On Family Rigidity Theorems I

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Abstract. In this paper, we first prove a local family version of the Atiyah-Bott-Segal-Singer Lefschetz fixed point formula, then we extend the famous Witten's rigidity Theorems to the family case. Several family vanishing theorems for elliptic genera are also proved.

0 Introduction. Let M, B be two compact smooth manifolds, and $\pi : M \to B$ be a submersion with compact fibre X. Assume that a compact Lie group G acts fiberwise on M, that is the action preserves each fiber of π. Let P be a family of elliptic operators along the fiber X, commuting with the action G. Then the family index of P is

$$\mathrm{Ind}(P) = \mathrm{Ker}P - \mathrm{Coker}P \in K_G(B). \tag{0.1}$$

Note that $\mathrm{Ind}(P)$ is a virtual G-representation. Let $\mathrm{ch}_g(\mathrm{Ind}(P))$ with $g \in G$ be the equivariant Chern character of $\mathrm{Ind}(P)$ evaluated at g.

In this paper we will first prove a family fixed point formula which expresses $\mathrm{ch}_g(\mathrm{Ind}(P))$ in terms of the geometric data on the fixed points X^g of the fiber of π. The by applying this formula, we generalize the Witten rigidity theorems and several vanishing theorems proved in [**Liu4**] for elliptic genera to the family case.

A family elliptic operator P is called *rigid on the equivariant Chern character level* with respect to this S^1-action, if $\mathrm{ch}_g(\mathrm{Ind}(P)) \in H^*(B)$ is independent of $g \in S^1$. When the base B is a point, we recover the classical rigidity and vanishing theorems. When B is a manifold, we get many nontrivial higher order rigidity and vanishing theorems by taking the coefficients of certain expansion of ch_g. For the history of the Witten rigidity theorems, we refer the reader to [**BT**], [**K**], [**L**], [**H**], [**Liu1**] and [**Liu2**]. The family vanishing theorems generalize those vanishing theorems in [**Liu4**], which in turn give us many higher order vanishing theorems in the family case. In a forthcoming paper, we will extend our results to general loop group representations and prove much more general family vanishing theorems which generalize the results in [**Liu4**]. We believe there should be some applications of our results to topology and geometry, which we hope to report on a later occasion.

This paper is organized as follows. In Section 1, we prove the equivariant family index theorem. In Section 2, we prove the family rigidity theorem. In the last part of Section 2, motivated by the family rigidity theorem, we state a conjecture. In Section 3, we generalize the family rigidity theorem to the nonzero anomaly case. As corollaries, we derive several vanishing theorems.

1 Equivariant family index theorem

The purpose of this section is to prove an equivariant family index theorem. As pointed out by Atiyah and Singer, we can introduce equivariant families by proceeding as in [**AS1, 2**]. Here we will prove it directly by using the local index theory as developed by Bismut.

This section is organized as follows: In Section 1.1, we state our main result, Theorem 1.1. In Section 1.2, by using the local index theory, we prove Theorem 1.1.

1.1 The index bundle

Let M, B be two compact manifolds, and $\pi : M \to B$ be a fibration with compact fibre X, and assume that $\dim X = 2k$. Let TX denote the relative tangent bundle. Let W be a complex vector bundle on M and h^W be an Hermitian metric on W.

Let h^{TX} be a Riemannian metric on TX and ∇^{TX} be the corresponding Levi-Civita connection on TX along the fibre X. Then the Clifford bundle $C(TX)$ is the bundle of Clifford algebras over M whose fibre at $x \in M$ is the Clifford algebra $C(T_xX)$ of (TX, h^{TX}).

We assume that the bundle TX is spin on M. Let $\Delta = \Delta^+ \oplus \Delta^-$ be the spinor bundle of TX. We denote by $c(\cdot)$ the Clifford action of $C(TX)$ on Δ.

Let ∇ be the connection on Δ induced by ∇^{TX}. Let ∇^W be a Hermitian connection on (W, h^W) with curvature R^W. Let $\nabla^{\Delta\otimes W}$ be the connection on $\Delta\otimes W$ along the fibre X:

$$\nabla^{\Delta\otimes W} = \nabla \otimes 1 + 1 \otimes \nabla^W. \tag{1.1}$$

For $b \in B$, we denote by $E_b, E_{\pm,b}$ the set of $\mathcal{C}^\infty$-sections of $\Delta \otimes W$, $\Delta_\pm \otimes W$ over the fiber X_b. We regard the E_b as the fibre of a smooth $\mathbf{Z}_2$-graded infinite dimensional vector bundle over B. Smooth sections of E over B will be identified to smooth sections of $\Delta \otimes W$ over M.

Let $\{e_i\}$ be an orthonormal basis of (TX, h^{TX}), let $\{e^i\}$ be its dual basis.

Definition 1.1 *Define the twisted Dirac operator to be*

$$D^X = \textstyle\sum_i c(e_i)\nabla^{\Delta\otimes W}_{e_i}. \tag{1.2}$$

Then D^X is a family Dirac operator which acts fiberwise on the fibers of π. For $b \in B$, D^X_b denote the restriction of D^X to the fibre E_b. D^X interchanges E_+ and E_-. Let $D^X_\pm$ be the restrictions of D^X to $E_\pm$. By Atiyah and Singer [**AS2**], the difference bundle over B

$$\mathrm{Ind}(D) = \mathrm{Ker} D_{+,b} - \mathrm{Ker} D_{-,b}. \tag{1.3}$$

is well-defined in the K-group $K(B)$.

Now, let G be a compact Lie group which acts fiberwise on M. We will consider that G acts as identity on B. Without loss of generality we can assume that G acts on (TX, h^{TX}) isometrically. We also assume that the action of G lifts to Δ and W, and that the G-action commutes with ∇^W.

In this case, we know that $\mathrm{Ind}(D^X) \in K_G(B)$. Now we start to give a proof of a local family fixed point formula which extends [**AS2**, Proposition 2.2]

Proposition 1.1 *There exist $V_j \in \widehat{G}$ with $j = 1, \cdots, r$, a finite number of sections $(s_{i_j+1}, \cdots, s_{i_{j+1}})$ with $i_{j+1} - i_j = \dim V_j$ of $\mathcal{C}^\infty(B, E_-)$ such that we can find a basis $\{e_{j,l}\}$ of V_j, under which the map $\overline{D}_{+,b} : \mathcal{C}^\infty(B, E_{+,b}) \oplus \oplus_{j=1}^r V_j \to \mathcal{C}^\infty(B, E_{-,b})$ given by*

$$\overline{D}_{+,b}^X(s + \Sigma_{j,l}\lambda_{j,l}e_{j,l}) = D_+^X s + \Sigma_{j,l}\lambda_{j,l}s_{i_j+l} \tag{1.4}$$

is G-equivariant and surjective. The vector spaces $\mathrm{Ker}\overline{D}_{+,b}^X$ form a G-vector bundle $\mathrm{Ker}\overline{D}_+^X$ on B, and the element $[\mathrm{Ker}\overline{D}_+^X] - \oplus_{j=1}^r V_j \in K_G(B)$ depends only on D^X and not on the choice of $\{V_j\}$ and the sections $\{s_i\}$.

Proof: Given $b_0 \in B$, we can find $a > 0$ and a ball $U(b_0) \subset B$ around b_0, such that for any $b \in U(b_0)$, a is not an eigenvalue of $D_b^{X,2}$.

Let $E_b^{[0,a[} = E_{+b}^{[0,a[} \oplus E_{-b}^{[0,a[}$ be the direct sum of the eigenspaces of $D_b^{X,2}$ associated to the eigenvalues $\lambda \in [0, a[$. By [**BeGeV**, Proposition 9.10], $E^{[0,a[}$ forms a finite dimensional sub-bundle $E^{[0,a[} \subset E$ over $U(b_0)$. Clearly, $E^{[0,a[}$ is a G-vector bundle on $U(b_0)$. By [**S**, Proposition 2.2], we have an isomorphism of vector bundles on B

$$E^{[0,a[} = \oplus_{V \in \widehat{G}} \mathrm{Hom}_G(V, E^{[0,a[}) \otimes V, \tag{1.5}$$

where $\widehat{G}$ denotes the space of all irreducible representations of G. We can also find $t_{i,k} \in \mathcal{C}^\infty(U(b_0), \mathrm{Hom}_G(V, E_-^{[0,a[}))$ such that for $b \in U(b_0)$, the elements $t_{i,l}$ form a basis of $\mathrm{Hom}_G(V, E_-^{[0,a[})_b$. Let $\{e_{i,l}\}$ be a basis of V_i. Then we can choose the sections $t_{i,k}e_{i,l} \in \mathcal{C}^\infty(B, E_-^{[0,a[})$ to be our s_i's. This proves the first part of the proposition locally.

The global version now follows easily by extending the above local sections of $\mathcal{C}^\infty(U(b_0), E_-)$ together with a use of the partition of unity argument. This is essentially the same as the proof of [**AS2**, Proposition 2.2]. ■

By [**S**, Proposition 2.2], we have

$$\mathrm{Ind}(D^X) = \oplus_{V \in \widehat{G}} \mathrm{Hom}_G(V, \mathrm{Ind}(D^X)) \otimes V \tag{1.6}$$

and $\mathrm{Hom}_G(V, \mathrm{Ind}(D^X)) \in K(B)$. We denote by $(\mathrm{Ind}(D^X))^G \in K(B)$ the G-invariant part of $\mathrm{Ind}(D^X)$.

By composing the action of G and the Chern character of $\mathrm{Hom}_G(V, \mathrm{Ind}(D^X))$, we get the equivariant Chern character $\mathrm{ch}_g(\mathrm{Ind}(D^X)) \in H^*(B)$.

Definition 1.2 *We say that the operator D^X is rigid on the equivariant Chern character level, if $\mathrm{ch}_g(\mathrm{Ind}(D^X))$ is constant on $g \in G$. More generally, we say D^X is rigid on the equivariant K-Theory level, if $\mathrm{Ind}(D^X) = (\mathrm{Ind}(D^X))^G$.*

In the rest of this paper, when we say D^X is rigid, we always mean D^X is rigid on the equivariant Chern character level.

