NOVEMBER 1999 MATH-PH/9911012

Two-term dilogarithm identities related to conformal field theory

ANDREI G. BYTSKO

Steklov Mathematics Institute,
Fontanka 27, St.Petersburg 191011, Russia

Abstract

We study 2×2 matrices A such that the corresponding TBA equations yield $c[A]$ in the form of the effective central charge of a minimal Virasoro model. Certain properties of such matrices and the corresponding solutions of the TBA equations are established. Several continuous families and a discrete set of admissible matrices A are found. The corresponding two-term dilogarithm identities (some of which appear to be new) are obtained. Most of them are proven or shown to be equivalent to previously known identities.

MSC 1991: 11Z05, 33B99, 41A60, 65H10, 81T40

Keywords: effective central charge, TBA, Virasoro characters, dilogarithm identities

1 Introduction.

The (normalized) Rogers dilogarithm is a transcendental function defined for $x \in [0,1]$ as follows

$$L(x) = \tfrac{6}{\pi^2}\Big(\sum_{n=1}^{\infty} \frac{x^n}{n^2} + \frac{1}{2}\ln x \ln(1-x)\Big) . \tag{1.1}$$

It is a strictly increasing continuous function satisfying the following functional equations:

$$L(x) + L(1-x) = 1 , \tag{1.2}$$

$$L(x) + L(y) = L(xy) + L\Big(\tfrac{x(1-y)}{1-xy}\Big) + L\Big(\tfrac{y(1-x)}{1-xy}\Big) . \tag{1.3}$$

Dilogarithm identities of the form

$$\sum_{k=1}^{r} L(x_k) = c , \tag{1.4}$$

where $c \geq 0$ is a rational number, and $x_k \in [0,1]$ are algebraic numbers (i.e. they are real roots of polynomial equations with integer coefficients) arise in different contexts in mathematics and theoretical physics (see e.g., [1] and references therein). In particular, they appear in the description of the asymptotic behaviour of infinite series $\chi(q)$ of the form

$$\chi(q) = q^{\text{const}} \sum_{\vec{m}=\vec{0}}^{\infty} \frac{q^{\vec{m}^{\,t} A \vec{m} + \vec{m}\cdot\vec{B}}}{(q)_{m_1} \ldots (q)_{m_r}} , \tag{1.5}$$

bytsko@pdmi.ras.ru

where $(q)_n = \prod_{k=1}^{n}(1-q^k)$ and $(q)_0 = 1$. Suppose that A and $\vec{B}$ are such that the sum in (1.5) involves only non-negative powers of q (hence $\chi(q)$ is convergent for $0<|q|<1$). Let $q = e^{2\pi i\tau}$, $\mathrm{Im}(\tau)>0$ and $\hat{q} = e^{-2\pi i/\tau}$. The saddle point analysis (see e.g., [2, 3]) shows that the asymptotics of $\chi(q)$ in the $\tau\to 0$ limit is $\chi(q)\sim \hat{q}^{-\frac{c}{24}}$ with c given by (1.4) and the numbers $0\le x_i\le 1$ satisfying the following equations

$$x_i \,=\prod_{j=1}^{r}(1-x_j)^{(A_{ij}+A_{ji})}\,, \qquad i=1,\ldots,r\,. \tag{1.6}$$

Let A be an $r\times r$ matrix with rational entries such that all x_i in (1.6) belong to the interval $[0,1]$. Introduce $c[A] = \sum_{i=1}^{r} L(x_i)$. We will call the matrix A admissible if $c[A]$ is rational. As seen from (1.6), it is sufficient to consider only symmetric A.

The principal aim of this work is to search for admissible 2×2 matrices A such that $c[A]$ has the form of the effective central charge c_{st} of a minimal Virasoro model $\mathcal{M}(s,t)$, i.e.

$$c_{st} \,= 1-\frac{6}{s\,t}\,, \tag{1.7}$$

where s and t are co-prime numbers.

The physical motivation for the formulated mathematical task is twofold. First, equations (1.4) and (1.6) arise in the context of the thermodynamic Bethe ansatz (TBA) approach to the ultra-violet limit of certain (1+1)-dimensional integrable systems [4]. In this case the matrix A is related to the corresponding S-matrix, $S(\theta)$, and c gives the value of the effective central charge of the ultra-violet limit of the model in question. Below we will refer to a system of equations of the type (1.6) as the TBA equations.

Second, equations (1.4) and (1.6) appear in the conformal field theory. Namely, the series (1.5) can be identified for certain A (upon choosing specific $\vec{B}$ and possibly imposing some restriction on the summation over $\vec{m}$) as characters (or linear combinations of characters) of irreducible representations of the Virasoro algebra (see [5] for characters of the minimal models). In this case c is the value of the effective central charge of the conformal model to which the character $\chi(q)$ belongs.

In addition, the search for admissible matrices A has a pure mathematical outcome. It allows us to find many dilogarithm identities and to make a step towards classification of the identities (1.4) for $r=2$ (the complete classification is an open problem that appears to be quite involved).

In the $r=1$ case there are only five algebraic numbers on the interval $[0,1]$ such that c in (1.4) is rational,

$$L(0)\,=0\,,\quad L(1-\rho)\,=\tfrac{2}{5}\,,\quad L(\tfrac{1}{2})\,=\tfrac{1}{2}\,,\quad L(\rho)\,=\tfrac{3}{5}\,,\quad L(1)\,=1\,. \tag{1.8}$$

Here $\rho=\frac{1}{2}(\sqrt{5}-1)$ is the positive root of the equation $x^2+x=1$. Notice that all the values of $c=L(x)$ listed in (1.8) have the form (1.7) (with $(s,t)=(2,3)$, $(2,5)$, $(3,4)$, $(3,5)$, and $st=\infty$ for $c=1$). They correspond, respectively, to

$$A=\infty\,,\quad 1\,,\quad \tfrac{1}{2}\,,\quad \tfrac{1}{4}\,,\quad 0\,. \tag{1.9}$$

