ON THE SPECTRAL THEORY OF SURFACES WITH CUSPS

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ABSTRACT. We are interested in the spectral properties of Dirac operators on noncompact surfaces. Under the assumption that 1) the ends of the given surface M are cusps as in the case of finite area surfaces of negative curvature and 2) the geometry of the Dirac bundle in question is closely related to the geometry of M we investigate the essential spectrum of the corresponding Dirac operator D and discuss its Fredholm index.

INTRODUCTION

Let M be a noncompact surface with a complete Riemannian metric. Assume that M has finitely many ends and that each end U of M is a cusp. By this we mean that U is diffeomorphic to $S \times (0, \infty)$, where $S = \mathbb{R}/\mathbb{Z}$ is the circle, the metric on U is of the form

$$f^2 ds^2 + dt^2$$

and has finite area, and the Gauss curvature K satisfies

$$-b^2 \le K = -\partial_t^2 f / f \le -a^2$$

on U, where a, b are some appropriate positive constants. These assumptions imply that M is conformally equivalent to a closed surface $\bar{M}$ with finitely many points deleted.

The most important examples are complete surfaces with finite area and globally pinched negative Gauss curvature. For that class to be included in our discussion, we have to be somewhat modest in the regularity assumption since then the above diffeomorphism need only be C^2, cf. [?], [?]. More precisely, we assume that f is C^1 and that the second partial derivative of f in the t–direction exists and is continuous. When necessary we label the ends by an index k.

Let E be a graded Dirac bundle over M in the sense of Gromov and Lawson, see [?]. That is, E is a bundle of left modules over the Clifford bundle $\mathrm{Cl}\, M$ of M with compatible Hermitian metric and connection together with a parallel field α of unitary involutions which anticommutes with Clifford multiplication by vector fields. These data determine a Dirac operator D and a decomposition

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$E = E^+ \oplus E^-$ into the eigenspaces of α with eigenvalue ±1. We are interested in the spectral properties of D as an unbounded and essentially self–adjoint operator in the space $L^2(M, E)$ of square integrable sections of E.

There is no reason to expect close relations between the geometry of M and the spectral properties of D if the underlying Dirac bundle is not closely related to the geometry of M. To establish such a relation we propose the following notion. We say that a vector bundle E over M with Hermitian metric and connection is *geometric* if there is a parallel twoform S with values in the bundle of skew Hermitian endomorphisms of E such that the curvature R of E satisfies

$$R(X, Y)u = KS(X, Y)u$$

for all vector fields X, Y of M and sections u of E. The class of geometric bundles over M admits all the standard operations on vector bundles and contains all flat bundles and all bundles which are associated to the Riemannian structure or a spin structure of M.

Now let E be a fixed graded Dirac bundle over M and assume that E is geometric. Then for any local orthonormal frame X, Y of TM, $C := XYS(X, Y)$ is a parallel field of Hermitian endomorphisms of E independent of the choice of X and Y and hence is globally defined. We have

$$XY \cdot R(X, Y)u = KCu$$

for any section u of E. Now the field α of involutions of E is parallel and anticommutes with Clifford multiplication by vector fields, hence $C(E^\pm) \subset E^\pm$. We set

$$C^\pm = C|E^\pm : E^\pm \to E^\pm .$$

Let $U = S \times (0, \infty)$ be an end of M. Let $T = \partial_t$ and $X = f^{-1}\partial_s$ and orient U by this frame; if M is oriented, then we assume that the orientations of U and M coincide. The oriented orthonormal frame T, X is a section of the $SO(2)$–principal bundle $SO(U)$ of oriented orthonormal frames on U and determines a trivialization $SO(U) = U \times SO(2)$. Let $\widetilde{SO}(U) = U \times \mathbb{R} \to U \times SO(2)$ be the corresponding lift, where we map $r \in \mathbb{R}$ to the counterclockwise rotation by the angle $2\pi r$. Then $\widetilde{SO}(U)$ is an $\mathbb{R}$–principal bundle over U and any bundle associated to $\widetilde{SO}(U)$ via a unitary representation of $\mathbb{R}$ comes with a Hermitian metric and connection. Now for $c \in \mathbb{R}$ and $w \in \mathbb{C}$ with $|w| = 1$ we let $L(c, w)$ be the complex line bundle associated to $\widetilde{SO}(U)$ via the unitary representation $\exp(2\pi icr)$ of $\widetilde{SO}(2)$, and with atlas given by the holonomy requirement $[s + 1, t, 0, u] = [s, t, 0, wu]$ along the circles $S \times \{t\}$. Note that $L(1, 1)$ is the tangent bundle of U with its natural complex structure determined by the chosen orientation.

The complex area element $\omega_{\mathbb{C}} = iTX$ defines a parallel field of unitary involutions of E over U which commutes with α and C. We set

$$E^\pm(c, \varepsilon) = \{u \in E^\pm|U \mid Cu = cu \, , \omega_{\mathbb{C}}u = \varepsilon u\} \, ,$$

where $c \in \mathbb{R}$ and $\varepsilon \in \{+1, -1\}$. We will show that $E^+(c, \varepsilon)$ is isomorphic to an orthogonal sum of line bundles $L(-\varepsilon c, w)$, and we denote by $m(c, \varepsilon, w)$ the number of times $L(-\varepsilon c, w)$ occurs in this sum. The spectral characteristics which we investigate in this paper are given in terms of C^+ and the integers $m(c, \varepsilon, w)$. All data associated to an end $U = U_k$ will be indexed by k if necessary.

Suppose $E = \operatorname{Cl} M$ is the Clifford bundle over M with the even–odd decomposition. Then $E^+|U = E^+(0, 1) \oplus E^+(0, -1)$, $m(0, \varepsilon, 1) = 1$ and $m(0, \varepsilon, w) = 0$ for $w \neq 1$. If M is oriented and E is the spinor bundle associated to the corresponding spin^c structure, then $E^+|U = E^+(0, 1)$, $m(0, 1, 1) = 1$ and $m(0, 1, w) = 0$ for $w \neq 1$. If E is a spinor bundle associated to a spin structure of M, then $E^+|U = E^+(1/2, 1)$. If the spin structure is the restriction of a spin structure of the closed surface $\bar{M}$, then $m(1/2, 1, -1) = 1$ and $m(1/2, 1, w) = 0$ for $w \neq -1$.

