WAVELET ANALYSIS ON MANIFOLDS I

DIETER KLUSCH

Institut für Mathematik
Humboldt Universität zu Berlin
SFB 288 'Differentialgeometrie und Quantenphysik'
Unter den Linden 6, D–10099 Berlin
email:klusch@math.fu-berlin.de

WAVELET ANALYSIS ON MANIFOLDS I

Dieter Klusch *

Abstract

The purpose of this note is to describe a new Cauchy wavelet analysis of selfadjoint operators in Hilbert spaces and its applications to fundamental problems of Global Analysis. We use the wavelet synthesis integral operator to introduce (i) a new wavelet calculus for power and exponential functions and (ii) a new class of generalized (fractal) zeta and eta functions. This extends the Seeley–Grubb functional calculus [GS96] and the abstract setting of the recent Brüning–Lesch spectral theory [BL99].
In the applications we describe how Cauchy wavelet analysis works in the theory of elliptic differential operators on manifolds. We show that the Duistermaat–Guillemin–Weinstein 'wave trace invariants' [Gui96] are actually 'fractal zeta function invariants'. We use the pseudodifferential resolvent analysis of Grubb and Seeley [GS95], [G99] to determine the full singularity structures of the fractal zeta and eta functions and the resulting heat trace expansions for pseudodifferential boundary problems for general elliptic systems of order $d \geq 1$ on compact manifolds. This generalizes recent fundamental trace expansion results of Grubb [G99], and – for the important special case of first order well-posed boundary problems for Dirac–type operators – of course the prominent results of Atiyah–Patodi–Singer [APS75], of Grubb and Seeley [GS95], [GS96], of Booss and Wojciechowski [BW93], of Müller [Mü94] and of Brüning and Lesch [BL97], [BL99].

1 Introduction

In a series of papers we will develop systematically a new Cauchy wavelet analysis of selfadjoint operators in Hilbert spaces with applications to fundamental problems of Global Analysis.
We extend the wavelet spectral geometry of [K97] and use Cauchy wavelets to introduce a new wavelet calculus and a new family of wavelet zeta and eta functions in a general functional analytic Hilbert space setting. This extends the abstract framework of the recent Brüning–Lesch spectral theory of boundary value problems for Dirac type operators (cf. [BL97], [BL99]). In particular, we combine the wavelet calculus with the fine pseudodifferential resolvent analysis of Seeley & Grubb [GS95] and of Grubb [G99] in order to generalize or to reprove zeta and heat trace expansions for pseudodifferential boundary problems for general elliptic systems (P, S_ϱ) of order $d \geq 1$ with pseudo–normal ψdo boundary conditions S [G99;§9]. We show that the results apply of course to first order well–posed boundary problems, especially to those prominent examples considered for the product or non–product case by Atiyah, Patodi and Singer [APS75], by Grubb and Seeley [GS95], [GS96], by Booss–Bavnbek and Wojciechowski [BW93] or by Brüning and Lesch [BL97], [BL99].

The purpose of this note is to describe

– a wavelet analysis associated to locally compact groups

– a new wavelet calculus for selfadjoint operators in a Hilbert space, defined from the continuous wavelet transform and its inverse with respect to Cauchy wavelets and combined with the holomorphic Seeley–Grubb calculus [GS95], [GS96]

*Supported by Deutsche Forschungsgemeinschaft, SFB 288 'Differentialgeometrie und Quantenphysik'. Humboldt Universität zu Berlin, D–10099 Berlin

– a new wavelet family of fractal zeta and eta functions, generalizing the fractal Riemann–Weierstrass functions of wavelet analysis [HT91], [H95], [JM96] as well as the zeta and eta functions of elliptic operators on manifolds [ABP73], [APS75], [GS96], [BL97]

– a new wavelet theoretic characterization of the Duistermaat–Guillemin–Weinstein "Wave Trace Invariants" [DG75], [Gui96] as wavelet zeta function invariants

– the singularitiy structures of the fractal zeta and eta functions and the complete heat trace expansions for pseudodifferential boundary problems for general elliptic systems (P, S_{ϱ}) on compact manifolds with specifications to first order well–posed boundary problems [G99].

Extensions of the results to the case of admissible manifolds [GK93] by use of Müller's theory of "Relative zeta functions" [Mü97], wavelet theoretic extensions of the recent Brüning–Lesch spectral theory [BL99], index theorems [BL99], [Sch98], a new wavelet spectral geometry of singular manifolds and the details of the material presented here will be given in subsequent papers.

2 Wavelet Analysis

2.1 General set–up

We here consider 'wavelet' analysis associated to square integrable representations of locally compact groups [H95], [Me90], [GMP85], [HW89].
Let $\mathcal{U}(H)$ be the group of unitary operators acting in some Hilbert space H. If G is a group, then $U \in Hom(G, \mathcal{U}(H))$ is called a unitary representation of G in H. For $g \in H$ the set of vectors $\{U(x)g|x \in G\}$ is called the orbit of g, and g is cyclic iff $\overline{span}\{U(x)g|x \in G\} = H$. A representation U is irreducible iff every vector $g \neq 0$ is cyclic.
Now consider the Hilbert space $L^2(G, d\mu_l)$ of square integrable functions over a locally compact group G with respect to the left–invariant Haar measure $d\mu_l$:

$$||f||^2_{L^2(G,d\mu_l)} = \int_G |f(x)|^2 d\mu_l(x) < \infty \ .$$

Definition 2.1
A representation U of G in H is called square integrable if there exists one vector $g \in H, g \neq 0$ such that

$$\langle U(x)g|g\rangle_H \in L^2(G, d\mu_l) \ ,$$

i.e.

$$\int_G |\langle U(x)g|g\rangle_H|^2 d\mu_l(x) < \infty \ .$$

We now can introduce the 'wavelet' transform for arbitrarily locally compact groups in an abstract functional analytic Hilbert space setting.

Definition 2.2
Let U be a square integrable representation of a locally compact group G acting in a Hilbert space H. Then the left–transform over G of a function $s \in H$ with respect to a vector $g \in H, g \neq 0$ is defined by the set of scalar products

$$\mathcal{L}_g s(x) = \langle U(x)g|s\rangle_H \quad (x \in G) .$$

Note that from the Schwarz inequality $|\mathcal{L}_g s(x)| \leq ||g||_H ||s||_H$ it follows that $\mathcal{L}_g$ is in general of higher regularity than the analysed function s itself.
Moreover, the left–transform $\mathcal{L}_g s$ with respect to an admissible analysing vector g conserves energy:

Definition 2.3
A vector $g \in H$ is called admissible if

$$c_g := \int_G |\mathcal{L}_g g(x)|^2 d\mu_l < \infty . \tag{2.1}$$

Theorem 2.4
Let $g \in H$ be an admissible vector. Then we have

$$\langle \mathcal{L}_g s|\mathcal{L}_g r\rangle_{L^2(G,d\mu_l)} = \int_G \overline{\mathcal{L}_g s(x)} \mathcal{L}_g r(x) d\mu_l = c_g \langle s|r\rangle_H ,$$

where $c_g > 0$ is given by (2.1).

