Massey products in symplectic manifolds

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Abstract

The paper is devoted to study of Massey products in symplectic manifolds. Theory of generalized and classical Massey products and a general construction of symplectic manifolds with nontrivial Massey products of arbitrary large order are exposed. The construction uses the symplectic blow-up and is based on the author results, which describe conditions under which the blow-up of a symplectic manifold X along its submanifold Y inherits nontrivial Massey products from X ot Y. This gives a general construction of nonformal symplectic manifolds.

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§1. Introduction

In this paper we resume our study of the formality problem for symplectic manifolds, which we started in [1, 2] where the first examples of nonformal simply connected symplectic compact manifolds were constructed and a method of constructing such manifolds by symplectic blow-ups was introduced.

A smooth manifold X is called symplectic if there is a nondegenerate closed 2-form ω on X. We denote this object by (X, ω). Symplectic manifolds first appeared in analytic mechanics but long ago became a subject of pure mathematics [3]. In this paper we consider only closed symplectic manifolds.

It is naturally to expect that the existence of a symplectic structure strongly restricts the topology of a manifold. A lack of a wide spectrum of examples only confirms this guess. The main examples of symplectic manifolds are Kähler manifolds and the existence of a Kähler structure really poses strong topological restrictions on a manifold.

The problem of existence of symplectic manifolds with no Kähler structure was posed in the early 70s and the first example of such a manifold was a 4-dimensional non-simply connected symplectic manifold constructed by Thurston [4]. It appeared later that this manifold is one of complex non-Kähler surfaces from the Kodaira classification list. Using this example McDuff constructed a simply connected symplectic manifold with no Kähler structure [5]. This implies that the class of Kähler manifolds is a subclass of symplectic manifold but the problem how narrow is it is far from a complete solution.

Recently Gompf showed that any finitely presented group is the fundamental group of some 4-dimensional symplectic manifold [6]. Notice that the first Betti number of a Kähler manifold is even.

One of the basic topological properties of Kähler manifolds is a formality established in [7]. The formality of a space means that its real homotopy type is completely defined by the real cohomology ring $H^*(X, \mathbb{R})$.

An existence of nonformal and, hence, non-Kähler symplectic manifolds was known quite long ago [8]. In particular, the Kodaira–Thurston manifold is nonformal although Thurston did not discuss that. All examples of nonformal symplectic manifolds known until recently were non-simply connected and their nonformality arises from certain properties of the fundamental group. For simply connected manifolds there was the conjecture by Lupton and Oprea, which stated that simply connected symplectic manifolds are formal and in this sense are similar to Kähler manifolds [9]. Some

partial results confirming that were obtained in [10].

This conjecture was disproved by us in [1, 2] where infinitely many nonformal simply connected symplectic manifolds of all even dimensions greater than 8 were presented. In particular, the 10-dimensional McDuff manifold [5] is nonformal. Any simply connected manifold of dimension less than equal 6 is formal [11, 12]. Therefore the problem on existence of nonformal simply connected symplectic manifolds is open only for 8-dimensional manifolds.

A well-known obstruction to formality of a space X is an existence of nontrivial higher, i.e., n-tuple for $n \geq 3$, Massey products in $H^*(X)$. It is a nontriviality of certain triple ordinary Massey products in some simply connected symplectic manifolds that was established in [2].

Independently of the formality problem an existence of higher Massey products reflects a complexity of the topology of a space. Higher Massey products, or operations, appear in some problems of homotopy topology, which are, for instance, a description of the cohomology Hurewicz homomorphism and a description of the differentials of the Eilenberg–Moore spectral sequence. One of the main aims of this paper is to show that symplectic and, in particular, simply connected symplectic manifolds may have nontrivial n-tuple Massey products for arbitrary large n.

The paper comprises two parts.

In the first part we introduce the most general Massey products. A particular case of them are the classical matrix Massey products introduced by May in 1969. Our approach to the definition of Massey products is based on considering solutions to the Maurer–Cartan equation

$$d\,A - \bar{A} \cdot A \equiv 0 \mod K(A). \tag{1}$$

on the algebra of matrix differential forms on the manifold. Here A is an upper triangular matrix whose entries are differential forms on the manifold and $K(A)$ is a certain submodule associated with this matrix.

It was noticed by May that the defining system for any matrix Massey product can be presented by a matrix satisfying the Maurer–Cartan equation (1) but this approach was not developed.

A solution A to the equation (1) may be treated as a formal connection. Then the corresponding generalized Massey products are the cohomology classes of "the curvature form"

$$\mu(A) = d\,A - \bar{A} \cdot A$$

of the connection A. This differential-geometric approach to the definition

of Massey products enables us to establish the basic properties of these products in a quite simple manner.

In the first part we consider in brief relations of Massey products to the Hurewicz and the suspension homomorphisms and to the Eilenberg–Moore spectral sequence.

In the second part of this paper we consider applications of Massey products to symplectic manifolds. There are three main methods of constructing new symplectic manifolds from given ones. These are the symplectic fibration, the fiber connected sum, and the symplectic blow-up. Although for constructing symplectic manifolds with nontrivial Massey products one may use the symplectic fibration as it was done, for instance, in the case of the Kodaira–Thurston manifold, it seems that for these reasons the symplectic blow-up [25, 5] is the most perspective construction.

Let (X, ω) be a symplectic manifold and $Y \subset X$ be its symplectic submanifold. We denote by $\widetilde{X}$ the symplectic blow-up of X along Y. The main problem studied in the second part is what may happen with higher Massey products under the symplectic blow-up. It splits into two parts:

1) "*A survival*". Let X have irreducible higher (matrix) Massey products. What conditions on Y guarantee that these products survive in $\widetilde{X}$?

2) "*An inheritance*". Assume that Y has nontrivial higher (matrix) Massey products. Under what conditions on X these products are inherited by $\widetilde{X}$?

Let us formulate the main results in this direction.

Theorem A *Let a simply connected symplectic manifold X have an irreducible generalized Massey product of dimension k. Then for any symplectic submanifold $Y \subset X$ with* $\operatorname{codim} Y > k$ *the corresponding symplectic blow-up $\widetilde{X}$ also has an irreducible generalized Massey product of dimension k.*

Theorem B *Let a symplectic manifold (Y, ω) have a nontrivial matrix n-tuple Massey product $\langle S_1, \dots, S_n \rangle$ where $S_i \in N(H^1(Y))$ are matrices of one-dimensional cohomology classes for $1 \leq i \leq n$. Then for any symplectic embedding $Y \subset X$ of codimension not less than $2(n+1)$ the corresponding symplectic blow-up $\widetilde{X}$ has a nontrivial n-tuple Massey product $\langle \widetilde{S}_1, \dots, \widetilde{S}_n \rangle$, where $\widetilde{S}_i \in N(H^3(\widetilde{X}))$, $1 \leq i \leq n$.*

Theorem C *Let a symplectic manifold (Y, ω) have a nontrivial triple matrix Massey product. Then for any symplectic embedding $Y \subset X$ of codimension greater or equal than 8 the corresponding symplectic blow up $\widetilde{X}$ also has a nontrivial triple matrix Massey product.*

A particular case of Theorem C was established by us in [1, 2] and moreover quite general reasonings applied to the blow-ups of $\mathbb{C}P^n$ along the

embedded Kodaira–Thurston manifolds give a bound for the codimension equal to 8 [1]. Later we showed that particularly for this manifold the bound for the codimension is decreased to 6.

In [2] we posed a problem of inheritance of nontrivial Massey products in a general case. It was discussed in [13] where Theorem C for ordinary Massey products was proved.

Theorem D *Let a symplectic manifold (Y,ω) have a strictly irreducible quadruple matrix Massey product $\langle S_1, S_2, S_3, S_4\rangle$. Then for any symplectic submanifold $Y \subset X$ such that*

$$\operatorname{codim} Y > 2 \operatorname{sdeg} \langle S_1, S_2, S_3, S_4\rangle$$

the corresponding symplectic blow-up $\widetilde{X}$ has a nontrivial quadruple matrix Massey product.

Theorems A, B, C, and D enables us to construct new manifolds with nontrivial Massey products by the symplectic blow-up of X along a submanifold Y. In this event it needs that X or Y have nontrivial Massey products. Therefore to make this method work we have to have an initial family of symplectic manifolds with nontrivial Massey products. An existence of such a family is guaranteed by Theorem 6, which we shall prove in §3:

For any k there exist symplectic manifolds with Massey products whose weights are strictly equal to $2k$.

Examples of such manifolds are constructed in Proposition 8.

These results show that an existence of a symplectic structure does not strongly restrict the homotopy type of a manifold. We would like to propose the following

Conjecture *For any finite polyhedron P and any natural N there are a symplectic manifold X and an embedding $f : P \to X$ such that*

$$f_* : \pi_k(P) \to \pi_k(X) \quad \textit{is a monomorphism for } k \leq N.$$

For $N = 1$ and $\dim P = 2$ the conjecture follows from Gompf's results [6].

The authors thank D. van Straten and A. Tralle for helpful discussions.

§2. Massey products and some constructions of the homotopy theory

2.1. The algebra of forms and its minimal model.

Let $\mathcal{A}$ be a differential graded algebra over a field k. This means that $\mathcal{A}$ is a direct sum of the subspaces $\mathcal{A}^l$ formed by homogeneous elements of degree $l \geq 0$:

$$\mathcal{A} = \oplus_{l \geq 0} \mathcal{A}^l, \quad \deg a = l \quad \text{for} \quad a \in \mathcal{A}^l,$$

and there are the following linear operations on A: an associative multiplication

$$\wedge : \mathcal{A}^l \times \mathcal{A}^m \to \mathcal{A}^{l+m}, \quad l, m \geq 0,$$

and a differential

$$d : \mathcal{A}^l \to \mathcal{A}^{l+1}, \quad l \geq 0.$$

Moreover it is assumed that the following conditions hold:

1) $a \wedge b = (-1)^{lm} b \wedge a$ for $a \in \mathcal{A}^l$, $b \in \mathcal{A}^m$;

2) $d(a \wedge b) = d\,a \wedge b + (-1)^l a \wedge d\,b$ for $a \in \mathcal{A}^l$ (the Leibniz rule);

3) $d^2 = 0$.

A homomorphism of differential graded algebras $(\mathcal{A}, d_{\mathcal{A}})$ and $(\mathcal{B}, d_B)$ over a field k is a k-linear mapping

$$f : \mathcal{A} \to \mathcal{B},$$

which respects the grading:

$$f(\mathcal{A}^l) \subset \mathcal{B}^l \quad \text{for} \quad l \geq 0,$$

and satisfies the following conditions:

$$f(a \wedge b) = f(a) \wedge f(b), \quad a, b \in \mathcal{A},$$

$$d_{\mathcal{B}} f(a) = f(d_{\mathcal{A}} a), \quad a \in \mathcal{A}.$$

The cohomologies of the algebra $(\mathcal{A}, d_{\mathcal{A}})$ are defined in the natural manner:

$$H^l(\mathcal{A}, d_{\mathcal{A}}) = \mathrm{Ker}\,(d_{\mathcal{A}} : \mathcal{A}^l \to \mathcal{A}^{l+1}) / \mathrm{Im}\,(d_{\mathcal{A}} : \mathcal{A}^{l-1} \to \mathcal{A}^l).$$

To every closed element (or cocycle) $a \in \mathcal{A}$, i.e. such that $d_{\mathcal{A}} a = 0$, there corresponds its cohomology class $[a] \in H^+(\mathcal{A}, d_{\mathcal{A}})$. If $[a] = 0$, then it is said that a is exact.

A homomorphism $f : (\mathcal{A}, d_{\mathcal{A}}) \to (\mathcal{B}, d_{\mathcal{B}})$ maps closed elements to closed ones and exact elements to exact ones. Therefore it induces a homomorphism

$$f^* : H^*(\mathcal{A}, d_{\mathcal{A}}) \to H^*(\mathcal{B}, d_{\mathcal{B}})$$

by the formula $f^*[a] = [f(a)]$.

An algebra $(\mathcal{A}, d)$ is connected if $H^0(\mathcal{A}, d) = k$. If in addition $H^1(\mathcal{A}, d) = 0$, then it is said that A is simply connected.

It is also assumed that $\mathcal{A}$ is augmented, which means that there is an epimorphism of differential graded algebras

$$\varepsilon : \mathcal{A} \to k$$

where k is formed by zero degree elements and is endowed with the zero differential. The ideal $I = \operatorname{Ker} \varepsilon$ is called the augmentation ideal.

In this paper the main example of such an algebra would be the algebra $\mathcal{E}^*(X)$ of smooth differential forms on a smooth manifold X. If X is simply connected then $\mathcal{E}^*(X)$ is also simply connected.

Let $i : x \to X$ be the embedding of the point $x \in X$ into X. The induced mapping defines an augmentation

$$\varepsilon = i^* : \mathcal{E}^*(X) \to \mathcal{E}^*(x) = k = \mathbb{R}.$$

The augmentation ideal consists of all forms of positive degree and smooth functions $\varphi : X \to \mathbb{R}$ such that $\varphi(x) = 0$.

In the sequel we assume for simplicity that $k = \mathbb{R}$.

A differential graded algebra $(\mathcal{M}, d_{\mathcal{M}})$ is called minimal if

1) $\mathcal{M}^0 = \mathbb{R}$, $d(\mathcal{M}^0) = 0$, and the multiplication by elements of $\mathcal{M}^0$ coincides with the multiplication by elements of the main field $k = \mathbb{R}$;

2) $\mathcal{M}^+ = \oplus_{l>0} \mathcal{M}^l$ is freely generated by homogeneous elements $x_1, \ldots, x_n, \ldots$:

$$\mathcal{M}^+ = \wedge(x_1, \ldots),$$

for any $l > 0$ there are finitely many such generators of degree l, and

$$\deg x_i \leq \deg x_j \quad \text{for} \quad i \leq j;$$

3) the differential d is reducible:

$$d\, x_i \in \wedge(x_1, \ldots, x_{i-1}) \quad \text{for } i \geq 1,$$

i. e., $d\, x_i$ is a polynomial in $x_1, \ldots, x_{i-1}$.

It is clear that a minimal algebra $\mathcal{M}$ is simply connected if and only of $\mathcal{M}^1 = 0$. In this case $\deg x_i \geq 2$ for $i \geq 1$ and the reducibility condition is written as

$$d(\mathcal{M}^+) \subset \mathcal{M}^+ \wedge \mathcal{M}^+$$

where $\mathcal{M}^+ \wedge \mathcal{M}^+$ is a linear subspace generated by reducible elements.