We first analyze the essential spectrum $\operatorname{spec}_e D$ of D which is localized on the ends of M by the decomposition principle. Recall that D is Fredholm if and only if $0 \notin \operatorname{spec}_e D$.

0.1. THEOREM. *1) If* $m_k(c, \varepsilon, 1) = 0$ *for all* k, c, ε, *then* $\operatorname{spec}_e D = \emptyset$.
2) If $m_k(1/2, \varepsilon, 1) = 0$ *for all* k *and* ε, *then* D *is Fredholm.*
3) If $m_k(1/2, \varepsilon, 1) \neq 0$ *and the directional derivative* $X(K)$ *is uniformly bounded on* U_k *for some* k, *then* $\operatorname{spec}_e D = \mathbb{R}$.

Note that the assumption on $X(K)$ in the third assertion holds if the end is warped, that is, if f does not depend on s.

Set $D^+ = D|L^2(M, E^+) \cap C^1(M, E^+)$. In the case when D is Fredholm, we also determine the index of D^+ explicitly.

0.2. THEOREM. *Let* $m = \operatorname{rank} E^+$ *and suppose that* $m_k(1/2, \varepsilon, 1) = 0$ *for all* k *and* ε. *Then*

$$\begin{aligned} \operatorname{ind} D^+ = (m/2 - \operatorname{tr} C^+)\, \chi(M) &+ \frac{1}{2} \sum_{k,c,\varepsilon} \operatorname{sign}(1/2 - c)\, m_k(c, \varepsilon, 1) \\ &- \frac{1}{2} \sum_{k,c,\varepsilon,w} \varepsilon\, \eta(w)\, m_k(c, \varepsilon, w)\,, \end{aligned}$$

where $\eta(1) = 0$ *and* $\eta(\exp(2\pi i \rho)) = 1 - 2\rho$ *for* $0 < \rho < 1$.

We emphasize that in our theorems, we do not need assumptions on derivatives of the curvature or, respectively, assumptions on the asymptotic behaviour of third or higher derivatives of the metric. Special cases of the above theorems were obtained in [**?**], [**?**], [**?**], and [**?**].

In the case when E is the Clifford bundle with the even–odd decomposition, then D^+ is called the *Gauss–Bonnet operator*. We get

$$\operatorname{ind} D^+ = \chi(M) + \#\{\text{ends}\} = \chi(\bar{M})\,.$$

When M is oriented and E is the spinor bundles associated to the spin^c structure of M, then D^+ is called the *Riemann–Roch operator*. In this case

$$\operatorname{ind} D^+ = \big(\chi(M) + \#\{\text{ends}\}\big)/2 = \chi(\bar{M})/2\,.$$

When M is oriented and E is the spinor bundle associated to a a spin structure which extends to a spin structure on $\bar{M}$, then $\operatorname{ind} D^+ = 0$.

Recall that M is conformally equivalent to a closed surface $\bar{M}$ with finitely many points deleted. Hence the Riemannian metric on M is conformally equivalent to a Riemannian metric with cylindrical ends. The latter were discussed in the work of Atiyah, Patodi and Singer [?], and therefore it is interesting to know whether the dimension of the space $\mathcal{H}^2(M,E)$ of square integrable harmonic sections of E is a conformal invariant. Now it is a rather straightforward consequence of our description of geometric bundles in Sec. 2 that $\dim \mathcal{H}^2(M,E^+(1))$ and $\dim \mathcal{H}^2(M,E^-(0))$ are conformally invariant, where

$$E^\pm(c) = \{u \in E^\pm \mid Cu = cu\}\,,$$

but in other cases conformal invariance may fail. Using the the conformal invariance of $\dim \mathcal{H}^2(M,E^-(0))$ the Gauss–Bonnet and Riemann–Roch formulas above are easy consequences of the results in [?], but also follow easily by a direct argument for surfaces with cylindrical ends as in Section 4 of [?].

The plan of the paper is as follows. In Section 1 we discuss the geometry of the ends in more detail. In Section 2 we characterize geometric bundles and determine their structure over the ends of M. In Section 3 we introduce operators which model the Dirac operator along the ends. In Section 4 we prove our main analytical lemmas. In Section 5 we investigate the essential spectrum and prove Theorem **??**. In Section 6 we compute the index formula from Theorem **??**.

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1. The Ends

By assumption each end U is diffeomorphic to $S \times (0,\infty)$, where $S = \mathbb{R}/\mathbb{Z}$ is the circle, and the metric is of the form $f^2 ds^2 + dt^2$, where $f = f(s,t)$ is continuously differentiable with respect to s and twice continuously differentiable with respect to t. We recall that we assume that the area of U is finite. We have $f'' + Kf = 0$, where the *Gauss curvature* $K = K(s,t)$ is continuous and bounded by $-b^2 \le K \le -a^2 < 0$ for appropriate positive constants a, b. Here and below the prime $'$ denotes differentiation with respect to the variable t.

The curves $c_s(.) = (s,.)$ are unit speed geodesics. The circles $S_t = S \times \{t\}$ are perpendicular to the geodesics c_s. The length and geodesic curvature of S_t are given by

$$\bar{f}(t) = \int_0^1 f(s,t)ds \tag{1.1}$$

and

$$\kappa(s,t) = -f'(s,t)/f(s,t)\,, \tag{1.2}$$

respectively. Recall that κ satisfies the *Riccati equation*

$$\kappa' = \kappa^2 + K\,. \tag{1.3}$$

By the assumption on finite area f is the *stable solution* of the Jacobi equation. Hence

$$\begin{gathered} f(s,0)\exp(-bt) \le f(s,t) \le f(s,0)\exp(-at)\,, \\ a \le \kappa(s,t) \le b\,. \end{gathered} \tag{1.4}$$

We also introduce the *averaged geodesic curvature*

$$\bar{\kappa}(t) = \frac{1}{\bar{f}(t)} \int_0^1 \kappa(s,t) f(s,t)ds\,. \tag{1.5}$$

By (**??**),

$$\bar{\kappa}(t) = -\bar{f}'(t)/\bar{f}(t)\,. \tag{1.6}$$

Furthermore,

$$\begin{gathered} \bar{f}(0)\exp(-bt) \le \bar{f}(t) \le \bar{f}(0)\exp(-at)\,, \\ a \le \bar{\kappa}(t) \le b\,. \end{gathered} \tag{1.7}$$