It follows from Theorem 2.4 that the left–transform is an isometry, i.e. we have

$$||\mathcal{L}_g s(x)||^2_{L^2(G,d\mu_l)} = c_g ||s||^2_H .$$

Thus $\mathcal{L}_g$ may be inverted by the adjoint. But more general inversion formulae exist. For this purpose one introduces the left–synthesis $\mathcal{L}_h^*$ with respect to an admissible vector h as the adjoint operator of the left–transform with respect to h, i.e.

$$\langle s|\mathcal{L}_h^* T\rangle_H = \langle \mathcal{L}_h s|T\rangle_{L^2(G,d\mu_l)} ,$$

where $T \in L^2(G, d\mu_l)$ and

$$\mathcal{L}_h^* T = \int_G T(x)U(x)h d\mu_l(x) .$$

Thus the left–systhesis $\mathcal{L}_h^* : L^2(G, d\mu_l) \to H$ is a bounded linear map with

$$||\mathcal{L}_h^* T||^2_H \leq c_h ||T||^2_{L^2(G,d\mu_l)} .$$

Note that the left–systhesis allows one to write functions in H as superpositions of group–translated functions $U(x)h$ with weight $T(x)$.
Moreover, the left–synthesis with respect to a reconstruction vector is actually the inversion of the left–transform:

Definition 2.5
A vector $h \in H$ is called a reconstructing vector for the analysing vector $g \in H$ iff

$$c_{g,h} := \frac{\langle \mathcal{L}_h g|\mathcal{L}_g h\rangle_{L^2(G,d\mu_l)}}{\langle g|h\rangle_H} \tag{2.2}$$

satisfies $0 < |c_{g,h}| < \infty$.

Theorem 2.6
Let $h \in H$ be an admissible reconstruction vector of an admissible vector $g \in H$. Then we have

$$\mathcal{L}_h^* \mathcal{L}_g = c_{g,h} I_H \ ,$$

where the constant $c_{g,h}$ is given by (2.2).

2.2 Wavelet transform

To define the desired wavelet calculus we need the following specifications of the general set–up.

(a) Let $G = (\mathbb{R}, +)$ and $H = L^2(\mathbb{R})$. Then the translation operator

$$T_b : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ s(t) \mapsto s(t-b)\ \ (b \in \mathbb{R})$$

is a unitary, but not irreducible representation of G in $L^2(\mathbb{R})$.

(b) Let $G = (\mathbb{R}^+, \cdot)$ and $H = L^2(\mathbb{R})$. Then the dilation operator

$$D_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})\ ,\ s(t) \mapsto a^{-1/2} s(t/a)\ \ (a \in \mathbb{R}^+)$$

is a unitary, but not irreducible representation of G in $L^2(\mathbb{R})$.

(c) Let $\mathbb{H} = \{(b,a) | b \in \mathbb{R}\ ,\ a \in \mathbb{R}^+\}$ be the upper half–plane and $G = (\mathbb{H}, \circ)$ be the affine group. Then a unitary representation of $\mathbb{H}$ on $L^2(\mathbb{R})$ is given by

$$U : \mathbb{H} \to \mathcal{U}(L^2(\mathbb{R}))\ ,\ (b,a) \mapsto U(b,a) = T_b D_a\ .$$

(d) Denote by $H_+^2(\mathbb{R})$ and $H_-^2(\mathbb{R})$ the closed subspaces of $L^2(\mathbb{R})$ consisting of those functions f of $L^2(\mathbb{R})$ that are progessive (i.e. $supp \hat{f} \subset \mathbb{R}_0^+$) or regressive (i.e. $supp \hat{f} \subset R^-$) respectively. Then $U = T_b D_a$ is irreducible on $H_+^2(\mathbb{R})$ and, by symmetry, also on $H_-^2(\mathbb{R})$. Thus $L^2(\mathbb{R})$ can be split into irreducible components $L^2(\mathbb{R}) = H_+^2(\mathbb{R}) \oplus H_-^2(\mathbb{R})$.

(e) Let $G = (\mathbb{H}, \circ)$ and $H = L^2(\mathbb{R})$. A function $g \in L^2(\mathbb{R})$ with $\hat{g}(0) = 0$ is called a wavelet. If g is admissible, then the affine group $\mathbb{H}$ has a square integrable representation on $L^2(\mathbb{R})$. In fact, let $U(b,a) : g \to a^{-1/2} g((t-b)/a)$, then

$$a^{-1/2} \int_{\mathbb{R}} \overline{g}\left(\frac{t-b}{a}\right) g(t) dt \in L^2(\mathbb{H}, d\mu_l)$$

with left invariant Haar measure $d\mu_l = da db/a^2$.
This leads to the following version of the L^2-left–transform.

Definition 2.7
Let $f \in L^2(\mathbb{R}\ )$ and g be a wavelet. The left-transform $a^{1/2} \mathcal{L}_g f(b,a)$ over $\mathbb{H}$ is called the continuous 1–D wavelet transform of f with respct to the wavelet g:

$$a^{1/2} \mathcal{L}_g f(b,a) := (L_g f)(b,a) := \frac{1}{a} \int_{\mathbb{R}} \overline{g}\left(\frac{t-b}{a}\right) f(t) dt\ . \tag{2.3}$$

2.3 Cauchy wavelets

To define the wavelet class of fractal zeta and eta functions in an abstract Hilbert space setting we need a wavelet calculus that substitutes the familiar Seeley–Grubb holomorphic functional calculus [GS95], [GS96] by a new wavelet transform technique. This is not based as usual on the Mellin transform and its inversion, but on the wavelet transform and the wavelet synthesis operator with respect to $1D$–Cauchy wavelets with complex parameter (also called 'Hyperbolic chirp wavelets' [H95]).
The familiar Cauchy wavelets are progressive wavelets with real–valued Fourier transform and are defined on $I\!R$ by

$$\phi_\beta(t) := \frac{1}{2\pi}\Gamma(\beta+1)(1-it)^{-\beta-1} \quad (\beta > 0) \tag{2.4}$$

with Fourier transform

$$\hat{\phi}_\beta(v) = \begin{cases} v^\beta \exp\{-v\} & (v > 0) \\ 0 & (v \leq 0) \end{cases} .$$

These wavelets have been at first investigated in the context of Quantum mechanics by T. Paul [Pau85], and in the wavelet analysis of the fractal Riemann–Weierstrass function by M. Holschneider and Ph. Tchamitchian [HT91], [H95] or S. Jaffard and Y. Meyer [JM96].
In the Global Anlysis the $1D$–Cauchy wavelets appear for the first time in connection with spectral geometry for the Laplacian on manifolds with cusps in [K97].
Recently, the class of directional $2D$–Cauchy wavelets was studied in connection with image processing in [AMV99].
The complex valued Cauchy wavelets are of high regularity, but have at most a polynomial decay at ∞, and for $\beta \in I\!N$ they are nothing but the βth derivate of the Cauchy kernel. Accordingly, since

$$2\pi a^{-1}\overline{\phi}_\beta\left(\frac{t-b}{a}\right) = a^\beta \Gamma(\beta+1)\left(\frac{i}{\omega - t}\right)^{\beta+1} ,$$

with $\omega = b + ia \in I\!H$, the wavelet transform (1.6) of $f \in L^2(I\!R)$ with respect to ϕ_β is – up to a prefactor – a holomorphic function on the Poincaré upper half–plane and can be written as (cf. [K97])

$$(L_{\phi_\beta} f)(b,a) = \frac{a^\beta}{2\pi}\Gamma(\beta+1)\int_{I\!R}(a+it)^{-\beta-1} f(t+b)dt . \tag{2.5}$$

A reconstruction wavelet of ϕ_β is any integrable function $h_\beta \in L^1(I\!R)$ with

$$\int_{I\!R^+} a^{\beta-1}\hat{h}_\beta(a)e^{-a}da = 1 \;,\; \hat{h}_\beta(0) = 0 . \tag{2.6}$$

¿From now on we consider the progressive hyberbolic chirp wavelets ϕ_s with $\beta > 0$ replaced by the complex variable s with $\mathrm{Re}\, s > 0$.