Now let us calculate the equivariant Chern character $\mathrm{ch}_g(\mathrm{Ind}(D^X))$ in terms of the fixed point data of g.

Let T^HM be a G-equivariant sub-bundle of TM such that

$$TM = T^HM \oplus TX. \tag{1.7}$$

Let P^{TX} denote the projection from TM to TX. If $U \in TB$, let U^H denote the lift of U in T^HM, so that $\pi_* U^H = U$.

Let h^{TB} be a Riemannian metric on B, and assume that W has the Riemannian metric $h^{TM} = h^{TX} \oplus \pi^* h^{TB}$. Note that our final results will be independent of g^{TB}. Let ∇^{TM}, ∇^{TB} denote the corresponding Levi-Civita connections on M and B. Put $\nabla^{TX} = P^{TX}\nabla^{TM}$ which is a connection on TX. As shown in [**B1**, Theorem 1.9], ∇^{TX} is independent of the choice of h^{TB}. Now the connection ∇^{TX} is well defined on TX and on M. Let R^{TX} be the corresponding curvature. We denote by ∇ and $\nabla^{\Delta\otimes W}$ the corresponding connections on Δ and $\Delta\otimes W$ induced by ∇^{TX} and ∇^W.

Take $g \in G$ and set

$$M^g = \{x \in M, gx = x\}. \tag{1.8}$$

Then $\pi : M^g \to B$ is a fibration with compact fibre X^g. By [**BeGeV**, Proposition 6.14], TX^g is naturally oriented in M^g.

Let N denote the normal bundle of M^g, then $N = TX/TX^g$. We denote the differential of g by dg which gives a bundle isometry $dg : N \to N$. Since g lies in a compact abelian Lie group, we know that there is an orthogonal decomposition $N = N(\pi) \bigoplus \bigoplus_{0<\theta<\pi} N(\theta)$, where $dg_{|N(\pi)} = -\mathrm{id}$, and for each $\theta, 0 < \theta < \pi$, $N(\theta)$ is a complex vector bundle on which dg acts by multiplication by $e^{i\theta}$, and $\dim N(\pi)$ is even. So $N(\pi)$ also is naturally oriented.

As the Levi-Civita connection ∇^{TM} preserves the decomposition $TM = TM^g \oplus_{0<\theta\leq\pi} N(\theta)$, the connection ∇^{TX} also preserves the decomposition $TX = TX^g \oplus_{0<\theta\leq\pi} N(\theta)$ on M^g. Let ∇^{TX^g}, ∇^N, $\nabla^{N(\theta)}$ be the corresponding induced connections on TX^g, N and $N(\theta)$, and et R^{TX^g}, R^N, $R^{N(\theta)}$ be the corresponding curvatures. Here we consider $N(\theta)$ as a real vector bundle. Then we have the decompositions:

$$R^{TX} = R^{TX^g} \oplus R^N, \qquad R^N = \oplus_\theta R^{N(\theta)}. \tag{1.9}$$

Definition 1.3 *For $0 < \theta \leq \pi$, we write*

$$\begin{aligned}
&\mathrm{ch}_g(W, \nabla^W) = \mathrm{Tr}\Big[g \exp(\tfrac{-R^W}{2\pi i})\Big],\\
&\widehat{A}(TX^g, \nabla^{TX^g}) = \det{}^{1/2}\Big(\frac{\frac{i}{4\pi}R^{TX^g}}{\sinh(\frac{i}{4\pi}R^{TX^g})}\Big),\\
&\widehat{A}_\theta(N(\theta), \nabla^{N(\theta)}) = \frac{1}{i^{\frac{1}{2}\dim N(\theta)}\det^{1/2}\Big(1 - g\exp(\frac{i}{2\pi}R^{N(\theta)})\Big)}
\end{aligned} \tag{1.10}$$

Let $\mathrm{ch}_g(W)$, $\widehat{A}(TX^g)$, $\widehat{A}_\theta(N(\theta))$ denote the corresponding cohomology classes on M^g.

If we denote by $\{x_j, -x_j\}$ $(j = 1, \cdots, l)$ the Chern roots of $N(\theta)$, TX^g such that Πx_j define the orientation of $N(\theta)$ and TX^g, then

$$
\begin{aligned}
\widehat{A}(TX^g) &= \Pi_j \frac{x_j}{2} / \sinh(\frac{x_j}{2}), \\
\widehat{A}_\theta(N(\theta)) &= 2^{-l} \Pi_{j=1}^l \frac{1}{\sinh \frac{1}{2}(x_j + i\theta)} = \Pi_{j=1}^l \frac{e^{\frac{1}{2}(x_j+i\theta)}}{e^{x_j+i\theta} - 1}.
\end{aligned} \tag{1.11}
$$

We denote by $\pi_* : H^*(M^g) \to H^*(B)$ the intergration along the fibre X^g.

Theorem 1.1 *We have the following identity in $H^*(B)$:*

$$
\mathrm{ch}_g(\mathrm{Ind}(D^X)) = \pi_* \left\{ \Pi_{0<\theta\le\pi} \widehat{A}_\theta(N(\theta)) \widehat{A}(TX^g) \mathrm{ch}_g(W) \right\}. \tag{1.12}
$$

1.2 A heat kernel proof of Theorem 1.1

As Atiyah and Singer indicated in the end of [**AS2**], we can proceed as in [**AS1,2**] to introduce an equivariant family, and then to find a formula for the equivariant Chern character of the index bundle. Here, we will use a different approach by combining the local relative index theory and the equivariant technique to give a direct proof of the local version of Theorem 1.1.

We denote by ${}^0\nabla = \nabla^{TX} \oplus \pi^* \nabla^{TB}$ the connection on TM. Let $S = \nabla^{TM} - {}^0\nabla$. By [**B1**, Theorem 1.9], $\langle S(.).,.\rangle_{h^{TM}}$ is a tensor independent of h^{TB}. For $U \in T^H M$, we define a horizontal 1-form k on M by

$$
k(U) = \sum_i \langle S(U)e_i, e_i \rangle. \tag{1.13}
$$

Definition 1.4 *Let ∇^E denote the connection on E such that if $U \in TB$ and s is a smooth section of E over B, then*

$$
\nabla_U^E s = \nabla_{U^H}^{\Delta\otimes W} s. \tag{1.14}
$$

If U, V are smooth vector fields on B, we write

$$
T(U^H, V^H) = -P^{TX}[U^H, V^H] \tag{1.15}
$$

which is a tensor.

Let $f_1, \cdots, f_m$ be a basis of TB, and $f^1, \cdots, f^m$ be the dual basis. Define

$$
c(T) = \frac{1}{2} \Sigma_{\alpha,\beta} f^\alpha f^\beta c(T(f_\alpha^H, f_\beta^H)). \tag{1.16}
$$

Definition 1.5 *For $t > 0$, let A_t be the Bismut superconnection constructed in [**B1**, §3],*

$$
A_t = \sqrt{t} D^X + (\nabla^E + \frac{1}{2}k) - \frac{1}{4\sqrt{t}} c(T). \tag{1.17}
$$

It is clear that A_t is also G-invariant.

Let dv_X denote the Riemannian volume element on the fiber X. Let Φ be the scaling homomorphism from $\Lambda(T^*B)$ into itself : $\omega \to (2\pi i)^{-(\deg \omega)/2}\omega$.

Theorem 1.2 *For any $t > 0$, the form $\Phi \mathrm{Tr}_s[g \exp(-A_t^2)]$ is closed and that its cohomology class is independent of t and represents $\mathrm{ch}_g(\mathrm{Ind}(D^X))$ in cohomology.*

Proof: Just proceed as in [**B1**, §2(d)]. ∎

Theorem 1.3 *We have the following identity*

$$\begin{aligned}(1.18)\qquad &\lim_{t\to 0} \Phi \mathrm{Tr}_s[g \exp(-A_t^2)] = \\ &\int_{X^g} \widehat{A}(TX^g, \nabla^{TX^g})\Pi_{0<\theta\leq\pi}\widehat{A}_\theta(N(\theta), \nabla^{N(\theta)})\mathrm{ch}_g(W, \nabla^W).\end{aligned}$$

Proof: If A is a smooth section of $T^*X \otimes \Lambda(T^*B) \otimes \mathrm{End}(\Delta \otimes W)$, we use the notation

$$(\nabla_{e_i}^{\Delta\otimes W} + A(e_i))^2 = \sum_{i=1}^{2k} (\nabla_{e_i}^{\Delta\otimes W} + A(e_i))^2 - \nabla^{\Delta\otimes W}_{\Sigma_{i=1}^{2k}\nabla_{e_i}^{TX}e_i} - A(\Sigma_{i=1}^{2k}\nabla_{e_i}^{TX}e_i).$$

Let ∇_t' be the connection on $\Lambda(T^*B)\widehat{\otimes}\Delta\otimes W$ on the fibre X.

$$(1.19)\qquad \nabla_t' = \nabla^{\Delta\otimes W} + \frac{1}{2\sqrt{t}}\left\langle S(.)e_j, f_\alpha^H\right\rangle c(e_j)f^\alpha + \frac{1}{4t}\left\langle S(.)f_\alpha^H, f_\beta^H\right\rangle f^\alpha f^\beta$$

Let K^X denote the scalar curvature of the fiber (X, h^{TX}). By the Lichnerowicz formula [**B1**, Theorem 3.5], we get

$$\begin{aligned}(1.20)\qquad A_t^2 = &-t(\nabla_{t,e_i}')^2 + \frac{t}{4}K^X + \frac{t}{2}c(e_i)c(e_j)R^W(e_i, e_j) \\ &+\sqrt{t}c(e_i)f^\alpha R^W(e_i, f_\alpha^H) + \frac{1}{2}f^\alpha f^\beta R^W(f_\alpha^H, f_\beta^H).\end{aligned}$$

Let $P_u(x, x', b)(b \in B, x, x' \in X_b)$ be the smooth kernel associated to $\exp(-A_t^2)$ with respect to $dv_X(x')$. Then

$$(1.21)\qquad \Phi\mathrm{Tr}_s[g\exp(-A_t^2)] = \int_X \Phi\mathrm{Tr}_s[gP_t(g^{-1}x, x, b)]dv_X(x).$$