Later it will be important to consider the *arc length* along S_t,

$$r(s,t) = \frac{1}{\bar{f}(t)} \int_0^s f(s,t)ds\,. \tag{1.8}$$

Then

$$r(0,t) = 0 \quad \text{and} \quad r(1,t) = 1\,. \tag{1.9}$$

Let ∇ be the Levi–Civita connection and define the orthonormal frame T, X as in the introduction. This frame is parallel in the T–direction and we have

$$\nabla_X T = -\kappa X \quad \text{and} \quad \nabla_X X = \kappa T \,. \tag{1.10}$$

Denote by $\operatorname{var} \kappa_t$ the *variance* of κ on S_t,

$$\operatorname{var} \kappa_t = \frac{1}{\bar{f}(t)} \int_S (\kappa(s,t) - \bar{\kappa}(t))^2 f(s,t) ds \,. \tag{1.11}$$

The following lemma is well known, compare §7 in [**?**], and gives a sufficient criterion for the eventual vanishing of the variance of κ.

1.12. LEMMA. *If the metric on U is C^3 and $X(K)$ is uniformly bounded, then*

$$\lim_{t \to \infty} \operatorname{var} \kappa_t = 0 \,.$$

2. THE BUNDLES

We fix an end $U = S \times (0,\infty)$ of M and let $SO(U)$ be the $SO(2)$–principal bundle of oriented orthonormal frames over U. Our preferred frame $F = (T, X)$ is a global section of $SO(U)$ and determines a trivialization

$$U \times SO(2) \to SO(U)\,, (s,t,A) \mapsto F(s,t)A\,,$$

and a lift

$$\widetilde{SO}(U) := U \times \mathbb{R} \xrightarrow[\mathrm{id}\times\pi]{} U \times SO(2) = SO(U)$$

of $SO(U)$ to an $\mathbb{R}$–principal bundle over U, where $\pi(r)$ is counterclockwise rotation by the angle $2\pi r$.

The tangent bundle with its natural parallel complex structure — counterclockwise rotation by a right angle — is the complex line bundle associated to the representation $\exp(2\pi i r)$ of $\mathbb{R}$ on $\mathbb{C}$.

The section $\tilde F(s,t) = (s,t,0)$ is a lift of the frame $F = F(s,t)$ and determines lifts $\tilde T = \tilde F_* T$ of T and $\tilde X = \tilde F_* X$ of X along the image of $\tilde F$. In terms of $\tilde T$ and $\tilde X$, the connection form ω and curvature form Ω of the Levi-Civita connection are given by

$$\omega(\tilde T) = 0\,, \quad \omega(\tilde X) = -\kappa i \quad \text{and} \quad \Omega(\tilde T, \tilde X) = -Ki\,.$$

Fix $c \in \mathbb{R}$ and $w \in \mathbb{C}$ with $|w| = 1$. Let $L(c,w)$ be the complex line bundle associated to $\widetilde{SO}(U)$ via the representation $\exp(2\pi i c r)$ of $\mathbb{R}$ on $\mathbb{C}$ with an atlas given by the holonomy relation $[s+1,t,0,u] = [s,t,0,wu]$. Since the representation of $\mathbb{R}$ is unitary, $L(c,w)$ has a natural Hermitian metric. Furthermore, ω determines a unitary connection on $L(c,w)$. It is easy to see that $L(c,w)$ is a geometric bundle; we prove a more precise statement in the lemma below. We also have

$$L(c,w) \otimes L(c',w') = L(c+c', ww')\,. \tag{2.1}$$

2.2. LEMMA. *The line bundle $L(c,w)$ has a section Φ of norm 1 such that*

$$\Phi(s+1,t) = w\Phi(s,t)\,, \quad \nabla_T \Phi = 0\,, \quad \nabla_X \Phi = -\kappa i c \Phi\,, \quad R(T,X)\Phi = -Kic\Phi\,.$$

Proof. In the standard notation for bundles associated to principal bundles let $\Phi = \Phi(s,t) = [s,t,0,1]$. Then Φ has norm 1 and the asserted holonomy. Now under the representation $\exp(2\pi i c r)$, the connection form ω and curvature form Ω are multiplied by c. Hence Φ also satisfies the remaining equations. ☐

We now describe the structure of geometric bundles over U. A similar description is also valid over all of M. However, for ease of presentation we stick to geometric bundles over U.

2.3. LEMMA. *Let E be a geometric bundle over U. Then E is isomorphic to an orthogonal sum of line bundles $L(c,w)$.*

Proof. Let S be the parallel tensor with $R(T,X) = KS(T,X)$ as in the definition of geometric bundles. Since S is parallel and skew symmetric in T and X, $S(T,X)$ is a parallel field of endomorphisms of E over U. Hence E is the direct sum of the parallel and pairwise perpendicular subbundles

$$E(c) = \{u \in E \mid S(T,X)u = -icu\}\,, \quad c \in \mathbb{R}\,.$$

Hence we may assume that $E = E(c)$ for some fixed c.

Let $t > 0$ and Φ, Ψ be sections of $E|S_t$ solving

$$\nabla_X \Phi = -\kappa ic\Phi\,, \quad \nabla_X \Psi = -\kappa ic\Psi\,.$$

Then

$$\begin{aligned} X\langle \Phi, \Psi\rangle &= \langle \nabla_X \Phi, \Psi\rangle + \langle \Phi, \nabla_X \Psi\rangle \\ &= \langle -\kappa ic\Phi, \Psi\rangle + \langle \Phi, -\kappa ic\Psi\rangle = 0\,. \end{aligned}$$

Now extend Φ by parallel translation in the T-direction and recall that $[T,X] = \kappa X$, see (**??**). We get

$$\nabla_T(\nabla_X \Phi) = \nabla_{[T,X]}\Phi + R(T,X)\Phi = -\kappa^2 ic\Phi - Kic\Phi\,.$$

Using (**??**) we also have

$$\nabla_T(-\kappa ic\Phi) = -\kappa' ic\Phi = -(\kappa^2 + K)ic\Phi\,.$$

Hence $\nabla_X \Phi$ and $-\kappa ic\Phi$ solve the same differential equation. Hence $\nabla_X \Phi = -\kappa ic\Phi$ on all of U.