These A allow us to construct Virasoro characters of the form (1.5). In particular, $A=\infty$ gives $\chi(q)=1$, which is the only character of the trivial $\mathcal{M}(2,3)$ model, and $A=0$ gives (for $B=0$) the eta-function $\eta(q)$. For the other A we have, for instance, (see [3] and references therein)

$$\chi^{2,5}_{1,1} \,= q^{\frac{11}{60}}\sum_{m=0}^{\infty}\frac{q^{m^2+m}}{(q)_m}\,, \qquad \chi^{3,4}_{1,2} \,= q^{\frac{1}{16}}\sum_{m=0}^{\infty}\frac{q^{\frac{1}{2}m^2+\frac{1}{2}m}}{(q)_m}\,, \tag{1.10}$$

$$\chi^{3,5}_{1,2}+\chi^{3,5}_{1,3} \,= q^{\frac{1}{40}}\sum_{m=0}^{\infty}\frac{q^{\frac{1}{4}m^2}}{(q)_m}\,. \tag{1.11}$$

The observation that all values of c obtained from the $r = 1$ TBA equations are of the form (1.7) motivates our choice of c for the $r = 2$ case. Notice however that in the latter case $0 \le c[A] \le 2$. Therefore, we allow st in (1.7) to acquire negative values (which makes sense in the light of Proposition 2 below), keeping the requirement that $|s|$ and $|t|$ are co-prime. It should be remarked here that another natural candidate for $c[A] \le 2$ is the central charge of the Z_n-parafermionic model [6],

$$c_n = \frac{2(n-1)}{n+2}, \qquad n = 2, 3, 4, \ldots \tag{1.12}$$

As we will see below, this form of c appears in the connection to the $r = 2$ TBA also quite often.

The paper is organized as follows. In section 2 certain properties of the solution to the $r = 2$ TBA equations are described (e.g., we find what classes of A correspond to $c = 1$, $c < 1$ and $c > 1$), and some continuous families of admissible matrices A are found. In section 3 various admissible matrices A (not belonging to continuous families) with $c[A]$ of the form (1.7) are presented. The corresponding dilogarithm identities are obtained and in most cases proven or shown to be equivalent to previously known identities. In section 4 we briefly discuss possible applications and remaining questions.

2 Properties of $r = 2$ TBA equations.

Our aim is to search for such admissible matrices $A = \left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right)$ that the value of $c[A] = L(x) + L(y)$ has the form (1.7) ($|s|$ and $|t|$ are co-prime numbers and st may be negative). Recall that $0 \le x, y \le 1$ satisfy the equations

$$\begin{aligned} x &= (1-x)^{2a}(1-y)^{2b} \\ y &= (1-x)^{2b}(1-y)^{2d}. \end{aligned} \tag{2.1}$$

Let us denote $D := ad - b^2 = \det A$ and introduce the functions $\kappa(t)$ and $\delta(t)$ defined for $t \ge 0$ as follows:

$$\kappa(t) = \xi, \quad \delta(t) = L(\xi), \qquad \text{where} \quad \xi = (1-\xi)^{2t}, \quad 0 \le \xi \le 1. \tag{2.2}$$

Since the summation in (1.5) is taken over non-negative numbers, it is too restrictive to require A to be positive definite. Instead, we impose weaker conditions ensuring that the sum in (1.5) involves only non-negative powers of q:

$$a, d \ge 0, \qquad b \ge -\min(a, d). \tag{2.3}$$

Notice that these are sufficient conditions for (2.1) to have a solution on the interval $[0, 1]$.

For $b = 0$ equations (2.1) decouple and $c[A] = \delta(a) + \delta(d)$. Then, taking the (finite) values of a and d from the list (1.9), we obtain

$$c = \tfrac{4}{5}, \quad \tfrac{9}{10}, \quad 1, \quad \tfrac{11}{10}, \quad \tfrac{6}{5}, \quad \tfrac{7}{5}, \quad \tfrac{3}{2}, \quad \tfrac{8}{5}, \quad 2. \tag{2.4}$$

The first two values are the effective central charges of the $\mathcal{M}(5,6)$ and $\mathcal{M}(5,12)$ minimal models, whereas the last four values correspond to the Z_8, Z_{10}, Z_{13} and Z_∞ parafermionic models.

Another possibility for the $b = 0$ case is to take a to be any positive (rational) number and put $d = (4a)^{-1}$. As seen from (2.1), this leads to $y = 1 - x$, and hence $c[A] = 1$ due to (1.2). In fact, it appears that the set (2.4) exhausts possible rational values of $c[A]$ for $b = 0$ (a rigorous proof of this statement would be desirable). Thus, the $b = 0$ case does not lead to non-trivial $r = 2$ dilogarithm identities. For the rest of the paper we will assume that $b \ne 0$.

Notice that the system (2.1) may in general have several solutions on the interval $[0, 1]$. For example, if $a > 0$, $\frac{1}{2} > b > 0$, $d = 0$ (notice that $\kappa(0) = 1$), the system (2.1) possesses the extra solution $x=0$, $y=1$. Such a situation is undesirable from the physical point of view (x_i in the TBA equations (1.6) are physical entities which should be defined uniquely). Therefore, in the present paper we will deal mainly with such matrices A that the solution of (2.1) is unique.

Proposition 1 *Suppose that A satisfies (2.3) and*

$$D \geq -\frac{1}{2}\max\Big\{d\Big(\frac{1}{\kappa(a)}-1\Big),\, a\Big(\frac{1}{\kappa(d)}-1\Big)\Big\}. \tag{2.5}$$

Then the system (2.1) possesses a unique solution on the interval $[0,1]$.

The proof of this and of the other propositions in this section is given in the Appendix. Equation (2.5) involves the function $\kappa(t)$ which cannot be expressed in terms of elementary functions. It can be reduced to more explicit (although weaker) estimates. For instance, employing the Bernoulli and a Jensen-type inequalities to estimate $\kappa(t)$, we derive that (2.5) holds if $D \geq -ad$ for $d \leq \frac{1}{2}$, $b>0$, and if $D \geq -(2ad)/(2d{+}1)$ for $d > \frac{1}{2}$, $b>0$.

