The Exact Quantum Sine-Gordon Field Equation and Other Non-Perturbative Results

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September 21, 1999

Abstract

Using the methods of the "form factor program" exact expressions of all matrix elements are obtained for several operators of the quantum sine Gordon model: all powers of the fundamental bose field, general exponentials of it, the energy momentum tensor and all higher currents. It is found that the quantum sine-Gordon field equation holds with an exact relation between the "bare" mass and the renormalized mass. Also a relation for the trace of the energy momentum is obtained. The eigenvalue problem for all higher conserved charges is solved. All results are compared with Feynman graph expansions and full agreement is found.

The classical sine-Gordon model is given by the wave equation

$$\Box\varphi(t,x) + \frac{\alpha}{\beta}\sin\beta\varphi(t,x) = 0.$$

Since Coleman [1] found the wonderful duality between the quantum sine-Gordon and the massive Thirring model a lot of effort has been made to understand this quantum field theoretic model. A further contribution in this direction is the present paper of which an extended version will be published elsewhere [2]. The "bootstrap" program (see e.g. [3]) for integrable quantum field theoretical models in 1+1 dimensions starts with the two particle sine-Gordon S-matrix for the scattering of fundamental bosons (lowest breathers) [4]

$$S(\theta) = \frac{\sinh\theta + i\sin\pi\nu}{\sinh\theta - i\sin\pi\nu}$$

where θ is the rapidity difference defined by $p_1 p_2 = m^2\cosh\theta$ and ν is related to the coupling constant by $\nu = \beta^2/(8\pi - \beta^2)$.

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From the S-matrix off-shell quantities as arbitrary matrix elements of local operators are obtained by means of the "form factor program" [5]. In particular we provide exact expressions for all matrix elements of all powers of the fundamental bose field $\varphi(t,x)$ and its exponential $\mathcal{N}\exp i\gamma\varphi(t,x)$ for arbitrary γ. Here and in the following $\mathcal{N}$ denotes normal ordering with respect to the physical vacuum which means in particular for the vacuum expectation value $\langle\, 0\,|\,\mathcal{N}\exp i\gamma\varphi(t,x)|\,0\,\rangle = 1$. For the exceptional value $\gamma=\beta$ we find that the operator $\Box^{-1}\mathcal{N}\sin\beta\varphi(t,x)$ is local. Moreover the quantum sine-Gordon field equation[1]

$$\Box\varphi(t,x)+\frac{\alpha}{\beta}\mathcal{N}\sin\beta\varphi(t,x)=0 \tag{1}$$

is fulfilled for all matrix elements, if the "bare" mass $\sqrt{\alpha}$ is related to the renormalized mass by[2]

$$\alpha = m^2\frac{\pi\nu}{\sin\pi\nu} \tag{2}$$

where m is the physical mass of the fundamental boson. The factor $\frac{\pi\nu}{\sin\pi\nu}$ modifies the classical equation and has to be considered as a quantum correction. For the sinh-Gordon model an analogous quantum field equation has been obtained in [23][3]. Note that in particular at the 'free fermion point' $\nu\to 1$ ($\beta^2\to 4\pi$) this factor diverges, a phenomenon which is to be expected by short distance investigations [8]. For fixed bare mass square α and $\nu\to 2,3,4,\ldots$ the physical mass goes to zero. These values of the coupling are known to be specific: 1. the Bethe ansatz vacuum in the language of the massive Thirring model shows phase transitions [9] and 2. the model at these points is related [10, 11, 12] to Baxters RSOS-models which correspond to minimal conformal models with central charge $c=1-6/(\nu(\nu+1))$ (see also [23]).

