' = g''k \text{ and } k \cdot x' = x''.$$

Clearly, $g = g''kh$, and since $k \cdot (h \cdot x)$ exists it follows by condition (PA2) that $(kh) \cdot x$ exists and $(kh) \cdot x = x''$. Thus $(g, x) \sim (g'', x'')$, and so $\sim$ is transitive. ∎

Denote the set of $\sim$-equivalence classes on $G \times Y$ by Y_G, and denote the $\sim$-equivalence class containing the element (g, x) by $[g, x]$. Define a function from $G \times Y_G$ to Y_G by $(h, [g, x]) \mapsto [hg, x]$ which we denote by $h \cdot [g, x]$.

Lemma 2 *This is a well-defined (global) group action.*

Proof Suppose that $[g, x] = [g', x']$. Then there exist $k \in G$ such that $g = g'k$ and $k \cdot x = x'$. But then $hg = hg'k$ and so $(hg, x) \sim (hg', x')$. The remainder of the proof is straightforward. ∎

Define $\iota: Y \to Y_G$ by $\iota(x) = [1, x]$.

Lemma 3 *The function ι is injective.*

Proof Suppose that $\iota(x) = \iota(x')$. Then $[1, x] = [1, x']$. By definition, there exist $h \in G$ such that $1 = 1h$ and $h \cdot x = x'$. Thus $h = 1$, and so $x = x'$ by condition (PA3). ∎

Our first main result is the following. We denote by ϕ the homomorphism from G to $I(Y_G)$ corresponding to the global action.

Theorem 4 *The action of G on Y_G is a globalisation of the partial action of G on Y. Furthermore, $G \cdot \iota(Y) = Y_G$.*

Proof Let $x \in Y$, and suppose that $g \cdot x = x'$ is defined. By definition $g \cdot \iota(x) = [g, x]$, and $\iota(x') = [1, g \cdot x] = [g, x] = g \cdot \iota(x)$. Thus $\theta(g) \subseteq \iota^{-1}\phi(g)\iota$. To prove the reverse inclusion, suppose that $g \cdot [1, x] \in \iota(Y)$. Then $[g, x] = [1, x']$ for some $x' \in X$. Thus there exists $h \in G$ such that $g = 1h$ and $h \cdot x = x'$. Hence $g = h$ and so $g \cdot x = x'$. The proof of the last statement is immediate. ∎

In fact, a stronger result is true: in some sense, the globalisation (ι, ϕ) is the best possible. We now make this assertion precise.

Theorem 5 *Let (κ, ψ), where $\psi: G \to I(Z)$, be any globalisation of the partial action $\theta: G \to I(Y)$. Then there exists a unique injection $\lambda: Y_G \to Z$ such that $\phi(g) = \lambda^{-1}\psi(g)\lambda$ for every $g \in G$.*

Proof Define $\lambda: Y_G \to Z$ by $\lambda[g, x] = \psi(g)\kappa(x)$. We show first that λ is well-defined. Suppose that $[g, x] = [g', x']$. Then there exists $h \in G$ such that $g = g'h$ and $h \cdot x = x'$. Thus $\psi(g)\kappa(x) = \psi(g'h)\kappa(x)$. But (κ, ψ) globalises θ and so $\kappa(x') = \kappa(h \cdot x) = \psi(h)\kappa(x)$. Since ψ is a homomorphism, $\psi(g'h) = \psi(g')\psi(h)$. Hence

$$\psi(g)\kappa(x) = \psi(g'h)\kappa(x) = \psi(g')\psi(h)\kappa(x) = \psi(g')\kappa(x').$$

Thus $\lambda[g, x] = \lambda[g', x']$ as required.