An algebra $(\mathcal{M}, d_{\mathcal{M}})$ is called a minimal model for an algebra $(\mathcal{A}, d_{\mathcal{A}})$ if

1) the algebra $(\mathcal{M}, d_{\mathcal{M}})$ is minimal;

2) there is a homomorphism $h : (\mathcal{M}, d_{\mathcal{M}}) \to (\mathcal{A}, d_{\mathcal{A}})$, which induces an isomorphism of the cohomology rings:

$$h^* : H^*(\mathcal{M}, d_{\mathcal{M}}) \to H^*(\mathcal{A}, d_{\mathcal{A}}).$$

In the sequel homomorphisms satisfying these conditions are called *quasi-isomorphisms.*

The fundamental theorem of Sullivan [14, 7] reads

Theorem 1 (*Sullivan*) *If $(\mathcal{A}, d_{\mathcal{A}})$ is a simply connected differential graded algebra such that $\dim H^l(\mathcal{A}) < \infty$ for any $l \geq 0$, then there is a minimal model for $\mathcal{A}$ which is unique up to isomorphism.*

An example of an algebra satisfying the hypothesis of the theorem is the algebra of smooth forms on a simply connected compact manifold.

If $\mathcal{A} = \mathcal{E}^*(X)$, then the minimal model $\mathcal{M}_X$ for $\mathcal{A}$ is also called the (real) minimal model for the space X. Since, by the de Rham, theorem the cohomologies of the algebra of smooth forms on X are isomorphic to the real cohomologies of X, the homomorphism

$$h : (\mathcal{M}_X, d_{\mathcal{M}}) \to (\mathcal{E}^*(X), d_X)$$

induces the isomorphism

$$h^* : H^*(\mathcal{M}_X) \to H^*(X)$$

where $H^*(X) = H^*(X, \mathbb{R})$.

Examples. 1) Let $X = S^{2n+1}$, $n \geq 1$. Then $\mathcal{M}_X^+$ is generated by an element x with $dx = 0$ and $\deg x = 2n + 1$.

2) For $X = S^{2n}$, $n \geq 1$, its minimal model $\mathcal{M}_X^+$ is generated by elements x and y with $\deg x = 2n$, $\deg y = 4n - 1$, $dx = 0$, and $dy = x^2$.

An important property of minimal models is given by the following statement [14, 7].

Theorem 2 (*Sullivan*) *Any smooth mapping*

$$f : X \to Y$$

induces a homomorphism of the minimal models

$$\widehat{f} : \mathcal{M}_Y \to \mathcal{M}_X$$

such that the diagram

$$\begin{array}{ccc} H^*(\mathcal{M}_X) & \xleftarrow{\widehat{f}^*} & H^*(\mathcal{M}_Y) \\ h_X^* \downarrow & & \downarrow h_Y^* \\ H^*(X) & \xleftarrow{f^*} & H^*(Y) \end{array}$$

is commutative.

Other properties of minimal models are exposed in [14, 7] and we note only one of them: there is the natural homomorphism

$$\mathrm{Hom}\,(\pi_l(X), \mathbb{R}) = \mathcal{M}_X^l / \mathcal{M}_X^+ \wedge \mathcal{M}_X^+,$$

i. e., irreducible elements from $\mathcal{M}_X^l$ are in a one-to-one correspondence with homomorphisms from $\pi_l(X)$ to $\mathbb{R}$. Here we assume that there is the minimal model for X and this model is constructed only for simply connected and nilpotent non-simply connected spaces.

In the non-simply connected case we confine to a description of the minimal models for nilmanifolds which are quotients of nilpotent Lie groups G with respect to uniform lattices $\Gamma \subset G$.

Let G be a simply connected nilpotent Lie group, let $\mathcal{G}$ be its Lie algebra, and let $\mathcal{G}^*$ be the algebra dual to $\mathcal{G}$, i. e., the algebra of linear functions $f : \mathcal{G} \to \mathbb{R}$.

Let $e^1, \ldots, e^n$ be a basis for $\mathcal{G}$ and let $\omega_1, \ldots, \omega_n$ be the dual basis for $\mathcal{G}^*$: $\omega_i(e^j) = \delta_i^j$. The structure constants c_k^{ij} are defined as follows:

$$[e^i, e^j] = \sum_{k=1}^{n} c_k^{ij} e^k$$

and, by the Mal'tsev theorem [15], the group G contains a discrete subgroup Γ with the compact quotient space G/Γ if and only if the structure constants are rational in a certain basis $e^1, \ldots, e^n$.

The group G is diffeomorphic to $\mathbb{R}^n$ and its elements may be identified with vectors $\xi = \sum_i \xi_i e^i \in \mathbb{R}^n$. The multiplication is defined by the Campbell–Hausdorff formula

$$\left(\sum_i \xi_i e^i\right) \times \left(\sum_j \eta_j e^j\right) = \sum_k P_k(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n) e^k$$

and it follows from the nilpotence of $\mathcal{G}$ that $P_1, \ldots, P_n$ are polynomials. If c_k^{ij} are rational, then $P_1, \ldots, P_n$ are polynomials with rational coefficients and the subgroup generated by the vectors $e^1, \ldots, e^n$ is a (uniform) lattice, i. e., it is a discrete subgroup with the compact quotient G/Γ.

Let $\mathcal{M}_X$ be the minimal model for $X = G/\Gamma$ where Γ is a lattice in G with the compact quotient G/Γ. Then $\mathcal{M}_X^+$ is generated by elements $a_1, \ldots, a_n$ where

$$\deg a_1 = \ldots = \deg a_n = 1,$$
$$d\, a_k = \sum_{i,j} c_k^{ij}\, a_i \wedge a_j.$$

The mapping $\mathcal{M}_X \to \mathcal{E}^*(X)$ has the form

$$a_k \to \omega_k$$

where ω_k is the left-invariant form on X generated by the element $\omega_k \in \mathcal{G}^*$. By the Nomizu theorem [16], this mapping induces the isomorphism

$$H^*(\mathcal{M}_X) \simeq H^*(X)$$

and the minimal model for X coincides with the complex generated by left-invariant 1-forms.

Notice that the minimal models for nilmanifolds are not simply connected.

Example. Let G be the group of all matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R}$$

and let $\Gamma = G_{\mathbb{Z}}$ be a subgroup of all elements with $x, y, z \in \mathbb{Z}$. Take the following basis for leftinvariant 1-forms:

$$\omega_1 = d\, x, \quad \omega_2 = d\, y, \quad \omega_3 = x\, dy - dz.$$

The minimal model $\mathcal{M}_X$ for the Heisenberg nilmanifold $X = G/G_{\mathbb{Z}}$ is generated by elements a_1, a_2, a_3 of degree 1 where

$$d\,a_1 = d\,a_2 = 0, \quad d\,a_3 = a_1 \wedge a_2.$$

2.2. Formality.

To every minimal algebra $(\mathcal{M}, d)$ there corresponds its cohomology ring $\mathcal{H}^*(\mathcal{M})$, which we consider as a differential graded algebra with the zero differential: $(\mathcal{H}^*(\mathcal{M}), 0)$.

If there is a homomorphism

$$f : (\mathcal{M}, d) \to (\mathcal{H}^*(\mathcal{M}), 0)$$

inducing an isomorphism of the cohomology rings

$$f^* : \mathcal{H}^*(\mathcal{M}) \simeq \mathcal{H}^*(\mathcal{M}),$$

then the minimal algebra $\mathcal{M}$ is called *formal*.

An existence of such an isomorphism means that the algebra $(\mathcal{M}, d)$ is the minimal model for its cohomology ring. The construction of the minimal model for any algebra $(\mathcal{A}, d_{\mathcal{A}})$ is done effectively by induction and formality means that $\mathcal{M}$ is reconstructed from $\mathcal{H}^*(\mathcal{M})$.

A differential graded algebra $(\mathcal{A}, d_{\mathcal{A}})$ is called *formal* if its minimal model $\mathcal{M}(\mathcal{A})$ is formal. If that holds the minimal model for $(\mathcal{A}, d_{\mathcal{A}})$ is reconstructed from $\mathcal{H}^*(\mathcal{A}, d_{\mathcal{A}}) = \mathcal{H}^*(\mathcal{M}(\mathcal{A}), d_{\mathcal{M}})$.

If the algebra $\mathcal{E}^*(X)$ of smooth forms on a smooth manifold X is formal, then the space X is called formal itself.

A notion of formality is generalized for compact simply connected and nilpotent polyhedra. In this case the algebra $\mathcal{E}^*(X)$ is replaced by the algebra of piecewise polynomial forms [14, 17].

Examples of formal spaces are compact symmetric spaces and compact Kähler manifolds [7], and also simply connected compact manifolds (without boundary) of dimension ≤ 6 [11, 12].

An example of a nonformal algebra is the minimal model for the three-dimensional Heisenberg nilmanifold exposed in §2.1. Indeed, if there is a homomorphism

$$f : (\mathcal{M}_X, d) \to (H^*(X), 0)$$

inducing the isomorphism of the cohomology rings, then

$$f(a_1) \neq 0, \quad f(a_3) = 0$$

and, therefore, $f(a_1 \wedge a_3) = 0$. But the element $a_1 \wedge a_3$ realizes a nontrivial cohomology class.

The formality criterion for minimal algebras was given in [7]. To explain it notice that a minimal algebra is a tensor product of exterior algebras $\Lambda(V_l)$ where V_l are the spaces generated by l-dimensional generators:

$$\mathcal{M} = \otimes_{l \geq 0} \Lambda(V_l).$$

In each space V_l let us take a subspace C_l of all closed elements.

Theorem 3 ([7]) *A minimal algebra $(\mathcal{M}, d)$ is formal if and only if in each V_l there is a complement N_l to C_l:*

$$V_l = C_l \oplus N_l$$

such that any closed element from the ideal $I_N = I(\oplus N_l)$ generated by $N_1, N_2, \ldots$ is exact.

It is not always convenient to use this criterion and hence the following test is used quite often: to find in $H^*(\mathcal{M}, d)$ nontrivial classes represented by Massey products. If there are such classes, then the algebra $(\mathcal{M}, d)$ is not formal. The converse statement is not true as is shown by an example from [18]. We shall discuss relations of Massey products to formality later after defining these operations.

2.3. The Maurer–Cartan equations in differential algebras and generalized Massey products.

Let $(\mathcal{A}, d)$ be a differential graded algebra over $k(= \mathbb{R})$ with an augmentation. Let $M(\mathcal{A})$ be a set of all upper triangular half-infinite matrices with entries from $\mathcal{A}$, zeroes at the diagonal and finitely many nonzero entries.

Hence, for any matrix

$$A = (a_{ij})_{i,j \geq 1} \in M(\mathcal{A})$$

we have

$$a_{ij} \in \mathcal{A}, \quad a_{ij} = 0 \text{ for } \quad j \leq i \text{ and } \quad i, j \geq n+1$$

for some n. Notice that the condition

$$a_{ij} = 0 \text{ for } \quad i, j \geq n+1$$

distinguishes in $M(\mathcal{A})$ a subset $M_n(\mathcal{A})$ consisting of all $(n \times n)$-matrices with entries from $\mathcal{A}$. We have a natural filtration

$$M_1(\mathcal{A}) \subset M_2(\mathcal{A}) \subset \ldots \subset M_n(\mathcal{A}) \subset \ldots M(\mathcal{A}).$$

There are the operations of addition and multiplication endowing $M(\mathcal{A})$ with an algebra structure over $k = \mathbb{R}$. The subsets $M_n(\mathcal{A})$, $n \geq 1$, are subalgebras of $M(\mathcal{A})$.

Let us define a differential d on $M(\mathcal{A})$ as

$$d\,A = (d\,a_{ij})_{i,j\geq 1}. \tag{2}$$

The mapping acting on homogeneous elements $a \in A^k$ as follows

$$a \to \bar{a} = (-1)^k a$$

generates an automorphism of order 2 (an involution) of $\mathcal{A}$. This automorphism is extended to an automorphism of $M(\mathcal{A})$ as $\bar{A} = (\bar{a}_{ij})_{i,j\geq 1}$ and the differential (2) satisfies the generalized Leibniz rule

$$d\,(AB) = (d\,A)B + \bar{A}(d\,B).$$

The algebra M is naturally bigraded:

$$M = \sum_{p\geq 1,\, k\geq 0} M^{p,k}.$$

Indeed, let $(a)_{ij}$ be a matrix with one and only one nonzero entry such that it is the (i,j) entry which equals $a \in \mathcal{A}$. Then define $M^{p,k}$ as a subspace of M generated by

$$(a)_{i,(i+p)}, \qquad a \in \mathcal{A}^k, \quad i = 1, 2, \ldots.$$

The multiplication acts as follows

$$M^{p,k} \otimes M^{q,l} \stackrel{\cdot}{\to} M^{p+q,k+l}$$

and is not graded commutative.

Another structure on M is the bigraded Lie brackets

$$M^{p,k} \otimes M^{q,l} \stackrel{[\,,]}{\longrightarrow} M^{p+q,k+l}$$

defined on homogeneous elements as

$$[A, B] = A \cdot B - (-1)^{kl} B \cdot A, \quad A \in M^{p,k}, \quad B \in M^{q,l}.$$

It is straightforwardly checked that these brackets endow M with a structure of a differential Lie superalgebra with standard properties:

1. $[A, B] = -(-1)^{kl}[B, A]$;
2. $d\,[A, B] = [d\,A, B] + [\bar{A}, d\,B]$;

3. $(-1)^{km}[[A,B],C]+(-1)^{lk}[[B,C],A]+(-1)^{lm}[[C,A],B]=0$ for $A\in M^{p,k}$, $B\in M^{q,l}$, $C\in M^{r,m}$.

Now for any (not only for homogeneous) matrix $A\in M$ define its *kernel*, $\operatorname{Ker} A$, as a certain two-sided $\mathcal{A}$-submodule of M. By the definition, $\operatorname{Ker} A$ is linearly generated as an $\mathcal{A}$-module by the matrices $(1)_{ij}$ such that $A\cdot(1)_{ij}=(1)_{ij}\cdot A=0$. This implies that

$$AB=BA=0$$

for any matrix $B\in\operatorname{Ker} A$.

Notice that generically $\operatorname{Ker} A$ is not an ideal with respect to either the multiplication, either the Lie brackets. But there is a submodule $\operatorname{Ker}' A$ of $\operatorname{Ker} A$ which is a two-sided ideal with respect to both these operations. Let us describe it. Let m be the number of the first nonzero column in A and let k be the number of the last nonzero row in A. Then define $\operatorname{Ker}' A$ as a $\mathcal{A}$-submodule of M generated by the matrices $(1)_{ij}$ with $i<m$ and $j>k$.