3 Wavelet calculus

In view of Theorem 2.6 the usual inversion formula for (2.3) gives rise to a linear operator M_h mapping arbitrary functions $F(b,a)$ over the half–plane $I\!H$ to functions over the real line $I\!R$. With $h_{b,a} := T_b D_a h$, h any reconstruction wavelet, one thus defines

$$(M_h F)(t) := \int_{I\!R^+}\int_{I\!R} F(b,a)a^{-1}h_{b,a}(t)dbda ,$$

whenever the double integral is absolutely convergent. Note that for $h \in L^1(I\!R)$ it is well–defined for all functions F in the Frechet space $S(I\!H)$ since they are rapidly decaying as $|b|$ or $a+(1/a)$ tends to infinity.
The wavelet synthesis operator M_h itself gives rise to the following wavelet calculus in an abstract functional analytic Hilbert space–setting.

Let Q be a lower bounded self–adjoint operator in a Hilbert space H with dense domain $D(Q)$ and compact resolvent $R_\lambda(Q) := (Q-\lambda)^{-1}$.

(1) For $\tau = a - ib, \operatorname{Re}\tau > 0$ the exponential function of Q can be defined by

$$e^{-\tau Q}(I - P_0(Q)) = \frac{i}{2\pi}\int_C e^{-\tau\lambda} R_\lambda(Q) d\lambda \,, \tag{3.1}$$

where P_0 is the orthogonal projection onto $\ker Q$ and the integration curve $C \subset \mathbb{C}$ is given by (cf. [GS96])

$$C := \{\lambda = re^{i\theta} | \infty > r \geq r_0\} \cup \{\lambda = r_0 e^{i\varphi} | \theta \geq \varphi \geq -\theta\} \cup \{\lambda = re^{i(2\pi-\theta)} | r_0 \leq r < \infty\}$$

with $0 < \theta < \pi/2$, and r_0 chosen so that $R_\lambda(Q)$ is holomorphic on $0 < |\lambda| \leq r_0$ with $\|R_\lambda(Q)\|_H = 0(|\lambda|^{-1})$, and meromorphic at $\lambda = 0$.

(2) We now define the wavelet power function of Q for $\operatorname{Re} s > 0$ and $x \in I\!R$ by

$$Z(Q,s,x) := \int_{I\!R^+}\int_{I\!R} a^{s-2} h_s\left(\frac{x-b}{a}\right) e^{-\tau Q} dbda \,, \tag{3.2}$$

where h_s is the reconstruction wavelet (2.3) for the Cauchy wavelet ϕ_s.
Clearly, the integral over the half–plane is well–defined as norm–convergent integral. If Q is invertible then $Z(Q,s,x) = \exp(ixQ)Q^{-s}$.
For $x = 0$ we have

$$Z(Q,s,0) := Z(Q,s) = \int_{I\!R^+}\int_{I\!R} a^{s-2} h_s\left(-\frac{b}{a}\right) e^{-\tau Q} dbda \,.$$

This coincides with the usual power function (cf. [GS96])

$$Z(Q,s) = \frac{i}{2\pi}\int_C \lambda^{-s} R_\lambda(Q) d\lambda \quad (\operatorname{Re} s > 0) \,,$$

with $Z(Q,s) = 0$ on the nullspace of Q, since $\int_C \lambda^{-s-1} d\lambda = 0$.
Moreover, it follows that

$$Y(Q,s,x) := QZ(Q^2, \frac{s+1}{2}, x) = \int_{I\!R^+}\int_{I\!R} a^{(s-3)/2} h_{(s+1)/2}\left(\frac{x-b}{a}\right) Q e^{-\tau Q^2} dbda \tag{3.3}$$

is well–defined for $\operatorname{Re} s > 0$ and $x \in I\!R$.
More generally, if $B : D(Q) \to H$ is any bounded linear operator with

$$P_0(Q) B P_0(Q) = 0 \,, \tag{3.4}$$

then the wavelet power function

$$Y(Q,B,s,x) := BZ(Q^2, \frac{s+1}{2}, x) = \int_{I\!R^+}\int_{I\!R} a^{(s-3)/2} h_{(s+1)/2}\left(\frac{x-b}{a}\right) B e^{-\tau Q^2} dbda \tag{3.5}$$

is well–defined for $\mathrm{Re}\, s > 0$ and $x \in I\!R$.

(3) We turn to wavelet space. It follows from the scalar formulas, valid in each eigenspace, that the exponential functions are related to the wavelet power functions by the following 'wavelet transforms' with respect to Cauchy wavelets:

$$(\mathrm{Re}\,\tau)^s e^{-\tau Q} \quad = \quad a^{-1} \int_{I\!R} \overline{\phi}_s \left(\frac{x-b}{a} \right) Z(Q,s,x) dx \ , \tag{3.6}$$

$$(\mathrm{Re}\,\tau)^{(s+1)/2} Q e^{-\tau Q^2} \quad = \quad a^{-1} \int_{I\!R} \overline{\phi}_{(s+1)/2} \left(\frac{x-b}{a} \right) Y(Q,s,x) dx \ , \tag{3.7}$$

$$(\mathrm{Re}\,\tau)^{(s+1)/2} B e^{-\tau Q^2} \quad = \quad a^{-1} \int_{I\!R} \overline{\phi}_{(s+1)/2} \left(\frac{x-b}{a} \right) Y(Q,B,s,x) dx \ , \tag{3.8}$$

$\tau = a - ib, \mathrm{Re}\,\tau > 0,\ b \in I\!R$ and $\mathrm{Re}\, s > 0$.

4 Fractal zeta and eta functions

We can now introduce a new wavelet family of fractal zeta and eta functions for the operators Q and B defined in the previous section. In addition, we assume that

$$(Q+i)^{-1} \in C_p(H) \tag{3.9}$$

for some $p > 0$, where $C_p(H)$ denotes the v. Neumann – Schatten class of order p (cf. [BL97]). Then the wavelet calculus immediately implies the following results:

Lemma 4.1
Let Q be a lower bounded self–adjoint operator in Hilbert space H with dense domain $D(Q)$ and compact resolvent $R_\lambda(Q)$. Assume that $(Q+i)^{-1} \in C_p(H)$, for some $p > 0$, and let $B : D(Q) \to H$ with (3.4) be a bounded linear operator in H.
Then, for $x \in I\!R$ and $\tau = a - ib, \mathrm{Re}\,\tau > 0, b \in I\!R$ we have:

(1) The fractal zeta function of $Q \geq 0$

$$\begin{aligned} \zeta(Q,s,x) := tr_H Z(Q,s,x) \quad &= \quad \int_{I\!R^+} \int_{I\!R} a^{s-2} h_s \left(\frac{x-b}{a} \right) tr_H e^{-\tau Q} dbda \\ &= \quad \sum_{\lambda \in spec(Q) \setminus \{0\}} \lambda^{-s} \exp\{i\lambda x\} \end{aligned}$$

is holomorphic for $\mathrm{Re}\, s >> 0$.

(2) The fractal eta function of Q

$$\begin{aligned} \eta(Q,s,x) := tr_H Y(Q,s,x) \quad &= \quad \int_{I\!R^+} \int_{I\!R} a^{(s-3)/2} h_{(s+1)/2} \left(\frac{x-b}{a} \right) tr_H Q e^{-\tau Q^2} dbda \\ &= \quad \sum_{\lambda \in spec(Q) \setminus \{0\}} sign(\lambda) |\lambda|^{-s} \exp\{i\lambda^2 x\} \end{aligned}$$

is holomorphic for $\mathrm{Re}\, s >> 0$.