We conclude that there is an orthonormal frame $\Phi_1, \dots, \Phi_m$ of E, where $m = \operatorname{rank} E$, such that $\nabla_T \Phi_j = 0$ and $\nabla_X \Phi_j = -\kappa ic\Phi_j$ for all j. It follows that the holonomy $H = (h_{ij})$, defined by

$$\Phi_j(s+1,t) = \sum h_{ij}\Phi_i(s,t)\,,$$

is independent of (s,t) and unitary. Now an appropriate unitary change of frame with constant coefficients diagonalizes H. ☐

We now turn to geometric Dirac bundles. Fix $c \in \mathbb{R}$, $\varepsilon \in \{-1,+1\}$ and $w \in \mathbb{C}$ with $|w| = 1$. Let

$$L(c,\varepsilon,w) = L(-\varepsilon c, w) \oplus L(\varepsilon(1-c), w)\,.$$

Viewing the Clifford bundle as the bundle associated to $\widetilde{SO}(U)$ via the representation $\exp(2\pi ir)$ on $\mathrm{Cl}\,\mathbb{R}^2$, we describe Clifford multiplication on $L(c,\varepsilon,w)$ by the following representation of $\mathrm{Cl}\,\mathbb{R}^2$ on $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \varepsilon i \\ \varepsilon i & 0 \end{pmatrix}, \quad \begin{pmatrix} -\varepsilon i & 0 \\ 0 & \varepsilon i \end{pmatrix}.$$

Here the matrices describe Clifford multiplication by $e_1, e_2, e_1e_2 \in \mathrm{Cl}\,\mathbb{R}^2$ respectively. Furthermore, let $\alpha = \varepsilon\omega_{\mathbb{C}}$, where $\omega_{\mathbb{C}} = iTX$ is the complex area element.

It is easy to see that $L(c,\varepsilon,w)$ is a graded Dirac bundle with $L^+(c,\varepsilon,w) = L(-\varepsilon c,w)$ and $L^-(c,\varepsilon,w) = L(\varepsilon(1-c),w)$. Moreover, it is immediate that

$$L(c,\varepsilon,w)\otimes L(c',w') = L(c-\varepsilon c',\varepsilon,ww')\,. \tag{2.4}$$

2.5. Lemma. *There are sections* Φ^+ *of* $L^+(c,\varepsilon,w)$ *and* Φ^- *of* $L^-(c,\varepsilon,w)$ *of norm* 1 *such that*

$$\begin{gathered}\Phi^\pm(s+1,t) = w\Phi^\pm(s,t)\,,\quad T\Phi^\pm = \pm\Phi^\mp\,,\quad X\Phi^\pm = \varepsilon i\Phi^\mp\,,\\ \nabla_T\Phi^\pm = 0\,,\quad X\nabla_X\Phi^+ = -\kappa c\Phi^-\,,\quad X\nabla_X\Phi^- = \kappa(1-c)\Phi^+\,,\\ TXR(T,X)\Phi^+ = Kc\Phi^+\,,\quad TXR(T,X)\Phi^- = K(1-c)\Phi^-\,.\end{gathered}$$

Proof. Let Φ^+ be a section of $L(-\varepsilon c,w)$ as in Lemma **??** and set $\Phi^- = T\Phi^+$. It is immediate from the definition of Clifford multiplication that these sections have the asserted properties. ☐

We now describe the structure of graded geometric Dirac bundles over U. As in the case of geometric bundles, there is a similar description for graded geometric Dirac bundles over all of M.

2.6. Lemma. *Let* E *be a graded geometric Dirac bundle over* U *with involution* α. *Then* E *is isomorphic to an orthogonal sum of bundles* $L(c,\varepsilon,w)$.

It is clear that the numbers $m(c,\varepsilon,w)$ from the introduction are equal to the number of times that the bundles $L(c,\varepsilon,w)$ occur in the decomposition of E over U in Lemma **??**.

Proof of Lemma **??**. As in the introduction, let $TXR(T,X) = KC$. We have

$$C(v\cdot u) = v\cdot(1-C)u\,,$$

for any $v\in TM$ and $u\in E$ with the same foot point [1]. Note that E^+ and E^- are invariant under R and Clifford multiplication by TX, hence $C(E^\pm)\subset E^\pm$.

The complex area element $\omega_{\mathbb{C}} = iTX$ commutes with the involution α of E and C and hence $E^\pm$ is the direct sum of the pairwise perpendicular and parallel subbundles

$$E^\pm(c,\varepsilon) = \{u\in E^\pm \mid C(u) = cu,\, \omega_{\mathbb{C}}u = \varepsilon u\}\,,$$

where $c\in\mathbb{R}$ and $\varepsilon\in\{+1,-1\}$. Hence E is the direct sum of the pairwise perpendicular and parallel graded Dirac subbundles

$$E(c,\varepsilon) = E^+(c,\varepsilon)\oplus E^-(1-c,-\varepsilon)\,.$$

Hence we may assume $E = E(c,\varepsilon)$. But then by Lemma **??**, E^+ is isomorphic to the orthogonal sum of line bundles $L(-\varepsilon c,w)$. It follows easily that E is isomorphic to the orthogonal sum of the corresponding graded Dirac bundles $L(c,\varepsilon,w)$. ☐

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[1] For this conclusion we need that $K(s,t)\neq 0$ for some $(s,t)\in U$.

2.7. Examples. 1) The Clifford bundle $\mathrm{Cl}\,U$ together with the even–odd decomposition is equal to $L(0,1,1) \oplus L(0,-1,1)$.

2) The chosen orientation determines a spin^c structure on U. The corresponding spinor bundle is equal to $L(0,1,1)$.

3) There are two spin structures on U. The *trivial spin structure* is $\widetilde{SO}(U)/2\mathbb{Z}$. The spinor bundle associated to the trivial spin structure is $L(1/2,1,1)$.

The correspondence $(s,t) \leftrightarrow \exp(-(t+is))$ identifies U with the unit ball B with midpoint removed. The restriction of the unique spin structure of B to U is called the *nontrivial spin structure.* The spinor bundle associated to the nontrivial spin structure of U is equal to $L(1/2,1,-1)$.