Proposition 2 *Suppose that A is a symmetric invertible $r\times r$ matrix such that the corresponding solution of (1.6) on the interval $[0,1]$ is unique. Then*

$$c\,[A] + c\,[\tfrac{1}{4}A^{-1}] = r\,. \tag{2.6}$$

This proposition explains why it makes sense to allow st in (1.7) to be negative. If $c[A] = 1-\frac{6}{st} > 1$, then $c[\frac{1}{4}A^{-1}] = 1+\frac{6}{st} < 1$. Furthermore, Proposition 2 shows also that it is sufficient to consider only such A that $b>0$. Indeed, if $b<0$, then (2.3) implies that $D>0$. Therefore, the off-diagonal entries of the 'dual' matrix $\frac{1}{4}A^{-1}$ are positive.

Proposition 3 *Suppose that A satisfies (2.3). Then*

$$c[A] > 1 \quad \text{if and only if} \quad b < \tfrac{1}{2} \text{ and } ad < (\tfrac{1}{2}-b)^2; \tag{2.7}$$
$$c[A] = 1 \quad \text{if and only if} \quad b \leq \tfrac{1}{2} \text{ and } ad = (\tfrac{1}{2}-b)^2; \tag{2.8}$$
$$c[A] < 1 \qquad \text{otherwise.} \tag{2.9}$$

Equation (2.8) implies that the solution of (2.1) satisfies the relation $x+y=1$ if and only if the matrix A has the form

$$A = \begin{pmatrix} a & \frac{1}{2}-\sqrt{ad} \\ \frac{1}{2}-\sqrt{ad} & d \end{pmatrix}, \qquad a,d \geq 0\,. \tag{2.10}$$

Notice that here $D=\sqrt{ad}-\frac{1}{4}$ and Proposition 1 cannot guarantee uniqueness of the solution of (2.1) for sufficiently small values of ad. However, as seen from the proof, even if (2.1) has several solutions all they satisfy the relation $x+y=1$.

Proposition 4 *Suppose that A is such that the corresponding solution of (2.1) on the interval $[0,1]$ is unique. Then this solution satisfies the relation $x=y$ if and only if $a=d$.*

This proposition implies that the value of $c[A]$ for a matrix of the form

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \tag{2.11}$$

depends only on $(a+b)$. Indeed, for $x=y$ and $a=d$ the system (2.1) turns into the pair of coinciding equations for one variable. Therefore, $x=y=\kappa(a+b)$ and $c[A]=2\delta(a+b)$.

Thus, the $r = 2$ dilogarithm identity for a matrix A of the form (2.11) reduces to an $r = 1$ identity. Therefore, the only values of $(a + b)$ in (2.11) that correspond to rational value of $c[A]$ are given by the set (1.9). Namely, for $(a + b) = 1,\ \frac{1}{2},\ \frac{1}{4},\ 0$ we obtain, respectively,

$$c = \tfrac{4}{5}, \quad 1, \quad \tfrac{6}{5}, \quad 2\,. \tag{2.12}$$

The value $c = 1$ here corresponds to a particular case ($d = a$, $b = \frac{1}{2} - a$) of the family (2.10). The value $c = \frac{4}{5}$ is the effective central charge of the $\mathcal{M}(5,6)$, $\mathcal{M}(3,10)$ and $\mathcal{M}(2,15)$ minimal models. The existence of the family of matrices (2.11) yielding this value of $c[A]$ was observed in [7]. The following realizations of (1.5) (with certain restriction on the summation) as Virasoro characters are known for this family: $a = \frac{2}{3}$, $b = \frac{1}{3}$ gives $\chi^{5,6}_{1,3}$ and $\chi^{5,6}_{1,1} + \chi^{5,6}_{1,5}$ [3]; $a = b = \frac{1}{2}$ gives $\chi^{5,6}_{1,2}$, $\chi^{5,6}_{1,4}$, $\chi^{5,6}_{2,2}$ and $\chi^{5,6}_{2,4}$ [7]; $a = 1$, $b = 0$ gives $\chi^{3,10}_{1,5}$ [7]. Let us remark that, according to Proposition 1, the solution of (2.1) for the $a+b = 1$ case of (2.11) is unique at least for $a > 0.25$. Numerical computations show that it becomes non-unique for $a < a_0 \approx 0.1$.

To complete the general discussion of the properties of solutions to the system (2.1) let us find some estimates for $c[A]$.

Proposition 5 *Suppose that A satisfies (2.3) and $a \geq d > 0$. Then the following lower and upper bounds on $c[A]$ hold:*

$$\delta(b+d) + L\Big(\big(\kappa(d)\big)^{\frac{a+b}{d}}\Big) \;\leq\; c[A] \;\leq\; \delta(a+b) + \delta(d)\,, \quad \text{for} \quad d \leq b\,; \tag{2.13}$$

$$\delta(b+d) + L\Big(\big(\kappa(\tfrac{D}{a-b})\big)^{\frac{a^2-b^2}{D}}\Big) \;\leq\; c[A] \;\leq\; \delta(a+b) + \delta(\tfrac{D}{a-b})\,, \quad \text{for} \quad d \geq b > 0\,; \tag{2.14}$$

$$\delta(a+b) + \delta(\tfrac{D}{a-b}) \;\leq\; c[A] \;\leq\; 2\delta(b+d)\,, \quad \text{for} \quad b < 0\,. \tag{2.15}$$

As an application of this proposition, we notice that if A is such that $a \geq b \geq d > \xi_0 \approx 3.75$, then $c[A]$ cannot be the effective central charge of a minimal model. Indeed, the smallest non-zero value of c_{st} is $\frac{2}{5}$ (recall that s and t in (1.7) are co-prime), whereas $c[A] \leq \delta(2\xi_0) + \delta(\xi_0) < \frac{2}{5}$.