Let us put the Maurer–Cartan operator $\mu: M\to M$ equal to

$$\mu(A)=d\,A-\bar{A}\cdot A.$$

Definition 1 *A matrix $A\in M$ is called the matrix of a formal connection if it satisfies the Maurer–Cartan equation in $\mathcal{A}$:*

$$d\,A-\bar{A}\cdot A\equiv 0 \mod \operatorname{Ker} A. \tag{3}$$

In other words, A is a formal connection if $\mu(A)\in\operatorname{Ker} A$.

Definition 2 *Let A be a formal connection. Then $\mu(A)$ is called the curvature matrix of $\mathcal{A}$.*

Example. Let A be an upper triangular $(n+1)\times(n+1)$-matrix with zeroes on the diagonal of the form

$$A=\begin{pmatrix} 0 & a_1 & * & \dots & * & * \\ 0 & 0 & a_2 & \dots & * & * \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 0 & a_n \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{4}$$

where $a_1,\dots,a_n$ do not vanish. Then $\operatorname{Ker} A\cap M_{n+1}(\mathcal{A})$ is generated over k by the matrix with one and only one nonzero entry, which is the $(n+1,n+1)$

entry and equals 1. The condition (3) takes the form

$$d\,A - \bar{A}\cdot A = \begin{pmatrix} 0 & \dots & 0 & \tau \\ 0 & \dots & 0 & 0 \\ & \dots & & \\ 0 & \dots & 0 & 0 \end{pmatrix}. \tag{5}$$

Lemma 1 (*The Bianci identity*) *For any matrix $A \in M$ we have*

$$d\,\mu(A) = \overline{\mu(A)}\cdot A - A\cdot\mu(A).$$

Proof. It follows from the definition of μ that

$$d\,\mu(A) = -d\,(\bar{A}\cdot A) = -d\,\bar{A}\cdot A - A\cdot d\,A = \overline{d\,A}\cdot A - A\cdot d\,A =$$

$$= \overline{(\mu(A) + \bar{A}\cdot A)}\cdot A - A(\mu(A) + \bar{A}\cdot A) =$$

$$= \overline{\mu(A)}\cdot A + A\cdot\bar{A}\cdot A - A\cdot\mu A - A\cdot\bar{A}\cdot A =$$

$$= \overline{\mu(A)}\cdot A - A\cdot\mu(A).$$

Lemma is proved.

Corollary 1 *The curvature matrix of a formal connection is closed.*

Proof. If A is a formal connection, then $\mu(A) \in \operatorname{Ker} A$, and, therefore, $\overline{\mu(A)} \in \operatorname{Ker} A$. Now the Bianci identity implies that $d\,\mu(A) = 0$.

Let us denote the cohomology class of a closed element $a \in \mathcal{A}$, $d\,a = 0$, by $[a] \in H^*(\mathcal{A})$. If $A \in M(\mathcal{A})$ is a closed matrix, $d\,A = 0$, then we denote by $[A] = ([a_{ij}])_{i,j\geq 1} \in M(H^*(\mathcal{A}))$ the corresponding matrix whose entries are the cohomology classes of a_{ij}.

Corollary 1 guarantees the correctedness of the following definition.

Definition 3 [1] *Let A be a solution to the Maurer–Cartan equation. Then the entries of the matrix $[\mu(A)]$ are called generalized Massey products.*

In other words, the generalized Massey products are the cohomology classes of the curvature matrices of formal connections. Speaking in the differential-geometric language, we say that the generalized Massey products

[1] Notice that the Massey products are defined in more general situation when $\mathcal{A}$ is an algebra over a commutative ring (for instance, $\mathbb{Z}$) and the multiplication is associative but not necessary graded commutative ($a\wedge b = (-1)^{pq} b\wedge a$). These are, for instance, the classical Massey products in $H^*(X,\mathbb{Z})$.

measure the deviations of connections from flat ones. If all generalized Massey products vanish, then the connection is flat (see §2.4).

Remarks.

1) The classical n-tuple Massey products (see §2.3) differ by sign from the corresponding generalized Massey products. We changed the sign in Definition 1 to stress that the constructions of abstract homological algebra and differential geometry are parallel.

2) It is shown that if $\alpha \in H^*(\mathcal{A})$ is a generalized Massey product, then $t\alpha$, where $t \in k$, is also a generalized Massey product. By the equality $[\mu(A)] = [d\,A - \bar{A} \cdot A] = -[\bar{A} \cdot A]$, we conclude that the generalized Massey products are the entries of the matrix $[\bar{A} \cdot A]$.

From two equivalent notions which are "a solution to the Maurer–Cartan equation" and "a formal connection" we shall use in the sequel only the first one.

Let $\varepsilon : \mathcal{A} \to k$ be an augmentation and let $I = \operatorname{Ker} \varepsilon$ be the augmentation ideal in $\mathcal{A}$. For a connected algebra $\mathcal{A}$, i. e. $\mathcal{A}^0 = k$, the augmentation ideal $I = \mathcal{A}^+$ is an ideal of elements of positive degree and this takes place, for instance, for a minimal algebra $\mathcal{A}$. By natural reasons of homological algebra, which are also used at the definition of the reduced bar-construction, we call a solution to the Maurer–Cartan equation *nontrivial* if it belongs to the ideal $M(I)$. Otherwise it is called trivial. In the sequel we shall consider only nontrivial solutions.

Examples.

1) Solutions of the Heisenberg type and the ideal $H^+(\mathcal{A}) \cdot H^+(\mathcal{A})$.

Let us consider a matrix

$$A = \begin{pmatrix} 0 & \bar{a}_1 & \bar{a}_2 & \dots & \bar{a}_l & c \\ 0 & 0 & & & 0 & b_1 \\ & & \ddots & & & b_2 \\ & & & \ddots & & \vdots \\ 0 & & \dots & & 0 & b_l \\ 0 & & \dots & & 0 & 0 \end{pmatrix} \tag{6}$$

where $a_i, b_j, c \in I$, $a_{ij} = 0$, $i, j > l + 1$. We say that this matrix is of the Heisenberg type keeping in mind an analogy with elements of the Lie algebra of the generalized Heisenberg group. Let us try to find a solution of the form (6) to the Maurer–Cartan equation. It follows from the definition that $d\,a_i = d\,b_j = 0$ and if α_i, β_j are the corresponding cohomology classes,

then the generalized Massey product defined by A is

$$[\mu(A)] = -\sum_{i=1}^{n} \alpha_i \beta_i \in H^+(\mathcal{A}) \cdot H^+(\mathcal{A}).$$

Hence we see that any element of the ideal $H^+(\mathcal{A}) \cdot H^+(\mathcal{A})$ is represented by a Massey product of the Heisenberg type. In the classical interpretation (see below) this is a double matrix product

$$\langle\, (\bar{\alpha}_1, \ldots, \bar{\alpha}_n), \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \rangle.$$

Definition 4 *Generalized Massey products belonging to $H^+(\mathcal{A}) \cdot H^+(\mathcal{A})$ are called completely reducible. A generalized Massey product is called completely irreducible if it has a nontrivial image under the projection*

$$H^*(\mathcal{A}) \to H^*(\mathcal{A})/H^+(\mathcal{A}) \cdot H^+(\mathcal{A}).$$

Remark. We shall see below that there are relations between Massey products which sometimes enables us to reduce complex products to more simple ones and, in particular, to ordinary products. As well as the notion of nontriviality of Massey products, which will be introduced below, the complete irreducibility characterizes a "nonsimplifiable" property of Massey products.

2) Triple (ordinary) Massey products.

Let α, β, and γ be the cohomology classes of closed elements $a \in \mathcal{A}^p$, $b \in \mathcal{A}^q$, and $c \in \mathcal{A}^r$. The triple Massey product $\langle \alpha, \beta, \gamma \rangle$ is defined if the Maurer–Cartan equation is solvable for

$$A = \begin{pmatrix} 0 & a & f & h \\ 0 & 0 & b & g \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This equation is equivalent to

$$d f = (-1)^p\, a \wedge b, \quad d g = (-1)^q\, b \wedge c \tag{7}$$

and that implies that the product $\langle \alpha, \beta, \gamma \rangle$ is defined if and only if

$$\alpha \cup \beta = \beta \cup \gamma = 0 \ \text{ in } \ H^*(\mathcal{A}).$$

The matrix $\mu(A)$ has the form (5) for $n = 3$ and defines the Massey product $[\mu(A)]$ which equals

$$\langle \alpha, \beta, \gamma \rangle = [\tau] = [(-1)^{p+1} a \wedge g + (-1)^{p+q} f \wedge c].$$

Since f and g are defined by (7) up to closed elements from $\mathcal{A}^{p+q}$ and $\mathcal{A}^{q+r}$, the triple Massey product $\langle \alpha, \beta, \gamma \rangle$ is defined modulo $\alpha \cdot H^{q+r}(\mathcal{A}) + \gamma \cdot H^{p+q}(\mathcal{A})$.

3) n-tuple ordinary Massey products [19].

Given $\alpha_1, \ldots, \alpha_n \in H^*(\mathcal{A})$, the n-tuple Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is defined if there are their representatives $a_1, \ldots, a_n$ such that the Maurer–Cartan equation is solvable for A of the form (4). In this event $\mu(A)$ takes the form (5) and the n-tuple (ordinary) Massey product is $\langle \alpha_1, \ldots, \alpha_n \rangle = [\tau]$. It is evident that for $n = 3$ we obtain a triple Massey product, which was already introduced and discussed in detail with explaining the type of multivaluedness.

Initial data.

Let A be a solution to the Maurer–Cartan equation. By the definition, the first diagonal of A consist only of closed elements, i.e., $d\, a_{i,i+1} = 0, \quad i = 1, 2, \ldots$. Other diagonals may also contain closed elements. This happens, for instance, for matrix Massey products which we shall discuss below. Consider the classes of all closed elements of A or a part of them and call them the initial data of the corresponding Maurer–Cartan problem.

The following statement, which is well-known for classical n-tuple ordinary or Massey products, shows that the set of the cohomology classes $[\mu(A)]$ of different solutions with the given initial data depends only on the initial data.

Proposition 1 *Let A be a solution to the Maurer–Cartan equation and let $\mathcal{K} = \operatorname{Ker} A$. Any replacement of the entry a_{ij} of the matrix A by $a_{ij} + d\, b$ is completed to a replacement of A by A' such that*

1. $A' = A + (d\, b)_{ij} \mod M^{j-i+1,*}$;
2. *A' is a solution to the Maurer–Cartan equation* $\mod \mathcal{K}$;
3. $[\mu(A')] = [\mu(A)]$.

Proof. A straightforward computation shows that the conditions 1 and 2 are satisfied by

$$A' = A + (d\, b)_{ij} + A \cdot (b)_{ij} - (\bar{b})_{ij} \cdot A.$$

It is also computed that

$$\mu(A') = \mu(A) + d\,((A \cdot (b)_{ij} - (\bar{b})_{ij} \cdot A) \cap \mathcal{K})$$

and, therefore, the condition 3 also holds for this choice of A'. This proves the proposition.

Naturality of generalized Massey products.

Let $f : \mathcal{A} \to \mathcal{B}$ be a homomorphism of differential graded algebras. It induces a mapping $\widehat{f} : M(\mathcal{A}) \to M(\mathcal{B})$ by the formula

$$\widehat{f}((a_{ij})_{i,j\geq 1}) = (f(a_{ij}))_{i,j\geq 1}.$$

It is clear that for any matrix $A \in M(\mathcal{A})$ we have

$$\widehat{f}(\operatorname{Ker} A) \subseteq \operatorname{Ker}(\widehat{f}(A)).$$

Therefore, any solution A to the Maurer–Cartan equation in $\mathcal{A}$ is mapped into a solution $\widehat{f}(A)$ of the Maurer–Cartan equation in $\mathcal{B}$. Hence we obtain a mapping of generalized Massey products:

$$f^*([\mu(A)]) = [\mu(\widehat{f}(A))]. \tag{8}$$

Standard reasonings of homological algebra prove the following statement.

Proposition 2 *If f is a quasiisomorphism of differential graded algebras, then the mapping* (8) *is one-to-one on generalized Massey products.*

For matrix Massey products this statement is proved in [20] (Theorem 1.5).

This implies that for the minimal model $\mathcal{M} = \mathcal{M}(\mathcal{A})$ for $\mathcal{A}$ the generalized Massey products in $H^*(\mathcal{A})$ may be computed from the minimal model.

2.4. The classical operations: n-tuple ordinary and matrix Massey products.

In this subsection we show how the classical Massey products introduced in the 60s in [19] and [20] fit into the general picture. It needs to mention that the relation between Massey products and the Maurer–Cartan equation was first noticed by May [20] but this analogy was not developed.

Let $\alpha_1, \dots, \alpha_n$, $\alpha_i \in H^{p_i}(\mathcal{A})$, $i = 1, \dots, n$, be cohomology classes and let $a_i \in \alpha_i$ be cocycles representing them in $\mathcal{A}$.

Definition 5 *A set $A = (a(i,j))$, $1 \leq i \leq j \leq n$, $(i,j) \neq (1,n)$, consisting of some elements of $\mathcal{A}$ is called a defining system for $\langle a_1, \dots, a_n \rangle$, if*

1. $a(i,i) = a_i$, $i = 1, \dots, n$;
2. $a(i,j) \in \mathcal{A}^{p(i,j)+1}$, $p(i,j) = \sum_{r=i}^{j}(p_r - 1)$;
3. $da(i,j) = \sum_{r=i}^{j-1} \bar{a}(i,r) \cdot a(r+1,j)$.

If these conditions hold, then

4. *the* $(p(1,n)+2)$*-dimensional cocycle*

$$c(A) = \sum_{r=1}^{n-1} \bar{a}(1,r)a(r+1,n)$$

is called the cocycle of a defining system A*.*

Definition 6 *An* n*-tuple product* $\langle a_1, \dots, a_n \rangle$ *is defined if there is at least one defining system for it. If this product is defined, then* $\langle \alpha_1, \dots, \alpha_n \rangle$ *consists of the cohomology classes* $[c(A)] \in H^{p(1,k)+2}(\mathcal{A})$ *of all defining systems* A *for* $\langle a_1, \dots, a_n \rangle$*.*

In these form these definitions appeared in [19]. We show they relate to the definitions given in the preceding subsection.

Let A be a defining system for $\langle a_1, \dots, a_n \rangle$. We write it as a matrix denoted by the same letter:

$$A = \begin{pmatrix} 0 & a(1,1) & a(1,2) & \dots & a(1,n-1) & a(1,n) \\ 0 & 0 & a(2,2) & \dots & a(2,n-1) & a(2,n) \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & a(n-1,n-1) & a(n-1,n) \\ 0 & 0 & 0 & \dots & 0 & a(n,n) \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

where $a(1,n)$ is an arbitrary element of $\mathcal{A}^{p(1,n)+2}$. It is easily seen that the conditions 1–3 of Definition 5 are equivalent to a condition that A satisfies the Maurer–Cartan equation. In this event the matrix $-\mu(A)$ has a single nonzero entry, which equals $c(A)$. The converse is also true: if A is a solution to the Maurer–Cartan equation with the initial data $a_{i\,,i+1} = a_i$, $i = 1, 2, \dots, n$, then the entries of A give a defining system for $\langle a_1, \dots, a_n \rangle$.