It follows from the above description of spinor bundles and (**??**) that we can factor graded geometric Dirac bundles with involution $\alpha = \omega_{\mathbb{C}}$ by spinor bundles,

$$L(c,1,w) = L(1/2,1,\pm 1) \otimes L(1/2-c,\pm w)\,. \tag{2.8}$$

3. The Model Operators

Lemma **??** implies that the properties of the Dirac operator on a graded Dirac bundle E over an end $U = S \times (0,\infty)$ are determined by the properties of the Dirac operators on the bundles $L(c,\varepsilon,w)$.

For $t > 0$ write

$$C^1(S_t)_w = \{u \in C^1([0,1] \times \{t\}) \mid u(1,t) = u(0,t)/w\}\,.$$

For an interval $I \subset (0,\infty)$ write correspondingly

$$C^1(S \times I)_w = \{u \in C^1([0,1] \times I) \mid u(1,t) = u(0,t)/w\}\,.$$

We let $C^1_0(S \times I)_w$ be the subspace of $u \in C^1(S \times I)_w$ which have compact support in $[0,1] \times I$.

We now transform the Dirac operators on the bundles $L(c,\varepsilon,w)$ to operators which are convenient for our purposes. Define matrices

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{3.1}$$

Then the map

$$\begin{aligned} \Psi : C^1_0\big(S \times (0,\infty)\big)_w \otimes \mathbb{C}^2 &\to C^1_0(U, L(c,\varepsilon,w))\,, \\ u = (u^+,u^-) &\mapsto u^+\Phi^+ + u^-\Phi^-\,, \end{aligned}$$

transforms the Dirac operator to the *model operator*

$$\begin{aligned} D_\infty := \Psi^{-1} D \Psi &= \gamma\big((\partial_t - \kappa/2) + (\varepsilon i X + \kappa(1/2 - c))\tau\big) \\ &= \gamma\big((\partial_t - \kappa/2) + A + \kappa B\big)\,, \end{aligned} \tag{3.2}$$

where

$$A = A(t) = \begin{pmatrix} A^+(t) & 0 \\ 0 & -A^+(t) \end{pmatrix} = \begin{pmatrix} \varepsilon i X & 0 \\ 0 & -\varepsilon i X \end{pmatrix} \tag{3.3}$$

and

$$B = (1/2 - c)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1/2 - c & 0 \\ 0 & c - 1/2 \end{pmatrix}\,. \tag{3.4}$$

The operator D_∞ depends on the Riemannian metric on $U = U_k$ and on the parameters c, ε and w.

We recall that the area element on U is $f ds dt$. For $t > 0$ and functions $u, v \in C^1(S_t)_w \otimes \mathbb{C}^2$ we let

$$(u,v)_t = \int_S \langle u(s,t), v(s,t)\rangle f(s,t) ds\,, \tag{3.5}$$

where $\langle .\,,.\rangle$ denotes the inner product in $\mathbb{C}^2$. For an interval $I \subset (0,\infty)$ and functions $u, v \in C^1(S \times I)_w \otimes \mathbb{C}^2$ we let

$$(u,v)_I = \int_I (u,v)_t dt = \int_I \int_S \langle u(s,t), v(s,t)\rangle f(s,t) ds dt\,, \tag{3.6}$$

denote the L^2–inner product of u and v. We use analogous notation for the corresponding norms. Note that Ψ extends to a unitary map

$$\Psi : L^2\big((S \times (0,\infty); f ds dt\big) \otimes \mathbb{C}^2 \to L^2(U, L(c,\varepsilon,w))\,.$$

We now turn to the family of operators $A = A(t)$ from (**??**) which splits into operators $A^+(t) = \varepsilon i X$ and $A^-(t) = -\varepsilon i X$ according to (**??**). We recall that the length of S_t is $\bar f(t)$ and that X has norm one. Hence we have the following lemma.

3.7. Proposition. *The operator iX is essentially self–adjoint as an operator in $L^2(S_t)$ with domain $C^1(S_t)_w$. Its spectrum is discrete and consists of the eigenvalues*

$$-2\pi(\rho + k)/\bar f(t)\,, \quad k \in \mathbb{Z}\,,$$

*where we write $w = \exp(2\pi i \rho)$. The corresponding eigenspaces are spanned by the functions $\exp(2\pi i(\rho + k)r/\bar f(t))$, where $r = r(s,t)$ denotes arc length along S_t as in (**??**).*

We note that 0 is in the spectrum of $A(t)$ if and only if $w = 1$ and that the nonzero eigenvalues grow like $1/\bar f(t) \to \infty$. Both these observations will be important below.

We let $P_0 = P_0(t)$ be the orthogonal projection onto the kernel of $A(t)$. If $w \neq 1$, then $P_0 = 0$ and if $w = 1$, then

$$P_0 u(s,t) = \bar u(t) = \frac{1}{\bar f(t)} \int_S u(s,t) f(s,t) ds\,. \tag{3.8}$$

Note that $\bar u$ is continuously differentiable if u is and that

$$\operatorname{supp} \bar u \subset \{t \mid \operatorname{supp} u \cap S_t \neq \emptyset\}\,. \tag{3.9}$$

In the case $w = 1$ we will also consider the following symmetric *model operator*

$$D_{\infty 0} := P_0 D_\infty P_0 = \gamma\big((\partial_t - \bar\kappa/2) + \bar\kappa(1/2 - c)\tau\big)\,, \tag{3.10}$$

with domain $C^1_0(0,\infty) \otimes \mathbb{C}^2$ in $L^2\big((0,\infty), \bar f dt\big) \otimes \mathbb{C}^2$. Here we identify functions in $C^1_0(0,\infty) \otimes \mathbb{C}^2$ with functions on $S \times (0,\infty)$ which do not depend on s, compare (**??**). Note that the substitution $\tilde u = \sqrt{\bar f} u$ defines a unitary equivalence of $L^2\big((0,\infty), \bar f dt\big) \otimes \mathbb{C}^2$ and $L^2\big((0,\infty), dt\big) \otimes \mathbb{C}^2$ and transforms $D_{\infty 0}$ into the Dirac system

$$\gamma\big(\partial_t + \bar\kappa(1/2 - c)\tau\big) \tag{3.11}$$

with domain $C^1_0(0,\infty) \otimes \mathbb{C}^2$.