3 Solutions of $r = 2$ TBA equations and corresponding dilogarithm identities.

Eqs. (2.11) (for $a+b{=}0, \frac{1}{4}, \frac{1}{2}, 1$) and (2.10) are examples of continuous families of admissible matrices A. Now we will present several other admissible matrices A having $c[A]$ in the form (1.7). For completeness, the previously known examples are also listed. Let us remind that, according to Proposition 2, the list of the matrices A below can be doubled by including their duals $\frac{1}{4}A^{-1}$, but this does not lead to new dilogarithm identities.

There exists the well-known representation of the type (1.5) for the characters of $\mathcal{M}(2, 2k{+}1)$ model with $\operatorname{rank} A = k{-}1$ (it provides the sum side of the Andrews-Gordon identities [8]). In the $k = 3$ case the corresponding matrix A is

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad c[A] = 4/7\,. \tag{3.1}$$

The corresponding dilogarithm identity is ($\lambda = 2\cos\frac{\pi}{7}$)

$$L(\tfrac{1}{\lambda^2}) + L(\tfrac{1}{(\lambda^2-1)^2}) \;=\; \tfrac{4}{7}\,. \tag{3.2}$$

The other known example is the following matrix that allows us to construct all characters of the $\mathcal{M}(3,7)$ (see [7], the case of $\chi^{3,7}_{1,2}$ was found earlier in [3])

$$A = \tfrac{1}{4}\begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}, \qquad c[A] = 5/7\,. \tag{3.3}$$

For instance,

$$\chi^{3,7}_{1,3+Q} = q^{\frac{1}{168}} \sum_{\substack{\vec{m}=\vec{0} \\ m_2=Q \bmod 2}}^{\infty} \frac{q^{m_1^2+\frac{3}{4}m_2^2+m_1m_2-\frac{1}{2}m_2}}{(q)_{m_1}(q)_{m_2}}\,, \qquad Q=0,1\,. \tag{3.4}$$

The corresponding dilogarithm identity is ($\lambda = 2\cos\frac{\pi}{7}$)

$$L(\tfrac{1}{\lambda^2}) + L(\tfrac{1}{1+\lambda}) \;=\; \tfrac{5}{7}\,. \tag{3.5}$$

Let us mention that both (3.2) and (3.5) can be derived from the Watson identities [9]

$$L(\alpha) - L(\alpha^2) = \tfrac{1}{7}\,, \quad L(\beta) + \tfrac{1}{2}L(\beta^2) = \tfrac{5}{7}\,, \quad L(\gamma) + \tfrac{1}{2}L(\gamma^2) = \tfrac{4}{7}\,, \tag{3.6}$$

where α, $-\beta$ and $-\gamma^{-1}$ are roots of the cubic

$$t^3 + 2t^2 - t - 1 \;=0 \tag{3.7}$$

such that $\lambda = 1+\alpha = \beta^{-1} = (1-\gamma)^{-1}$. The equivalence of (3.2) to the second equation in (3.6) was shown in [1]. Exploiting the Abel's duplication formula (which follows from (1.3))

$$\tfrac{1}{2}L(x^2) \;= L(x) - L(\tfrac{x}{1+x})\,, \tag{3.8}$$

we establish the equivalence of (3.5) to the second equation in (3.6):

$$L(\tfrac{1}{\lambda^2}) + L(\tfrac{1}{1+\lambda}) \;= L(\beta^2) + L(\tfrac{\beta}{1+\beta}) \;= L(\beta^2) + L(\beta) - \tfrac{1}{2}L(\beta^2) \;= L(\beta) + \tfrac{1}{2}L(\beta^2)\,.$$

Next we describe admissible matrices A obeying a specific pattern. Let us mention that the $a=1$ case was found in [3] and the $a=\frac{1}{2}$, $a=2$ cases in [7].

Proposition 6 *Among the matrices of the form*

$$A \;= \tfrac{1}{2}\begin{pmatrix} 2a & 1 \\ 1 & 1 \end{pmatrix}, \qquad a \geq 0 \tag{3.9}$$

only those with $a = 0,\ \frac{1}{2},\ 1,\ 2, \infty$ *have rational value of* $c[A]$. *These values are, respectively,* $c = 1,\ \frac{4}{5},\ \frac{3}{4},\ \frac{7}{10}, \frac{1}{2}$.

Proof. Denote $u = 1-x$, $v = 1-y$. In these variables equations (2.1) corresponding to (3.9) look as follows:

$$v \;= 1 - uv\,, \qquad 1 - u^2 \;= (u^2)^a\,. \tag{3.10}$$

Using the first of these relations and employing the formulae (1.2)-(1.3), we obtain

$$\begin{aligned} L(x) + L(y) \;&= 2 - L(u) - L(v) \\ &= 2 - L(1-v) - L(u^2) - L(1-u) \;= 2 - L(x) - L(y) - L(u^2)\,, \end{aligned}$$

and hence

$$c[A] \;= L(x) + L(y) \;= 1 - \tfrac{1}{2}L(u^2)\,. \tag{3.11}$$

Thus, $c[A]$ is rational only if $L(u^2)$ belongs to the list (1.8), i.e. $u^2 = 0,\ 1-\rho,\ \frac{1}{2},\ \rho,\ 1$. Noticing that for $w = u^2$ the second equation in (3.10) takes the form $w = (1-w)^{1/a}$, we obtain the possible values of $2a$ as inverse to these in (1.9) (cf. Proposition 2).

For $a=0$ the matrix (3.9) is a particular case of (2.10). For $a=\infty$ the corresponding series (1.5) contains no summation over the first variable and thus reduces to the $r=1$ case giving characters of the $\mathcal{M}(3,4)$ minimal model (for instance, the second character in (1.10)). For $a=\frac{1}{2}$ the matrix (3.9) is a particular case of (2.11). It allows us to construct several characters of the $\mathcal{M}(5,6)$ minimal model [7]. For instance,

$$\chi^{5,6}_{2,2+2Q} = q^{\frac{-1}{120}} \sum_{\substack{\vec{m}=\vec{0} \\ m_2=Q \bmod 2}}^{\infty} \frac{q^{\frac{1}{2}(m_1^2+m_2^2)+m_1m_2+\frac{1}{2}m_1}}{(q)_{m_1}(q)_{m_2}}, \quad Q=0,1. \tag{3.12}$$

The corresponding dilogarithm identity is $2L(1-\rho)=\frac{4}{5}$.