(3) The fractal eta function of Q and B

$$\begin{aligned}\eta(Q,B,s,x) := tr_H Y(Q,B,s,x) &= \int_{I\!R+}\int_{I\!R} a^{(s-3)/2} h_{(s+1)/2}\left(\frac{x-b}{a}\right) tr_H B e^{-\tau Q^2}\, dbda \\ &= \sum_{\lambda\in spec(Q)\setminus\{0\}} tr(P_\lambda B)|\lambda|^{-s-1}\exp\{i\lambda^2 x\}\end{aligned}$$

is holomorphic for $\mathrm{Re}\, s >> 0$. Here P_λ is the orthogonal projection onto the λ–eigenspace of Q, and $tr(P_\lambda B) =: tr_{\ker(Q-\lambda)} B$.

Remark:
The fractal zeta and eta functions of Lemma 4.1 can also be represented in Mellin space. In particular, note that

$$\eta(Q,B,s,x) = \frac{1}{\Gamma((s+1)/2)}\int_{I\!R+} a^{(s-1)/2} tr_H B e^{-\tau Q^2}\, da$$

with $\tau = a - ix, \mathrm{Re}\,\tau > 0, x \in I\!R$ is holomorphic for large $\mathrm{Re}\, s >> 0$.
The special fractal eta function $\eta(Q,B,s,0) := \eta(Q,B,s)$, recently studied by Brüning and Lesch [BL97] in connection with the gluing law and non–local boundary value problems, has a meromorphic extension to the whole s–plane provided that the heat trace $tr_H B\exp(-aQ^2)$ has an asymptotic expansion as $a \to 0+$.

In wavelet space we obtain the corresponding 'wavelet traces' as wavelet transforms of the fractal zeta and eta functions with respect to Cauchy wavelets:

Lemma 4.2
Under the assumptions of Lemma 4.1 we have for $\mathrm{Re}\, s >> 0$ *and* $\tau = a - ib, \mathrm{Re}\,\tau > 0, b \in I\!R$*:*

$$\begin{aligned}(1)\quad (L_{\phi_s}\zeta(Q,s,x))(\tau) &= (\mathrm{Re}\,\tau)^s tr_H e^{-\tau Q} = (\mathrm{Re}\,\tau)^s \sum_{\lambda\in spec(Q)\setminus\{0\}} e^{-\tau\lambda}\,, \\ (2)\quad (L_{\phi_{(s+1)/2}}\eta(Q,s,x))(\tau) &= (\mathrm{Re}\,\tau)^{(s+1)/2} tr_H Q e^{-\tau Q^2} = (\mathrm{Re}\,\tau)^{(s+1)/2} \sum_{\lambda\in spec(Q)\setminus\{0\}} \lambda e^{-\tau\lambda^2}\,, \\ (3)\quad (L_{\phi_{(s+1)/2}}\eta Q,B,s,x))(\tau) &= (\mathrm{Re}\,\tau)^{(s+1)/2} tr_H B e^{-\tau Q^2} = (\mathrm{Re}\,\tau)^{(s+1)/2} \sum_{\lambda\in spec(Q)\setminus\{0\}} tr(P_\lambda B) e^{-\tau\lambda^2}\,.\end{aligned}$$

Remarks:
1. A prominent example of a fractal zeta function is the classical 'non–differentiable' fractal Riemann–Weierstrass function defined by

$$W_\beta(t) = 2\sum_{n\geq 1} n^{-2\beta}\exp\{in^2 t\} \quad (\beta > 1/2)\,,$$

(cf. [H95],[HT91]).
It is associated to the scalar Laplacian $\partial_\theta^2 : L^2(S^1) \to L^2(S^1), S^1 = \{z \in \mathbb{C}\,||z| = 1\}$, by encoding the discrete spectrum $spec(\partial_\theta^2) = \{n^2\}_{n\in\mathbb{Z}}$. By Lemma 4.1, we have the scale space representation

$$W_\beta(t) = \zeta(\partial_\theta^2,\beta,t) = \int_{I\!R+}\int_{I\!R} a^{\beta-2} h_\beta\left(\frac{t-b}{a}\right)\left[tr_{L^2} e^{-\tau\partial_\theta^2} - \dim\ker\partial_\theta^2\right] dbda\,,$$

and, by Lemma 4.2, the wavelet trace

$$(L_{\phi_\beta}\zeta(\partial_\theta^2,\beta,t))(\tau) = (\mathrm{Re}\,\tau)^\beta\left[tr_{L^2} e^{-\tau\partial_\theta^2} - \dim\ker\partial_\theta^2\right] = (\mathrm{Re}\,\tau)^\beta\left[\vartheta(\tau) - 1\right]\,,$$

where $\vartheta(\tau) = \sum_{-\infty}^{+\infty} \exp\{-n^2\tau\}, \tau = a - ib, \operatorname{Re}\tau > 0$ is a classical Jacobi theta function.
It is a very remarkable fact that the pointwise differentiability of $W_\beta(t)$ at selected points in the orbits of 0 and 1 with respect to the modular group G_ϑ can be deduced from general results of fractal analysis through wavelet transforms, as was shown by an analysis of the theta function near the imaginary axis in [HT91], [H95] and [JM96].

2. The concept of generalized fractal Riemann–Weierstrass functions was at first introduced in [K97] with applications to spectral geometry, analytic number theory and signal analysis. This new unified wavelet analytic approach to fundamental results of harmonic analysis and of differential geometry extends the results of [K91] and [K92].

In the next sections we describe fundamental applications of the fractal zeta and eta functions in Global Analysis.

5 Fractal zeta function invariants

Let M be a compact $(n+1)$–dimensional manifold and $Q > 0$ a self–adjoint elliptic pseudodifferential operator, $ord\, Q = 1$, operating on the space of smooth half-densities $C^\infty(M, \Omega^{1/2})$. Consider the fractal zeta funtion of Q

$$\zeta(Q, s, x) = tr Z(Q, s, x) = \int_{I\!R^+} \int_{I\!R} a^{s-2} h_s \left(\frac{x-b}{a}\right) tr\, e^{-\tau Q}\, dbda\ , \tag{5.1}$$

holomorphic for $\operatorname{Re} s >> 0$. Then the associated 'Wave trace' [Gui96]

$$e(t) := \sum_{\lambda \in spec(Q)} e^{i\lambda t} \tag{5.2}$$

is a tempered distribution with the following properties [Ch73], [DG75]:
(a) let $\sigma(Q)(z, \xi)$ be the leading symbol of Q, and let E be the Hamiltonian vector field on $T^*M\backslash 0$ associated with $\sigma(Q)$:

$$E := \sum \frac{\partial}{\partial \xi_i}\sigma(Q)\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_i}\sigma(Q)\frac{\partial}{\partial \xi_i}\ . \tag{5.3}$$

Then a necessary condition for $T \in I\!R$ to be in the singular support of $e(t)$ is that there exists a T–periodic trajectory γ of E.

(b) If γ is nondegenerate, it contributes to the wave trace (5.2) a singularity of the form

$$e_\gamma(t) \sim \sum_{r \geq 1} c_r (t - T + i0)^{-2+r} \log(t - T + i0)\ , \tag{5.4}$$

and the coefficient, c_1 of the leading term in (5.4) is given by Duistermaat–Guillemin formula

$$c_1(Q, T) = \frac{T_\gamma}{2\pi} i^{\sigma_\gamma} |\det(I - P_\gamma)|^{-1/2} \exp\left[i \int_0^T \sigma_{sub}(Q)(\gamma) dt\right]\ , \tag{5.5}$$

where T_γ is the primitive period of γ, σ_γ is the Maslov index of γ, P_γ is the linearized Poincaré map about $\gamma, \sigma_{sub}(Q)$ is the subprinciple symbol of Q.
The coefficients c_r are called 'Wave trace invariants'.