To show that λ is injective, suppose that $\lambda[g,x] = \lambda[h,y]$. Then $\psi(g)\kappa(x) = \psi(h)\kappa(y)$. But ψ is a homomorphism and so $\psi(h^{-1}g)\kappa(x) = \kappa(y)$. Thus $(\kappa^{-1}\psi(h^{-1}g)\kappa)(x) = y$. But ψ is a globalisation of θ and $x, y \in Y$ so $\theta(h^{-1}g)(x) = y$. Hence

$$(g,x) = (h(h^{-1}g), x) \sim (h, (h^{-1}g) \cdot x) = (h,y).$$

It is immediate that $\lambda\iota = \kappa$.

We now show that $\phi(g) = \lambda^{-1}\psi(g)\lambda$. By definition $\phi(g)[h,x] = [gh,x]$. However $\lambda[h,x] = \psi(h)\kappa(x)$, and

$$\psi(g)\lambda[h,x] = \psi(gh)\kappa(x) = \lambda[gh,x],$$

since ψ is a homomorphism. Thus $\phi(g) \subseteq \lambda^{-1}\psi(g)\lambda$. To prove the reverse inclusion, suppose that $(\lambda^{-1}\psi(g)\lambda)[h,x]$ is defined. Then it equals $[gh,x]$, as required.

Finally, we prove that λ is unique with the stated properties. Let $\mu\colon Y_G \to Z$ be any injection such that $\mu\iota = \kappa$ and $\phi(g) = \mu^{-1}\psi(g)\mu$ for every $g \in G$. We prove that $\mu = \lambda$. Let $[g,x] \in Y_G$. Then

$$\mu[g,x] = \mu(g \cdot [1,x]) = \mu(\phi(g)([1,x])).$$

But

$$\phi(g)([1,x]) = \phi(g)(\iota(x)) = (\mu^{-1}\psi(g)\mu)(\iota(x)) = \mu^{-1}(\psi(g)\kappa(x)) = \mu^{-1}(\lambda[g,x]).$$

Hence $\mu[g,x] = \lambda[g,x]$. ■

Globalisations of partial actions arise naturally in group theory. Let G be a group acting globally on the set X, and let $Y \subseteq X$ such that $G \cdot Y = X$. In Example 2 of Section 1, we observed that G acts partially on the set Y. It is evident that the action of G on X is a globalisation of the partial action of G on Y. In fact, more is true: by Theorem 5, there is an embedding λ of Y_G in X. However, $G \cdot Y = X$, and so λ is also surjective. It follows that the action of G on X is isomorphic to the action of G on Y_G. Hence X is the globalisation of Y under the action of G. Triples such as (G, X, Y) are the basis of Macbeath's paper [10]. Under additional topological assumptions, he shows how a presentation for G can be obtained from the partial action of G on Y.

2.2 Möbius transformations

As a simple illustration of our globalisation theorem, we show that at the algebraic level it enables us to connstruct the Riemann sphere from the partial action of $PGL(2,\mathbb{C})$ on the complex plane.

A *Möbius transformation* is a partial function α of the complex plane having the form: $\alpha(z) = \frac{az+b}{cz+d}$ where a, b, c, d are complex numbers and $ad - bc \neq 0$.[2] Möbius transformations are *partial* functions rather than functions because

[2] Of course, there are many ways to write the function α because multiplying a, b, c and d by a non-zero complex number results in a different form but the same function. This must always be borne in mind.

of their denominators; if the coefficient c is non-zero then there is one value of z satisfying $cz + d = 0$. Thus the transformations are of two types: the functions in which $c = 0$, and the partial functions in which $c \neq 0$. Of course, the partial transformations are 'only just' partial; the domain of the function omits the point $\frac{-d}{c}$ and the image omits the point $\frac{a}{c}$. The condition $ad - bc \neq 0$ ensures that in both cases the transformations are injective. Thus Möbius transformations are either bijections or partial bijections of $\mathbb{C}$. The inverse transformation is also a Möbius transformation, namely $\alpha^{-1}(z) = \frac{dz-b}{-cz+a}$.