By using standard estimates on the heat kernel, for $b \in B$, we can reduce the problem of calculating the limit of (1.21) when $t \to 0$ to an open neighbourhood $\mathcal{U}_\varepsilon$ of X_b^g in X_b. Using normal geodesic coordinates to X_b^g in X_b, we will identify $\mathcal{U}_\varepsilon$ to an ε-neighbourhood of X^g in $N_{X^g/X}$. We know that, if $(x, z) \in N_{X^g/X}$ with $x \in X^g$, then

$$(1.22)\qquad g^{-1}(x, z) = (x, g^{-1}z).$$

Let $dv_{X^g}(x), dv_{N_{X^g/X,x}}$ with $x \in X^g$ be the corresponding volume forms on TX^g and $N_{X^g/X}$ induced by h^{TX}. Let $k(x, z)(x \in X^g, z \in N_{X^g/X}, |z| < \varepsilon)$ be defined by

$$(1.23)\qquad dv_X = k(x, z)dv_{X^g}(x)dv_{N_{X^g/X}}(z).$$

Then it is clear that

$$k(x,0)=1.$$

By the discussion following (1.21), (1.23), we get

$$\begin{aligned}
&\lim_{t\to 0}\Phi\mathrm{Tr}_s[g\exp(-A_t^2)] = \lim_{t\to 0}\int_{\mathcal{U}_{\frac{\varepsilon}{8}}}\Phi\mathrm{Tr}_s\Big[gP_t(g^{-1}x,x)\Big]dv_X(x)\\
(1.24)\qquad &= \lim_{t\to 0}\int_{x\in X^g}\int_{\substack{|Y|\le\varepsilon/8,\\ Y\in N_{X^g/X}}}\Phi\mathrm{Tr}_s\Big[gP_t\Big(g^{-1}(x,Y),(x,Y)\Big)\Big]k(x,Y)\\
&\qquad\qquad dv_{X^g}(x)dv_{N_{X^g/X}}(Y).
\end{aligned}$$

By taking $x_0\in X_b^g$ and using the finite propagation speed as in [**B2**, § 11b)], we may assume that in X_b we have the identification $(TX)_{x_0}\simeq\mathbf{R}^{2k}$ with $0\in\mathbf{R}^{2k}$ representing x_0 and that the extended fibration over $\mathbf{R}^{2k}$ coincides with the given fibration restricted to $B(0,\varepsilon)$.

Take any vector $Y\in\mathbf{R}^{2k}$. We can trivialize $\Lambda(T^*B)\widehat{\otimes}\Delta\otimes W$ by parallel transport along the curve $u\to uY$ with respect to ∇_t'.

Let $\rho(Y)$ be a $\mathcal{C}^\infty$-function over $\mathbf{R}^{2k}$ which is equal to 1 if $|Y|\le\frac{\varepsilon}{4}$, and equal to 0 if $|Y|\ge\frac{\varepsilon}{2}$. Let Δ^{TX} be the ordinary Laplacian operator on $(TX)_{x_0}$. Let H_{x_0} be the vector space of smooth sections of the bundle $(\Lambda(T^*B)\widehat{\otimes}\Delta\otimes W)_{x_0}$ over $(TX)_{x_0}$. For $t>0$, let L_t^1 be the operator acting on H_{x_0}:

$$L_t^1=(1-\rho^2(Y))(-t\Delta^{TX})+\rho^2(Y)A_t^2. \tag{1.25}$$

For $t>0$, $s\in H_{x_0}$, we write

$$\begin{aligned} F_ts(Y)&=s(\tfrac{Y}{\sqrt{t}}),\\ L_t^2&=F_t^{-1}L_t^1F_t.\end{aligned} \tag{1.26}$$

Let $\{e_1,\cdots,e_{2l'}\}$ be an orthonormal basis of $(TX^g)_{x_0}$, and let $\{e_{2l'+1},\cdots,e_{2k}\}$ be an orthonormal basis of $N_{X^g/X,x_0}$. Let L_t^3 be the operator obtained from L_t^2 by replacing the Clifford variables $c(e_j)$ with $1\le j\le 2l'$ by the operators $\frac{e^j}{\sqrt{t}}-\sqrt{t}i_{e_j}$.

Let $P_t^i(Y,Y')$ with $Y,Y'\in(TX)_{x_0}$ and $|Y'|<\frac{\varepsilon}{4}$, $i=1,2,3$ be the smooth kernel associated to $\exp(-L_t^i)$ with respect to the volume element $dv_{TX_{x_0}}(Y')$. By using the finite propagation speed method, there exist $c,C>0$, such that for $Y\in N_{X^g/X,x_0}$, $|Y|\le\frac{\varepsilon}{8}$ and $t\in]0,1]$, we have

$$\Big|P_t(g^{-1}Y,Y)k(x_0,Y)-P_t^1(g^{-1}Y,Y)\Big|\le c\exp(-\frac{C}{t^2}). \tag{1.27}$$

For $\alpha\in\mathbf{C}(e^j,i_{e_j})_{(1\le j\le 2l')}$, let $[\alpha]^{\max}\in\mathbf{C}$ be the coefficient of $e^1\wedge\cdots\wedge e^{2l'}$ in the expansion of α. Then as in [**B2**, Proposition 11.12], if $Y\in N_{X^g/X}$

$$\mathrm{Tr}_s\Big[gP_t^1(g^{-1}Y,Y)\Big]=(-2i)^{\frac12\dim X^g}t^{-\frac12\dim N_{X^g/X}}\mathrm{Tr}_s\Big[gP_t^3\Big(\frac{g^{-1}Y}{\sqrt{t}},\frac{Y}{\sqrt{t}}\Big)\Big]^{\max}. \tag{1.28}$$

Let $R^{TX}_{|M^g}, R^W_{|M^g}, \cdots$ be the corresponding restrictions of $R^{TX}, R^W, \cdots$ to M^g. Let ∇_{e_j} be the ordinary differentiation operator on $(TX)_{x_0}$ in the direction e_j. By [**ABoP**, Proposition 3.7], and (1.20), we have, as $t \to 0$,

$$L^3_t \to L^3_0 == -\sum_{j=1}^{2k} \left(\nabla_{e_j} + \frac{1}{4}\left\langle R^{TX}_{|M^g} Y, e_j \right\rangle\right)^2 + R^W_{|M^g}. \tag{1.29}$$

By proceeding as in [**B2**, §11g)- §11i)], we obtain the following: there exist some constants $\gamma > 0, c > 0, C > 0, r \in \mathbf{N}$ such that for $t \in]0,1]$ and $Y, Y' \in (TX)_{x_0}$, we have

$$\begin{aligned} \left|P^3_t(Y,Y')\right| &\leq c(1+|Y|+|Y'|)^r \exp(-C|Y-Y'|^2), \\ \left|(P^3_t - P^3_0)(Y,Y')\right| &\leq ct^\gamma (1+|Y|+|Y'|)^r \exp(-C|Y-Y'|^2). \end{aligned} \tag{1.30}$$

From (1.28) and (1.30), we get

$$\begin{aligned} &\lim_{t\to 0} \int_{\substack{|Y|\leq \frac{\epsilon}{8} \\ Y\in N_{X^g/X}}} \Phi \mathrm{Tr}_s[gP^1_t(g^{-1}Y,Y)] dv_{N_{X^g/X}}(Y) \\ &= \lim_{t\to 0} \int_{\substack{|Y|\leq \epsilon/8\sqrt{t} \\ Y\in N_{X^g/X}}} (-2i)^{\frac{1}{2}\dim X^g} \Phi \mathrm{Tr}_s[gP^3_t(g^{-1}Y,Y)] dv_{N_{X^g/X}}(Y) \\ &= \int_{N_{X^g/X}} (-2i)^{\frac{1}{2}\dim X^g} \Phi \mathrm{Tr}_s[gP^3_0(g^{-1}Y,Y)]^{\max} dv_{N_{X^g/X}}(Y). \end{aligned} \tag{1.31}$$

Now we define

$$A = -\sum_{j=1}^{2k} \left(\nabla_{e_j} + \frac{1}{4}\left\langle R^{TX}_{|M^g} Y, e_j \right\rangle\right)^2 \tag{1.32}$$

By Mehler's formula [**G**], the smooth kernel $q(Y,Y')$ for $Y, Y' \in TX$, associated to $\exp(-A)$ is given by

$$\begin{aligned} q(Y,Y') = (4\pi)^{-k} \det{}^{1/2}\Big(\frac{R^{TX}/2}{\sinh(R^{TX}/2)}\Big) \exp\Big\{ &-\frac{1}{4}\left\langle \frac{R^{TX}/2}{\tanh(R^{TX}/2)} Y, Y\right\rangle \\ -\frac{1}{4}\left\langle \frac{R^{TX}/2}{\tanh(R^{TX}/2)} Y', Y'\right\rangle &+ \frac{1}{2}\left\langle \frac{R^{TX}/2}{\sinh(R^{TX}/2)} e^{R^{TX}/2} Y, Y'\right\rangle \Big\} \end{aligned} \tag{1.33}$$

From (1.9) and (1.33), we deduce, for $Y \in N_{X^g/X}$,

$$\begin{aligned} q(g^{-1}Y,Y) = (4\pi)^{-k} &\det{}^{1/2}\Big(\frac{R^{TX}/2}{\sinh(R^{TX}/2)}\Big) \\ &\exp\Big\{ -\frac{1}{2}\left\langle \frac{R^N/2}{\sinh(R^N/2)}\Big(\cosh(R^N/2) - e^{R^N/2}g^{-1}\Big)Y, Y\right\rangle \Big\}. \end{aligned} \tag{1.34}$$

On the other hand, for $Y \in N(\theta)$, we have

$$\left\langle \frac{R^N e^{R^N/2}}{2\sinh(R^N/2)} g^{-1} Y, Y\right\rangle = \left\langle \frac{R^N/2}{\sinh(R^N/2)} \frac{1}{2}(e^{R^N/2}g^{-1} + e^{-R^N/2}g)Y, Y\right\rangle. \tag{1.35}$$