4. The Weapons

We fix an end $U = S \times (0, \infty)$ and consider a model operator D_∞ as in (**??**). We recall that the area element on U is $f\,ds\,dt$. The following lemma is now immediate from the definition of the L^2–inner product (**??**) and partial integration.

4.1. Lemma. *Let $I = [t_0, t_1] \subset (0, \infty)$ and assume that $u, v \in C^1(S \times I)_w \otimes \mathbb{C}^2$. Then*

$$(\partial_t u, v)_I = (u, -\partial_t v + \kappa v)_I + \{(u, v)_{t_1} - (u, v)_{t_0}\}\,.$$

In particular, $\gamma(\partial_t - \kappa/2)$ is symmetric on $C_0^1\big(S \times (0, \infty)\big)_w \otimes \mathbb{C}^2$.

Now P_0 is self–adjoint, see (**??**). Hence the following formulas are immediate from Lemma **??**.

4.2. Lemma. *On $C^1\big(S \times (0, \infty)\big)_w \otimes \mathbb{C}^2$ we have*

$$P_0 \partial_t P_0 = \partial_t P_0 \quad \text{and} \quad P_0 \kappa P_0 = \bar\kappa P_0\,.$$

Therefore we also get

$$\begin{aligned} &(1 - P_0)\partial_t P_0 = 0\,, && (1 - P_0)\kappa P_0 = (\kappa - \bar\kappa)P_0\,, \\ &P_0 \partial_t (1 - P_0) = P_0(\kappa - \bar\kappa)\,, && P_0 \kappa (1 - P_0) = P_0(\kappa - \bar\kappa)\,. \end{aligned}$$

Our first weapon is the following lemma.

4.3. Lemma. *On $C^1\big(S \times (0, \infty)\big)_w \otimes \mathbb{C}^2$ we have*

$$\begin{aligned} P_0 D_\infty P_0 &= \gamma\big((\partial_t - \bar\kappa/2) + \bar\kappa B\big)P_0\,, \\ (1 - P_0)D_\infty P_0 &= \gamma(\kappa - \bar\kappa)(B - 1/2)P_0\,, \\ P_0 D_\infty (1 - P_0) &= \gamma P_0(\kappa - \bar\kappa)(B + 1/2)(1 - P_0)\,. \end{aligned}$$

Proof. By definition, $AP_0 = P_0 A = 0$. Furthermore, $[P_0, \gamma] = [P_0, \tau] = 0$ and $B\gamma + \gamma B = 0$. The formulas are now immediate from Lemma **??**. ☐

By $P_>(t)$ and $P_<(t)$ we denote the spectral projection in $L^2(S_t)_w \otimes \mathbb{C}^2$ corresponding to the eigenvalues of $A(t)$ which are > 0 and < 0, respectively. The following lemma is our second weapon.

4.4. Lemma. *Let $I = [t_0, t_1] \subset (0, \infty)$ be a compact interval and suppose $u \in C^1(S \times I)_w \otimes \mathbb{C}^2$ satisfies $P_>(t_0)u_{t_0} = 0$ and $P_<(t_1)u_{t_1} = 0$. Then*

$$\|Au\|_I \le \|D_\infty u\|_I + C_1\,\|u\|_I\,,$$

where $C_1 = b\max(|c|, |1 - c|)$.

Proof. By approximation we may assume that u is C^2. Now by (**??**), $[\partial_t, X] = \kappa X$. Therefore $\partial_t A u = A\partial_t u + \kappa A u$ and hence, by Lemma **??**,

$$(\partial_t u, Au)_I = (u, -A\partial_t u)_I + \{(u, Au)_{t_1} - (u, Au)_{t_0}\}\,.$$

Now A is symmetric. Therefore our assumption on u implies

$$2\,\mathrm{Re}(\partial_t u, Au)_I = \{(u,Au)_{t_1} - (u,Au)_{t_0}\} \geq 0\,.$$

Hence

$$\begin{aligned}\|D_\infty u\|_I \|Au\|_I &\geq \mathrm{Re}(D_\infty u, \gamma Au)_I \\ &= \mathrm{Re}(\partial_t u, Au)_I + \|Au\|_I^2 + \mathrm{Re}(\kappa(B-1/2)u, Au)_I \\ &\geq \|Au\|_I^2 - b\max(|c|,|1-c|)\|u\|_I\|Au\|_I\,.\end{aligned}$$

☐

Recall that $P_0 = 0$ if $w \neq 1$. We assume now that $w=1$. Then for $u = (u^+,u^-) \in C_0^1\big(S\times(0,\infty)\big)\otimes\mathbb{C}^2$ we have $P_0 u = (\bar u^+, \bar u^-)$, where $\bar u^\pm$ is defined as in (**??**). We set $P_0^+ u = (\bar u^+, 0)$ and $P_0^- u = (0, \bar u^-)$ so that $P_0 = P_0^+ + P_0^-$. Now our third and last weapon reads as follows.

4.5. LEMMA. *Assume* $w=1$ *and* $c \neq 1/2$. *Let* $I = [t_0,t_1] \subset (0,\infty)$ *be a compact interval and suppose* $u \in C^1(S\times I)\otimes\mathbb{C}^2$ *satisfies*

$$\begin{aligned} &P_0^-(t_0)u_{t_0} = 0\,, P_0^+(t_1)u_{t_1} = 0 \quad \text{if } 1/2 < c\,, \\ &P_0^+(t_0)u_{t_0} = 0\,, P_0^-(t_1)u_{t_1} = 0 \quad \text{if } 1/2 > c\,. \end{aligned}$$

Then

$$a|1/2-c|\,\|P_0 u\|_I \leq \|P_0 D_\infty u\|_I + C_2\,\|(1-P_0)u\|_I\,,$$

where $C_2 = b|1/2-c| + (b-a)/2$.