Recently, V. Guillemin [Gui96] has proved the conjecture of A. Weinstein that the higher wave trace invariants c_r determine the entire Birkhoff canonical form (see [Gui96] for definition and proof).
One crucial point in Guillemin's proof is the characterization of the wave trace invariants as 'zeta function invariants', due to S. Zelditch (see [Gui96, App. A.]).
The following theorem gives a new interpretation of this result in the wavelet analytic setting.

Theorem 5.1
Let M be a compact $(n+1)$-dimensional manifold and $Q > 0$ a self–adjoint elliptic pseudodifferential operator, $ord\, Q = 1$, operating on the space of smooth half–densities $C^\infty(M, \Omega^{1/2})$. Let $e(t)$ be the 'wave trace' (5.2) *with the properties (a) and (b).*
Let T be the period of a nondegenerate trajectory of the Hamilton vector field (5.3) *and h_s be the reconstructing wavelet* (2.6) *for the Cauchy wavelet ϕ_s. Then the fractal zeta function at $x = T$*

$$\zeta(Q,s,T) = tr\, Z(Q,s,T) = \int_{I\!R^+}\int_{I\!R} a^{s-2} h_s\left(\frac{T-b}{a}\right) tr\, e^{-\tau Q}\, dbda = \sum_{\lambda \in spec(Q)} \lambda^{-s} e^{i\lambda T}$$

is holomorphic for $\mathrm{Re}\, s >> 0$, *and extends meromorphically on $\mathbb{C}$ with simple poles at $s = 1, 0, -1, \ldots$. The residues at these points are the wave trace invariants $c_r(Q,T)$, i.e. the full singularity structure of the fractal zeta function is described by*

$$\zeta(Q,s,T) \sim \sum_{k=0}^{\infty} c_{k+1}(Q,T)[s-(1-k)]^{-1}$$

and $c_1(Q,T)$ is given by (5.5).

Remarks:
1. Theorem 5.1 includes the fact that the wave trace invariants at $T = 0$ ("big" singularity) are the usual zeta function invariants, as was shown by Duistermaat & Guillemin in [DG75].

2. Recently, J. Jorgenson and S. Lang [JL99] have studied relations between the heat operator $\exp(-t\Delta)$ and the wave operator $\exp(-it\sqrt{\Delta})$, Δ the Laplacian on a Riemannian manifold.
Note that the methods and results in [JL99] are also of wavelet analytic nature, since – what is overlooked by the authors – the basic Jorgenson–Lang G–transform [JL99,§2] is essentially a continuous wavelet transform with respect to Marr–Wavelets [Ma82], [H95]. We will return to this really new aspect in part III.

6 Fractal zeta functions and pseudodifferential boundary problems

6.1 General elliptic systems

(i) Let M be a n–dimensional compact C^∞ manifold with boundary $\partial M = N$, and $P : C^\infty(M, E_1) \to C^\infty(M, E_2)$ an elliptic differential operator of order $d \geq 1$ between sections of Hermitian vector bundles $E_i, \dim E_i = m$.
As usual, let $U_c = N \times [0, c]$ be a collar neighborhood of the boundary $N, x_n \in [0, c]$ a normal coordinate with normal derivate ∂_{x_n}. We assume that in U_c the E_i are isomorphic

to the pull backs of the $E_i^* := E_i|_N$.
Let $H^s(M, E_i)$ be the Sobolev space of order s of sections of E_i and set

$$K^s(E_i^{*d}) := \prod_{0\le j<d} H^{s-j-1/2}(E_i^*) \ , \quad E_i^{*d} := \bigoplus_{0\le j<d} E_i^* \ .$$

Then, for $s > d - 1/2$, the operator $\varrho := \{\gamma_0, \dots, \gamma_{d-1}\}$ with $\gamma_j u := (-i\partial_{x_n})^j u|_{x_n=0}$ maps $H^s(E_i)$ into $K^s(E_i^{*d})$.
The sections u of E_1 and w of E_2 in H^s satisfy the Green's formula

$$\langle Pu, w\rangle_M - \langle u, P^* w\rangle_M = \langle A\varrho u, \varrho w\rangle_N \ ,$$

where $A := (A_{jk})_{j,k=0,\dots,d-1}$ is upper skew–triangular with differential operators A_{jk}, $ord\,(A_{jk}) = d-1-j-k$, $A_{jk} = 0$ for $k > d-1-j$ and the A_{jk} with $k = d-1-j$ are isomorphisms.
Let S be an operator on $K^s(E_i^{*d})$, then the boundary condition

$$S\varrho u = 0 \tag{6.1}$$

determines a realization P_S of P with domain

$$D(P_S) = \{u \in H^d(M, E_1) | S\varrho u = 0\} \tag{6.2}$$

(ii) We assume that S defines a *pseudo-normal* boundary condition in the sense of Grubb [G99; Ass.2.1]:
$S = (S_{jk})_{j,k=0,\dots,d-1}$ is a matrix of admissible classical $\psi do's$ with $S_{jk} : E_1^* \to F_j, F_j$ admissible bundles over N, and $ord(S_{jk}) = j - k, S_{jk} = 0$ for $j < k$. Moreover, S_{jj} is surjective and uniformly surjectively elliptic.

(iii) To define the fractal zeta and eta functions of P_S by means of the wavelet calculus we need the following *resolvent growth conditions* in the sense of Grubb [G99;Ass.2.2]:
Let $E_1 = E_2 = E$ and $J \subset [0, 2\pi]$ an open interval. Then there is an open sector $\Gamma = \{\lambda \in \mathbb{C} \setminus \{0\} | \arg\lambda \in J\}$, such that the following holds:
(G1): P is elliptic, and for the principal symbol p° of $P, p^\circ(y,\xi)$–λ is invertible for all (y, ξ, λ) with $\lambda \in \Gamma \cup \{0\}, |\xi|^2 + |\lambda|^{2/d} \ge 1$, the inverse being $0(|\xi|^d + |\lambda|^{-1})$ on closed subsectors $\Gamma' \subset \Gamma$, uniformly in $y \in U_c, y := (y', x_n)$.
(G2): $\dim[F := \bigoplus_{j=0}^{d-1} F_j] = (md)/2$, the system $\{P, S\varrho\}$ is elliptic, and for any closed subsector Γ' there exists an $r \ge 0$ such that the resolvent $R_\lambda(P_S) = 0(\lambda^{-1})$ for $\lambda \in \Gamma'_r := \{\lambda \in \Gamma' || \lambda| \ge r\}$.