For $a=1$ the matrix (3.9) allows us to construct all characters of the $\mathcal{M}(3,8)$ (see [7], the case of $\chi^{3,8}_{1,2}$ was found earlier in [3]). For instance,

$$\chi^{3,8}_{1,4} = q^{\frac{1}{8}} \sum_{\vec{m}=\vec{0}}^{\infty} \frac{q^{m_1^2+\frac{1}{2}m_2^2+m_1m_2+m_1+\frac{1}{2}m_2}}{(q)_{m_1}(q)_{m_2}}. \tag{3.13}$$

The corresponding dilogarithm identity is

$$L(1-\tfrac{1}{\sqrt{2}}) + L(\sqrt{2}-1) = \tfrac{3}{4}, \tag{3.14}$$

or, equivalently, $L(\frac{1}{\sqrt{2}}) - L(\sqrt{2}-1) = \frac{1}{4}$. The latter relation is just a particular case, $x=\frac{1}{\sqrt{2}}$, of the Abel's duplication formula (3.8). Let us remark that the dual matrix gives $c[\frac{1}{4}A^{-1}]=\frac{5}{4}$ which is the central charge of the Z_6 parafermionic model.

For $a=2$ the matrix (3.9) allows us to construct some characters of the $\mathcal{M}(4,5)$ [7]. For instance,

$$\chi^{4,5}_{2,2} = q^{\frac{1}{120}} \sum_{\vec{m}=\vec{0}}^{\infty} \frac{q^{2m_1^2+\frac{1}{2}m_2^2+m_1m_2+\frac{1}{2}m_2}}{(q)_{m_1}(q)_{m_2}}. \tag{3.15}$$

The corresponding dilogarithm identity is $L(1-\sqrt{\rho}) + L(1-\frac{1}{1+\sqrt{\rho}}) = \frac{7}{10}$, or, equivalently,

$$L(\sqrt{\rho}) + L(\tfrac{1}{1+\sqrt{\rho}}) = \tfrac{13}{10}. \tag{3.16}$$

This identity was found in [7] as a consequence of the formula (3.15). The proof of Proposition 6 provides an algebraic derivation for (3.16) based on the functional relation (1.3).

Now we present a list of admissible matrices A with $c[A]$ in the form (1.7) that have not appeared in the literature before. These are results of computer based search performed bearing in the mind the general properties of $r=2$ TBA equations discussed in the previous section. For some of the corresponding dilogarithm identities we give an explicit algebraic proof or show that they are equivalent to certain known identities. The cases where such a proof is lacking were checked numerically (with a precision of order 10^{-15}).

The effective central charge of the $\mathcal{M}(3,5)$ model is produced by

$$A = \tfrac{1}{4}\begin{pmatrix} 5 & 4 \\ 4 & 4 \end{pmatrix}, \qquad c[A] = 3/5. \tag{3.17}$$

Notice that $c[\frac{1}{4}A^{-1}]=\frac{7}{5}$ is the central charge of the Z_8 parafermionic model. Solving (2.1) for (3.17), we find that $x=1-\delta^2$ and $y=(1+\delta)^{-2}$ where δ is the positive root of the quartic

$$\delta^4 + 2\delta^3 - \delta - 1 = 0. \tag{3.18}$$

Applying the Ferrari's method, we reduce this equation to

$$\delta^2 + \delta = \rho + 1. \tag{3.19}$$

The solution is $\delta = \frac{1}{2}(\sqrt{3+2\sqrt{5}}-1) = \frac{1}{2}(\sqrt{4\rho+5}-1)$. The corresponding dilogarithm identity reads

$$L(1-\delta^2) + L(\tfrac{1}{(1+\delta)^2}) \;= L\Big(\tfrac{1}{2}\sqrt{4\rho+5} - \tfrac{1}{2} - \rho\Big) + L\Big(\tfrac{1}{2} + \tfrac{1}{2}\rho - \tfrac{1}{2}\sqrt{5\rho-2}\Big) \;= \tfrac{3}{5}\,. \tag{3.20}$$

Gordon and McIntosh proved in [10] for the same δ the following identity

$$L(\delta) - L(\delta^3) \;= \tfrac{1}{5}\,. \tag{3.21}$$

Let us show that (3.20) and (3.21) are equivalent. Using (1.2) and (3.8) several times, we find

$$\begin{aligned}
&L(1-\delta^2) + L(\tfrac{1}{(1+\delta)^2}) \;= 1 - L(\delta^2) + L(\tfrac{1}{(1+\delta)^2}) \\
&= 1 - 2L(\delta) + 2L(\tfrac{\delta}{1+\delta}) + 2L(\tfrac{1}{1+\delta}) - 2L(\tfrac{1}{2+\delta}) \;= 1 - 2L(\delta) + 2 - 2L(\tfrac{1}{2+\delta}) \\
&= 3 - 2L(\delta) - 2L(1-\delta^3) \;= 1 - 2\Big(L(\delta) - L(\delta^3)\Big) \;= \tfrac{3}{5}\,.
\end{aligned}$$

In the last line we used that $(2+\delta)^{-1} = 1-\delta^3$ holds due to (3.19).