Also we calculate all matrix elements of all higher local currents $J_M^\mu(t,x)$ ($M=\pm1,\pm3,\ldots$) fulfilling $\partial_\mu J_M^\mu(t,x)=0$ which is characteristic for integrable models. The higher charges fulfill the eigenvalue equation

$$\left(\int dx J_M^0(x)-\sum_{i=1}^n\left(p_i^+\right)^M\right)|\,p_1,\ldots,p_n\rangle^{in}=0. \tag{3}$$

In particular for $M=\pm1$ the currents yield the energy momentum tensor $T^{\mu\nu}=T^{\nu\mu}$ with $\partial_\mu T^{\mu\nu}=0$. We find that its trace fulfills

$$T^\mu_{\ \mu}(t,x)=-2\frac{\alpha}{\beta^2}\left(1-\frac{\beta^2}{8\pi}\right)\left(\mathcal{N}\cos\beta\varphi(t,x)-1\right). \tag{4}$$

This formula is consistent with renormalization group arguments [13, 14]. In particular this means that $\beta^2/4\pi$ is the anomalous dimension of $\cos\beta\varphi$. Again this operator equation

[1] In the framework of constructive quantum field theory quantum field equations where considered in [6, 7].

[2] One of the authors (H.B.) has learned from Al.B. Zamolodchikov that this mass relation is consistent with his results in [18].

[3] It should be obtained from (1) by the replacement $\beta\to ig$. However the relation between the bare and the renormalised mass in [23] differs from the analytic continuation of (2) by a factor which is $1+O(\beta^4)\neq 1$.

is modified by a quantum correction $(1-\beta^2/8\pi)$. Obviously for fixed bare mass square α and $\beta^2 \to 8\pi$ the model will be conformal invariant which is related to a Berezinski-Kosterlitz-Thouless phase transition [4, 16, 17][4]. The proofs of the statements (1) – (4) is sketched in the following together with some checks in perturbation theory. The complete proofs will be published elsewhere [2].

Let $\mathcal{O}(x)$ be a local scalar operator. The generalized form factors are defined by the vacuum – n-particle matrix elements

$$\langle\, 0\,|\,\mathcal{O}(x)\,|\,p_1,\ldots,p_n\,\rangle^{in} = e^{-ix(p_1+\ldots+p_n)}\, f_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n)\ ,\quad \text{for}\ \ \theta_1>\ldots>\theta_n$$

where the θ_i are the rapidities of the particles $p_i^\mu = m(\cosh\theta_i,\sinh\theta_i)$. In the other sectors of the variables the functions $f_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n)$ are given by analytic continuation with respect to the θ_i. General matrix elements are obtained from $f_n^{\mathcal{O}}(\underline{\theta})$ by crossing which means in particular analytic continuation $\theta_i \to \theta_i \pm i\pi$.

A form factor of n fundamental bosons (lowest breathers) is of the form [5]

$$f_n^{\mathcal{O}}(\underline{\theta}) = N_n^{\mathcal{O}} K_n^{\mathcal{O}}(\underline{\theta}) \prod_{1\le i<j\le n} F(\theta_{ij})$$

where $N_n^{\mathcal{O}}$ is a normalization constant, $\theta_{ij}=\theta_i-\theta_j$ and $F(\theta)$ is the two particle form factor function. It fulfills Watson's equations

$$F(\theta) = F(-\theta)S(\theta) = F(2\pi i-\theta)$$

with the S-matrix given above. Explicitly it is given by the integral representation [5]

$$F(\theta) = N \exp \int_0^\infty \frac{dt}{t}\, \frac{\left(\cosh\frac{1}{2}t - \cosh(\frac{1}{2}+\nu)t\right)\left(1-\cosh t(1-\frac{\theta}{i\pi})\right)}{\cosh\frac{1}{2}t\,\sinh t}$$

normalized such that $F(\infty)=1$. The K-function $K_n^{\mathcal{O}}(\underline{\theta})$ is meromorphic, symmetric and periodic (under $\theta_i \to \theta_i + 2\pi i$).