Matrix Massey products were first introduced by May [20]. Since the matrix entries have to stay homogeneous, we need to define *multipliable* matrices.

Definition 7 *Let* $N(I)$ *be the set of all finite rectangular matrices whose entries are homogeneous elements from the augmentation ideal* I*. Two matrices* $X, Y \in N(I)$ *of sizes* $p \times q$ *and* $r \times s$ *are multipliable (in this order) if*

1. $q = r$*;*
2. $\deg x_{ij} + \deg y_{jk}$ *is independent of* j *for all* (i,k)*,* $1 \le i \le p$*,* $1 \le k \le s$*.*

To any matrix $X = (x_{ij})_{\substack{i=1\\j=1}}^{p\,q} \in N(I)$ there corresponds an integer matrix $\mathcal{D}(X) = (\deg x_{ij})$. If X and Y are multipliable, then

$$\mathcal{D}(X \cdot Y) = \mathcal{D}(X) * \mathcal{D}(Y) = (\deg x_{ij} + \deg y_{jk})_{ik}.$$

The rest part of the definition of matrix Massey products is completely analogous to the definition of ordinary products. Let V_1, V_2, ..., V_n be matrices from $N(H^+(\mathcal{A}))$ such that V_i and V_{i+1} are multipliable for $i = 1, \ldots, n-1$. Consider matrices $X_i \in N(I)$ consisting of cocycles representing matrices V_i, $i = 1, \ldots, n$, i. e.,

$$d\,X_i = 0 \qquad [X_i] = V_i, \quad i = 1, \ldots, n.$$

A sequence X_i, $i = 1, \ldots, n$, is multipliable by its construction and $\mathcal{D}(X_i) = \mathcal{D}(V_i)$, $i = 1, \ldots, n$. Let us denote by $K + m$ the matrix obtained from the integer matrix $K = (k_{ij})$ by adding to each its entry an integer number m: $K + m = (k_{ij} + m)$.

Definition 8 *A subset $A = (X(i,j))$, $1 \le i, j \le n$, $(i,j) \ne (1,n)$, of $N(I)$ is called a defining system for $\langle X_1, \ldots, X_n \rangle$ if*

1. $X(i,i) = X_i$, $i = 1, \ldots, n$;
2. $\mathcal{D}(X(i,j)) = (\mathcal{D}(X_i) - 1) * (\mathcal{D}(X_{i+1}) - 1) * \ldots * (\mathcal{D}(X_j) - 1) + 1$;
3. $d\,X(i,j) = \sum_{r=i}^{j-1} \bar{X}(i,r) \cdot X(r+1,j)$.

If these condition hold, then

4. *the matrix cocycle*

$$c(A) = \sum_{r=1}^{n-1} \bar{X}(1,r) \cdot X(r+1,n)$$

of degree $\mathcal{D}(1,p) = \mathcal{D}(X_1) * \ldots * \mathcal{D}(X_n) - n + 2$ is called the cocycle of a defining system A.

Definition 9 *An n-tuple matrix product $\langle X_1, \ldots, X_n \rangle$ is defined if there is at least one defining system for it. If this product is defined, then $\langle V_1, \ldots, V_n \rangle$ consists of the matrix cohomology classes $[c(A)]$ of all defining systems A for $\langle X_1, \ldots, X_n \rangle$.*

As in the case of ordinary products to every defining system A there corre-

sponds a block matrix

$$A = \begin{pmatrix} 0 & X(1,1) & X(1,2) & \dots & X(1,n-1) & X(1,n) \\ 0 & 0 & X(2,2) & \dots & X(2,n-1) & X(2,n) \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & X(n-1,n-1) & X(n-1,n) \\ 0 & 0 & 0 & \dots & 0 & X(n,n) \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

where $X(i,j)$ are blocks described by Definition 8 and $X(1,n)$ is an arbitrary element of $\operatorname{Ker} A$. The matrix A satisfies the Maurer–Cartan equation and $-\mu(A)$ has one and only one nonzero block entry. It is the $(1,n)$ entry, which equals $c(A) - d\,X(1,n)$:

$$-[\mu(A)] = [c(A)].$$

The inverse is also valid: if A is a solution to the Maurer–Cartan equation whose initial data are block matrices realizing the matrix cohomology classes $V_1, \dots, V_n$, then $-[\mu(A)] \in \langle V_1, \dots, V_n \rangle$.

An analogue of Proposition 1 for n-tuple products was proved in [19] for ordinary products and in [20] for matrix products.

Proposition 3 *Let an n-tuple matrix Massey product $\langle V_1, \dots, V_n \rangle$ be defined. Then its values depends only on matrix cohomology classes $V_1, \dots, V_n$.*

Indeterminacy and nontriviality of Massey products.

Matrix Massey products are not a formal generalization of ordinary n-tuple operations and appear quite naturally, for instance, at describing the indeterminacy of an n-tuple product. This effect is almost nonvisible for triple products $\langle v_1, v_2, v_3 \rangle$ but it is already spectacular for quadruple products.

Definition 10 *Let an n-tuple matrix Massey product $\langle V_1, \dots, V_n \rangle$ is defined. Then it is called trivial if $0 \in \langle V_1, \dots, V_n \rangle$.*

In accord with Definition 4 we give the following

Definition 11 *An n-tuple (matrix) Massey product $\langle V_1, V_2, \dots, V_n \rangle$ is called completely reducible if*

$$\langle V_1, \dots, V_n \rangle \subset H^+(\mathcal{A}) \cdot H^+(\mathcal{A}),$$

otherwise it is called irreducible. Such a product is called strictly irreducible if

$$\langle V_1, \dots, V_n \rangle \cap H^+(\mathcal{A}) \cdot H^+(\mathcal{A}) = \emptyset.$$

Now assume that $\langle V_1, \ldots, V_n\rangle$ and $\langle W_1, \ldots, W_n\rangle$ are defined and have equal multidegrees, i. e., $\mathcal{D}(V_1, \ldots, V_n) = \mathcal{D}(W_1, \ldots, W_n)$. Put

$$\langle V_1, \ldots, V_n\rangle + \langle W_1, \ldots, W_n\rangle =$$

$$= \{x + y;\ x \in \langle V_1, \ldots, V_n\rangle,\ y \in \langle W_1, \ldots, W_n\rangle\},$$

$$\lambda\langle V_1, \ldots, V_n\rangle = \{\lambda x;\ x \in \langle V_1, \ldots, V_n\rangle\}.$$

Definition 12 *The set*

$$\mathrm{In}\,\langle V_1, \ldots, V_n\rangle = \{x - y;\ x, y \in \langle V_1, \ldots, V_n\rangle\}$$

is called the indeterminacy of the Massey product $\langle V_1, \ldots, V_n\rangle$.

Hence, if $\langle V_1, \ldots, V_n\rangle$ is trivial, then

$$\langle V_1, \ldots, V_n\rangle \subset \mathrm{In}\,\langle V_1, \ldots, V_n\rangle.$$

The following statement proved in [20] describes the indeterminacy of n-tuple products in terms of $(n-1)$-tuple products.

Proposition 4 *Let a product* $\langle V_1, \ldots, V_n\rangle$ *be defined. Then*

$$\mathrm{In}\,\langle V_1, \ldots, V_n\rangle \subset \cup_{(X_1,\ldots,X_{n-1})}\langle W_1, \ldots, W_{n-1}\rangle, \quad \textit{where}$$

$$W_1 = (V_1 \quad X_1), \quad W_k = \begin{pmatrix} V_k & X_k \\ 0 & V_{k+1} \end{pmatrix}, \quad 2 \le k \le n-2,$$

$$W_{n-1} = \begin{pmatrix} X_{n-1} \\ V_n \end{pmatrix}.$$

The union is taken over all $(n-1)$*-tuples* $X_1, \ldots, X_{n-1} \in N(H^*(\mathcal{A}))$ *such that* $\mathcal{D}(X_k) = \mathcal{D}(V_k, V_{k+1}) - 1$. *Moreover, for* $n = 3$ *we have*

$$\mathrm{In}\,\langle V_1, V_2, V_3\rangle = \cup_{(X_1,X_2)}\langle W_1, W_2\rangle = \cup_{(X_1,X_2)}(\bar{V}_1 \cdot X_2 + \bar{X}_1 \cdot V_3).$$

Remark. The properties of Massey products to be nontrivial and to be irreducible are independent. A Massey product may be completely reducible but not trivial (see §2.3) and if $n > 3$, then n-tuple Massey products may be trivial but not completely irreducible. If $n = 3$, then triviality implies reducibility.

Strictly defined Massey products.

To obtain all elements of an n-tuple ordinary or matrix product it needs to consider all defining systems for this product or, which is equivalent, to consider all solutions to the Maurer–Cartan equation with given initial data. Defining systems are inductively constructed and such a process leads to multivaluedness of the product. Moreover it a situation may appear when partially constructed defining system can not be completed. In difference with a multivaluedness the last effect is not well-known because it takes place not for triple but for n-tuple products with $n \geq 4$. A potential impossibility of completing partially constructed defining systems gives rise to the following definition, which first appeared in [21] (see also [20]).

Definition 13 *Let a matrix product $\langle V_1, \ldots, V_n \rangle$ be defined. Then it is called strictly defined if any product*

$$\langle V_i, \ldots, V_j \rangle, \quad 1 \leq j - i \leq n - 2$$

contains only the zero matrix.

It is clear that triple products are strictly defined. For many reasons strictly defined products are more useful for applications. The proof of the following simple lemma may be found in [20].

Lemma 2 *A matrix product $\langle V_1, \ldots, V_n \rangle$ is strictly defined if and only if any its partially determined defining system $\{X(i,j);\ j - i \leq k\}$, $1 \leq k \leq n-2$, can be extended to a defining system A for $\langle V_1, \ldots, V_n \rangle$.*

2.5. Massey products as obstructions to formality.

The most important for this paper property of Massey operations is given by

Theorem 4 *Let a differential graded algebra $\mathcal{A}$ is formal. Then*

a) generalized Massey products in $H^*(\mathcal{A})$ are completely reducible;

b) n-tuple Massey products in $H^*(\mathcal{A})$ are trivial for $n \geq 3$.

Proof. Let $\mathcal{M} = (\mathcal{M}, d)$ be the minimal model for $\mathcal{A}$ and let $H^* = H^*(\mathcal{A}, d)$. By the formality of A, there is a quasiisomorphism

$$h : (\mathcal{M}, d) \to (H^*, 0). \tag{9}$$

By Proposition 2, we can compute the generalized Massey products in $H^*(\mathcal{A})$ only from $(\mathcal{M}, d)$.

a) Let A be a solution to the Maurer–Cartan equation and $[\mu(A)]$ be the corresponding Massey product. By (8), we obtain

$$h^*([\mu(A)]) = [\mu(\widehat{h}(A))] = -[\widehat{h}(\bar{A}) \cdot \widehat{h}(A)] \in M(H^+ \cdot H^+).$$

Since h^* is an isomorphism, we have $[\mu(A)] \in \mathcal{M}(H^+ \cdot H^+)$ which proves a).

b) Let $\langle V_1, \ldots, V_n \rangle$ be an n-tuple matrix Massey product. It follows from Proposition 2 that

$$\langle V_1, \ldots, V_n \rangle = \langle V_1, \ldots, V_n \rangle_H \tag{10}$$

where the symbol $\langle\ \rangle_H$ denotes a Massey product in the differential graded algebra $(H^*, 0)$. Since $d \equiv 0$, , it is clear that the family of matrices $A = (X(i,i) = V_i;\ X(i,j) = 0;\ 1 \le i,j \le n,\ j - i > 0 : (i,j) \ne (1,n))$ is a defining system for $\langle V_1, \ldots, V_n \rangle_H$. For $n \ge 3$ we have $c(A) = 0$ and, therefore, $0 \in \langle V_1, \ldots, V_n \rangle_H$. The equality (10) proves b) and hence finishes the proof of the theorem.

Corollary 2 *If a differential graded algebra $(\mathcal{A}, d)$ has irreducible or non-trivial matrix Massey products, then the algebra is nonformal.*

This reasoning enables us to construct obstructions to formality.

Example. Let us consider the generalized Heisenberg group, which is the group of all matrices of the form

$$\begin{pmatrix} 1 & x_1 & x_2 & \ldots & x_n & z \\ 0 & 1 & 0 & \ldots & 0 & y_1 \\ & & 1 & & & y_2 \\ \vdots & & & \ddots & & \vdots \\ & & & & 1 & y_n \\ 0 & & \ldots & & 0 & 1 \end{pmatrix}$$

where $x_i, y_j, z \in \mathbb{R}$, $1 \le i,j \le n$. Take for a uniform lattice the subgroup of integer matrices and denote the corresponding nilmanifold by X_n. The minimal model $\mathcal{M}_n = \mathcal{M}(X_n)$ for X_n coincides with the complex generated by left-invariant 1-forms [16]. There is a basis for such forms, which in the coordinates $(\bar{x}, \bar{y}, z)$ is written as

$$\alpha_i^+ = d\,x_i, \quad \alpha_i^- = d\,y_i, \quad 1 \le i \le n; \quad \beta = -d\,z + \sum_{i=1}^{n} x_i\, d\,y_i.$$

Therefore, we have

$$\mathcal{M}_n = \Lambda(\alpha_i^{\pm},\ 1 \le i \le n,\ \beta;\ d\,\alpha_i^{\pm} = 0,\ d\,\beta = \alpha_1^+ \wedge \alpha_1^- + \ldots + \alpha_n^+ \wedge \alpha_n^-).$$

Using this representation one can compute $H^*(X_n, \mathbb{R})$, but we only notice that the first nontrivial Massey products appear at the dimension $n+1$ and are of the form

$$[\beta \wedge \alpha_1^{\varepsilon_1} \wedge \alpha_2^{\varepsilon_2} \wedge \ldots \wedge \alpha_n^{\varepsilon_n}] \tag{11}$$

where $\varepsilon_i \in \{\pm 1\}$, $i = 1, \ldots, n$. The elements (11) do not lie in $H^+ \cdot H^+$ and it is clear that

$$[\beta \wedge \alpha_1^{\varepsilon_1} \wedge \ldots \wedge \alpha_n^{\varepsilon_n}] =$$

$$= \langle (\varepsilon_1 \alpha_1^{-\varepsilon_1}, \varepsilon_2 \alpha_2^{-\varepsilon_2}, \ldots, \varepsilon_n \alpha_n^{-\varepsilon_n}), \begin{pmatrix} \alpha_1^{\varepsilon_1} \\ \alpha_2^{\varepsilon_2} \\ \ldots \\ \alpha_n^{\varepsilon_n} \end{pmatrix}, \alpha_1^{\varepsilon_1} \wedge \alpha_2^{\varepsilon_2} \wedge \ldots \wedge \alpha_3^{\varepsilon_3} \rangle.$$

Hence, the cohomology classes (11) are represented by nontrivial and, therefore, irreducible triple matrix Massey products.