It is easy to construct examples to show that the composition of two Möbius transformations is not in general a Möbius transformation. However, Möbius transformations can be 'multiplied' by an operation which, as we have seen above, cannot be composition of partial functions. If $\alpha(z) = \frac{az+b}{cz+d}$ and $\beta(z) = \frac{Az+B}{Cz+D}$ are Möbius transformations then their product, which we shall denote by $\alpha \odot \beta$, is defined to be $\frac{(aA+bC)z+(aB+bD)}{(cA+dC)z+(cB+dD)}$. The result is certainly a Möbius transformation, as can easily be checked. Consequently, $\odot$ is a well-defined binary operation on the set of Möbius transformations. With respect to this operation the Möbius transformations form a group, called the *Möbius group*. It follows from [7], that if α and β are Möbius transformations, then $\alpha \odot \beta$ is the unique Möbius transformation γ such that $\alpha \circ \beta \subseteq \gamma$. The Möbius group is isomorphic to the projective general linear group $PGL(2, \mathbb{C})$ [4]. It follows, that we have a partial group action of $PGL(2, \mathbb{C})$ on the set $\mathbb{C}$.

Before proving our main result we need the following lemma.

Lemma 6 *Let $\theta \colon G \to I(Y)$ be a unital partial action of the group G on the set Y. Suppose that for each $g \in G$ the set $dom\,\theta(g)$ is equal to Y itself, or omits at most one point from Y, which we denote by x_g; in this case we say that g* omits *x_g. Suppose that for all $g, h \in G$ such that g omits x_g, and h omits x_h we have that $(h^{-1}g) \cdot x_g = x_h$. Then Y_G is obtained from $\iota(Y)$ by the adjunction of exactly one extra point.*

Proof We need to consider two cases. Suppose first that $[g, x] \in Y_G$ and g does not admit x from its domain. Then $g \cdot x$ is defined and $[g, x] = [1, g \cdot x] \in \iota(Y)$. Now let $[g, x], [h, y] \in Y_G$ be any two elements such that g omits x from its domain, and h omits y from its domain. Then by assumption, $(h^{-1}g) \cdot x = y$. Thus

$$[g, x] = [h(h^{-1}g), x] = [h, (h^{-1}g) \cdot x] = [h, y].$$

This shows that there is exactly one element in Y_G which is not in $\iota(Y)$. ■

Continuing with the assumptions of Lemma 10, we may describe exactly how the globalisation behaves. Denote by ∞ the unique point adjoined to $\iota(Y)$ to obtain Y_G. Suppose that g omits x_g from its domain. Then from the proof of Lemma 10, we have that $\infty = [g, x_g]$. Let ϕ be the globalisation of θ. Then $\phi(g)(\iota(x_g)) = \phi(g)[1, x_g] = \infty$; thus ι of the point omitted by $\theta(g)$ from its domain is sent to ∞ by $\phi(g)$. To determine the value taken by $\phi(g)(\infty)$, observe that $\theta(g)$ and $\theta(g^{-1})$ are mutually inverse partial bijections. Thus $\theta(g^{-1})$ omits the point $x_{g^{-1}}$ from its domain. By the proof of Lemma 10, we have that $\infty = [g^{-1}, x_{g^{-1}}]$. Thus $\phi(g)(\infty) = [gg^{-1}, x_{g^{-1}}] = [1, x_{g^{-1}}] = \iota(x_{g^{-1}})$, which is ι of the point omitted by $\theta(g^{-1})$. We therefore know exactly how the globalisation behaves.

We now return to Example 1 of Section 1, and show that the classical theory of Möbius transformations is a special case of our construction.