It is easy to see that

$$(1.36)\qquad \cosh(R^N/2) - \frac{1}{2}(e^{R^N/2}g^{-1} + e^{-R^N/2}g) = \frac{1}{2}(1-g^{-1})(e^{R^N/2} - e^{-R^N/2}g).$$

From (1.9), (1.34)- (1.36), we get

$$(1.37)\qquad \begin{aligned} &\int_{N_{X^g/X}} q(g^{-1}Y,Y)dv_{N_{X^g/X}}(Y) = (4\pi)^{-\frac{1}{2}\dim X^g} \\ &\det{}^{1/2}\Big(\frac{R^{TX^g}/2}{\sinh(R^{TX^g}/2)}\Big)\Big[\det{}^{1/2}(1-g^{-1}_{|N})\det{}^{1/2}\Big(1-ge^{-R^N}\Big)\Big]^{-1}. \end{aligned}$$

We may and will assume that on the basis $\{e_m\}_{2l'+1\le m\le 2k}$, the matrix of g has diagonal blocks

$$\left[\begin{array}{cc} \cos(\theta_j) & -\sin(\theta_j) \\ \sin(\theta_j) & \cos(\theta_j) \end{array}\right], 0<\theta_j\le\pi.$$

Then one verifies easily that the action of g on Δ is given by

$$(1.38)\qquad g = \Pi_{l'+1\le j\le k}\Big(\cos(\theta_j/2) + \sin(\theta_j/2)c(e_{2j-1})c(e_{2j})\Big).$$

By (1.29) and (1.38), we know that

$$(1.39)\ \mathrm{Tr}_s\Big[gP_0^3(g^{-1}Y,Y)\Big] = \Pi_{l'+1\le j\le k}\Big(-2i\sin(\theta_j/2)\Big)\mathrm{Tr}\Big[g\exp(-R^W_{|M^g})\Big]q(g^{-1}Y,Y).$$

From (1.24), (1.27), (1.31), (1.37) and (1.39), we finally arrive at the wanted formula (1.18). ■

By Theorems 1.2 and 1.3, we now have the complete proof of Theorem 1.1. ■

2 Family Rigidity Theorem

This section is organized as follows. In Section 2.1, we state our main theorem of the paper: the family rigidity theorem. In Section 2.2, we prove it by using the equivariant family index theorem and the modular invariance. In Section 2.3, motivated by the family Witten rigidity theorem, we state a conjecture about a K-theory level rigidity theorem for elliptic genera.

Throughout this section, we use the notations of Section 1, and take $G = S^1$.

2.1 Family rigidity theorem

Let $\pi : M \to B$ be a fibration of compact manifolds with fiber X and $\dim X = 2k$. We assume that the S^1 acts fiberwise on M, and TX has an S^1-equivariant spin structure. As in [**AH**], by lifting to the double cover of S^1, the second condition is always satisfied. Let V be a real vector bundle on M with structure group $Spin(2l)$. Similarly we can assume that V has an S^1-equivariant spin structure without loss of generality.

The purpose of this part is to prove the elliptic operators introduced by Witten [**W**] are rigid in the family case, at least at the equivariant Chern character level. Let us recall them more precisely.

For a vector bundle E on M, let

$$S_t(E) = 1 + tE + t^2S^2E + \cdots,\qquad (2.1)$$
$$\Lambda_t(E) = 1 + tE + t^2\Lambda^2E + \cdots,$$

be the symmetric and exterior power operations in $K(M)[[t]]$. Let

$$\begin{aligned}\Theta'_q(TX) &= \otimes_{n=1}^{\infty}\Lambda_{q^n}(TX)\otimes_{m=1}^{\infty} S_{q^m}(TX),\\ \Theta_q(TX) &= \otimes_{n=1}^{\infty}\Lambda_{-q^{n-1/2}}(TX)\otimes_{m=1}^{\infty} S_{q^m}(TX),\\ \Theta_{-q}(TX) &= \otimes_{n=1}^{\infty}\Lambda_{q^{n-1/2}}(TX)\otimes_{m=1}^{\infty} S_{q^m}(TX).\end{aligned}\qquad (2.2)$$

We also define the following elements in $K(M)[[q^{1/2}]]$:

$$\begin{aligned}\Theta'_q(TX|V) &= \otimes_{n=1}^{\infty}\Lambda_{q^n}(V)\otimes_{m=1}^{\infty} S_{q^m}(TX),\\ \Theta_q(TX|V) &= \otimes_{n=1}^{\infty}\Lambda_{-q^{n-1/2}}(V)\otimes_{m=1}^{\infty} S_{q^m}(TX),\\ \Theta_{-q}(TX|V) &= \otimes_{n=1}^{\infty}\Lambda_{q^{n-1/2}}(V)\otimes_{m=1}^{\infty} S_{q^m}(TX),\\ \Theta^*_q(TX|V) &= \otimes_{n=1}^{\infty}\Lambda_{-q^n}(V)\otimes_{m=1}^{\infty} S_{q^m}(TX).\end{aligned}\qquad (2.3)$$

Let $p_1(\cdot)_{S^1}$ denote the first S^1-equivariant Pontrjagin class and $\Delta(V) = \Delta^+(V)\oplus\Delta^-(V)$ be the spinor bundle of V.

In the following, we denote by $D^X\otimes W$ the Dirac operator on $\Delta\otimes W$ as defined in Section 1. We also write $d_s^X = D^X\otimes\Delta(TX)$. The following theorem is the family analogue of the Witten rigidity theorems as proved in [**BT**], [**T**] and [**Liu2**].

Theorem 2.1 *(a) The family operators $d_s^X\otimes\Theta'_q(TX)$, $D^X\otimes\Theta_q(TX)$ and $D^X\otimes\Theta_{-q}(TX)$ are rigid.*

(b) If $p_1(V)_{S^1} = p_1(TX)_{S^1}$, then $D^X\otimes\Delta(V)\otimes\Theta'_q(TX|V)$, $D^X\otimes(\Delta^+(V)-\Delta^-(V))\otimes\Theta^*_q(TX|V)$, $D^X\otimes\Theta_q(TX|V)$ and $D^X\otimes\Theta_{-q}(TX|V)$ are rigid.

2.2 Proof of the family rigidity theorem

For $\tau\in\mathbf{H} = \{\tau\in\mathbf{C}; \mathrm{Im}\tau > 0\}$, $q = e^{2\pi i\tau}$, let

$$\begin{aligned}\theta_3(v,\tau) &= c(q)\Pi_{n=1}^{\infty}(1+q^{n-1/2}e^{2\pi iv})\Pi_{n=1}^{\infty}(1+q^{n-1/2}e^{-2\pi iv}),\\ \theta_2(v,\tau) &= c(q)\Pi_{n=1}^{\infty}(1-q^{n-1/2}e^{2\pi iv})\Pi_{n=1}^{\infty}(1-q^{n-1/2}e^{-2\pi iv}),\\ \theta_1(v,\tau) &= c(q)q^{1/8}2\cos(\pi v)\Pi_{n=1}^{\infty}(1+q^{n}e^{2\pi iv})\Pi_{n=1}^{\infty}(1+q^{n}e^{-2\pi iv}),\\ \theta(v,\tau) &= c(q)q^{1/8}2\sin(\pi v)\Pi_{n=1}^{\infty}(1-q^{n}e^{2\pi iv})\Pi_{n=1}^{\infty}(1-q^{n}e^{-2\pi iv}).\end{aligned}\qquad (2.4)$$

be the classical Jacobi theta functions [**Ch**], where $c(q) = \Pi_{n=1}^{\infty}(1-q^n)$.

Let $g = e^{2\pi it}\in S^1$ be a generator of the action group. Let $\{M_\alpha\}$ be the fixed submanifolds of the circle action. Then $\pi: M_\alpha\to B$ be a submersion with fibre X_α. We have the following equivariant decomposition of TX

$$TX_{|M_\alpha} = N_1\oplus\cdots\oplus N_h\oplus TX_\alpha,\qquad (2.5)$$

Here N_γ is a complex vector bundle such that g acts on it by $e^{2\pi i m_\gamma t}$. We denote the Chern roots of N_γ by $\{2\pi i x_\gamma^j\}$, and the Chern roots of $TX_\alpha \otimes_{\mathbf{R}} \mathbf{C}$ by $\{\pm 2\pi i y_j\}$. We will write $\dim_{\mathbf{C}} N_\gamma = d(m_\gamma)$ and $\dim X_\alpha = 2k_\alpha$.

Similarly, let

$$V_{|M_\alpha} = V_1 \oplus \cdots \oplus V_{l_0}, \tag{2.6}$$

be the equivariant decomposition of V restricted to M_α. Assume that g acts on V_v by $e^{2\pi i n_v t}$, where some n_v may be zero. We denote the Chern roots of V_v by $2\pi i u_v^j$. Let us write $\dim_{\mathbf{R}} V_v = 2d(n_v)$.