(iv) We now turn to the fractal zeta and eta functions of P_S. For this purpose let $[\pi/2, 3/2\pi]$ be in the interior of J, and let R_λ exist on the keyhole region $R := \{\lambda || \lambda| \le r$ or $|\arg\lambda - \pi| \le \pi/2 + \varepsilon\}$. Then the exponential function of P_S can be defined by

$$e^{-\tau P_S} = \frac{i}{2\pi}\int_{\partial R} e^{-\tau\lambda}(P_S - \lambda)^{-1} d\lambda \ ,$$

$\tau = a - ib, \mathrm{Re}\,\tau > 0, b \in \mathbb{R}$. This implies

$$e^{-\tau P_S} = \frac{i}{2\pi}\int_{\partial R} \partial_\lambda^m (P_S - \lambda)^{-1} (-\tau)^{-m} e^{-\tau\lambda} d\lambda$$

for $(m+1)d > n = \dim M$.
By the wavelet calculus the fractal power function of P_S is well defined for $\mathrm{Re}\, s > 0, x \in I\!R$, and is given by

$$Z(P_S, s, x) = \frac{i}{2\pi} \int_{\partial R} \partial_\lambda^m (P_S - \lambda)^{-1} \int_{I\!R^+} \int_{I\!R} a^{s-2} h_s \left(\frac{x-b}{a} \right) (-\tau)^{-m} e^{-\tau\lambda} dbdad\lambda, .$$

Let φ be any compactly supported morphism in E. Then $\varphi \partial_\lambda^m R_\lambda$ maps $L^2(E)$ into $H^{(m+1)d}(E|_{M_1})$, where M_1 is a smooth compact neighborhood of supp φ in M such that $\partial M_1 = M_1 \cap N$ is a neighborhood of supp $\varphi \cap N$ in N. Thus the operator $\varphi \partial_\lambda^m R_\lambda$ is trace class with continuous kernel, and the trace is the integral of the fiber trace of the kernel on the diagonal with integration over M_1.
It follows that the fractal zeta function

$$tr(\varphi Z(P_S, s, x)) = \frac{i}{2\pi} \int_{\partial R} tr(\varphi \partial_\lambda^m (P_S - \lambda)^{-1}) \int_{I\!R^+} \int_{I\!R} a^{s-2} h_s \left(\frac{x-b}{a} \right) (-\tau)^{-m} e^{-\tau\lambda} dbdad\lambda \tag{6.3}$$

with $\tau = a - ib, x \in I\!R$ is holomorphic for $\mathrm{Re}\, s >> 0$, and the same holds for the fractal eta function

$$tr(\varphi Y(P_S, s, x)) =$$

$$\frac{i}{2\pi} \int_{\partial R} tr(\varphi P_S \partial_\lambda^m (P_S^* P_S - \lambda)^{-1}) \int_{I\!R^*} \int_{I\!R} a^{(s-3)/2} h_{(s+1)/2} \left(\frac{x-b}{a} \right) (-\tau)^{-m} e^{-\tau\lambda} dbdad\lambda . \tag{6.4}$$

Since M is compact the resolvent has a pole at $\lambda = 0$ if $\ker P_S \neq 0$. Thus $r(\lambda) := tr(\varphi \partial_\lambda^m (P_S - \lambda)^{-1})$ is meromorphic at $\lambda = 0$ with $r(\lambda) = \Gamma(m+1)(-\lambda)^{-m-1} tr(\varphi P_0(P_S)) + h(\lambda)$, $h(\lambda)$ holomorphic at $\lambda = 0$.

Now the wavelet calculus combined with the fundamental resolvent expansion of Grubb [G99;Theorem 9.1)] yields the following trace expansions which seem to be *not* available by the familiar Mellin transform technique of the heat equation method used, e.g., in [ABP73], [Gi95], [BW93], [Mü94], [GS95], [GS96], [BL97] or [BL99].
The passage from the resolvent expansion to the fractal zeta and eta function asymptotics is described by the following general wavelet transition formula.

Proposition 6.1
Let $\mathcal{B}$ be a Banach space and $r : \mathbb{C} \to \mathcal{B}$ be meromorphic at $\lambda = 0$ with Laurent expansion

$$r(\lambda) = \sum_{-k}^{\infty} b_j (-\lambda)^j \ , \ \ |\lambda| \le \rho \ .$$

Let r be holomorphic in $\Gamma_{\delta_0} = \{\lambda \in \mathbb{C} \,||\, \arg(\lambda - \pi|) < \delta_0 \le \pi\}$ and $r(\lambda) = 0(|\lambda|^{-\alpha})$, some $\alpha \in (0,1]$, as $\lambda \to \infty$, uniformly in each sector Γ_δ for $\delta < \delta_0$.
Let C be a Laurent loop and define for $\mathrm{Re}\, s > 1 - \alpha, x \in I\!R$ and $\tau = a - ib, \mathrm{Re}\, \tau > 0$

$$\zeta(s, x) := \int_{I\!R^+} \int_{I\!R} a^{s-2} h_s \left(\frac{x-b}{a} \right) \frac{i}{2\pi} \int_C r(\lambda) e^{-\tau\lambda} d\lambda dbda \ ,$$

where h_s is a reconstruction wavelet for the Cauchy wavelet ϕ_s. Suppose that $r(\lambda)$ has an asymptotic expansion as $\lambda \to \infty$

$$r(-\lambda) \sim \sum_{j \ge 0} \sum_{\beta=0}^{n_j} a_{j\alpha} \lambda^{-\alpha_j} (\log \lambda)^\beta \ , \quad \alpha_j \in I\!R^+, \lim_{j \to \infty} \alpha_j = +\infty, n_j \in I\!N_0 \ ,$$

uniformly for $-\lambda$ in Γ_δ, for each $\delta < \delta_0$.
Then $\Gamma(s)\zeta(s,x)$ is meromorphic on $\mathbb{C}$ with singularity structure

$$\Gamma(s)\zeta(s,x) \sim \sum_{j=-k}^{-1} \frac{-\overline{b}_j(x)}{s-j-1} + \sum_{j\geq 0}\sum_{\beta=0}^{n_j} \frac{\overline{a}_{j\beta}(x)\beta!}{(s+\alpha_j-1)^{\beta+1}},$$

in the sense that for large N, the left–hand side minus the sums for $j \leq N$ in the right–hand side is holomorphic for $1-\alpha_N < \operatorname{Re} s < N+1$.

¿From [G99;(9.1)], (6.3), (6.4) and Proposition 6.1 we can deduce the following new result.

Theorem 6.2
Let P_S be a realization (6.2) defined from a differential operator P, ord $P = d \geq 1$, in a bundle E over a compact manifold M such that S defines a pseudo–normal boundary condition (6.1). Suppose that the resolvent growth conditions (G1) and (G2) hold. Let $(m+1)d > n = \dim M, x \in \mathbb{R}$ and φ be any compactly supported morphism in E. Then the fractal zeta and eta functions have singularity structures described by:

$$\begin{aligned}\Gamma(s)tr(\varphi Z(P_S,s,x)) \;\sim\;& tr(\varphi P_0(P_S))s^{-1} + \frac{\overline{a}_0(P,S,x)}{s-n/d}\\ &+ \sum_{k\geq 1} \frac{\overline{a}_k(P,S,x)+\overline{b}_k(P,S,x)}{s-(n-k)/d}\\ &+ \sum_{k\geq 0}\left(\frac{\overline{c}_k(P,S,x)}{(s+k/d)^2} + \frac{\overline{c}^*_k(P,S,x)}{s+k/d}\right),\end{aligned} \tag{6.5}$$

$$\begin{aligned}\Gamma(s)tr(\varphi Y(P_S,2s-1,x)) \;\sim\;& \frac{\overline{a}_0(P,S,x)}{s-(n+1)/d} + \sum_{k\geq 1}\frac{\overline{a}_k(P,S,x)+\overline{b}_k(P,S,x)}{s-(n-k+1)/d}\\ &+\sum_{k\geq 0}\left(\frac{\overline{c}_k(P,S,x)}{(s+(k-1)/d)^2} + \frac{\overline{c}^*_k(P,S,x)}{s+(k-1)/d}\right).\end{aligned} \tag{6.6}$$

For $x = 0$ the coefficients are related to those in $[G99; Th.9.1]$ by suitable gamma factors.