2.6. Spheric cocycles and the suspension homomorphism.

Let X be a finite CW-complex. An element $q \in H^k(X, \mathbb{R})$ is called *spheric* if there is a mapping

$$f : S^k \to X \tag{12}$$

such that $f^*(q) \neq 0$.

Since $f^*(H^+ \cdot H^+) = 0$ for any mapping (12), the image of a spheric class under the projection

$$H^+ \xrightarrow{r} H^+/H^+ \cdot H^+.$$

does not equal zero. Let $h_* : \pi_*(X) \to H_*(X, \mathbb{R})$ be the real Hurewicz homomorphism. There is a commutative diagram

$$\begin{array}{ccccc} H^*(X, \mathbb{R}) & \simeq & \mathrm{Hom}_{\mathbb{R}}(H_*(X, \mathbb{R}), \mathbb{R}) & \xrightarrow{h^*} & \mathrm{Hom}_{\mathbb{R}}(\pi_*(X), \mathbb{R}) \\ & \searrow^{r} & & \swarrow_{\hat{h}^*} & \\ & & H^*/H^+ \cdot H^+ & & \end{array} \tag{13}$$

Take a subspace $S \subset H^*(X, \mathbb{R})$ complement to $\mathrm{Ker}\, h$ such that $r|_S$ is a monomorphism. Any such a subspace S is called a *subspace of spheric generators* in $H^*(X, \mathbb{R})$.

It appears a natural question: "When $H^*(X)$ is generated by spheric generators ?" It follows from the diagram (13) that that holds if and only if $\mathrm{rk}\, h_* = \dim(H^*/H^+ \cdot H^+)$. For fields of the zero characteristic this problem

is well studied. Different conditions, which are equivalent to the condition that the cohomology ring is generated by spheric generators, may be found in [18]. We shall mention only the following one.

Proposition 5 *If X is formal, then $H^*(X)$ is generated by spheric generators.*

The proof of this proposition is also given in [18]. Notice that the inverse to this statement is not true. From this proposition we derive the following obstruction to formality.

Corollary 3 *If*

$$\operatorname{rk} h_* < \dim(H^*/H^+ \cdot H^+)$$

where h_* is the Hurewicz homomorphism, then X is nonformal.

A subspace of spheric generators S and a subspace generated by all Massey products including ordinary ones are complementary subspaces of $H^*(X, \mathbb{R})$. To make this statement rigorous let us define the cohomology suspension homomorphism

$$\Omega^* : H^*(X, \mathbb{R}) \to H^{*-1}(\Omega X, \mathbb{R}) \tag{14}$$

where ΩX is the loop space of X. Denote by $[X, Y]$ the set of homotopy classes of mappings from X to Y and define (14) as follows:

$$H^n(X, \mathbb{R}) \simeq [X, K(\mathbb{R}, n)] \stackrel{\Omega}{\to} [\Omega X, \Omega K(\mathbb{R}, n)] = [\Omega X, K(\mathbb{R}, n-1)] \simeq$$

$$\simeq H^{n-1}(\Omega X, \mathbb{R}).$$

An alternative definition comes from considering the fibration $p : EX \stackrel{\Omega X}{\to} X$ and the exact cohomology sequence for $(EX, \Omega X)$. There are homomorphisms

$$H^{*-1}(\Omega X) \stackrel{\partial}{\to} H^*(EX, \Omega X) \stackrel{p^*}{\leftarrow} H^*(X, pt)$$

and, since EX is contractible, ∂ is an isomorphism. Hence, the homomorphism

$$\partial^{-1} \circ p^* : H^*(X, pt) \to H^{*-1}(\Omega X)$$

is correctly defined and $\Omega^* = \partial^{-1} \circ p^*$.

There is a commutative diagram

$$\begin{array}{ccc} H^*(X, \mathbb{R}) & \stackrel{\Omega^*}{\longrightarrow} & H^{*-1}(\Omega X, \mathbb{R}) \\ h^* \downarrow & & \downarrow h^* \\ \operatorname{Hom}_{\mathbb{Z}}(\pi_*(X), \mathbb{R}) & \stackrel{\delta^*}{\longrightarrow} & \operatorname{Hom}_{\mathbb{Z}}(\pi_{*-1}(\Omega X), \mathbb{R}) \end{array}$$

where δ^* is induced by the isomorphism $\partial^{-1} : \pi_{k-1}(\Omega X) \to \pi_k(X)$ from the exact homotopy sequence for the fibration $EX \stackrel{\Omega X}{\to} X$. We derive from this diagram that

$$\operatorname{Ker} h^* = \operatorname{Ker} \Omega^*.$$

Therefore, there is a decomposition

$$H^+(X, \mathbb{R}) = S \oplus \operatorname{Ker} \Omega^*.$$

The following statement by May is considered by specialists to be proved although the proof was never published.

Theorem 5 *The subspace* $\operatorname{Ker} \Omega^* \subset H^+(X, \mathbb{R})$ *is generated by all matrix Massey products including ordinary products.*

It implies the following useful

Corollary 4 *Given a space* X,

$$\operatorname{Ker} h_* < \dim(H^+ / H^+ \cdot H^+)$$

if and only if there are irreducible matrix Massey products in $H^*(X, \mathbb{R})$.

2.7. Massey products and the Eilenberg–Moore spectral sequence.

Here we expose in brief and in adapted for our study form the basic constructions related to the Eilenberg–Moore spectral sequence. At the end we shall show a relation between the cohomology suspension homomorphism and this sequence. Complete algebraic details of that may be found in the paper by Smith [22] (see also [18]).

Let I be the augmentation ideal of a differential graded algebra $(\mathcal{A}, d)$. The reduced bar-construction for $\mathcal{A}$ is defined as follows. Put

$$B\mathcal{A} = \sum_{n \geq 0} \otimes^n I$$

where the tensor product is considered over the ground field k. If $a_i \in I$ is a homogeneous element of degree p_i for $i = 1, \ldots, n$, then the product $a_1 \otimes a_2 \otimes \ldots \otimes a_n$ is denoted by $[a_1|a_2|\ldots|a_n]$. The module $B\mathcal{A}$ has a natural bigrading

$$\operatorname{bideg} [a_1|a_2|\ldots|a_n] = (-n, \sum_{i=1}^{n} p_i)$$

and the total degree is defined as

$$\mathrm{tdeg}\,[a_1|a_2|\dots|a_n] = \sum_{i=1}^{n} p_i - n.$$

By the construction, the module $B\mathcal{A}$ is a Hopf algebra with the diagonal

$$\Delta([a_1|a_2|\dots|a_n]) = \sum_{p=0}^{n} [a_1|\dots|a_p] \otimes [a_{p+1}|\dots|a_n]$$

and the multiplication

$$[a_1|\dots|a_k]\cdot[a_{k+1}|\dots|a_{k+n}] = \sum_{\sigma} \varepsilon_\sigma [a_{\sigma(1)}|a_{\sigma(2)}|\dots|a_{\sigma(k+n)}]$$

where the summation is taken over all ordered permutations $(1,\dots,k+n)$, i. e., $\sigma(1)<\sigma(2)<\dots<\sigma(k)$ and $\sigma(k+1)<\sigma(k+2)<\dots<\sigma(k+n)$. The signature ε_σ of σ is defined in the usual manner via transpositions but any simple transposition $(i,i+1)$ of neighboring indices has the signature $(-1)^{(p_i-1)(p_{i+1}-1)}$.

There are two differentials on $B\mathcal{A}$: the inner differential

$$d_{\mathcal{A}}([a_1|a_2|\dots|a_n]) = \sum_{i=1}^{n} (-1)^i [\bar{a}_1|\bar{a}_2|\dots|\bar{a}_{i-1}|d\,a_i|a_{i+1}|\dots|a_n],$$

and the combinatorial differential

$$\delta([a_1|a_2|\dots|a_n]) = \sum_{i=1}^{n} (-1)^i [\bar{a}_1|\bar{a}_2|\dots|\bar{a}_{i-1}a_i|a_{i+1}|\dots|a_n].$$

These differentials are bihomogeneous and of bidegrees $(1,0)$ and $(0,1)$. Moreover, $d_{\mathcal{A}}\delta + \delta\, d_{\mathcal{A}} = 0$ and, therefore, $\nabla = d_{\mathcal{A}} + \delta$ is a differential of bidegree $(1,1)$ on $B\mathcal{A}$. The Hopf algebra $B\mathcal{A}$ with the differential ∇ is called the *bar-construction* for $(\mathcal{A}, d)$.

The decreasing filtration $\mathcal{F}_n(B\mathcal{A}) = \left(\sum_{j\geq n} B\mathcal{A}^{j,*}\right)$ is defined by the first (tensor) grading and the spectral sequence corresponding to this filtration is called the Eilenberg–Moore spectral sequence. It "lives" in the second quadrant. Although generally this filtration is infinite, the term $\{E_\infty^{p,q} = \lim\limits_{\to\, i} E_i^{p,q}\}^{p,q}$ is correctly defined and associated to $H(B(\mathcal{A}),\nabla)$. It is well-known [23] that the bar-construction endowed only with the combinatorial differential δ defines the homologies of $\mathrm{Tor}_{\mathcal{A}}^{**}(k,k)$ with coefficients in the $\mathcal{A}$-module k where the $\mathcal{A}$-action is defined by the augmentation:

$a(k) = \varepsilon(a) \cdot k$. It is easily derived that the term E_2 of the Eilenberg–Moore spectral sequence is

$$E_2^{*,*} \simeq \mathrm{Tor}^{\,*}_{H^*(\mathcal{A})}(k,k).$$

The sequence is natural with respect to homomorphisms of differential graded algebras. This means that for a given homomorphism $\varphi : (\mathcal{A}, d_{\mathcal{A}}) \to (\mathcal{B}, d_{\mathcal{B}})$ the mapping of the bar-constructions induces a homomorphism of the spectral sequences

$$\varphi^* : (E_r(\mathcal{A})^{*,*}, d_r) \to (E_r^{*,*}(\mathcal{B}), d_r), \quad r = 1, 2, \ldots .$$

If $\mathcal{A}$ is of a topological origin, i. e., if $(\mathcal{A}, d) = (\mathcal{E}^*(X), d)$ where X is a smooth simply connected manifold, then for the corresponding Eilenberg–Moore spectral sequence we have

1) $E_2^{*,*} \simeq \mathrm{Tor}^{\,*}_{H^*(X,\mathbb{R})}(k,k)$;

2) $E_r \Rightarrow H^*(\Omega X, \mathbb{R})$.

The cohomology suspension homomorphism is related to the Eilenberg–Moore spectral sequence as follows. Consider a mapping $\Sigma : \mathcal{A} \to B(\mathcal{A})$ defined as

$$\Sigma(a) = [a]. \tag{15}$$

This mapping commutes with the differentials and we obtain the induced mapping

$$\Sigma^* : H^+(X) \to H(B(\mathcal{A}), \nabla) \simeq H^*(\Omega X, \mathbb{R}) \tag{16}$$

It seems that the following proposition was first proved by Moore [24] (see also [22]).

Proposition 6 *There is an equality*

$$\Sigma^* = \Omega^*.$$

It follows from (15) that Σ^* descends through the Eilenberg–Moore spectral sequence and we have a sequence of mappings

$$\Sigma_r^* : H^+(X, \mathbb{R}) \to E_r^{-1,*}.$$

In particular, $\Sigma_1^* : H^+(X, \mathbb{R}) \to E_1^{-1,*} = B^{-1}(H^*(X, \mathbb{R}), \delta)$ is defined as $\Sigma_1^*(\alpha) = [\alpha]$, and there is the natural projection

$$\Sigma_2^* : H^+(X, \mathbb{R}) \to E_2^{-1,*} = \mathrm{Tor}^{\,1}_{H^*(X,\mathbb{R})}(k,k) \simeq H^+(X, \mathbb{R})/H^+ \cdot H^+.$$

Corollary 5 *If* $\mathrm{rk}\, h_* < \dim H^+(X, \mathbb{R})/H^+ \cdot H^+$, *then the corresponding Eilenberg–Moore spectral sequence does not stabilize at the term* E_2.

§3. Higher Massey operations in symplectic manifolds

3.1. Symplectic manifolds and symplectic blow-ups.

A smooth manifold X with a given 2-form ω is called symplectic if this form meets two conditions:

1) ω is nondegenerate, i. e., for any nonzero vector $\xi \in T_x X$ there is a vector $\eta \in T_x X$ such that $\omega(\xi, \eta) \neq 0$;

2) $d\omega = 0$.

Such a form ω is called symplectic.

By the Darboux theorem, near any point $x \in X$ there are local coordinates $x^1, \ldots, x^{2n}$ such that the symplectic form equals

$$\omega = \sum_{j=1}^{n} d\, x^j \wedge d\, x^{j+n}.$$

This implies that a symplectic manifold X is even-dimensional and the form ω^n is the volume form:

$$\omega^n = n!\, d\, x^1 \wedge \ldots \wedge d\, x^n.$$

Take a Riemannian metric on X and define an operator A linearly acting in the fibers of the tangent bundle TX as

$$(A\xi, \eta) = g_{jk}(A\xi)^j \eta^k = \omega(\xi, \eta).$$

The operator A is skew-symmetric and, therefore, $A^*A = -A^2$ is symmetric and positive. Take the positive square root of A^*A:

$$Q = \sqrt{-A^2} > 0$$

and put $J = AQ^{-1}$. Then we have

$$J^2 = -1$$

and J defines an almost complex structure on X. This structure is compatible with the symplectic structure, which means that the form

$$\langle \xi, \eta \rangle = (\xi, J\eta)$$

is Hermitian and positive-definite.

In the sequel we shall consider only compact symplectic manifolds because any noncompact almost complex manifold has a compatible symplectic

structure [25]. For compact manifolds the condition $[\omega^n] \neq 0 \in H^{2n}(X, \mathbb{R})$ shows that not any compact complex manifold has a symplectic structure.

Examples of symplectic manifolds are Kähler manifolds, .i e., complex manifolds with a Hermitian metric $h_{jk}\, d\, z^j d\, \bar{z}^k$ such that the form

$$\omega = h_{jk}\, d\, z^j \wedge d\, \bar{z}^k$$

is symplectic.