Proof. We set $\tilde\tau = \mathrm{sign}(1/2-c)\,\tau$, with τ from (**??**). Now we have $P_0 A = AP_0 = 0$ by the definition of P_0. Hence

$$\begin{aligned}(P_0 D_\infty u, \gamma\tilde\tau P_0 u)_I &= ((\partial_t - \kappa/2)P_0 u, \tilde\tau P_0 u)_I + ((\partial_t-\kappa/2)(1-P_0)u, \tilde\tau P_0 u)_I \\ &\quad + (\kappa B P_0 u, \tilde\tau P_0 u)_I + (\kappa B(1-P_0)u, \tilde\tau P_0 u)_I\,.\end{aligned}$$

Next we estimate the four terms on the right hand side. By Lemma **??**, the first term is

$$\begin{aligned}((\partial_t-\kappa/2)P_0 u, \tilde\tau P_0 u)_I &= -(P_0 u, (\partial_t - \kappa/2)\tilde\tau P_0 u)_I \\ &\quad + (P_0 u, \tilde\tau P_0 u)_{t_1} - (P_0 u, \tilde\tau P_0 u)_{t_0}\,.\end{aligned}$$

By our assumption

$$(P_0 u, \tilde\tau P_0 u)_{t_1} - (P_0 u, \tilde\tau P_0 u)_{t_0} = \begin{cases} \|\bar u^-\|_{t_1} + \|\bar u^+\|_{t_0} & \text{if } 1/2 < c\,, \\ \|\bar u^+\|_{t_1} + \|\bar u^-\|_{t_0} & \text{if } 1/2 > c\,, \end{cases}$$

so the above computation gives

$$\mathrm{Re}((\partial_t - \kappa/2)P_0 u, \tilde\tau P_0 u)_I \geq 0\,.$$

By Lemma **??**, the second term can be estimated as follows,

$$\begin{aligned}|((\partial_t - \kappa/2)(1-P_0)u, \tilde{\tau}P_0u)_I| &= |(P_0\frac{\kappa-\bar{\kappa}}{2}(1-P_0)u, \tilde{\tau}P_0u)_I| \\ &\leq \frac{b-a}{2}\|(1-P_0)u\|_I\|P_0u\|_I\,.\end{aligned}$$

Recall that $B=(1/2-c)\tau$. Hence the absolute value of the fourth term can be estimated by

$$|(\kappa B(1-P_0)u, \tilde{\tau}P_0u)_I| \leq b|1/2-c|\|(1-P_0)u\|_I\|P_0u\|_I\,.$$

The crucial estimate involves the third term, but this is immediate from the definition of B and $\tilde{\tau}$:

$$(\kappa BP_0u, \tilde{\tau}P_0u)_I \geq a|1/2-c|\|P_0u\|_I^2\,.$$

We conclude that

$$\begin{aligned}&\|P_0D_\infty u\|_I\|P_0u\|_I \geq \\ &\qquad a|1/2-c|\|P_0u\|_I^2 - (b|1/2-c| + (b-a)/2)\|(1-P_0)u\|_I\|P_0u\|_I\,.\end{aligned}$$

This finishes the proof of the lemma. ☐

5. The Essential Spectrum

Since M is complete, the Dirac operator D with domain $C_0^1(M,E)$ is essentially self–adjoint in $L^2(M,E)$, see [?, p.117]. We denote by $\bar{D}$ the closure of D and recall that the *spectrum* $\operatorname{spec}\bar{D}$ of $\bar{D}$ is contained in $\mathbb{R}$.

5.1. Definition. A real number λ belongs to the *discrete spectrum* $\operatorname{spec}_d\bar{D}$ of $\bar{D}$ if λ is an eigenvalue of finite multiplicity of $\bar{D}$ and an isolated point of $\operatorname{spec}\bar{D}$. The complement $\operatorname{spec}_e\bar{D} = \operatorname{spec}\bar{D}\setminus\operatorname{spec}_d\bar{D}$ is called the *essential spectrum* of $\bar{D}$.

In particular, $\bar{D}$ is Fredholm if and only if $0\notin\operatorname{spec}_e\bar{D}$. An easy consequence of Weyl's Criterion and the Rellich Lemma, the so–called Decomposition Principle, says that $\operatorname{spec}_e\bar{D}$ does not depend on compact parts of M, see below.

We say that a sequence (u_n) in $C_0^1(M,E)$ is a *special Weyl sequence (for E)* if

$$\lim_{n\to\infty}\|u_n\| = 1 \quad\text{and}\quad \operatorname{supp}u_n\to\infty \quad\text{as } n\to\infty\,. \tag{5.2}$$

Now the following well known characterization of $\operatorname{spec}_e\bar{D}$ by special Weyl sequences is a version of the Decomposition Principle.

5.3. Lemma. *$\lambda\in\operatorname{spec}_e\bar{D}$ if and only if there is a special Weyl sequence (u_n) with*

$$\lim_{n\to\infty}\|(D-\lambda)u_n\| = 0\,.$$

5.4. Corollary. *Assume that there is a compact subset K of M and a constant $c_K>0$ such that $\|Du\|\ge c_K\|u\|$ for all $u\in C_0^1(M\setminus K,E)$. Then*

$$\operatorname{spec}_e\bar{D}\cap(-c_K,c_K)=\emptyset\,.$$

It is convenient to introduce the following notation. Let $\tilde{E}$ be a Hermitian vector bundle over a complete Riemannian manifold $\tilde{M}$ with compact boundary. Let $\tilde{D}$ be a symmetric first order differential operator on $C_0^1(\tilde{M},\tilde{E})$. Set

$$\begin{aligned}\operatorname{spec}_\infty\tilde{D} = \{\lambda\in\mathbb{R}\mid{}&\text{there is a special Weyl sequence } (u_n)\\ &\text{for } \tilde{E} \text{ with } \lim\nolimits_{n\to\infty}\|(\tilde{D}-\lambda)u_n\|=0\}\,.\end{aligned} \tag{5.5}$$

Then $\operatorname{spec}_e\bar{D} = \operatorname{spec}_\infty D$ by Lemma **??**, but, in general, (**??**) ignores possible contributions from the boundary. The following result is now obvious from Lemma **??**.

5.6. Lemma. *We have $\operatorname{spec}_e\bar{D}=\cup\operatorname{spec}_\infty D_\infty$, where the union is over all k,c,ε, and w with $m_k(c,\varepsilon,w)>0$.*

It remains to discuss $\operatorname{spec}_\infty D_\infty$. In the case $w=1$, the model operator $D_{\infty 0}$ from (**??**) comes into play.