For $f(x)$ a holomorphic function, we denote by $f(y)(TX^g) = \Pi_j f(y_j)$, the symmetric polynomial which gives characteristic class of TX^g, and we use the same notation for N_γ. Now we define some functions on $\mathbf{C} \times \mathbf{H}$ with values in $H^*(B)$,

$$\begin{aligned}
F_{d_s}(t,\tau) &= \Sigma_\alpha \pi_* \Big[\Big(2\pi y \frac{\theta_1(y,\tau)}{\theta(y,\tau)}\Big)(TX^g)\Pi_\gamma\Big(i^{-1}\frac{\theta_1(x_\gamma + m_\gamma t,\tau)}{\theta(x_\gamma + m_\gamma t,\tau)}\Big)(N_\gamma)\Big],\\
F_D(t,\tau) &= \Sigma_\alpha \pi_* \Big[\Big(2\pi y \frac{\theta_2(y,\tau)}{\theta(y,\tau)}\Big)(TX^g)\Pi_\gamma\Big(i^{-1}\frac{\theta_2(x_\gamma + m_\gamma t,\tau)}{\theta(x_\gamma + m_\gamma t,\tau)}\Big)(N_\gamma)\Big],\\
F_{-D}(t,\tau) &= \Sigma_\alpha \pi_* \Big[\Big(2\pi y \frac{\theta_3(y,\tau)}{\theta(y,\tau)}\Big)(TX^g)\Pi_\gamma\Big(i^{-1}\frac{\theta_3(x_\gamma + m_\gamma t,\tau)}{\theta(x_\gamma + m_\gamma t,\tau)}\Big)(N_\gamma)\Big],\\
F^V_{d_s}(t,\tau) &= i^{-k}\Sigma_\alpha \pi_* \Big[\Big(\frac{2\pi i y}{\theta(y,\tau)}\Big)(TX^g)\frac{\Pi_v\theta_1(u_v + n_v t,\tau)(V_v)}{\Pi_\gamma\theta(x_\gamma + m_\gamma t,\tau)(N_\gamma)}\Big],\\
F^V_D(t,\tau) &= i^{-k}\Sigma_\alpha \pi_* \Big[\Big(\frac{2\pi i y}{\theta(y,\tau)}\Big)(TX^g)\frac{\Pi_v\theta_2(u_v + n_v t,\tau)(V_v)}{\Pi_\gamma\theta(x_\gamma + m_\gamma t,\tau)(N_\gamma)}\Big],\\
F^V_{-D}(t,\tau) &= i^{-k}\Sigma_\alpha \pi_* \Big[\Big(\frac{2\pi i y}{\theta(y,\tau)}\Big)(TX^g)\frac{\Pi_v\theta_3(u_v + n_v t,\tau)(V_v)}{\Pi_\gamma\theta(x_\gamma + m_\gamma t,\tau)(N_\gamma)}\Big],\\
F^V_{D^*}(t,\tau) &= i^{-k+l}\Sigma_\alpha \pi_* \Big[\Big(\frac{2\pi i y}{\theta(y,\tau)}\Big)(TX^g)\frac{\Pi_v\theta(u_v + n_v t,\tau)(V_v)}{\Pi_\gamma\theta(x_\gamma + m_\gamma t,\tau)(N_\gamma)}\Big].
\end{aligned} \tag{2.7}$$

By Theorem 1.1 and [**LaM**, p238], we get, for $t \in [0,1] \setminus \mathbf{Q}$ and $g = e^{2\pi i t}$,

$$\begin{aligned}
F_{d_s}(t,\tau) &= \mathrm{ch}_g\Big(\mathrm{Ind}(d_s^X \otimes \Theta'_q(TX))\Big),\\
F_D(t,\tau) &= q^{-k/8}\mathrm{ch}_g\Big(\mathrm{Ind}(D^X \otimes \Theta_q(TX))\Big),\\
F_{-D}(t,\tau) &= q^{-k/8}\mathrm{ch}_g\Big(\mathrm{Ind}(D^X \otimes \Theta_{-q}(TX))\Big),\\
F^V_{d_s}(t,\tau) &= c(q)^{l-k}q^{(l-k)/8}\mathrm{ch}_g\Big(\mathrm{Ind}(D^X \otimes \Delta(V) \otimes \Theta'_q(TX|V))\Big),\\
F^V_D(t,\tau) &= c(q)^{l-k}q^{-k/8}\mathrm{ch}_g\Big(\mathrm{Ind}(D^X \otimes \Theta_q(TX|V))\Big)\\
F^V_{-D}(t,\tau) &= c(q)^{l-k}q^{-k/8}\mathrm{ch}_g\Big(\mathrm{Ind}(D^X \otimes \Theta_{-q}(TX|V))\Big),\\
F^V_{D^*}(t,\tau) &= (-1)^l c(q)^{l-k}q^{(l-k)/8}\mathrm{ch}_g\Big(\mathrm{Ind}(D^X \otimes (\Delta^+(V) - \Delta^-(V))\\
&\qquad \otimes \Theta^*_q(TX|V))\Big).
\end{aligned} \tag{2.8}$$

Considered as functions of (t,τ), we can obviously extend these F's and F^V's to meromorphic functions on $\mathbf{C} \times \mathbf{H}$. Note that these functions are holomorphic in τ. The rigidity theorems are equivalent to the statement that these F's and F^V's are independent of t. As explained in [**Liu2**], we will prove it in two steps: i) we show that these F, F^V are doubly periodic in t; ii) we prove they are holomorphic in t. Then it is trivial to see that they are constant in t.

Lemma 2.1 *(a) For $a, b \in 2\mathbf{Z}$, $F_{d_s}(t,\tau), F_D(t,\tau)$ and $F_{-D}(t,\tau)$ are invariant under the action*

$$U : t \to t + a\tau + b \tag{2.9}$$

(b) If $p_1(V)_{S^1} = p_1(TX)_{S^1}$, then $F^V_{d_s}(t,\tau), F^V_D(t,\tau)$, $F^V_{-D}(t,\tau)$ and $F^V_{D^*}(t,\tau)$ are invariant under U.

Proof: Recall that we have the following transformation formulas of theta-functions [**Ch**]:

$$\begin{aligned} &\theta(t+1,\tau) = -\theta(t,\tau), && \theta(t+\tau,\tau) = -q^{-1/2}e^{-2\pi it}\theta(t,\tau), \\ &\theta_1(t+1,\tau) = -\theta_1(t,\tau), && \theta_1(t+\tau,\tau) = q^{-1/2}e^{-2\pi it}\theta_1(t,\tau), \\ &\theta_2(t+1,\tau) = \theta_2(t,\tau), && \theta_2(t+\tau,\tau) = -q^{-1/2}e^{-2\pi it}\theta_2(t,\tau), \\ &\theta_3(t+1,\tau) = \theta_3(t,\tau), && \theta_3(t+\tau,\tau) = q^{-1/2}e^{-2\pi it}\theta_3(t,\tau). \end{aligned} \tag{2.10}$$

From these, for $\theta_v = \theta, \theta_1, \theta_2, \theta_3$ and $(a,b) \in (2\mathbf{Z})^2$, $l \in \mathbf{Z}$, we get

$$\theta_v(x + l(t + a\tau + b), \tau) = e^{-\pi i(2lax + 2l^2at + l^2a^2\tau)}\theta_v(x + lt, \tau) \tag{2.11}$$

which proves (a).

To prove (b), note that, since $p_1(V)_{S^1} = p_1(TX)_{S^1}$, we have

$$\Sigma_{v,j}(u_v^j + n_v t)^2 = \Sigma_j (y_j)^2 + \Sigma_{\gamma,j}(x_\gamma^j + m_\gamma t)^2. \tag{2.12}$$

This implies the equalities:

$$\begin{aligned} \Sigma_{v,j} n_v u_v^j &= \Sigma_{\gamma,j} m_\gamma x_\gamma^j, \\ \Sigma_\gamma m_\gamma^2 d(m_\gamma) &= \Sigma_v n_v^2 d(n_v), \end{aligned} \tag{2.13}$$

which together with (2.11) proves (b). ∎

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, we define its modular transformation on $\mathbf{C} \times \mathbf{H}$ by

$$g(t,\tau) = \left(\frac{t}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right). \tag{2.14}$$

The two generators of $SL_2(\mathbf{Z})$ are

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.15}$$

which act on $\mathbf{C} \times \mathbf{H}$ in the following way:

$$S(t,\tau) = \left(\frac{t}{\tau}, -\frac{1}{\tau}\right), \; T(t,\tau) = (t, \tau + 1). \tag{2.16}$$

Let Ψ_τ be the scaling homomorphism from $\Lambda(T^*B)$ into itself : $\beta \to \tau^{\frac{1}{2}\deg\beta}\beta$.

Lemma 2.2 *(a) We have the following identities:*

$$F_{d_s}(\tfrac{t}{\tau},-\tfrac{1}{\tau})=i^k\Psi_\tau F_D(t,\tau),\quad F_{d_s}(t,\tau+1)=F_{d_s}(t,\tau),$$
$$F_{-D}(\tfrac{t}{\tau},-\tfrac{1}{\tau})=i^k\Psi_\tau F_{-D}(t,\tau),\quad F_D(t,\tau+1)=F_{-D}(t,\tau)e^{-\frac{\pi i}{4}k}. \tag{2.17}$$

(b) If $p_1(V)_{S^1}=p_1(TX)_{S^1}$, then we have

$$F^V_{d_s}(\tfrac{t}{\tau},-\tfrac{1}{\tau})=(\tfrac{\tau}{i})^{\frac{l-k}{2}}i^k\Psi_\tau F^V_D(t,\tau),\quad F^V_{d_s}(t,\tau+1)=e^{-\frac{\pi i}{4}(k-l)}F^V_{d_s}(t,\tau),$$
$$F^V_{-D}(\tfrac{t}{\tau},-\tfrac{1}{\tau})=(\tfrac{\tau}{i})^{\frac{l-k}{2}}i^k\Psi_\tau F^V_{-D}(t,\tau),\quad F^V_D(t,\tau+1)=e^{-\frac{\pi i}{4}k}F^V_{-D}(t,\tau), \tag{2.18}$$
$$F^V_{D^*}(\tfrac{t}{\tau},-\tfrac{1}{\tau})=(\tfrac{\tau}{i})^{\frac{l-k}{2}}i^{k-l}\Psi_\tau F^V_{D^*}(t,\tau),\quad F^V_{D^*}(t,\tau+1)=e^{-\frac{\pi i}{4}(k-l)}F^V_{D^*}(t,\tau).$$