Other examples of simply connected symplectic manifolds of dimension greater than four are constructed from known ones by the symplectic fibration [4], the symplectic blow-up [25, 5], and the fiber connected sum [6]. We shall consider symplectic blow-ups and first recall their construction.

Symplectic blow-up. Let $f : Y \to X$ be a symplectic embedding, i. e., ω be the symplectic form on X and $f^*\omega$ be the symplectic form on Y. Therefore we identify Y with the submanifold $f(Y) \subset X$.

To every point $x \in Y \subset X$ there corresponds a subspace $E_x \subset T_x X$, which is the symplectic normal complement to $T_x Y$ in $T_x X$:

$$\omega(\xi, \eta) = 0 \quad \xi \in T_x Y,\ \eta \in E_x.$$

The spaces E_x are pasted into a vector bundle

$$E \to Y,$$

which is isomorphic to the normal bundle to Y.

Since Y is a symplectic submanifold, the restrictions of ω on E_x are nondegenerate and there is a family of almost complex structures J_x on E_x smoothly depending on x and compatible with $\omega|_{E_x}$. Hence, the structure group of E reduces to $U(k) = SO(2k) \cap Sp(k)$ where $k = \frac{1}{2}(\dim X - \dim Y)$.

We identify the fibers of E with $\mathbb{C}^k$ and consider another fibration

$$\widetilde{E} \to Y,$$

whose fibers are isomorphic to the fibers of the canonical line bundle $L \to \mathbb{C}P^{k-1}$ and which has the same associated principal $U(k)$-bundle as $E \to Y$. The fibration

$$L \to \mathbb{C}P^{k-1},$$

where

$$L = \{(\xi, l) \in \mathbb{C}^k \times \mathbb{C}P^{k-1} | \xi \in l\},$$

parameterizes pairs (l, ξ) where l is a line in $\mathbb{C}^k$ and $\xi \in l$. The projection onto the base is

$$(\xi, l) \to l \in \mathbb{C}P^{k-1}$$

and the structure group $U(k)$ acts on the fibration as follows

$$A \cdot (\xi, l) = (A\xi, Al).$$

The fibers of E and, therefore, of $\widetilde{E}$ are endowed with the Hermitian metric

$$(\xi, \eta) = \omega(\xi, J\eta).$$

We denote by E_r and $\widetilde{E}_r$ the submanifolds of E and $\widetilde{E}$ defined by the inequality $|\xi| \leq r$.

Let us construct now the symplectic blow-up $\widetilde{X}$ of X along Y. For that take a closed tubular neighborhood V of Y and identify it with E_1. Remove the interior of V from X and glue $\widetilde{E}_1$ to the obtained manifold with boundary. This gluing is defined by the isomorphism $\partial E_1 = \partial \widetilde{E}_1 = \partial V$. The resulted manifold

$$\widetilde{X} = \overline{(X \backslash V)} \cup_{\partial V} \widetilde{E}_1$$

is called the symplectic blow-up. In fact, it is a result of the fiberwise blow-ups of the fibers E_x at the points $\xi = 0$.

There is the projection

$$\pi : \widetilde{X} \to X,$$

which is a diffeomorphism outside Y

$$\widetilde{X} \backslash \pi^{-1}(Y) \to X \backslash Y$$

and its restriction over Y is the fibration

$$\pi^{-1}(Y) \to Y$$

with $\mathbb{C}P^{k-1}$-fibers.

As shown in [25] and [5], there is a symplectic structure on $\widetilde{X}$, which coincides with $\pi^*(\omega)$ outside a small neighborhood of $\pi^{-1}(Y)$.

3.2. Survival of higher Massey products.

The main aim of this subsection is a proof of the following statement.

Theorem A *Let a simply connected symplectic manifold X have an irreducible generalized Massey product of dimension k. Then for any symplectic submanifold $Y \subset X$ with* $\operatorname{codim} Y > k$ *the corresponding symplectic blow-up $\widetilde{X}$ also has an irreducible generalized Massey product of dimension k.*

Corollary 6 *Under the hypothesis of the Theorem the manifolds X and $\widetilde{X}$ are nonformal.*

It follows from Theorem A and Corollary 4 that

Corollary 7 *Let the cohomology ring of a symplectic manifold X have a nonspheric generator of dimension k. Then the symplectic blow-up $\widetilde{X}$ of X along any symplectic submanifold $Y \subset X$, with $\operatorname{codim} Y > k$, also has a nonspheric k-dimensional generator and, therefore, is nonformal.*

From that using Corollary 3 we obtain

Corollary 8 *Let a simply connected symplectic manifold X satisfies the inequality*

$$\operatorname{rk} h_k < \dim H^k(X)/H^k(X) \cap (H^+(X))^2.$$

Then the symplectic blow-up $\widetilde{X}$ of X along any symplectic submanifold $Y \subset X$, with $\operatorname{codim} Y > k$, is nonformal.

Let us prove Theorem A. Let $\pi : \widetilde{X} \to X$ be the natural projection. By assumption, there is a generalized irreducible Massey product $u \in H^k(X)$. We shall show that $\pi^*(u) \in H^k(\widetilde{X})$ is also a generalized irreducible Massey product.

Let $A \in M(\mathcal{E}^*(X))$ be a solution to the Maurer–Cartan equation such that u is an entry of the matrix $[d\,A - \bar{A} \cdot A]$. Since solutions to the Maurer–Cartan equation are natural, $\pi^*(A)$ is a solution to this equation in $\mathcal{E}^*(\widetilde{X})$. Moreover the corresponding entry of the matrix

$$[d\,\pi^*(A) - \overline{\pi^*(A)} \cdot \pi^*(A)] = \pi^*([d\,A - \bar{A} \cdot A])$$

equals $\pi^*(u)$. It remains to prove that $\pi^*(u)$ is irreducible. This is implied by the following statement.

Lemma 3 *Let $\pi : \widetilde{X} \to X$ be the natural projection of the symplectic blow-up of X along $Y \subset X$. The induced mapping*

$$\pi^* : \operatorname{Tor}_{1,k}^{H^*(X)}(\mathbb{R}, \mathbb{R}) \to \operatorname{Tor}_{1,k}^{H^*(\widetilde{X})}(\mathbb{R}, \mathbb{R}) \tag{17}$$

is a monomorphism for $k < \operatorname{codim} Y$.

Indeed, for the algebra $H = H^*(X)$ there is an isomorphism

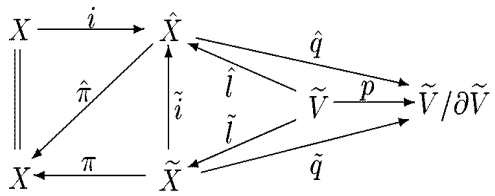
$$\operatorname{Tor}_{1,*}^{H}(k, k) \simeq H^+/H^+ \cdot H^+$$

and the monomorphism (17) implies that $\pi^*(u) \neq 0 \bmod H^+(\widetilde{X}) \cdot H^+(\widetilde{X})$, which means that $\pi^*(u)$ is irreducible.

Proof of the lemma. Let $V \subset X$ be a tubular neighborhood of Y and let $\widetilde{V}$ be its symplectic blow-up along Y. In this notation, we have $\widehat{X} = (X \backslash \dot{V}) \cup_{\partial \widetilde{V}} \widetilde{V}$ where $\dot{V} = V \backslash \partial V$ and the boundaries of ∂V and $\partial \widetilde{V}$ are identified in the natural manner. Consider the space

$$\widehat{X} = X \cup_{\partial \widetilde{V}} \widetilde{V} = \widetilde{X} \cup_{\partial V} V.$$

There is a commutative diagram



(18)

where i, $\widetilde{i}$, $\widehat{l}$, and $\widetilde{l}$ are the embeddings, p, $\widehat{q}$, $\widetilde{q}$, π, and $\widehat{\pi}$ are the projections and, moreover, $\widehat{\pi}$ is a retraction.

Consider the ideal $I = \operatorname{Ker} i^*$. Since X is a retract of $\widehat{X}$, there is a multiplicative isomorphism

$$H^*(\widehat{X}) \simeq H^*(X) \oplus I \tag{19}$$

where $H^*(X)$ is identified with its image under the retraction $\widehat{\pi}^*$. Since I is an ideal, by (19), we conclude that

$$\widehat{\pi}^* : H^+(X)/(H^+(X))^2 \longrightarrow H^+(\widehat{X})/(H^+(\widehat{X}))^2 \tag{20}$$

is a monomorphism.

Now from the exact cohomology sequence for $(\widehat{X}, X)$ derive that $\widehat{q}^*$ maps $H^+(\widetilde{V}/\partial \widetilde{V})$ isomorphically onto I. Let $\widetilde{a} \in H^2(\widetilde{V}/\partial \widetilde{V})$ be the projective Thom class. Using the fibration

$$r : \widetilde{V} \stackrel{\dot{\mathbb{C}P}^m}{\longrightarrow} Y \tag{21}$$

where r is the projection, $m = \frac{1}{2} \operatorname{codim} Y$, and $\dot{\mathbb{C}P}^m = \mathbb{C}P^m \backslash D^{2m}$, we describe $H^+(\widetilde{V}/\partial \widetilde{V})$ as a $H^*(Y)$-module:

$$H^+(\widetilde{V}/\partial \widetilde{V}) \simeq \widetilde{a} H^*(Y) \oplus \widetilde{a}^2 H^*(Y) \oplus \ldots \oplus \widetilde{a}^m H^*(Y).$$

Put $\widehat{a} = \widehat{q}^*(\widetilde{a})$ and applying $\widehat{q}^*$ we obtain

$$I \simeq \widehat{a} H^*(Y) \oplus \widehat{a}^2 H^*(Y) \oplus \ldots \oplus \widehat{a}^m H^*(Y). \tag{22}$$

The fibration (21) defines a structure of a $H^*(Y)$-module on $H^*(\widetilde{V})$:

$$H^*(\widetilde{V}) \simeq H^*(Y) \oplus aH^*(Y) \oplus \ldots \oplus a^{m-1}H^*(Y) \tag{23}$$

where $a \in H^2(\widetilde{V})$ is the two-class, which is mapped into the generator $\mathbb{C}P^m \subset \widetilde{V}$. Notice that $a = \widehat{l}^*(\widehat{a}) = p^*(\widetilde{a})$ and $\widetilde{a} = \widehat{q}^*(\widehat{a})$. Moreover the decompositions (22) and (23) are related by the homomorphism induced by the embedding $\widehat{l}$. Finally we put

$$I_1 = \widehat{a}H^*(Y) \oplus \ldots \oplus \widehat{a}^{m-1}H^*(Y), \quad I_2 = \widehat{a}^m H^*(Y)$$

and show that $\widetilde{i}^*|_{I_1}$ is a monomorphism. Define a $H^*(Y)$-submodule $H^*(\widetilde{V})$ as follows

$$H^*_+(\widetilde{V}) = aH^*(Y) \oplus a^2H^*(Y) \oplus \ldots \oplus a^{m-1}H^*(Y).$$

As shown in [5] there is an exact short sequence of modules

$$0 \to H^*(X) \xrightarrow{\pi^*} H^*(\widetilde{X}) \xrightarrow{\widetilde{l}^*} H^*_+(\widetilde{V}) \to 0. \tag{24}$$

It follows from the diagram (18) that $\widehat{l}^*|_{I_1}$ is an isomorphism and $\widetilde{i}^*|_{I_1}$ is a monomorphism, which splits $\widetilde{l}^*$.

Now using the decomposition (19) and keeping in mind that $\widetilde{i}^*$ is a multiplicative isomorphism in dimensions less or equal than $\operatorname{codim} Y$ we obtain

$$(H^+(\widetilde{X}) \cdot H^+(\widetilde{X}))^k \subseteq (H^+(X) \cdot H^+(X)) \oplus (i^*(I))^k, \quad k < \operatorname{codim} Y.$$

From that we conclude that

$$\pi^* : H^k(X)/H^k(X) \cap (H^+(X))^2 \to H^k(\widetilde{X})/H^k(\widetilde{X}) \cap (H^+(\widetilde{X}))^2$$

is a monomorphism for $k < \operatorname{codim} Y$. This proves the lemma.

Remark. For the simplest case, for triple Massey products, the notions of nontriviality and irreducibility are very closed. By a simple modification of the given proof of Lemma 3 one can obtain the following statement.

Proposition 7 *Let X be a simply connected symplectic manifold with a nontrivial triple Massey product of dimension k. Then for any its symplectic blow-up $\widetilde{X}$ along $Y \subset X$, with $\operatorname{codim} Y > k$, also has a nontrivial Massey product of dimension 3.*

3.3. Symplectic manifolds of arbitrary large weight

Let $\langle X_1, \ldots, X_k \rangle$ be a k-tuple matrix Massey product in $H^*(M)$. We say that the weight of $\langle X_1, \ldots, X_k \rangle$ is *strictly equal to* k if there is an element $a \in \langle X_1, \ldots, X_k \rangle$, which does not belong to the linear subspace generated by all Massey products $\langle Y_1, \ldots, Y_l \rangle$ with $l < k$. We shall see below (Remark 3) that nontrivial relations between cohomologies classes sometimes may imply some nontrivial k-tuple products are expressed in terms of Massey products of less weight and of another elements. The main result of this subsection is the following

Theorem 6 *For any k there exist symplectic manifolds with Massey products whose weights are strictly equal to $2k$.*

For proving this theorem we shall use the symplectic nilmanifolds $M(2n)$ introduced by us in [2]. Let us recall in brief their construction.

Take the Witt algebra $W(1)$ of formal vector fields on the line. It has a basis

$$e_i = x^{i+1}\frac{d}{dx}, \quad i = -1, 0, 1, \ldots,$$

with the commutation relations

$$[e_i, e_j] = (j - i)e_{i+j}, \quad i, j \geq -1. \tag{25}$$

Denoting by $\mathcal{L}(\ldots)$ the linear span of the corresponding vectors, we consider the subalgebras

$$\mathcal{L}_k(1) = \mathcal{L}(e_k, e_{k+1}, \ldots), \quad k = 1, 2, \ldots.$$

There is a natural filtration

$$\ldots \subset \mathcal{L}_1(1) \subset \mathcal{L}_0(1) \subset \mathcal{L}_{-1} = W(1),$$

which is useful for study $W(1)$ and its subalgebras and quotients.

Consider the finite-dimensional nilpotent Lie algebras

$$\mathcal{V}_n = \mathcal{L}_1(1)/\mathcal{L}_{n+1}(1), \quad n = 1, 2, \ldots,$$

and denote the corresponding nilpotent Lie groups by V_n, $n = 1, 2, \ldots$. By (25), $\mathcal{V}_n$ is a nilpotent n-dimensional Lie algebra with a basis $\{e_1, \ldots, e_n\}$ and the Lie brackets

$$[e_i, e_j] = \begin{cases} (j - i)e_{i+j}, & i + j \leq n, \\ 0, & i + j > n. \end{cases} \tag{26}$$

By the nilpotence, we identify $\mathcal{V}_n$ and V_n as sets and define the multiplication $\times$ by the Campbell–Hausdorff formula. Another interpretation of V_n as a group of polynomial transformations of the line was recently used in [26].