Theorem 7 *The globalisation of the partial action of $PGL(2,\mathbb{C})$ on $\mathbb{C}$ is precisely the usual globalisation obtained by adjoining ∞ to $\mathbb{C}$ and extending the Möbius transformations in the usual way.*

Proof We use Lemma 10 to show first that the globalisation is obtained by adjoining one point to $\mathbb{C}$. Let g be the Möbius transformation $\frac{az+b}{cz+d}$ where $c \neq 0$, and let h be the Möbius transformation $\frac{\alpha z+\beta}{\gamma z+\delta}$ where $\gamma \neq 0$. Then g omits the point $\frac{-d}{c}$ and h omits the point $\frac{-\delta}{\gamma}$. The Möbius transformation $h^{-1} \odot g$ is given by

$$\frac{(\delta a - \beta c)z + (\delta b - \beta d)}{(\alpha c - \gamma a)z + (\alpha d - \gamma b)}.$$

It is easy to check that $(h^{-1} \odot g)(\frac{-d}{c}) = \frac{-\delta}{\gamma}$. The conditions of Lemma 10 are therefore satisfied. Thus the globalisation is obtained by adjoining exactly one extra point, which we denote by ∞. The argument following the proof of Lemma 10 now shows that our globalisation of the partial action of the Möbius group is algebraically identical to the globalisation obtained by embedding the complex plane in the Riemann sphere. ■

3 The P-theorem

We observed in Section 2.1, that if G is a group acting globally on a set X and Y is a subset of X satisfying $G \cdot Y = X$ then the pair (G, X) is a globalisation of the partial action determined by the pair (G, Y). This situation is reminiscent of one of the basic constructions of inverse semigroup theory, which we now recall. A *McAlister triple* (G, X, Y) consists of a group G, a partially ordered set X, a subsemilattice and order ideal Y of X, together with an action of G on X by order automorphisms such that $G \cdot Y = X$, and for every $g \in G$ we have that $g \cdot Y \cap Y \neq \emptyset$. Such triples are used to construct E-unitary semigroups. In this section, we explain the exact connection between globalisations of partial actions of groups and the theory of E-unitary inverse semigroups. As a consequence, we shed light on Munn's proof of the P-theorem [12]. To do this, we shall need to consider a special class of partial actions of groups: namely, partial actions of groups on semilattices, together with their globalisations. We begin by making precise the definition of such actions.

Let Y be a meet semilattice, and let $\mathcal{I}(Y)$ denote the inverse monoid consisting of order isomorphisms between non-empty order ideals of Y. Observe that the intersection of two non-empty order ideals of Y is again non-empty because Y is a meet semilattice. A unital dual prehomomorphism from a group G to $\mathcal{I}(Y)$ will be said to define a *unital partial action of the group G on the semilattice Y.*

The first result is essentially a restatement of the contents of the paper [14] in terms of partial actions. Reecall that σ is the minimum group congruence on an inverse seemigroup S which is defined by $s\,\sigma\,t$ if an only if there exists $a \in S$ such that $a \leq s, t$.

Theorem 1 *The theory of unital partial actions of the group G on the semilattice Y is equivalent to the theory of E-unitary inverse semigroups S where the semilattice of S is isomorphic to Y, and where S/σ is isomorphic to G.*

Proof We sketch out the proof and refer the reader to [14] for details.

(Construction 1). Let $\theta\colon G \to \mathcal{I}(Y)$ be a unital dual prehomomorphism. Let

$$S = \{(y, g) \in Y \times G\colon y \in \operatorname{im}\theta(g)\}$$

equipped with the product

$$(y, g)(y', g') = (g \cdot (g^{-1} \cdot y \wedge y'), gg').$$

Then by Lemma 2.3 of [14], S is an E-unitary inverse semigroup with set of idempotents $\{(y, 1)\colon y \in Y\}$ isomorphic to Y and S/σ is isomorphic to G.

(Construction 2). Let S be an E-unitary inverse semigroup. Put $G = S/\sigma$ and $Y = E(S)$. Define a function $\theta\colon G \to \mathcal{I}(Y)$ as follows: for each $g \in G$ let

$$\operatorname{dom}\theta(g) = \{a^{-1}a\colon a \in g\}$$

and define $\theta(g)(a^{-1}a) = aa^{-1}$. Then by Lemma 3.1 of [14], θ is a unital dual prehomomorphism.

If we begin with the inverse semigroup S, and apply (Construction 1) followed by (Construction 2), then by Theorem 3.2 of [14] we end up with an inverse semigroup isomorphic to S. Consequently, every E-unitary semigroup is isomorphic to one obtained from a partial action of a group on a semilattice by means of (Construction 1).