Proof: By [**Ch**], we have the following transformation formulas for the Jacobi theta-functions:

$$\theta(\tfrac{t}{\tau},-\tfrac{1}{\tau})=\tfrac{1}{i}\sqrt{\tfrac{\tau}{i}}e^{\frac{\pi i t^2}{\tau}}\theta(t,\tau),\quad \theta(t,\tau+1)=e^{\frac{\pi i}{4}}\theta(t,\tau),$$
$$\theta_1(\tfrac{t}{\tau},-\tfrac{1}{\tau})=\sqrt{\tfrac{\tau}{i}}e^{\frac{\pi i t^2}{\tau}}\theta_2(t,\tau),\quad \theta_1(t,\tau+1)=e^{\frac{\pi i}{4}}\theta_1(t,\tau), \tag{2.19}$$
$$\theta_2(\tfrac{t}{\tau},-\tfrac{1}{\tau})=\sqrt{\tfrac{\tau}{i}}e^{\frac{\pi i t^2}{\tau}}\theta_1(t,\tau),\quad \theta_2(t,\tau+1)=\theta_3(t,\tau),$$
$$\theta_3(\tfrac{t}{\tau},-\tfrac{1}{\tau})=\sqrt{\tfrac{\tau}{i}}e^{\frac{\pi i t^2}{\tau}}\theta_3(t,\tau),\quad \theta_3(t,\tau+1)=\theta_2(t,\tau).$$

The action of T on the functions F and F^V are quite simple, and we leave the proof to the reader. Here we only check the action of S. By (2.19), we get

$$F_{d_s}(\frac{t}{\tau},-\frac{1}{\tau})=\Sigma_\alpha\pi_*\Big[\Big(2\pi y\frac{\theta_1(y,-\frac{1}{\tau})}{\theta(y,-\frac{1}{\tau})}\Big)(TX^g)\Pi_\gamma\Big(i^{-1}\frac{\theta_1(x_\gamma+m_\gamma\frac{t}{\tau},-\frac{1}{\tau})}{\theta(x_\gamma+m_\gamma\frac{t}{\tau},-\frac{1}{\tau})}\Big)(N_\gamma)\Big]$$
$$=\Sigma_\alpha i^k\tau^{-k_\alpha}\pi_*\Big[\Big(2\pi\tau y\frac{\theta_1(\tau y,\tau)}{\theta(\tau y,\tau)}\Big)(TX^g)\Pi_\gamma\Big(i^{-1}\frac{\theta_1(\tau x_\gamma+m_\gamma t,\tau)}{\theta(\tau x_\gamma+m_\gamma t,\tau)}\Big)(N_\gamma)\Big] \tag{2.20}$$

If α is a differential form on B, we denote by $\{\alpha\}^{(p)}$ the component of degree p of α. It is easy to see that (2.17)for F_{d_s} follows from the following identity:

$$\tau^{-k_\alpha}\Big\{\pi_*\Big[\Big(\tau y\frac{\theta_1(\tau y,\tau)}{\theta(\tau y,\tau)}\Big)(TX^g)\Pi_\gamma\Big(i^{-1}\frac{\theta_1(\tau x_\gamma+m_\gamma t,\tau)}{\theta(\tau x_\gamma+m_\gamma t,\tau)}\Big)(N_\gamma)\Big]\Big\}^{(2p)}$$
$$=\tau^p\Big\{\pi_*\Big[\Big(y\frac{\theta_1(y,\tau)}{\theta(y,\tau)}\Big)(TX^g)\Pi_\gamma\Big(i^{-1}\frac{\theta_1(x_\gamma+m_\gamma t,\tau)}{\theta(x_\gamma+m_\gamma t,\tau)}\Big)(N_\gamma)\Big]\Big\}^{(2p)} \tag{2.21}$$

By looking at the degree $2(p+k_\alpha)$ part, that is the $(p+k_\alpha)$-th homogeneous terms of the polynomials in x's and y's, on both sides, we immediately get (2.21).

From (2.7), (2.20) and (2.21), we obtain

$$\{F_{d_s}(\frac{t}{\tau},-\frac{1}{\tau})\}^{(2p)}=i^k\tau^p\{F_D(t,\tau)\}^{(2p)}, \tag{2.22}$$

which completes the proof of (2.17) for F_{d_s}. The other identities in (2.17) can be verified in the same way.

By using (2.12), (2.19) and the same trick as in the proof of (2.17), we can obtain the identities in (2.18). This completes the proof of Lemma 2.2. ■

The following lemma implies that the index theory comes in to cancel part of the poles of the functions F's and F^V's.

Lemma 2.3 *If TX and V are spin, then all of the F's and F^V's above are holomorphic in (t, τ) for $(t, \tau) \in \mathbf{R} \times \mathbf{H}$.*

Proof: Let $z = e^{2\pi it}$ and $l' = \dim M$. We will consider these F's and F^V's as meromorphic functions of two complex variables (z, q) with values in $H^*(B)$.

i) Let $N = \max_{\alpha,\gamma}|m_\gamma|$. Denote by $D_N \subset \mathbf{C}^2$ the domain

$$|q|^{1/N} < |z| < |q|^{-1/N}, 0 < |q| < 1. \tag{2.23}$$

Let f_α be the contribution of the component M^α in the functions F's and $c(q)^{k-l}F^V$'s. Then in D_N, by (2.4), (2.7), it is easy to see that f_α has expansions of the form

$$q^{-a/8}\Pi_\gamma(z^{m_\gamma} - 1)^{-l'd(m_\gamma)}\Sigma_{n=1}^{\infty} b_{\alpha,n}(z)q^n, \tag{2.24}$$

where a is an integer and $h_\alpha(z, q) = \Sigma_{n=1}^{\infty} b_{\alpha,n}(z)q^n$ is a holomorphic function of $(z, q) \in D_N$, and $b_{\alpha,n}(z)$ are polynomial functions of z. So as meromorphic functions, these F's and $c(q)^{k-l}F^V$'s have expansions of the form

$$q^{-a/8}\Sigma_{n=1}^{\infty} b_n(z)q^n. \tag{2.25}$$

with $b_n(z)$ rational function of z, which can only have poles on the unit circle $|z| = 1 \subset D_N$.

Now, if we multiply these F's and $c(q)^{k-l}F^V$'s by

$$f(z) = \Pi_\alpha\Pi_\gamma(1 - z^{m_\gamma})^{l'd(m_\gamma)}, \tag{2.26}$$

we get holomorphic functions which have convergent power series expansions of the form

$$q^{-a/8}\Sigma_{n=1}^{\infty} c_n(z)q^n. \tag{2.27}$$

with $\{c_n(z)\}$ polynomial functions of z in D_N.

By comparing the above two expansions, we get for $n \in \mathbf{N}$,

$$c_n(z) = f(z)b_n(z). \tag{2.28}$$

ii) On the other hand, we can expand the Witten element Θ's into formal power series of the form $\Sigma_{n=0}^{\infty} R_n q^n$ with $R_n \in K(M)$. So, for $t \in [0, 1] \setminus \mathbf{Q}$, $z = e^{2\pi it}$, we get a formal power serie of q for these F's and $c(q)^{k-l}F^V$'s:

$$q^{-a/8}\Sigma_{n=0}^{\infty} \mathrm{ch}_z(\mathrm{Ind}(D^X \otimes R_n))q^n \tag{2.29}$$

with $a \in \mathbf{Z}$.

By (1.6), we know

$$\mathrm{ch}_z(\mathrm{Ind}(D^X \otimes R_n)) = \Sigma_{m=-N(n)}^{N(n)} a_{m,n}z^m. \tag{2.30}$$

with $a_{m,n} \in H^*(B)$, and $N(n)$ some positive integer depending on n.

By comparing (2.7), (2.25) and (2.30), we get for $t \in [0, 1] \setminus \mathbf{Q}$, $z = e^{2\pi it}$,

$$b_n(z) = \Sigma_{m=-N(n)}^{N(n)} a_{m,n}z^m. \tag{2.31}$$

Since both sides are analytic functions of z, this equality holds for any $z \in \mathbf{C}$.

By (2.28), (2.31), and the Weierstrass preparation theorem, we deduce that

$$q^{-a/8}\Sigma_{n=1}^{\infty} b_n(z) q^n = \frac{1}{f(z)} q^{-a/8}\Sigma_{n=1}^{\infty} c_n(z) q^n. \tag{2.32}$$

is holomorphic in D_N. Obviously $\mathbf{R} \times \mathbf{H}$ lies inside this domain. The proof of Lemma 2.3 is complete. ■

Proof of the family rigidity theorem for spin manifolds: We will prove that these F's and F^V's are holomorphics on $\mathbf{C} \times \mathbf{H}$, which implies the rigidity theorem we want to prove.

We denote by F one of the functions: F's, F^V's, $\Psi_\tau F$'s and $\Psi_\tau F^V$'s. From their expressions, we know the possible polar divisors of F in $\mathbf{C} \times \mathbf{H}$ are of the form $t = \frac{n}{l}(c\tau + d)$ with n, c, d, l intergers and $(c, d) = 1$ or $c = 1$ and $d = 0$.

We can always find intergers a, b such that $ad - bc = 1$. Then the matrix $g = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in SL_2(\mathbf{Z})$ induces an action

$$F(g(t, \tau)) = F\Big(\frac{t}{-c\tau + a}, \frac{d\tau - b}{-c\tau + a}\Big) \tag{2.33}$$

Now, if $t = \frac{n}{l}(c\tau + d)$ is a polar divisor of $F(t, \tau)$, then one polar divisor of $F(g(t, \tau))$ is given by

$$\frac{t}{-c\tau + a} = \frac{n}{l}\Big(c\frac{d\tau - b}{-c\tau + a} + d\Big), \tag{2.34}$$

which exactly gives $t = n/l$.

But by Lemma 2.2, we know that, up to some constant, $F(g(t, \tau))$ is still one of these F's, F^V's, $\Psi_\tau F$'s and $\Psi_\tau F^V$'s. This contradicts Lemma 2.3, therefore s completes the proof of Theorem 2.1. ■

2.3 A conjecture

Motivated by the family rigidity theorem, Theorem 2.1, we and Zhang would like to make the following conjecture

Conjecture: *The operators considered in Theorem 2.1 are rigid on the equivariant K-theory level.*

This means that, as elements in $K_G(B)$, the equivariant index bundles of those elliptic operators actually lie in $K(B)$. Note that this conjecture is more refined than Theorem 2.1, since the equivariant Chern character map is not an isomorphism. In [**Z**], Zhang proved this for the canonical $Spin^c$-Dirac operator on almost complex manifolds.