The central charge of the $\mathcal{M}(3,4)$ model is produced by the following matrices

$$A \;= \tfrac{1}{2}\begin{pmatrix} 4 & 3 \\ 3 & 3 \end{pmatrix}, \qquad c[A] \;= 1/2\,, \tag{3.22}$$

$$A \;= \tfrac{1}{2}\begin{pmatrix} 8 & 3 \\ 3 & 2 \end{pmatrix}, \qquad c[A] \;= 1/2\,. \tag{3.23}$$

Notice that $c[\frac{1}{4}A^{-1}] \;= \frac{3}{2}$ is the central charge of the Z_{10} parafermionic model. Solving (2.1) for (3.22), we find: $x \;= \frac{1}{4}(3-\sqrt{5}) \;= \frac{1}{2}(1-\rho)$, $y \;= \sqrt{5}-2 \;= 2\rho-1$ and the corresponding dilogarithm identity reads

$$L(\tfrac{1}{2} - \tfrac{1}{2}\rho) + L(2\rho-1) \;= \tfrac{1}{2}\,. \tag{3.24}$$

To prove it we introduce $u \;= 1-x$, $v \;= 1-y$ and notice that $u \;= \frac{1}{2}(1+\rho) \;= 1/(2\rho)$ and $v = 2 - u^{-1} = 2(1-\rho)$. Employing (1.2) and (1.3), we obtain:

$$L(u) + L(v) = L(2u-1) + L(\tfrac{1}{2}) + L(\tfrac{v}{2}) = L(\rho) + \tfrac{1}{2} + L(1-\rho) = \tfrac{3}{2}\,,$$

which is equivalent to (3.24) due to (1.2).

Equations (2.1) for (3.23) can be transformed to the form:

$$x^4 - 6x^3 + 13x^2 - 10x + 1 \;= 0\,, \qquad y^4 + 6y^3 - 11y^2 + 6y - 1 \;= 0 \tag{3.25}$$

and $y(3-2x) = (1-x)$. Applying the Ferrari's method, we reduce these equations to

$$x^2 + (\sqrt{2}-3)x \;= 2\sqrt{2}-3\,, \qquad y^2 + 3(\sqrt{2}+1)y \;= \sqrt{2}+1\,. \tag{3.26}$$

The solution is $x = \frac{1}{2}(3-\sqrt{2}) - \frac{1}{2}\sqrt{2\sqrt{2}-1}$, which leads to the following dilogarithm identity:

$$L\Big(\tfrac{3}{2} - \tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}\sqrt{2\sqrt{2}-1}\Big) + L\Big((\tfrac{3}{2}+\sqrt{2})\sqrt{2\sqrt{2}-1} - \tfrac{3}{2} - \tfrac{3}{2}\sqrt{2}\Big) \;= \tfrac{1}{2}\,. \tag{3.27}$$

The effective central charge of the $\mathcal{M}(2,5)$ model is produced by

$$A \;= \tfrac{1}{2}\begin{pmatrix} 8 & 5 \\ 5 & 4 \end{pmatrix}, \qquad c[A] \;= 2/5\,. \tag{3.28}$$

Notice that $c[\frac{1}{4}A^{-1}] = \frac{8}{5}$ is the central charge of the Z_{13} parafermionic model. Solving (2.1) for (3.28), we find that $x = 1 - u_+$ and $y = u_-(u_- - 1)^{-1}$, where $u_+ > 0$ and $u_- < 0$ are the real roots of the quartic

$$u^4 + u^3 + 3u^2 - 3u - 1 = 0. \tag{3.29}$$

Applying the Ferrari's method, we reduce this equation to

$$u^2 - \rho u = 2\rho - 1. \tag{3.30}$$

The solution is $u_\pm = \frac{1}{2}\rho \pm \frac{1}{2}\sqrt{7\rho - 3}$, which leads to the following dilogarithm identity:

$$L\Big(1 - \tfrac{1}{2}\rho - \tfrac{1}{2}\sqrt{7\rho - 3}\Big) + L\Big(\tfrac{1}{2}\sqrt{28\rho + 45} - 2\rho - \tfrac{5}{2}\Big) = \tfrac{2}{5}. \tag{3.31}$$

To prove it we employ (1.2) and (1.3):

$$\begin{aligned} L(x) + L(y) &= L(1 - u_+) + L(1 - \tfrac{1}{1-u_-}) \\ &= 2 - L(u_+) - L(\tfrac{1}{1-u_-}) = 2 - L(\tfrac{u_+}{1-u_-}) - L(\rho) - L(\tfrac{1-u_+}{1-\rho}) \\ &= \tfrac{7}{5} - L(\tfrac{u_+}{1-u_-}) - L(\tfrac{1-\rho+u_-}{1-\rho}) = \tfrac{2}{5} - L(\tfrac{u_+}{1-u_-}) + L(\tfrac{-u_-}{1-\rho}) = \tfrac{2}{5}. \end{aligned}$$

In the last line we used that the relations $u_+ + u_- = \rho$, $u_+u_- = \rho^3$ and $(1-\rho)u_+ = -(1-u_-)u_-$ hold due to (3.30).

The central charge of the $\mathcal{M}(6,7)$ minimal model is produced by (this was noticed earlier by M. Terhoeven (unpublished))

$$A = \tfrac{1}{6}\begin{pmatrix} 8 & 1 \\ 1 & 2 \end{pmatrix}, \qquad c[A] = 6/7. \tag{3.32}$$

Notice that $c[\frac{1}{4}A^{-1}] = \frac{8}{7}$ is the central charge of the Z_5 parafermionic model. Solving (2.1) for (3.32), we derive that $x = \mu^{-1}$ and $y = 1 - \nu$, where $0 < \nu < 1$ and $\mu > 1$ are the real roots of the following equation

$$t^6 - 7t^5 + 19t^4 - 28t^3 + 20t^2 - 7t + 1 = 0. \tag{3.33}$$

The corresponding dilogarithm identity reads $L(\mu^{-1}) + L(1 - \nu) = \frac{6}{7}$, or equivalently

$$L(\nu) - L(\tfrac{1}{\mu}) = \tfrac{1}{7}. \tag{3.34}$$

It would be interesting to clarify whether this identity is related to the Watson identities.