It remains to show that if we start with a unital dual prehomomorphism $\theta\colon G \to \mathcal{I}(Y)$ and apply (Construction 1) followed by (Construction 2), then we end up essentially with θ. To this end, we say that two partial actions $\theta\colon G \to \mathcal{I}(Y)$ and $\phi\colon G \to \mathcal{I}(Z)$ are *isomorphic* if there is an isomorphism of semilattices $\iota\colon Z \to Y$ such that $\iota(\phi(g)(z)) = \theta(g)\iota(z)$ for all $g \in G$. We can now prove the result. Let $\theta\colon G \to \mathcal{I}(Y)$ be a unital dual prehomomorphism. Form the E-unitary semigroup S according to (Construction 1). We now describe the result of applying (Construction 2) to S; we denote the corresponding dual prehomomorphism by θ'. Let $g \in G$. The σ-class of S corresponding to g, is the set of all elements of the form $(y, g) \in S$. Now $(y, g)^{-1}(y, g) = (g^{-1} \cdot y, 1)$. Thus $\operatorname{dom}\theta'(g) = \{(g^{-1} \cdot y, 1)\colon y \in \operatorname{im}\theta(g)\}$. But $y \in \operatorname{im}\theta(g)$ if and only if $g^{-1} \cdot y \in \operatorname{dom}\theta(g)$. By (Construction 2), we have that

$\theta'(g)(g^{-1} \cdot y, 1) = (y, g)(y, g)^{-1} = (y, 1)$. On the other hand, $\theta(g)(g^{-1} \cdot y) = y$. Define $\iota\colon Y \times \{1\} \to Y$ by $\iota(y, 1) = y$. Then θ is isomorphic to θ', as required.■

As an aside, it is interesting to note that in [14] congruences on an E-unitary inverse semigroup are constructed in terms of the corresponding partial action on the semilattice.

Let $\theta\colon G \to \mathcal{I}(Y)$ be a unital partial action on a semilattice. We now investigate the properties of the globalisation $\phi\colon G \to I(Y_G)$ obtained in Theorem 2.4.

Lemma 2 *With the above notation:*

(i) *If* $[g, x] = [g, y]$ *then* $x = y$.

(ii) *Let* $[g, x] = [h, y]$ *and suppose that* $x' \leq x$, *then there exists* $y' \leq y$ *such that* $[g, x'] = [h, y']$.

(iii) *Define the relation* $\leq$ *on* Y_G *by* $[h, y] \leq [g, x]$ *if and only if there exists* $(g', x') \in [g, x]$ *and* $y' \leq x'$ *such that* $(g', y') \in [h, y]$. *Then* $\leq$ *is a partial order.*

(iv) *The semilattice* Y *is order-isomorphic with the partially ordered set* $\iota(Y)$ *with the order* $\leq$ *induced from* Y_G.

Proof (i) By assumption, $(g, x) \sim (g, y)$ so that $1 \cdot x = y$. But by condition (PA3), this implies that $x = y$.

(ii) By assumption, there exist $k \in G$ such that $g = hk$ and $k \cdot x = y$. Now $x' \leq x$ and by definition the domain of $\theta(k)$, is an order ideal of Y. Thus $\exists k \cdot x$ implies $y' = \exists k \cdot x'$. Furthermore, $\theta(k)$ is order-preserving and so $y' = k \cdot x' \leq k \cdot x = y$. It is immediate that $[g, x'] = [h, y']$.

(iii) We begin with an observation. Suppose that $[h, y] \leq [g, x]$. Then we claim that for every $(k, u) \in [g, x]$ there exists $v \leq u$ such that $(k, v) \in [h, y]$. By assumption there exists $(g', x') \in [g, x]$ and $y' \leq x'$ such that $(g', y') \in [h, y]$. Now $[g', x'] = [k, u]$ and $y' \leq x'$. Thus by (ii) above there exists $v \leq u$ such that $[g', y'] = [k, v]$. But then $(k, v) \in [h, y]$, as required.