3 Jacobi forms and vanishing theorems

In this Section, we generalize the rigidity theorems in the previous Section to the nonzero anomaly case, from which we derive a family of holomorphic Jacobi forms. As corollaries, we get many family vanishing theorems, especially a family $\widehat{\mathfrak{A}}$-vanishing theorem for loop space. This section generalizes some results of [**Liu4**, §3] to the family case.

This section is organized as follows: In Section 3.1, we state the generalization of the rigidity theorems to the nonzero anomaly case. In Section 3.2, we prove this result. In Section 3.3, as corollaries, we derive several family vanishing theorems.

We will keep the notations of Section 2.

3.1 Nonzero anomaly

Recall that the equivariant cohomology group $H^*_{S^1}(M, \mathbf{Z})$ of M is defined by

$$H^*_{S^1}(M, \mathbf{Z}) = H^*(M \times_{S^1} ES^1, \mathbf{Z}). \tag{3.1}$$

where ES^1 is the universal S^1-principal bundle over the classifying space BS^1 of S^1. So $H^*_{S^1}(M, \mathbf{Z})$ is a module over $H^*(BS^1, \mathbf{Z})$ induced by the projection $\overline{\pi} : M \times_{S^1} ES^1 \to BS^1$. Let $p_1(V)_{S^1}, p_1(TX)_{S^1} \in H^*_{S^1}(M, \mathbf{Z})$ be the equivariant first Pontrjagin classes of V and TX respectively. Also recall that

$$H^*(BS^1, \mathbf{Z}) = \mathbf{Z}[[u]] \tag{3.2}$$

with u a generator of degree 2.

In this section, we suppose that there exists some integer $n \in \mathbf{Z}$ such that

$$p_1(V)_{S^1} - p_1(TX)_{S^1} = n \cdot \overline{\pi}^* u^2. \tag{3.3}$$

As in [**Liu4**], we call n the anomaly to rigidity.

Following [**Liu2**], we introduce the following elements in $K(M)[[q^{1/2}]]$:

$$\begin{aligned}
\Theta'_q(TX|V)_v &= \otimes_{n=1}^{\infty} \Lambda_{q^n}(V - \dim V) \otimes_{m=1}^{\infty} S_{q^m}(TX - \dim X),\\
\Theta_q(TX|V)_v &= \otimes_{n=1}^{\infty} \Lambda_{-q^{n-1/2}}(V - \dim V) \otimes_{m=1}^{\infty} S_{q^m}(TX - \dim X),\\
\Theta_{-q}(TX|V)_v &= \otimes_{n=1}^{\infty} \Lambda_{q^{n-1/2}}(V - \dim V) \otimes_{m=1}^{\infty} S_{q^m}(TX - \dim X),\\
\Theta^*_q(TX|V)_v &= \otimes_{n=1}^{\infty} \Lambda_{-q^n}(V - \dim V) \otimes_{m=1}^{\infty} S_{q^m}(TX - \dim X).
\end{aligned} \tag{3.4}$$

For $g = e^{2\pi i t}, q = e^{2\pi i \tau}$, with $(t, \tau) \in \mathbf{R} \times \mathbf{H}$, we denote the equivariant Chern character of the index bundle of $D^X \otimes \Delta(V) \otimes \Theta'_q(TX|V)_v$, $D^X \otimes \Theta_q(TX|V)_v$, $D^X \otimes \Theta_{-q}(TX|V)_v$, and $D^X \otimes (\Delta^+(V) - \Delta^-(V)) \otimes \Theta^*_q(TX|V)_v$ by $2^l F^V_{d_s,v}(t,\tau), F^V_{D,v}(t,\tau)$, $F^V_{-D,v}(t,\tau)$ and $(-1)^l F^V_{D^*,v}(t,\tau)$ respectively. Similarly, we denote by $H(t,\tau)$ the equivariant Chern character of the index bundle of

$$D \otimes \otimes_{m=1}^{\infty} S_{q^m}(TX - \dim X).$$

Later we will consider these functions as the extensions of these functions from the unit circle with variable $e^{2\pi i t}$ to the complex plane with values in $H^*(B)$. For α a differential form on B, we denote by $\{\alpha\}^{(p)}$ the degree p component of α.

The purpose of this section is to prove the following theorem which generalizes the family rigidity theorems to the nonzero anomaly case.

Theorem 3.1 *Assume $p_1(V)_{S^1} - p_1(TX)_{S^1} = n \cdot \overline{\pi}^* u^2$ with $n \in \mathbf{Z}$. Then for $p \in \mathbf{N}$, $\{F^V_{d_s,v}(t,\tau)\}^{(2p)}$, $\{F^V_{D,v}(t,\tau)\}^{(2p)}$, $\{F^V_{-D,v}(t,\tau)\}^{(2p)}$ are holomorpic Jacobi forms of index $n/2$ and weight $k+p$ over $(2\mathbf{Z})^2 \rtimes \Gamma$ with Γ equal to $\Gamma_0(2), \Gamma^0(2), \Gamma_\theta$ respectively, and $\{F^V_{D^*,v}(t,\tau)\}^{(2p)}$ is a holomorphic Jacobi form of index $n/2$ and weight $k-l+p$ over $(2\mathbf{Z})^2 \rtimes SL_2(\mathbf{Z})$.*

See Section 3.2 for the definitions of these modular subgroups, $\Gamma_0(2), \Gamma^0(2)$ and Γ_θ.

3.2 Proof of Theorem 3.1

Recall that a (meromorphic) Jacobi form of index m and weight l over $L \rtimes \Gamma$, where L is an integral lattice in the complex plane $\mathbf{C}$ preserved by the modular subgroup $\Gamma \subset SL_2(\mathbf{Z})$, is a (meromorphic) function $F(t,\tau)$ on $\mathbf{C} \times \mathbf{H}$ such that

$$\begin{aligned} &F(\frac{t}{c\tau+d}, \frac{a\tau+b}{c\tau+d}) = (c\tau+d)^l e^{2\pi i m(ct^2/(c\tau+d))} F(t,\tau), \\ &F(t+\lambda\tau+\mu, \tau) = e^{-2\pi i m(\lambda^2\tau+2\lambda t)} F(t,\tau), \end{aligned} \tag{3.5}$$

where $(\lambda,\mu) \in L$, and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. If F is holomorphic on $\mathbf{C} \times \mathbf{H}$, we say that F is a holomorphic Jacobi form.

Now, we start to prove Theorem 3.1. Let $g = e^{2\pi i t} \in S^1$ be a generator of the action group. For $\alpha = 1,2,3$, let

$$\theta'(0,\tau) = \frac{\partial}{\partial t}\theta(t,\tau)|_{t=0}, \qquad \theta_\alpha(0,\tau) = \theta_\alpha(t,\tau)|_{t=0} \tag{3.6}$$

By applying Theorem 1.1, we get

$$\begin{aligned} &F^V_{d_s,v}(t,\tau) = (2\pi)^{-k} \frac{\theta'(0,\tau)^k}{\theta_1(0,\tau)^l} F^V_{d_s}(t,\tau), \\ &F^V_{D,v}(t,\tau) = (2\pi)^{-k} \frac{\theta'(0,\tau)^k}{\theta_2(0,\tau)^l} F^V_{D}(t,\tau), \\ &F^V_{-D,v}(t,\tau) = (2\pi)^{-k} \frac{\theta'(0,\tau)^k}{\theta_3(0,\tau)^l} F^V_{-D}(t,\tau), \\ &F^V_{D^*,v}(t,\tau) = (2\pi)^{l-k} \theta'(0,\tau)^{k-l} F^V_{D^*}(t,\tau), \\ &H(t,\tau) = (2\pi i)^{-k} \Sigma_\alpha \pi_* \Big[\Big(\frac{2\pi i y}{\theta(y,\tau)} \Big)(TX^g) \frac{\theta'(0,\tau)^k}{\Pi_\gamma \theta(x_\gamma + m_\gamma t, \tau)(N_\gamma)} \Big]. \end{aligned} \tag{3.7}$$

Lemma 3.1 *If $p_1(V)_{S^1} - p_1(TX)_{S^1} = n \cdot \overline{\pi}^* u^2$, we have*

$$\begin{aligned} &F^V_{d_s,v}(\tfrac{t}{\tau}, -\tfrac{1}{\tau}) = \tau^k e^{\pi i n t^2/\tau} \Psi_\tau F^V_{D,v}(t,\tau), \quad F^V_{d_s,v}(t,\tau+1) = F^V_{d_s,v}(t,\tau), \\ &F^V_{-D,v}(\tfrac{t}{\tau}, -\tfrac{1}{\tau}) = \tau^k e^{\pi i n t^2/\tau} \Psi_\tau F^V_{-D,v}(t,\tau), \quad F^V_{D,v}(t,\tau+1) = F^V_{-D,v}(t,\tau), \\ &F^V_{D^*,v}(\tfrac{t}{\tau}, -\tfrac{1}{\tau}) = \tau^{k-l} e^{\pi i n t^2/\tau} \Psi_\tau F^V_{D^*,v}(t,\tau), \quad F^V_{D^*,v}(t,\tau+1) = F^V_{D^*,v}(t,\tau). \end{aligned} \tag{3.8}$$

If $p_1(TX)_{S^1} = -n \cdot \overline{\pi}^* u^2$, then

$$H(\frac{t}{\tau}, -\frac{1}{\tau}) = \tau^k e^{\pi i n t^2/\tau} \Psi_\tau H(t,\tau), \quad H(t,\tau+1) = H(t,\tau). \tag{3.9}$$

Proof: First recall that the condition on the first equivariant Pontrjagin classes implies the equality

$$\Sigma_{v,j}(u_v^j+n_vt)^2-\Big(\Sigma_j(y_j)^2+\Sigma_{\gamma,j}(x_\gamma^j+m_\gamma t)^2\Big)=n\cdot t^2 \tag{3.10}$$

which gives the equalities:

$$\begin{aligned}&\Sigma_v n_v^2 d(n_v)-\Sigma_\gamma m_\gamma^2 d(m_\gamma)=n, \quad \Sigma_{v,j}n_v u_v^j=\Sigma_{\gamma,j}m_\gamma x_\gamma^j,\\ &\Sigma_{v,j}(u_v^j)^2=\Sigma_j(y_j)^2+\Sigma_{\gamma,j}(x_\gamma^j)^2.\end{aligned} \tag{3.11}$$