The list is completed with two matrices A such that $d = 0$. As was remarked above, in such a case equations (2.1) have an extra solution $x = 0$, $y = 1$. We however will focus on the 'regular' solution, $0 < x, y < 1$.

$$A = \tfrac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad c[A] = 8/7. \tag{3.35}$$

Solving the corresponding equations (2.1), we find that y satisfies the cubic (3.7) and $x = 1 - y^2$. Therefore, $y = \alpha$, $x = 1 - \alpha^2$ and the dilogarithm identity yielding the value of $c[A]$ in (3.35) is equivalent to the first identity in (3.6):

$$L(x) + L(y) = L(1 - \alpha^2) + L(\alpha) = 1 + L(\alpha) - L(\alpha^2) = \tfrac{8}{7}. \tag{3.36}$$

Notice that this is the central charge of the Z_5 parafermionic model. Let us remark that the dual matrix would have $c[\frac{1}{4}A^{-1}] = \frac{6}{7}$ (which is the central charge of the $\mathcal{M}(6,7)$ minimal model) but it does not satisfy (2.3) and thus Proposition 2 is not applicable.

$$A = \tfrac{1}{18}\begin{pmatrix} 8 & 3 \\ 3 & 0 \end{pmatrix}, \qquad c[A] = 6/5. \tag{3.37}$$

Solving the corresponding equations (2.1), we find that y satisfies the quartic (3.18) and $x = 1 - y^3$. Therefore, $y = \delta$, $x = 1 - \delta^3$ and the dilogarithm identity yielding the value of $c[A]$ in (3.37) is equivalent to the Gordon-McIntosh identity (3.21):

$$L(x) + L(y) = L(1-\delta^3) + L(\delta) = 1 + L(\delta^3) - L(\delta) = \tfrac{6}{5}\,. \tag{3.38}$$

The dual matrix would have $c[\frac{1}{4}A^{-1}] = \frac{4}{5}$ (which is the central charge of the $\mathcal{M}(5,6)$ minimal model) but it does not satisfy (2.3) and thus Proposition 2 is not applicable.

4 Discussion.

To summarize, we studied admissible 2×2 matrices A such that $c[A]$ (or $c[\frac{1}{4}A^{-1}] = 2 - c[A]$) computed via the corresponding TBA equations (2.1) is the effective central charge (1.7) of a minimal Virasoro model. Certain properties of such matrices have been established. In particular, we have described classes of A that have $c[A]$ less, equal or bigger than 1. Some upper and lower bounds for $c[A]$ have been obtained. Several continuous families and a 'discrete' set of admissible matrices A have been found. The corresponding two-term dilogarithm identities have been obtained. Some of them ((3.16), (3.20), (3.27), (3.31), (3.34)) are quite non-trivial and appear to be new. All the found identities but (3.27) and (3.34) have been proved directly by exploiting the functional dilogarithm relations or shown to be equivalent to the Watson and Gordon-McIntosh identities. This serves as a proof that the matrices presented in section 3 (some of them were found by computer based search) are indeed admissible. What the two unproven identities concern, the structure of (3.27) suggests that it presumably can be treated by the standard technique, whereas the status of (3.34) is less clear.

The presented set of matrices A presumably exhausts admissible matrices with not very fractional entries having $c[A]$ of the form (1.7). This can be claimed thanks to the Proposition 5 and the fact that the spectrum of c_{st} is separated from 0 and 2. However, the question whether the set is complete remains open. If the set is complete (or can be completed), it can be used for a classification of massive (1+1)-dimensional integrable models with diagonal scattering by the admissible values of the effective central charge c_{eff} for the corresponding S-matrices. In particular, our results imply that such a model with two massive particles may have in the ultra-violet limit (if the standard TBA analysis applies) c_{eff} of the form (1.7) given by (2.4) or $c = \frac{2}{5}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{7}{10}, \frac{5}{7}, \frac{3}{4}, \frac{6}{7}, \frac{8}{7}$.

Let us remark that a search for $r = 2$ admissible matrices corresponding to other forms of $c[A]$ will be more involved. For instance, the spectrum of c_n given by (1.12) is 'gapless' (i.e., not separated from 2). Therefore, according to Propositions 2 and 3, we will have to consider A with very small and very large entries.

It is interesting to understand whether the found admissible matrices can be employed in (1.5) to construct Virasoro characters. This would allow us to apply the quasi-particle representations [3] to the corresponding conformal models.

Acknowledgments: I am grateful to K. Kokhas for helpful discussions. This work has been completed during the workshop "Applications of integrability" at the Erwin Schrödinger Institute, Vienna and my visit to the Institut für Theoretische Physik, Freie Universität Berlin.

I thank the organizers of the workshop, the members of the ESI and the members of the ITP, FU-Berlin for warm hospitality. This work was supported in part by the grant RFFI-99-01-00101.

Appendix.

Proof of Proposition 1. Eliminating x in (2.1), we obtain

$$y^{\frac{1}{2b}}(1-y)^{-\frac{d}{b}} + y^{\frac{a}{b}}(1-y)^{-\frac{2}{b}D} = 1\,. \qquad \text{(A.1)}$$

Let $f(y)$ denote the l.h.s. of (A.1). For $D \geq 0$ the uniqueness of the solution is obvious since $f(y)$ is monotonic (strictly increasing for $b > 0$ and strictly decreasing for $b < 0$) on the interval $[0,1]$. Consider now the case of $D < 0$ (which implies $b > 0$ because of (2.3)). We have $f(0) = 0$, $f(1) = \infty$ and $f(y)$ is a smooth (but not necessarily monotonic) function for $0 < y < 1$. Eq. (A.1) can have several solutions if $f'(y) \equiv df(y)/dy$ has roots on this interval. The explicit form of $f'(y)$ shows that this can occur only for $y > y_{\min} = a(a-2D)^{-1}$. Furthermore, if (A.1) has several solutions, then among the roots of $f'(y)$ there must be at least one, denote it y_0, such that $f(y_0) < 1$. As seen from (A.1), the necessary condition for this is $y_0 < \kappa(d)$. If this relation is incompatible with the condition $y_0 > y_{\min}$, i.e. $2D \geq -a(\frac{1}{\kappa(d)} - 1)$, then the solution of (A.1) and hence of (2.1) is unique. Considering in the same way the counterpart of (A.1) for x, we obtain the condition $2D \geq -d(\frac{1}{\kappa(a)} - 1)$. Clearly, we can take the lowest of the two bounds.