We now turn to wavelet space. From the wavelet calculus we obtain the wavelet traces

$$(\operatorname{Re}\tau)^s tr(\varphi e^{-\tau P_S}) = a^{-1}\int_{\mathbb{R}} \overline{\phi}_s\left(\frac{x-b}{a}\right) tr(\varphi Z(P_S,s,x))dx$$

and

$$(\operatorname{Re}\tau)^{(s+1)/2} tr(\varphi P_S e^{-\tau P_S^* P_S}) = a^{-1}\int_{\mathbb{R}} \overline{\phi}_{(s+1)/2}\left(\frac{x-b}{a}\right) tr(\varphi Y(P_S,s,x))dx$$

$\tau = a - ib, \operatorname{Re}\tau > 0, b \in \mathbb{R}$ and $\operatorname{Re} s >> 0$.
The dependency on the parameter x is now 'killed' by integration, and for $b = 0$ we obtain special wavelet theoretic 'zooms' (cf. (2.5))

$$tr(\varphi e^{-aP_S}) = \frac{\Gamma(s+1)}{2\pi}\int_{\mathbb{R}}(a+it)^{-s-1} tr(\varphi Z(P_S,s,t))dt \tag{6.7}$$

and

$$tr(\varphi P_S e^{-aP_S^*P_S}) = \frac{\Gamma((s+3)/2)}{2\pi} \int_{\mathbb{R}} (a+it)^{-(s+3)/2} tr(\varphi Y(P_S, s, t))dt \ , \tag{6.8}$$

which represent the heat traces of P_S resp. $P_S^* P_S$.
Inserting in (6.7) the fractal zeta expansion (6.5) we can reprove the heat trace expansion of Grubb [G99;Cor.9.2] for the compact case.

Theorem 6.3
Under the assumptions of Theorem 6.2 for P_S and φ, the 'zoom' $\exp(-aP_s)$ has the asymptotic behaviour for $a \to 0$:

$$tr(\varphi e^{-aP_S}) \sim \alpha_0 a^{-n/d} + \sum_{k \geq 0} (\alpha_k + \beta_k) a^{(k-n)/d} + \sum_{k \geq 0} (\gamma_k \log a + \gamma_k^*) a^{k/d} \ ,$$

here the coefficients are proportional to those in [G99;(9.1)] by universal factors.

Remark:
To extend Theorems 6.2 and 6.3 to the case of admissible manifolds in the sense of Grubb–Kokholm [GK93] one can use W. Müller's [Mü97] theory of 'Relative zeta functions'. This will be discussed in part III.

6.2 First order well-posed problems

The results also hold for injectively elliptic realisations D_B and D_B^* of first order differential operators D with well-posed boundary conditions $B\gamma_0 u = 0$ in the sense of Seeley [S69]. Such realisations can be imbedded into a truly elliptic system, which can be treated by use of the Calderon projector and by the calculus of weakly polyhomogenuous ψdo's [G99;§5]. Let $\Psi_{cl}^0(E_1^*)$ be the algebra of classical ψdo's in E_1^* of order 0 with holomorphic functional calculus [S67]. Then Theorem 6.2 yields the following generalisations of the zeta expansions of Grubb [G99].

Theorem 6.4
Let $D : C^\infty(M, E_1) \to C^\infty(M, E_2)$ be a first order elliptic differential operator with a well–posed boundary condition in the sense of Seeley

$$B\gamma_0 u = 0 \ ,$$

$B \in \Psi_{cl}^0(E_1^*)$ and $\gamma_0 u = u|_N$.
Let φ and ψ be compactly supported morphisms in E_i resp. from E_k to $E_i; i, k = 1, 2$. Let D_B be the realisation of D and set

$$\Delta_1 := D_B^* D_B, \quad \Delta_2 := D_B D_B^* \ .$$

Then the fractal zeta and eta functions have singularity structures described by:

$$\begin{aligned}
\Gamma(s)tr(\varphi Z(\Delta_i, s, x)) &\sim s^{-1}tr(\varphi P_0(D_B)) + \sum_{j=0}^{n-1} \frac{a_{i,j-n}(D,B,x)}{s+1/2(j-n)} \\
&+ \sum_{k\geq 0}\left(\frac{-a_{i,k}(D,B,x)}{(s+1/2\,k)^2} + \frac{a^*_{i,k}(D,B,x)}{s+1/2\,k}\right) , \\
\Gamma(s)tr(\psi D_B Z(\Delta_1, s, x)) &\sim \sum_{j=0}^{n-1} \frac{b_{1,j-n}(D,B,x)}{s+1/2(j-n-1)} \\
&+ \sum_{k\geq 0}\left(\frac{-b_{1,k}(D,B,x)}{(s+1/2(k-1))^2} + \frac{b^*_{s,k}(D,B,x)}{s+1/2(k-1)}\right) , \\
\Gamma(s)tr(\psi D^*_B Z(\Delta_2, s, x)) &\sim \sum_{j=0}^{n-1} \frac{b_{2,j-n}(D,B,x)}{s+1/2(j-n-1)} \\
&+ \sum_{k\geq 0}\left(\frac{-b_{2,k}(D,B,x)}{s+1/2(k-1))^2} + \frac{b^*_{2,k}(D,B,x)}{s+1/2(k-1)}\right) .
\end{aligned}$$

For $x = 0$ we obtain the following results of Grubb [G99;(9.11)].

Corollary 6.5
In Theorem 6.4 let $x = 0$. Then the familiar zeta and eta functions have singularity structures described by

$$\begin{aligned}
\Gamma(s)tr(\varphi\Delta_i^{-s}) &\sim -s^{-1}tr(\varphi P_0(D_B)) \\
&+ \sum_{j=0}^{n-1} \frac{a_{i,j-n}(D,B,0)}{s+1/2(j-n)} \\
&+ \sum_{k\geq 0}\left(\frac{-a_{i,k}(D,B,0)}{(s+1/2k)^2} + \frac{a^*_{i,k}(D,B,0)}{s+1/2\,k}\right) , \quad i=1,2\,; \\
\Gamma(s)tr(\psi D_B\Delta_1^{-s}) &\sim \sum_{j=1}^{n-1} \frac{b_{1,j-n}(D,B,0)}{s+1/2(j-n-1)} \\
&+ \sum_{k\geq 0}\left(\frac{-b_{1,k}(D,B,0)}{(s+1/2(k-1))^2} + \frac{b^*_{1,k}(D,B,0)}{s+1/2(k-1)}\right) , \\
\Gamma(s)tr(\psi D^*_B\Delta_2^{-s}) &\sim \sum_{j=1}^{n-1} \frac{b_{2,j-n}(D,B,0)}{s+1/2(j-n-1)} \\
&+ \sum_{k\geq 0}\left(\frac{-b_{2,k}(D,B,0)}{(s+1/2(k-1))^2} + \frac{b^*_{2,k}(D,B,0)}{s+1/2(k-1)}\right) .
\end{aligned}$$

We now turn to wavelet space (cf. (6.7–6.8)). For Re $s >> 0$ and $a > 0$ the corresponding 'zooms' are given by the formulas

$$tr(\varphi e^{-a\Delta_i}) = \frac{\Gamma(s+1)}{2\pi} \int_{I\!R} (a+it)^{-s-1} tr(\varphi Z(\Delta_i, s, t)) dt \,, \quad i = 1,2 \,; \tag{6.9}$$

$$tr(\psi D_B e^{-a\Delta_1}) = \frac{\Gamma((s+3)/2)}{2\pi} \int_{I\!R} (a+it)^{-(s+3)/2} tr(\psi Y(D_B, s, t)) dt \,, \tag{6.10}$$

$$tr(\psi D_B^* e^{-a\Delta_2}) = \frac{\Gamma((s+3)/2)}{2\pi} \int_{I\!R} (a+it)^{-(s+3)/2} tr(\psi Y(D_B^*, s, t)) dt \,. \tag{6.11}$$

Inserting in (6.9)–(6.11) the wavelet zeta expansions of Theorem 6.4 we obtain the following recent results of Grubb [G99;(9.10)].