The formulas (26) show that the structure constants of $\mathcal{V}_n$ are rational and, therefore, by the Mal'tsev theorem [15], V_n has uniform lattices. Among such lattices we choose the group Γ_n generated by $\{e_1, \ldots, e_n\}$ with respect to $\times$. We obtain a family of nilmanifolds

$$M(n) = V_n/\Gamma_n, \quad n = 1, 2, \ldots .$$

Notice that V_3 is the three-dimensional Heisenberg group $\mathcal{H}$, $M(3) \simeq \mathcal{H}/\mathcal{H}_{\mathbb{Z}}$, and $M(3) \times S^1$ is the Kodaira–Thurston manifold.

The nilmanifolds $M(n)$ with $n \geq 3$ have many prominent properties and, in particular, for even n they have natural symplectic structures [2]. Let $\{\omega_1, \ldots, \omega_n\}$ be the basis for left-invariant 1-forms onV_n dual to $\{e_1, \ldots, e_n\}$. It follows from (26) that

$$d\,\omega_k = (k-2)\,\omega_1 \wedge \omega_{k-1} + (k-4)\,\omega_2 \wedge \omega_{k-2} + \ldots . \tag{27}$$

Moreover, as shown in [2] the left-invariant 2-form

$$\Omega_{2m} = (2m-1)\,\omega_1 \wedge \omega_{2m} + (2m-3)\,\omega_2 \wedge \omega_{2m-1} + \ldots + \omega_m \wedge \omega_{m+1}$$

defines a left-invariant symplectic structure on V_{2m}. The factorization descends this structure to $M(2m)$ where it defines an integer symplectic form, which we denote also by Ω_{2m}. Let $[\Omega_{2m}] \in H^2(M(2m), \mathbb{Z})$ the cohomology class of this form. Theorem 6 is implied by the following statement.

Proposition 8 *The symplectic class $[\Omega_{2m}] \in H^2(M(2m), \mathbb{Z})$ is a $2m$-tuple matrix Massey product whose weight is strictly equal $2m$.*

Proof. Consider an exterior algebra $\Lambda_{2m} = \Lambda(\omega_1, \omega_2, \ldots, \omega_{2m}; d)$ where $d\omega_1 = d\omega_2 = 0$ and $d\omega_k$ are defined by the formula (27) for $k \geq 3$. By the Nomizu theorem [16], the embedding

$$\Lambda_{2m} \longrightarrow \mathcal{E}^*(M(2m)),$$

induces by the embedding of left-invariant forms into the complex of all forms on $M(2m)$ is a weak isomorphism, Hence, we may compute Massey products in $H^*(M(2m))$ from Λ_{2m}. Notice that Λ_{2m} is isomorphic to the minimal model for $M(2m)$.

Let us represent Ω_{2m} by a $2m$-tuple Massey product. There is an inclusion

$$[\Omega_{2m}] \in \langle -m\omega_1, -(m-1)\omega_1, \ldots, -2\omega_1, (-\omega_1\ 0), \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}, 2\omega_1, \ldots, (m-1)\omega_1, \omega_1 \rangle. \tag{28}$$

For proving (28) consider the $(2m+2) \times (2m+2)$-matrix A of the form

$$\begin{pmatrix} 0 & -m\omega_1 & (1-m)\omega_2 & \ldots & (k-m)\omega_{1+k} & \ldots & (m-1)\omega_{2m} & 0 \\ 0 & 0 & (1-m)\omega_1 & \ldots & (k-m)\omega_k & \ldots & (m-1)\omega_{2m-1} & \omega_{2m} \\ & \ldots & & & & & \ldots & \\ 0 & 0 & 0 & \ldots & 0 & \ldots & (m-1)\omega_1 & \omega_2 \\ 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & \omega_1 \\ 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

Straightforward computations show that A satisfies the Maurer–Cartan equation and, therefore, is a defining system for the product (28). These computations also show that the matrix $\bar{A} \wedge A - d\,A$ has one and only one nonzero entry. It is the $(1, 2m+2)$ entry, which equals

$$-m\bar{\omega}_1 \wedge \omega_{2m} - (m-1)\bar{\omega}_2 \wedge \omega_{2m-1} - \ldots - \bar{\omega_m} \wedge \omega_{m+1} + \bar{\omega}_{m+2} \wedge \omega_{m-1} + \ldots +$$

$$+(m-1)\bar{\omega}_{2m} \wedge \omega_1 = \sum_{i=1}^{m} [2(m-i)+1]\omega_i \wedge \omega_{2m-i+1} = \Omega_{2m}.$$

The last equality proves the inclusion (28).

Now show that Ω_{2m} can not be expressed as a linear combination of k-tuple Massey products with $k < 2m$. For that we notice that the algebra Λ_{2m} is bigraded. The gradings are defined on the generators

$$\deg_1(\omega_i) = 1, \quad \deg_2(\omega_i) = i, \quad i = 1, 2, \ldots, 2m,$$

and are extended by the multiplication. It follows from (27) that the bidegree of the differential d is $(1, 0)$. Notice, that the degree rules used for the definition of multituple Massey products and introduced in §2.3 are applied only to $\deg_1$.

The algebra Λ_{2m} also has an increasing filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_{2m} = \Lambda_{2m}$$

where $\mathcal{F}_k = \Lambda(\omega_1, \ldots, \omega_k)$, $k = 1, 2, \ldots, 2m$. This filtration meets the following conditions:

(1) $f(\alpha \wedge \beta) \leq \max(f(\alpha), f(\beta))$.

(2) $f(d\,\alpha) < f(\alpha)$, and, moreover, if $\deg_1(\alpha) = 1$, then $f(d\,\alpha) = f(\alpha) - 1$. The last property follows from (27).

Suppose that

$$\Omega_{2m} = \sum_s \alpha_s \langle X_1^{(s)}, \ldots, X_{k(s)}^{(s)} \rangle, \quad k(s) < 2m, \quad \alpha_s \in \mathbb{R} \tag{29}$$

and $X_j^{(s)}$ are matrices whose entries are closed elements from Λ_{2m}. Since $\deg_1 \Omega = 2$, the degree rules imply that $\deg_1 X_j^{(s)} = 1$. Since $X_j^{(s)}$ is closed, all its entries are linear combinations of ω_1 and ω_2 and, therefore,

$$f(X_j^{(s)}) \leq 2 \quad \text{for all} \quad s \text{ and } \quad j. \tag{30}$$

Let $A^{(s)} = (X_{(i,j)}^{(s)})$ be a defining system for $\langle X_1^{(s)}, \ldots, X_{k(s)}^{(s)} \rangle$. Applying the degree rules, we obtain $\deg_1 X_{(i,j)}^{(s)} = 1$ for all s, i, j. Hence, it follows from (30) and (2) that $f(X_{(i,i+1)}^{(s)}) \leq 3$. Now applying the equality 3 from Definition 8 we obtain, by induction, that

$$f(X_{(i,j)}^{(s)}) \leq j - i + 2.$$

The last step of this inductive process together with the formula 4 from Definition 8 implies

$$f(c(A^{(s)})) \leq k(s)$$

where $c(A^{(s)})$ is the cocycle of the defining system. But from another side we have $f(\Omega(2m)) = 2m$ and, therefore, the equality (29) does not hold if $k(s) < 2m$ for all s. This contradiction proves the proposition.

Remarks. 1. We prove Proposition 8 by presenting an explicit formula (28) for the product representing $[\Omega_{2m}]$. This product is not uniquely defined. For instance, for $M(4)$ there is another inclusion except (28):

$$[\Omega_4] \in \langle (0 \quad 6\omega_2), \begin{pmatrix} \omega_1 & \omega_2 \\ 0 & \omega_1 \end{pmatrix}, \begin{pmatrix} \omega_1 & \omega_2 \\ 0 & \omega_1 \end{pmatrix}, \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} \rangle.$$

We are leave that for the reader to find the corresponding solution to the Maurer–Cartan equation.

2. In $M(4)$ the symplectic class is a sum of two cohomology classes

$$[\Omega_4] = [3\omega_1 \wedge \omega_4] + [\omega_2 \wedge \omega_3].$$

It is easy to check that the first component is represented by a quadruple Massey product as follows

$$[3\omega_1 \wedge \omega_4] \in \langle 6\omega_2, \omega_1, \omega_1, \omega_1 \rangle \tag{31}$$

and its weight is strictly equal to 4. Another component is represented in the form

$$[\omega_2 \wedge \omega_3] \in \langle -\omega_1, \omega_2, \omega_2 \rangle \tag{32}$$

and the weight of this product is strictly equal to 3.

3. For $m > 2$ the both classes $[3\omega_1 \wedge \omega_4]$ and $[\omega_2 \wedge \omega_3]$ do not vanish in $H^2(M(2m))$, but, by the equality

$$d\,\omega_5 = \Omega_4 = 3\omega_1 \wedge \omega_4 + \omega_2 \wedge \omega_3,$$

their sum equals zero. Hence, for $m > 2$ there is a nontrivial relation between the quadruple product (31) and the triple product (32) in $H^*(M(2m))$. Therefore, for $m > 2$ $[3\omega_1 \wedge \omega_4] \in H^2(M(2m))$ is not a cohomology class whose weight is strictly equal to 4.

3.4. Inheritance of higher Massey products.

Let (X, ω) be a symplectic manifold and let $Y \subset X$ be its submanifold.

In this subsection we find sufficient conditions, which guarantee that some nontrivial higher matrix Massey products in $H^*(Y)$ are inherited after the blow-up, i.e., generate nontrivial matrix Massey products in $H^*(\widetilde{X})$.

Consider matrix cohomology classes $S_1, \ldots, S_n \in N(H^+(\widetilde{X}))$ and assume that the product $\langle S_1, \ldots, S_n \rangle$ is defined and nontrivial. Our aim is to find some corresponding n-tuple Massey product $\langle \widetilde{S}_1, \ldots, \widetilde{S}_n \rangle$, with $\widetilde{S}_i \in N(H^+(Y))$, which is nontrivial in the symplectic blow-up of X along $Y \subset X$. If $n \geq 5$ and matrix classes S_i, $1 \leq i \leq n$, are taken arbitrary, then it is already a problematic question can one effectively achieve that for $Y \subset \mathbb{C}P^N$ with arbitrary large N. The reasons for that are the structure of Massey products for $n \geq 5$ and possible relations between Massey products of different orders in $H^*(Y)$. But there are two important cases, which we shall consider in the sequel:

1) S_i are matrices of one-dimensional classes, i.e., $S_i \in N(H^1(Y))$, $1 \leq i \leq n$, where n is arbitrary;

2) S_i are arbitrary matrices for $1 \leq i \leq n$ and $n = 3, 4$.

Let us formulate the main results of this subsection.

Theorem B *Let a symplectic manifold (Y, ω) have a nontrivial matrix n-tuple Massey product $\langle S_1, \ldots, S_n \rangle$ where $S_i \in N(H^1(Y))$ are matrices of one-dimensional cohomology classes for $1 \leq i \leq n$. Then for any symplectic*

embedding $Y \subset X$ of codimension not less than $2(n+1)$ the corresponding symplectic blow-up $\widetilde{X}$ has a nontrivial n-tuple Massey product $\langle \widetilde{S}_1, \ldots, \widetilde{S}_n \rangle$, where $\widetilde{S}_i \in N(H^3(\widetilde{X}))$, $1 \leq i \leq n$.

Corollary 9 *For any k there exists a simply connected symplectic manifold X of dimension $6k+2$ with a nontrivial $2k$-tuple matrix Massey product in $H^{4k+2}(X)$.*

Indeed, consider the symplectic manifold $M(2k)$ defined in §3.3. Let $f : M(2k) \to \mathbb{C}P^{3k+1}$ be a symplectic embedding, which exists by the Gromov–Tischler theorem [27]. The corollary follows from Proposition 8 and Theorem B applied to the pair $M(2k) \subset \mathbb{C}P^{3k+1}$. Corollary 9 is established.

Theorem C *Let a symplectic manifold (Y, ω) have a nontrivial triple matrix Massey product. Then for any symplectic embedding $Y \subset X$ of codimension greater or equal than 8 the corresponding symplectic blow up $\widetilde{X}$ also has a nontrivial triple matrix Massey product.*

In the sequel the following definition will be useful.

Definition 14 *Let $\mathcal{A}$ be a differential graded algebra and let $H \in N(\mathcal{A})$. The upper degree of H is*

$$\operatorname{sdeg} H = \max_{i,j} \deg h_{ij}$$

where $h_{ij} \in \mathcal{A}$ are the matrix entries of H.

Theorem D *Let a symplectic manifold (Y, ω) have a strictly irreducible quadruple matrix Massey product $\langle S_1, S_2, S_3, S_4 \rangle$. Then for any symplectic submanifold $Y \subset X$ such that*

$$\operatorname{codim} Y > 2 \operatorname{sdeg} \langle S_1, S_2, S_3, S_4 \rangle$$

the corresponding symplectic blow-up $\widetilde{X}$ has a nontrivial quadruple matrix Massey product.

Let us prove these theorems. First we construct from the given Massey product $\langle S_1, \ldots, S_n \rangle$ in $H^*(Y)$ another Massey product $\langle \widetilde{S}_1, \ldots, \widetilde{S}_n \rangle$ in $H^*(\widetilde{X})$. These parts of the proofs are similar for all theorems.

Let $S_1, \ldots, S_n \in N(H^*(Y))$ be multipliable matrices and the Massey product $\langle S_1, \ldots, S_n \rangle$ be defined. If $\pi : \widetilde{V} \to V \sim Y$ is the natural projection, then $\pi^* : H^*(Y) \to H^*(\widetilde{V})$ is a monomorphism and, hence, we shall not differ

S_i and $\pi^*(S_i)$ for $1 \le i \le n$. Consider the matrix classes $a \cup S_i \in N(H^*(\widetilde{V}))$, $1 \le i \le n$, where $a \in H^2(\widetilde{V})$ is the same as in §3.2. It is shown in §2.2 that

$$p^* : H^*(\widetilde{V}/\partial\widetilde{V}) \to H^*(\widetilde{V})$$

maps $\widetilde{a} \otimes H^*(Y) \subset H^*(\widetilde{V}/\partial\widetilde{V})$ isomorphically onto the corresponding submodule $a \cup H^*(Y) \subset H^*(\widetilde{V})$ where $p^*(\widetilde{a}) = a$. Therefore, the elements $S_i' \in N(H^*(\widetilde{V}))$ such that $p^*(S_i') = a \cup S_i$ are correctly defined for $1 \le i \le n$. Finally, we put

$$\widetilde{S}_i = \widetilde{q}^*(S_i'), \quad 1 \le i \le n \tag{33}$$

and notice that, since the diagram (18) commutes, we have

$$\widetilde{l}^*(\widetilde{S}_i) = a \cup S_i = p^*(S_i'), \quad 1 \le i \le n.$$

The theorems are implied by the following statement.