We now show that $\leq$ defines a partial order on Y_G. It is clear that $\leq$ is reflexive. Suppose that $[h, y] \leq [g, x]$ and $[g, x] \leq [h, y]$. Then by the observation above, given (g, x) there exists $x' \leq x$ such that $(g, x') \in [h, y]$. Similarly, given (g, x') there exists $x'' \leq x'$ such that $(g, x'') \in [g, x]$. However, this implies that $[g, x''] = [g, x]$, and so $x'' = x$ by (i). But $x'' \leq x' \leq x$ and so $x' = x$. Thus $(g, x) = (g, x') \sim (h, y)$ and so $[g, x] = [h, y]$. Transitivity follows readily from our initial observation.

(iv) It is immediate from the definition of $\leq$ that ι defines an order-preserving function from Y to $\iota(Y)$. Suppose that $\iota(y) \leq \iota(x)$. Then $[1, y] \leq [1, x]$. Thus there exists $x' \leq x$ such that $[1, x'] = [1, y]$. But then $x' = y$ by (i). Hence $y \leq x$. ■

The main result of this section is the following.

Theorem 3 *Let $\theta\colon G \to \mathcal{I}(Y)$ be a unital dual prehomomorphism where Y is a semilattice. Then $(G, Y_G, \iota(Y))$ is a McAlister triple.*

Proof Firstly, $\iota(Y)$ is an order ideal of Y_G. For suppose that $[g,x] \leq [1,y] \in \iota(Y)$. Then there exists $y' \leq y$ such that $(1,y') \in [g,x]$. Thus $[g,x] = [1,y'] = \iota(y')$. We proved in Lemma 2(iv) that $\iota(Y)$ is order isomorphic with the semilattice Y. In Theorem 2.4, we proved that $G \cdot \iota(Y) = Y_G$.

To show that the action of G on Y_G is by order automorphisms, suppose that $[g,x] \leq [h,y]$ and that $k \in G$. By definition of $\leq$, given (h,y) there exists $y' \leq y$ such that $(h,y') \in [g,x]$. But then

$$k \cdot [g,x] = k \cdot [h,y'] = [kh,y'] \leq [kh,y] = k \cdot [h,y].$$

Finally, let $g \in G$. Then by definition $\theta(g)$ has domain a non-empty order ideal of Y. Let $y \in \operatorname{dom}\theta(g)$. It follows that $g \cdot y \in Y$. But now

$$g \cdot \iota(y) = g \cdot [1,y] = [g,y] = [1, g \cdot y] = \iota(g \cdot y).$$

Thus $g \cdot \iota(Y) \cap \iota(Y) \neq \emptyset$. ■

The McAlister triple constructed in Theorem 3 is nothing other than the McAlister triple of the E-unitary inverse semigroup associated with the partial action by means of (Construction 1) in the proof of Theorem 1. The sequence which begins with an E-unitary inverse semigroup, applies (Construction 2) of Theorem 1, and then proves Lemma 2 and Theorem 3 is identical to the proof of the P-theorem to be found in Munn's paper [12]. At the set-theoretic level, therefore, Munn was essentially proving the globalisation theorem for a group acting partially on the set Y.

In globalising the partial action on a semilattice, we are in general only able to construct a partially ordered set. We can determine when the globalisation is also a semilattice by using inverse semigroup theory. An E-unitary inverse semigroup is constructed from a McAlister triple (G, X, Y) in which X is a semilattice precisely when the semigroup is of a type known as *F-inverse* (see Section 7.4 of [8] for more information and references). We may therefore deduce the following result from Theorem 3.

Theorem 4 *Let $\theta\colon G \to \mathcal{I}(Y)$ be a unital partial action of the group G on the semilattice Y. Then the universal globalisation of θ is a semilattice if and only if the E-unitary inverse semigroup constructed from θ is F-inverse.* ■

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