Proof of Proposition 2. Taking logarithm of the equations in (1.6), multiplying the resulting system with $\frac{1}{2}A^{-1}$ from the left, taking exponents of the new equations, and replacing all x_i by $(1-x_i)$, we obtain exactly equations (1.6) for $\frac{1}{4}A^{-1}$. Exploiting the property (1.2), we infer that $c[\frac{1}{4}A^{-1}] = \sum_{i=1}^{r} L(1-x_i) = \sum_{i=1}^{r}(1-L(x_i)) = r - c[A]$.

Proof of Proposition 3. In the case of $b > \frac{1}{2}$ we have $x < (1-x)^{2a}(1-y) \leq 1-y$. Therefore $c[A] = L(x) + L(y) < L(1-y) + L(y) = 1$. The analogous consideration for $b = \frac{1}{2}$ shows that $x + y = 1$ (and hence $c[A] = 1$) only if $a = 0$ or $d = 0$. Otherwise $x + y < 1$ and hence $c[A] < 1$.

Consider now the $b < \frac{1}{2}$ case. Let $4ad = (2b-1)^2$. Divide the first equation in (2.1) by $(1-y)$ and take its $(2b-1)$-th power. Divide the second equation in (2.1) by $(1-x)$ and take its $2a$-th power. The r.h.s. of the resulting equations coincide. Thus, we obtain

$$\Big(\frac{1-y}{x}\Big)^{1-2b} = \Big(\frac{y}{1-x}\Big)^{2a}, \qquad \text{(A.2)}$$

where the powers on both sides are positive. An assumption that $1-y > x$ leads to a contradiction since then the l.h.s. and the r.h.s. of (A.2) are, respectively, greater and smaller than 1. An assumption that $1-y < x$ leads to analogous contradiction. Thus, we conclude that $1-y = x$. Moreover, any matrix A such that $c[A] = 1$ necessarily satisfies (2.8). Indeed, $c[A] = 1$ implies the relation $x + y = 1$. Substituting it into (2.1), we obtain the conditions $4ad = (1-2b)^2$ and $b \leq \frac{1}{2}$ (the latter one guaranties existence of a solution on the interval $[0,1]$).

The hyperbola $4ad = (1-2b)^2$ divides the quadrant $a \geq 0$, $d \geq 0$ into two disjoint parts. Since $c[A]$ is continuous function of a and d, we infer that $c[A] < 1$ for $4ad > (1-2b)^2$ (because x and y are small for large a and d) and $c[A] > 1$ for $4ad < (1-2b)^2$ (because $x \approx 1$ and $y \approx 1$ for small a and d).

Proof of Proposition 4. Equation (A.1) in the $a = d$ case coincides with its x counterpart, that is x and y obey the same equation. This implies $x = y$ since we required the uniqueness of the solution. The 'only if' part of the proposition is obvious, it suffices to substitute the relation $x = y$ into (2.1).

Proof of Proposition 5. Let $b > 0$. Notice that $a \geq d$ implies $x \leq y$. Indeed, for d and b finite and $a >> d$, it follows from (2.1) that $x \approx 0$ whereas y is finite. Together with Proposition 4 this implies that $x < y$ for all $a > d$ since x and y are continuous functions of a, b, d (cf. (A.1)). Thus, we have $1 - x \geq 1 - y$. Substituting this inequality into (2.1), we obtain

$$(1-y)^{2(a+b)} \leq x \leq \kappa(a+b)\,, \qquad \kappa(b+d) \leq y \leq (1-x)^{2(b+d)}\,. \tag{A.3}$$

This provides the upper bound for x and the lower bound for y. In order to find an upper bound for y we can simply notice that the second equation in (2.1) implies $y < \kappa(d)$. Alternatively, we can first employ (2.1) to express y as follows: $y = (1-y)^{2D/a} x^{b/a}$. Together with $x < y$ this yields $y < \kappa(\frac{D}{a-b})$. Comparing the values of $\frac{D}{a-b}$ and d, we infer that the first upper bound for y is better if $d < b$. Now, if $y < \kappa(t)$, then the definition (2.2) implies also that $1 - y > \kappa(t)^{\frac{1}{2t}}$. Substituting this relation (with $t = d$ or $t = \frac{D}{a-b}$) into the first inequality in (A.3), we obtain the corresponding lower bounds for x. Having found the upper and lower bounds for x and y, we obtain the estimates (2.13) and (2.14) simply exploiting that $L(t)$ and hence $\delta(t)$ are strictly monotonic.

The estimates in (2.15) are derived by similar considerations in the $b < 0$ case.

References

[1] A.N. Kirillov, *Progr. Theor. Phys. Suppl.* **118** (1995) 61.

[2] W. Nahm, A. Recknagel, M. Terhoeven, *Mod. Phys. Lett.* **A8** (1993) 1835.

[3] R. Kedem, T.R. Klassen, B.M. McCoy, E. Melzer, *Phys. Lett.* **B304** (1993) 263.

[4] T.R. Klassen, E. Melzer, *Nucl. Phys.* **B338** (1990) 485.

[5] A. Rocha-Caridy, in *Vertex operators in mathematics and physics*, ed. J. Lepowsky et al (Springer, Berlin, 1985).

[6] V.A. Fateev, A.B. Zamolodchikov, *Sov. Phys. JETP* **62** (1985) 215.

[7] A.G. Bytsko, J. Phys. **A32** (1999) 8045.

[8] G.E. Andrews, *The theory of partitions* (Cambridge U. Press, Cambridge, 1984).

[9] G.N. Watson, *Quart. J. Math. Oxford*, Ser. **8** (1937) 39.

[10] B. Gordon, R.J. McIntosh, *Ramanujan J.* **1** (1997) 431.