For an odd number of solitons and anti-solitons [19] and for chargeless operators [2] a general integral representation of generalized form factors has been proposed. Using this integral representations and the fusion 'soliton + anti-soliton $\to$ breather' the following general formula for the K-function has been derived in [2][5]. It turns out to be of the form

$$K_n^{\mathcal{O}}(\underline{\theta}) = \sum_{l_1=0}^{1}\ldots\sum_{l_n=0}^{1}(-1)^{l_1+\ldots+l_n}\prod_{1\le i<j\le n}\left(1+(l_i-l_j)\frac{i\sin\pi\nu}{\sinh\theta_{ij}}\right) p_n^{\mathcal{O}}(\underline{\theta},\underline{z}) \tag{5}$$

where $z_i = \theta_i - \frac{i\pi}{2}(1+(2l_i-1)\nu)$. The dependence on the operator is encoded in the 'p-function' $p_n^{\mathcal{O}}$. It is separately symmetric with respect to the variables $\underline{\theta}$ and $\underline{z}$ and has

[4]This was pointed out to us by S.A. Bulgadaev.

[5]Bosonic sine-Gordon form factors have also been derived in [20]. For the sinh-Gordon model form factors where obtained in [21, 22, 23] in terms of determinants of symmetric polynomials. They are related to those of this paper by the analytic continuation $\beta \to ig$.

to fulfill some simple conditions in order that the form factor function $f_n^{\mathcal{O}}$ fulfill some properties [5, 20]. These properties follow (see [19]) from general LSZ-assumptions and in additions specific features typical for integrable field theories. In particular the recursion relation holds

$$\operatorname*{Res}_{\theta_{12}=i\pi} f_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n) = 2i\, f_{n-2}^{\mathcal{O}}(\theta_3,\ldots,\theta_n)\,(\mathbf{1} - S(\theta_{2n})\ldots S(\theta_{23}))\,. \tag{6}$$

Here we will not provide more details but only give some examples of operators and their corresponding p-functions:

1. The correspondence of exponentials of the field and their p-function[6] is

$$\mathcal{N} e^{i\gamma\varphi} \leftrightarrow \prod_{i=1}^{n} e^{(2l_i-1)i\pi\nu\gamma/\beta} \tag{7}$$

   for an arbitrary constant γ.

2. Taking derivatives of this formula with respect to γ we get for the field and its powers

$$\mathcal{N}\varphi^N \leftrightarrow \left(\sum_{i=1}^{r}(2l_i-1)\right)^N . \tag{8}$$

3. Higher currents (for $M = \pm 1, \pm 3, \ldots$) correspond to the p-functions

$$J_M^{\pm} \leftrightarrow \sum_{i=1}^{n} e^{\pm\theta_i} \sum_{i=1}^{n} e^{Mz_i} \tag{9}$$

   for $n =$ even and zero for $n =$ odd. For $M = \pm 1$ we get the light cone components of the energy momentum tensor $T^{\rho\sigma} = J_\sigma^\rho$ with $\rho,\sigma = \pm$ (see also [23]).

In order to prove equations as for example (1) and (4) we consider the corresponding p-functions and their K-functions defined by (5). The K-functions are rational functions of the $x_i = e^{\theta_i}$. We analyze the poles and the asymptotic behavior and find identities by induction and Liouville's theorem. Transforming these identities to the corresponding form factors one finds the field equation (1) and the trace equation (4) up to normalizations.

Normalization constants are obtained in the various cases by the following observations:

a) The normalization a field annihilating a one-particle state is given by the vacuum one-particle matrix element, in particular for the fundamental bose field one has

$$\langle\, 0 \,|\, \varphi(0) \,|\, p\,\rangle = \sqrt{Z^\varphi} \quad \text{with} \quad Z^\varphi = (1+\nu)\frac{\frac{\pi}{2}\nu}{\sin\frac{\pi}{2}\nu}\exp\left(-\frac{1}{\pi}\int_0^{\pi\nu}\frac{t}{\sin t}dt\right)$$

[6]For the sinh-Gordon model an analogous representation as (5) together with this p-function was obtained in [24] by different methods.

where $Z^\varphi = 1+O(\beta^4)$ is the finite wave function renormalization constant calculated in [5]. This gives the normalization constant

$$N_1^{(1)} = \sqrt{Z^\varphi}/2 \tag{10}$$

for the form factors of the fundamental bose field and which are obtained from the p-function of (8) for $N = 1$.