Corollary 6.6
Under the assumptions of Theorem 6.4 the 'zooms' have asymptotic expansions for $a \to 0$:

$$\begin{aligned} tr(\varphi e^{-a\Delta_i}) &\sim \sum_{j=0}^{n-1} a_{i,j-n}(0) a^{(j-n)/2} + \sum_{k\geq 0} (a_{i,k}(0)\log a + a^*_{i,k}(0)) a^{k/2}, \quad i = 1,2 \,; \\ tr(\psi D_B e^{-a\Delta_1}) &\sim \sum_{j=0}^{n-1} b_{1,j-n}(0) a^{(j-n-1)/2} + \sum_{k\geq 0} (b_{1,k}(0)\log a + b^*_{1,k}(0)) a^{(k-1)/2} \,, \\ tr(\psi D_B^* e^{-a\Delta_2}) &\sim \sum_{j=0}^{n-1} b_{2,j-n}(0) a^{(j-n-1)/2} + \sum_{k\geq 0} (b_{2,k}(0)\log a + b^*_{2,k}(0)) a^{(k-1)/2} \,. \end{aligned}$$

The coefficients are proportional to those in Corollary 6.5 by universal factors.

Remark:
The results of our wavelet machinery apply of course to prominent examples of well–posed problems for Dirac–type operators in the product or non–product case as considered, e.g. by Atiyah, Patodi and Singer [APS75], by Grubb and Seeley [GS95], [GS96], by Booss–Bavnbek and Wojchiechowski [BW93], by Müller [Mü94], by Brüning and Lesch [BL97], [BL99] or by Grubb [G99;§4]. This will be discussed in part II.

Since index $D_B = tr\, e^{-a\Delta_1} - tr\, e^{-a\Delta_2}$ is constant in $a > 0$ [G96;Sec.4.3] we finally obatin from Corollary 6.6 the following index theorem of Grubb [G99;Cor.9.7].

Corollary 6.7
In Corollary 6.6 let $\varphi = 1$. Then the index of D_B equals

$$index D_B = a^*_{1,0}(0) - a^*_{2,0}(0) \,.$$

Moreover, all the other coefficients coincide for $i = 1$ and 2, i.e. $a_{1,k}(0) = a_{2,k}(0)$ for all $k \geq -n$ and $a^*_{1,k}(0) = a^*_{2,k}(0)$ for all $k > 0$.

Acknowledgement: I thank J. Brüning, P.B. Gilkey, G. Grubb, W. Müller and B.–W. Schulze for interesting suggestions.

References

[ABP73] Atiyah, M.F., Bott, E., Patodi, V.K., On the heat equation and the index theorem, Invent. Math. 19 (1973), 279-330.

[AMV99] Antoine, J.P., Murenzi, R., Vandergheyst, P., The Cauchy wavelets, a new class of 2D–directional wavelets, Preprint: Inst. de Phys. Theorique, Universite Catholique de Louvin (1999).

[APS75] Atiyah, M.F., Patodi, V.K., Singer, I.M., Spectral asymmetry and Riemannian Geometry I, II. Math. Proc. Cambridge Philos. Soc., I: 77 (1975), 43–69; II: 78 (1975), 405–432.

[BL97] Brüning, J., Lesch, M., On the eta invariant of certain non–local boundary value problems, to appear in Duke Math. J.

[BL99] Brüning, J., Lesch, M., Spectral theory of boundary value problems for Dirac type operators, Preprint Nr. 368, SFB 288, Humboldt Universität zu Berlin (1999).

[BW93] Booss–Bavnbek, B., Wojchiechowski, K., Elliptic boundary problems for Dirac operators, Birkhäuser, Basel (1993).

[Ch73] Chazarain, J., Formule de Poisson pour les variétés riemanniennes Invent. Math. 24 (1974), 65–82.

[D92] Daubechies, I., Ten Lectures on Wavelets, SIAM, Philadelphia, PA (1992).

[DG75] Duistermaat, J., Guillemin, V., The spectrum of positive elliptic operators and periodic bicharacteristics, Invent. Math. 29 (1975), 39–79.

[G92] Grubb, G., Heat operator trace expansions and index for general Atiyah–Patodi–Singer problems, Comm. Partial Differential Equations 17 (1992), 2031–2077.

[G99] Grubb, G., Trace expansions for pseudodifferential boundary problems for Dirac–type operators and more general systems, Ark. Math. 37 (1999), 45–86.

[Gi95] Gilkey, P.B., Invariance theory, the heat equation, and the Atiyah–singer index theorem, 2nd edn. CRC Press, Boca Raton (1995).

[GK93] Grubb, G., Kokholm, N.J., A global calculus of parameter–dependent pseudodifferential boundary problems in L_p Sobolev spaces, Acta Math. 171 (1993), 165–229.

[GMP85] Grossmann, A., Morlet, J., Paul, T., Transforms associated to square integrable group representation I, J. Math. Phys. 26 (1996), 2473–9.

[GS95] Grubb, G., Seeley, R.T., Weakly parametric pseudodifferential operators and Atiyah–Patodi–Singer boundary problems, Invent. Math. 121 (1995), 481–529.

[GS96] Grubb, G., Seeley, R.T., Zeta and eta functions for Atiyah–Patodi–Singer operators, J. Geom. Anal. 6 (1996), 31–77.

[Gui96] Guillemin, V., Wave trace invariants, Duke Math. J. 83 (1996), 287–350.

[H95] Holschneider, M., Wavelets–An Analysis Tool, Clarendon Press, Oxford (1995).

[HT91] Holschneider, M., Tchamitchian, Ph., Pointwise analysis of Riemann's 'non-differentiable' function, Invent. Math. 105 (1991), 157–175.

[HW89] Heil, C.E., Walnut, D.F., Continuous and discrete wavelet transforms, SIAM Review 31 (1989), 628–666.

[JL99] Jorgenson, J., Lang, S., Obtaining wave kernels via heat kernels and Bessel function formalism, Preprint: Yale University, New Haven (1999).

[JM96] Jaffard, S., Meyer, Y., Wavelet Methods for Pointwise Regularity and Local Oscillations of Functions, Mem. Amer. Math. Soc. Vol. 123 (1996).

[K91] Klusch, D., The sampling theorem, Dirichlet series and Bessel functions, Math. Nachr. 154 (1991), 129–139.

[K92] Klusch, D., The sampling theorem, Dirichlet series and Hankel transforms, J. Comp. Appl. Math. 44 (1992), 261–273.

[K97] Klusch, D., Wavelets, Zeta Functions and Heat Operator Traces, Preprint Nr. A/22/97, FU Berlin (1997).

[Ma82] Marr, D., Vision, Freeman and Co., San Francisco (1982)

[Me90] Meyer, Y., Ondelettes et Operateurs I, II and III, Herrmann, Paris (1990).

[Mü94] Müller, W., Eta Invariants and Manifolds with Boundary, J. Diff. Geom. 40 (1994), 311–377.

[Mü97] Müller, W., Relative Zeta Functions, Relative Determinants and Scattering Theory, Preprint no. 494, SFB 256, Rheinische Friedrich–Wilhelms–Universität Bonn (1997).

[Pau85] Paul, T., Ondelletes et Mécanique Quantique, Thèse de doctorat d'état, Université d'Aix Marseille II (1985).

[S67] Seeley, R.T., Topics in pseudodifferential operators, (C.I.M.E., Stresa, 1968), pp. 167–305.

[S69] Seeley, R.T., Complex powers of an elliptic operator, Proc. Symp. Pure Math. 10 (1967), 288–307.

[Sch98] Schulze, B.–W., Boundary value problems and singular pseudo–differential operators, J. Wiley, Chichester (1998).