The action of T on the functions F and F^V are quite easy to check and we leave the detail to the reader. We only check the action of S. By (2.7), (2.19), (3.7) and (3.11), we have

$$\begin{aligned}F_{d_s,v}^V(\frac{t}{\tau},-\frac{1}{\tau})&=(2\pi i)^{-k}\\ &\Sigma_\alpha\pi_*\Big[\frac{\theta'(0,-\frac{1}{\tau})^k}{\theta_1(0,-\frac{1}{\tau})^l}\Big(\frac{2\pi iy}{\theta(y,-\frac{1}{\tau})}\Big)(TX^g)\frac{\Pi_v\theta_1(u_v+n_v\frac{t}{\tau},-\frac{1}{\tau})(V_v)}{\Pi_\gamma\theta(x_\gamma+m_\gamma\frac{t}{\tau},-\frac{1}{\tau})(N_\gamma)}\Big]\\ &=(2\pi i)^{-k}\tau^k e^{\pi i(nt^2/\tau)}\Sigma_\alpha\pi_*\Big[\frac{\theta'(0,\tau)^k}{\theta_1(0,\tau)^l}\Big(\frac{2\pi iy}{\theta(\tau y,\tau)}\Big)(TX^g)\frac{\Pi_v\theta_1(\tau u_v+n_vt,\tau)(V_v)}{\Pi_\gamma\theta(\tau x_\gamma+m_\gamma t,\tau)(N_\gamma)}\Big].\end{aligned} \tag{3.12}$$

As in (2.21), by comparing the $(p+k_\alpha)$-th homogeneous terms of the polynomials in x's, y's and u's on both side, we find the following equation

$$\begin{aligned}&\Big\{\pi_*\Big[\Big(\frac{2\pi iy}{\theta(\tau y,\tau)}\Big)(TX^g)\frac{\Pi_v\theta_1(\tau u_v+n_vt,\tau)(V_v)}{\Pi_\gamma\theta(\tau x_\gamma+m_\gamma t,\tau)(N_\gamma)}\Big]\Big\}^{(2p)}\\ &=\tau^p\pi_*\Big[\Big(\frac{2\pi iy}{\theta(y,\tau)}\Big)(TX^g)\frac{\Pi_v\theta_1(u_v+n_vt,\tau)(V_v)}{\Pi_\gamma\theta(x_\gamma+m_\gamma t,\tau)(N_\gamma)}\Big]\Big\}^{(2p)}.\end{aligned} \tag{3.13}$$

By (3.12), (3.13), we get the equation (3.8) for $F_{d_s,v}^V$. We leave the other cases to the reader. ■

Recall the three modular subgroups:

$$\begin{aligned}\Gamma_0(2)&=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbf{Z})\middle| c\equiv 0(\mathrm{mod}2)\right\},\\ \Gamma^0(2)&=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbf{Z})\middle| b\equiv 0(\mathrm{mod}2)\right\},\\ \Gamma_\theta&=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbf{Z})\middle|\begin{pmatrix}a&b\\c&d\end{pmatrix}\equiv\begin{pmatrix}1&0\\0&1\end{pmatrix}\text{ or }\begin{pmatrix}0&1\\1&0\end{pmatrix}(\mathrm{mod}2)\right\}.\end{aligned} \tag{3.14}$$

Lemma 3.2 *If $p_1(V)_{S^1}-p_1(TX)_{S^1}=n\cdot\overline{\pi}^*u^2$, then for $p\in\mathbf{N}$, $\{F_{d_s,v}^V(t,\tau)\}^{(2p)}$ is a Jacobi form over $(2\mathbf{Z})^2\rtimes\Gamma_0(2)$; $\{F_{D,v}^V(t,\tau)\}^{(2p)}$ is a Jacobi form over $(2\mathbf{Z})^2\rtimes\Gamma^0(2)$; $\{F_{-D,v}^V(t,\tau)\}^{(2p)}$ is a Jacobi form over $(2\mathbf{Z})^2\rtimes\Gamma_\theta$. If $p_1(TX)_{S^1}=-n\overline{\pi}^*u^2$, then $\{H(t,\tau)\}^{(2p)}$ is a Jacobi form over $(2\mathbf{Z})^2\rtimes SL_2(\mathbf{Z})$. All of them are of index $\frac{n}{2}$ and weight $k+p$.*

The function $\{F_{D^*,v}^V(t,\tau)\}^{(2p)}$ is a Jacobi form of index $\frac{n}{2}$ and weight $k-l+p$ over $(2\mathbf{Z})^2\rtimes SL_2(\mathbf{Z})$.

Proof: By (2.19), (3.7), we know that these F^V's and H satisfy the second equation of the definition of Jacobi forms (3.5).

Recall that T and ST^2ST generate $\Gamma_0(2)$, and also $\Gamma^0(2)$ and Γ_θ are conjugate to $\Gamma_0(2)$ by S and TS respectively. By Lemma 3.1, and the above discussion, for F^V's and H, we easily get the first equation of (3.5). ■

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, let us use the notation

$$F(g(t,\tau))|_{m,l} = (c\tau+d)^{-l} e^{-2\pi i m c t^2/(c\tau+d)} F\left(\frac{t}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right). \tag{3.15}$$

to denote the action of g on a Jacobi form F of index m and weight l.

By Lemma 3.1, for any function in $F \in \{\{F^V\}^{(2p)}, H^{(2p)}\}$, its modular transformation $\{F\}^{(2p)}(g(t,\tau))|_{\frac{n}{2},k+p}$ (or $\{F\}^{(2p)}(g(t,\tau))|_{\frac{n}{2},k-l+p}$) is still one of the $\{\{F^V\}^{(2p)}\}$'s and $H^{(2p)}$. Similar to Lemma 2.3, we have

Lemma 3.3 *For any $g \in SL_2(\mathbf{Z})$, let $F(t,\tau)$ be one of the $\{F^V\}^{(2p)}$'s or $H^{(2p)}$, Then $F(g(t,\tau))|_{\frac{n}{2},k+p}$ is holomorphic in (t,τ) for $t \in \mathbf{R}$ and $\tau \in \mathbf{H}$.*

As in Lemma 2.3, this is the place where the index theory comes in to cancel part of the poles of these functions. Of course, to use the index theory, we must use the spin conditions on TX and V.

Now, we recall the following result [**Liu4**, Lemma 3.4]:

Lemma 3.4 *For a (meromorphic) Jacobi form $F(t,\tau)$ of index m and weight k over $L \rtimes \Gamma$, assume that F may only have polar divisors of the form $t = (c\tau+d)/l$ in $\mathbf{C} \times \mathbf{H}$ for some integers c, d and $l \neq 0$. If $F(g(t,\tau))|_{m,k}$ is holomorphic for $t \in \mathbf{R}$, $\tau \in \mathbf{H}$ for every $g \in SL_2(\mathbf{Z})$, then $F(t,\tau)$ is holomorphic for any $t \in \mathbf{C}$ and $\tau \in \mathbf{H}$.*

Proof of Theorem 3.1: By Lemmas 3.1, 3.2, 3.3, we know that the $\{F^V\}^{(2p)}$'s and $H^{(2p)}$ satisfy the assumptions of Lemma 3.4. In fact, all of their possible polar divisors are of the form $l = (c\tau+d)/m$ where c, d are integers and m is one of the exponents $\{m_j\}$. The proof of Theorem 3.1 is complete. ■

3.3 Family vanishing theorems for loop space

The following lemma is established in [**EZ**, Theorem 1.2]:

Lemma 3.5 *Let F be a holomorphic Jacobi form of index m and weight k. Then for fixed τ, $F(t,\tau)$, if not identically zero, has exactly $2m$ zeros in any fundamental domain for the action of the lattice on $\mathbf{C}$.*

This tells us that there are no holomorphic Jacobi forms of negative index. Therefore, if $m < 0$, F must be identically zero. If $m = 0$, it is easy to see that F must be independent of t.

Combining Lemma 3.5 with Theorem 3.1, we have the following result.

Corollary 3.1 *Let M, B, V and n be as in Theorem 3.1, If $n=0$, the equivariant Chern characters of the index bundle of the elliptic operators in Theorem 3.1 are independent of $g \in S^1$. If $n<0$, then these equivariant Chern characters are identically zero; in particular, the Chern character of the index bundle of these elliptic operators are zero.*

Another quite interesting consequence of the above discussions is the following family $\widehat{\mathfrak{A}}$-vanishing theorem for loop space.

Theorem 3.2 *Assume M is connected and the S^1-action is nontrivial. If $p_1(TX)_{S^1} = n \cdot \overline{\pi}^* u^2$ for some integer n, then the equivariant Chern character of the index bundle, especially the Chern character of the index bundle, of $D \otimes \otimes_{m=1}^{\infty} S_{q^m}(TX - \dim X)$ is identically zero.*

Proof: In fact, by (3.11), we know that

$$\Sigma_j m_j^2 d(m_j) = n. \tag{3.16}$$

So the case $n<0$ can never happen. If $n=0$, then all the exponents $\{m_j\}$ are zero, so the S^1-action cannot have a fixed point. By (2.7)and (3.7), we know that $H(t,\tau)$ is zero. For $n>0$, one can apply Lemmas 3.1, 3.4 and 3.5 to get the result. ■

As remarked in [**Liu4**], the fact that the index of $D \otimes \otimes_{m=1}^{\infty} S_{q^m}(TX - \dim X)$ is zero may be viewed as a loop space analogue of the famous $\widehat{\mathfrak{A}}$-vanishing theorem of Atiyah and Hirzebruch [**AH**] for compact connected spin manifolds with non-trivial S^1-action. The reason is that, this operator corresponds to the Dirac operator on loop space LX, while the condition on $p_1(TX)_{S^1}$ is a condition for the existence of an equivariant spin structure on LX. This property is one of the most interesting and surprising properties of loop space. Now, under the condition of Theorem 3.2, a very interesting question is to know when the index bundle of this elliptic operator is zero in $K(B)$.

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