b) If a local operator is connected to an observable like a charge $Q = \int dx\, \mathcal{O}(x)$ we use the relation

$$\langle\, p' \,|\, Q \,|\, p \,\rangle = q \langle\, p' \mid p \,\rangle.$$

For example for the higher charges we obtain

$$N_2^{J_M} = \frac{i^M m^{M+1}}{2 \sin \pi\nu \sin \frac{M}{2}\pi\nu F(i\pi)} \quad \text{with} \quad \frac{1}{F(i\pi)} = Z^\varphi \frac{\beta^2}{8\pi\nu}\frac{\sin \pi\nu}{\pi\nu}.$$

c) We use Weinberg's power counting theorem for bosonic Feynman graphs [2][7]. For the exponentials of the boson field $\mathcal{O} = \mathcal{N} e^{i\gamma\varphi}$ this yields in particular the asymptotic behavior

$$f_n^{\mathcal{O}}(\theta_1,\theta_2,\ldots) = f_1^{\mathcal{O}}(\theta_1)\, f_{n-1}^{\mathcal{O}}(\theta_2,\ldots) + O(e^{-\operatorname{Re}\theta_1})$$

as $\operatorname{Re}\theta_1 \to \infty$ in any order of perturbation theory. This behavior is also assumed to hold for the exact form factors. Applying this formula iteratively we obtain from (5) with (7) the following relation for the normalization constants of the operators $\mathcal{N} e^{i\gamma\varphi}$

$$N_n^\gamma = (N_1^\gamma)^n \quad (n \geq 1).$$

d) The recursion relation (6) relates N_{n+2} and N_n. For all p-functions mentioned above we obtain

$$N_{n+2} = N_n \frac{2}{\sin \pi\nu F(i\pi)} \quad (n \geq 1).$$

Using c) and d) we calculate the normalization constants for the exponential of the field $\mathcal{N} e^{i\gamma\varphi}$ and obtain

$$N_1^\gamma = \sqrt{Z^\varphi}\frac{\beta}{2\pi\nu}. \tag{11}$$

The normalization constants (10) and (11) now yield the field equation (1) with the mass relation (2). The statement (4) is proved similarly. The eigen value equation (3) is obtained by taking the scalar products with arbitrary out-states and by using a general crossing formula [2].

[7]This type of arguments has been also used in [5, 21, 22, 23].

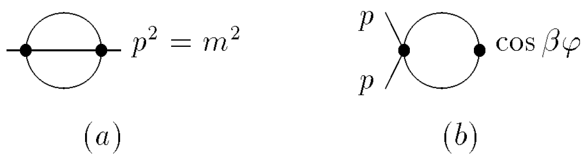

Figure 1: Feynman graphs

All the results may be checked in perturbation theory by Feynman graph expansions. In particular in lowest order the relation between the bare and the renomalized mass (2) is given by Figure 1 (a). It had already been calculated in [5] and yields

$$m^2 = \alpha\left(1 - \frac{1}{6}\left(\frac{\beta^2}{8}\right)^2 + O(\beta^6)\right)$$

which agrees with the exact formula above. Similarly we check the quantum corrections of the trace of the energy momentum tensor (4) by calculating the Feynman graph of Figure 1 (b) with the result again taken from [5] as

$$\langle\, p \,|\, \mathcal{N} \cos\beta\varphi - 1 \,|\, p \,\rangle = -\beta^2\left(1 + \frac{\beta^2}{8\pi}\right) + O(\beta^6).$$

This again agrees with the exact formula above since the normalization given by eq. (3) implies $\langle\, p \,|\, T^{\mu}_{\ \mu} \,|\, p \,\rangle = 2m^2$. All other equations have also been checked in perturbation theory [2].

Acknowledgments: We thank J. Balog, S.A. Bulgadaev, R. Flume, A. Fring, R.H. Poghossian, F.A. Smirnov, R. Schrader, B. Schroer and Al.B. Zamolodchikov for discussions. One of authors (H.B.) was supported by DFG, Sonderforschungsbereich 288 'Differentialgeometrie und Quantenphysik' and partially by INTAS grant 96-524.

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