Lemma 4 *Under the hypotheses of Theorems B, C, and D the matrix Massey $\langle \widetilde{S}_1, \dots, \widetilde{S}_n \rangle$ is defined and nontrivial.*

Since Massey products are natural and the diagram (18) is commutative, Lemma 4 follows from these two lemmas.

Lemma 5 *Under the hypotheses of Theorems B, C, and D the Massey product $\langle S_1', \dots, S_n' \rangle$ is defined.*

Lemma 6 *Under the hypotheses of Theorems B, C, and D the Massey product $\langle a \cup S_1, \dots, a \cup S_n \rangle$ is nontrivial.*

Notice that it is clear that the latter product is defined.

Proof of Lemma 5. Let $A = (X(i,j))$, where $1 \le i \le j \le n$ and$(i,j) \ne (1,n)$, be a defining system for $\langle S_1, \dots, S_n \rangle$. We have $X(i,j) \in N(\mathcal{E}^+(Y))$ and $[X(i,i)] = S_i$ for $1 \le i \le n$. Denote by $X'(i,j)$ the matrix differential forms $\pi^*(X(i,j))$ where $1 \le i \le j \le n$. Let α be a 2-form representing the class $a \in H^2(\widetilde{V})$. It can be taken such that $\alpha|_U = 0$ for U, a small tubular neighborhood of $\partial\widetilde{V}$. Indeed, $j^*(a) = 0$ where $j : \partial\widetilde{V} \to \widetilde{V}$ is the embedding and this means that $\alpha = d\,\beta$ in U' for some tubular neighborhood $U' \supset U$, which contains U. Take a cut-off function φ, which equals 0 in U and equals 1 outside U' and consider a form α_1 such that $\alpha_1 = d\,(\varphi\beta)$ in U' and $\alpha_1 = \alpha$ outside U'. Since $\alpha_1 \sim \alpha$, we obtain a form representing a and with the necessary properties.

It is clear that the family of matrix forms

$$(X''(i,j) = \alpha^{j-i+1} \wedge X'(i,j)), \quad 1 \le i \le j \le n,$$

form a defining system for $\langle a \cup S_1, \ldots, a \cup S_n \rangle$. Since $X''(i,j)|_U \equiv 0$ for $1 \le i \le j \le n$ these forms may be extended by zero forms onto $\widetilde{V} \cup_{\partial \widetilde{V}} C\partial\widetilde{V} \simeq \widetilde{V}/\partial\widetilde{V}$. Hence we obtain a family of simplicial forms on $\widetilde{V}/\partial\widetilde{V}$. We preserve for these forms the notation $A'' = (X''(i,j))$, $1 \le i \le j \le n$. By the construction, A'' is a defining system for $\langle S_1', \ldots, S_n' \rangle$. This proves Lemma 5.

The proof of Lemma 6 splits into three cases, which imply Theorems B, C, and D.

Lemma 6 B *Under the hypothesis of Theorem B the matrix Massey product* $\langle a \cup S_1, \ldots, a \cup S_n \rangle$ *is nontrivial.*

Proof. In difference with the previous proof it is more convenient to use in this case minimal algebras but not the algebras of forms. Let $(\mathcal{M}_Y, d_Y)$ be the minimal model for Y and let $m = \frac{1}{2}\operatorname{codim} Y$. Then the minimal model for $\widetilde{V}$ is as follows [2]:

$$(\mathcal{M}, d) \simeq \mathcal{M}_Y \otimes_d \{x, y\}$$

where $\deg x = 2$, $\deg y = 2m-1$, $d|_{\mathcal{M}_Y} = d_Y$, $d\,x = 0$,

$$d\,y = x^m + c_1 \wedge x^{m-1} + c_2 \wedge x^{m-2} + \ldots + c_{m-1} \wedge x + c_m, \qquad (34)$$

$c_i \in \mathcal{M}_Y$, $1 \le i \le m$. Notice that $[x] = a$.

Choose a representative $R_i \in N(\mathcal{M}_Y^+)$, $[R_i] = S_i$, for any cohomology class S_i where $1 \le i \le n$. We have

$$[x \wedge R_i] = a \cup S_i, \quad 1 \le i \le n.$$

Suppose that the Massey product is trivial:

$$0 \in \langle a \cup S_1, \ldots, a \cup S_n \rangle$$

and $A = (X(i,j))$, $1 \le i \le j \le n$, is a defining system for the product. Since the entries of the matrices S_i are one-dimensional cohomology classes, we have

$$\deg X(i,j) = 2(j-i) + 3, \quad 1 \le i \le j \le n, \quad (i,j) \ne (1,n).$$

Here each of the matrices contains entries of the same degree and hence the degrees of matrices are correctly defined.

Expand $X(i,j)$ into powers of x:

$$X(i,j) = \sum_{l=0}^{j-i} x^l \wedge X_l(i,j) + x^{j-i+1} \wedge X_{j-i+1}(i,j) \tag{35}$$

where $X_l(i,j) \in \mathcal{M}_Y$. Since $X(i,i) = x \wedge R_i$ for $1 \leq i \leq n$, we have

$$X_1(i,i) = R_i \quad (X_0(i,i) = 0), 1 \leq i \leq n. \tag{36}$$

By the hypothesis of Theorem B, we have $j - i + 1 \leq n - 1 < m$ for all $X(i,j)$. Therefore the equalities

$$d\,X(i,j) = \sum_{r=i}^{j-1} \overline{X(i,r)} \wedge X(r+1,j)$$

and (34) imply that

$$d\,X_{j-i+1}(i,j) = \sum_{r=i}^{j-1} \overline{X_{r-i+1}(i,r)} \wedge X_{r+1}(r+1,j).$$

The latter equality together with (36) means that $A_1 = (X_{j-i+1}(i,j))$, with $1 \leq i \leq j \leq n$, is a defining system for $\langle S_1, \ldots, S_n \rangle$. Applying to $c(A)$ the analogous computation with taking (35) into account, we obtain

$$c(A) = \sum_{r=1}^{n-1} \overline{X(1,r)} \wedge X(r+1,n) =$$

$$= x^n \wedge \sum_{r=1}^{n-1} \overline{X_r(1,r)} \wedge X_{n-r}(r+1,n) + \ldots = x^n \wedge c(A_1) + \ldots,$$

where dots denote terms whose degree in x is less than n.

Since $n < m$, the latter equality means that $[c(A)] = 0$ implies $[c(A_1)] = 0$ and, therefore, the product $\langle S_1, \ldots, S_n \rangle$ is trivial. We arrives at the contradiction, which proves Lemma and Theorem B.

Lemma 6 C *Under the hypothesis of Theorem C the Massey product* $\langle a \cup S_1, a \cup S_2, a \cup S_3 \rangle$ *is nontrivial.*

Proof. Let $A_1 = (R(i,j))$, where $1 \leq i \leq j \leq 3$ and $R(i,j) \in \mathcal{M}_Y$, be a defining system for $\langle S_1, S_2, S_3 \rangle$. Then $A = (x^{j-i+1} \wedge R(i,j))$ with $1 \leq i \leq j \leq 3$ is a defining system for $\langle a \cup S_1, a \cup S_2, a \cup S_3 \rangle$. It is clear that $c(A) = x^3 \wedge c(A_1)$, i. e., $[c(A)] = a^3 \cup [c(A_1)]$.

The indeterminacy of a triple Massey product is simple (see Proposition 4) and we conclude that any element of $u \in \langle a \cup S_1, a \cup S_2, a \cup S_3 \rangle$ has the form

$$u = [c(A)] + H \cup a \cup S_3 + a \cup \overline{S}_1 \cup K \tag{37}$$

where $H, K \in N(H^+(\widetilde{V}))$ are arbitrary matrices with appropriate sizes and multidegrees.

Since $H^*(\widetilde{V})$ is a free $H^*(Y)$-module with the basis $1, a, \ldots, a^{m-1}$, expanding $a \cup H$ and $a \cup K$ in this basis we derive from (37) that

$$u = a^3 \cup [c(A_1)] + \left(\sum_{i=0}^{m-1} a^i \cup H_i \right) \cup S_3 + \overline{S}_1 \cup \left(\sum_{i=0}^{m-1} a^i \cup K_i \right). \tag{38}$$

We have $S_i \in N(H^+(Y))$ and $m \geq 4$ and hence $u = 0$ together with (38) implies

$$[c(A_1)] + H_3 \cup S_3 + \overline{S}_1 \cup K_3 = 0.$$

This means that $0 \in \langle S_1, S_2, S_3 \rangle$, which contradicts to the initial hypothesis. Hence the lemma and Theorem C are proved.

Lemma 6 D *Under the hypothesis of Theorem D the Massey product* $\langle a \cup S_1, a \cup S_2, a \cup S_3, a \cup S_4 \rangle$ *is nontrivial.*

Proof. Suppose that the converse is true, i.e.,

$$0 \in \langle a \cup S_1, a \cup S_2, a \cup S_3, a \cup S_4 \rangle.$$

Let $A = (X(i,j))$, with $1 \leq i \leq j \leq 4$, be a defining system for the product. Represent the elements of the defining system in the minimal model $\mathcal{M}$ for $\widetilde{V}$ as polynomials in x. We have

$$\begin{array}{l} X(i,i) = x \wedge R_i, \quad [R_i] = S_i, \quad 1 \leq i \leq 4, \\ X(i,j) = \sum_{l=0}^{K(i,j)} x^l \wedge X_l(i,j), \quad 1 \leq j - i \leq 2. \end{array} \tag{39}$$

Notice that the hypothesis of the theorem guarantees that $\operatorname{sdeg} X(i,j) < 2m - 1 < 2\operatorname{codim} Y$, and, therefore, the generator $y \in \mathcal{M}$ does not come into the expansions (39), which means that $X_l(i,j) \in \mathcal{M}_Y$ for all i, j, l.

For the defining system A we derive the following equalities

$$\left\{ \begin{array}{ll} 1) & d\,X_l(i,i+1) = \overline{R}_i \wedge R_{i+1}, \quad l = 2, \quad i = 1,2,3, \\ 2) & d\,X_l(i,i+1) = 0, \qquad\qquad\quad l \neq 2, \quad i = 1,2,3, \\ 3) & d\,X_{l+1}(i,i+2) = \overline{R}_i \wedge X_l(i+1,i+2) + \\ & \overline{X_l(i,i+1)} \wedge R_{i+2}, l \geq 0, \quad i = 1,2, \\ 4) & d\,X_0(i,i+2) = 0, \quad i = 1,2. \end{array} \right. \tag{40}$$

This implies that $A_1(X_{j-i+1}(i,j))$, with $1 \le i \le j \le 4$, is a defining system for $\langle S_1, S_2, S_3, S_4 \rangle$.

Finally we obtain

$$c(A) = \overline{X(1,1)} \wedge X(2,4) + \overline{X(1,2)} \wedge X(3,4) + \overline{X(1,3)} \wedge X(4,4) =$$
$$= \sum_{l=0}^{3} x^l \wedge P_l + x^4 \wedge \{\overline{X_1(1,1)} \wedge X_3(2,4) + \overline{X_2(1,2)} \wedge X_2(3,4) +$$
$$+ \overline{X_3(1,3)} \wedge X_1(4,4) + \overline{X_0(1,2)} \wedge X_4(3,4) + \overline{X_1(1,2)} \wedge X_3(3,4) +$$
$$+ \overline{X_3(1,2)} \wedge X_1(3,4) + \overline{X_4(1,2)} \wedge X_0(3,4)\} + \sum_{l \ge 5} x^l \wedge P_l =$$
$$= \sum_{l \ne 4} x^l \wedge P_l + x^4 \wedge \{c(A_1) + \overline{X_0(1,2)} \wedge X_4(3,4) +$$
$$+ \overline{X_1(1,2)} \wedge X_3(3,4) + \overline{X_3(1,2)} \wedge X_1(3,4) + \overline{X_4(1,2)} \wedge X_0(3,4)\}$$

where P_l, $l \ne 4$, are cocycles from $\mathcal{M}_Y$. The cocycles P_l are expressed in terms of $X_r(i,j)$ but for $l \ne 4$ their explicit forms do not matter.

Since $\operatorname{sdeg} c(A) < 2m$, $c(A)$ is exact if and only if the cocycles P_l are exact for $l = 0, 1, \ldots$. In particular, we have

$$P_4 = c(A_1) + \overline{X_0(1,2)} \wedge X_4(3,4) + \overline{X_1(1,2)} \wedge X_3(3,4) +$$
$$\overline{X_3(1,2)} \wedge X_1(3,4) + \overline{X_4(1,2)} \wedge X_0(3,4) = d\,Q, \quad Q \in \mathcal{M}_Y.$$

It follows from the equality 2) of the system (40) that no one of elements $X_l(i, i+1)$ is a cocycle for $l \ne 2$ and $i = 1, 2, 3$. Passing to the cohomology classes we obtain

$$[c(A_1)] = -([\overline{X_0(1,2)}] \cup [X_4(3,4)] + [\overline{X_1(1,2)}] \cup [X_3(3,4)] +$$
$$+ [\overline{X_3(1,2)}] \cup [X_1(3,4)] + [\overline{X_4(1,2)}] \cup [X_0(3,4)]).$$

Therefore, $[c(A_1)] \in N(H^+(Y) \cdot H^+(Y))$ and that contradicts to the irreducibility of $\langle S_1, S_2, S_3, S_4 \rangle$. This proves the lemma and Theorem D.

Remark. The matrix cohomology classes $[X_l(1,2)]$ and $[X_l(3,4)]$ where $l = 0, 1, 3, 4$ are not arbitrary. The equalities (40) imply relations

$$\overline{S}_1 \cup [X_l(2,3)] + [\overline{X_l(1,2)}] \cup S_3 = 0,$$
$$\overline{S}_2 \cup [X_l(3,4)] + [\overline{X_l(2,3)}] \cup S_4 = 0, \qquad l = 0, 1, 3, 4.$$

The classes $[X_l(2,3)]$ with $l = 0, 1, 3, 4$ are restricted only by the degrees of their matrix entries.

Acknowledgement. The authors were supported by Russian Foundation for Basic Researches (the grants 99-01-01202-a (I. K. B.), 96-15-96877 and 98-01-00749 (I. A. T.)).

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