5.7. Theorem. *Fix an end U and constants c,ε, and w. Then we have:*
1) If $w\neq 1$ then $\operatorname{spec}_\infty D_\infty=\emptyset$.
2) If $w=1$ and $\lambda\in\operatorname{spec}_\infty D_\infty$ then $|\lambda|\ge a|1/2-c|$. Furthermore, $\operatorname{spec}_\infty D_\infty\subset\operatorname{spec}_\infty D_{\infty 0}$ with equality if $\limsup\operatorname{var}(\kappa_t)=0$.

Proof. Let $\lambda \in \operatorname{spec}_\infty D_\infty$ and (u_n) be a special Weyl sequence for D_∞ with $(D_\infty - \lambda)u_n \to 0$. Now for any $N > 0$ there is a constant $t_0 > 0$ such that the nonzero eigenvalues of $A(t)$ are of absolute value $\geq N$ for all $t \geq t_0$. Hence $(1 - P_0)u_n \to 0$, by Lemma **??**. This completes the proof of the first assertion.

Assume now that $w = 1$. By what we just said we have $(1 - P_0)u_n \to 0$, hence $\|P_0 u_n\| \to 1$ and therefore, by Lemma **??**,

$$\liminf_{t\to\infty} \|D_\infty u_n\| \geq \liminf_{t\to\infty} \|P_0 D_\infty u_n\| \geq a|1/2 - c|\,.$$

This establishes the inequality in the second assertion. Furthermore, by Lemma **??**,

$$P_0(D_\infty - \lambda)(1 - P_0)u_n = \gamma P_0(\kappa - \bar\kappa)(B + 1/2)(1 - P_0)u_n \to 0\,.$$

Therefore

$$(P_0 D_\infty P_0 - \lambda)P_0 u_n = P_0(D_\infty - \lambda)P_0 u_n \to 0\,.$$

But then $(P_0 u_n)$ is a special Weyl sequence with $(D_{\infty 0} - \lambda)P_0 u_n \to 0$. Hence λ is in $\operatorname{spec}_\infty D_{\infty 0}$, hence $\operatorname{spec}_\infty D_\infty \subset \operatorname{spec}_\infty D_{\infty 0}$.

Suppose now that $w = 1$ and $\limsup_{t\to\infty} \operatorname{var}(\kappa_t) = 0$. Let $\lambda \in \operatorname{spec}_\infty D_{\infty 0}$ and (u_n) be a special Weyl sequence for $D_{\infty 0}$ with $(D_{\infty 0} - \lambda)u_n \to 0$. Then $u_n = P_0 u_n$ and hence

$$\begin{aligned}(D_\infty - \lambda)u_n &= (D_\infty - \lambda)P_0 u_n \\ &= P_0(D_\infty - \lambda)P_0 u_n + (1 - P_0)(D_\infty - \lambda)P_0 u_n \\ &= (D_{\infty 0} - \lambda)P_0 u_n + \gamma(\kappa - \bar\kappa)(B - 1/2)P_0 u_n\,,\end{aligned}$$

by Lemma **??**. Now by assumption, the first term on the right hand side tends to zero as n tends to infinity. The second term tends to zero since $\|P_0 u_n\|$ is uniformly bounded, $\operatorname{supp} u_n \to \infty$, and $\limsup_{t\to\infty} \operatorname{var}(\kappa_t) = 0$. Hence λ is in $\operatorname{spec}_\infty D_\infty$, hence $\operatorname{spec}_\infty D_\infty = \operatorname{spec}_\infty D_{\infty 0}$. ☐

The lower bound on the essential spectrum in part two of the theorem follows in certain cases from the work of McKean [**?**].

We recall that $\bar D$ is Fredholm if and only if $0 \notin \operatorname{spec}_e \bar D$.

5.8. Corollary. *If $m_k(1/2, \varepsilon, 1) = 0$ for all k and ε, then $\bar D$ is Fredholm. On the other hand, if for some k and ε we have $m_k(1/2, \varepsilon, 1) > 0$ and $\operatorname{var}(\kappa_t) \to 0$ as $t \to \infty$, then $\operatorname{spec}_e \bar D = \mathbb{R}$.*

Theorem **??** and Corollary **??** imply Theorem **??** from the introduction.

6. The Index

We are all set for the discussion of the index of D^+. As usual, the ends of M come with an index k. For each end $U = U_k$, we also enumerate the line bundles $L(c, \varepsilon, w)$ which occur in the decomposition of $E|U$ by an additional index j, $L_{kj} = L(c_{kj}, \varepsilon_{kj}, w_{kj})$, where $1 \leq j \leq m = \dim E^+$. These indices will also be used for other objects attached to L_{kj} if necessary.

Throughout this section we assume that $m_k(1/2, \varepsilon, 1) = 0$ for all k. In other words, we assume that $w_{kj} \neq 1$ whenever $c_{kj} = 1/2$. Then the Dirac operator $\bar{D}$ is Fredholm, by Theorem **??** or Corollary **??**, respectively.

Consider an end $U = U_k = S \times (0, \infty)$ and set

$$U_t = U_{k,t} = S \times (t, \infty)\,, \quad S_t = S_{k,t} = S \times \{t\}\,. \tag{6.1}$$

Furthermore set

$$V_t = \cup_k U_{k,t}\,, \quad M_t = M \setminus V_t\,. \tag{6.2}$$

Then M_t is a compact surface with smooth boundary ∂M_t, consisting of the pairwise disjoint circles $S_{k,t}$. We let $\bar{U}_{k,t}$ and $\bar{V}_t$ denote the closures of $U_{k,t}$ and V_t, respectively.

Our aim is to reduce the computation of the index of D^+ to a boundary value problem on M_t, where $t > 0$ is sufficiently large.

By an *elliptic boundary value problem* we mean a pseudodifferential boundary condition which is well–posed in the sense of Seeley [**?**].

6.3. Lemma. *Let $t > 0$. Then any self–adjoint extension of D over M_t or $\bar{V}_t$, defined by an elliptic boundary value problem at ∂M_t is Fredholm.*

Proof. In Theorem 4.1 of [**?**] Brüning and Lesch characterize elliptic boundary value problems in terms of orthogonal projections and show that they are Fredholm. This implies the result for M_t.

The local analysis used for the result just mentioned implies a 'decomposition principle' for self–adjoint extensions of D over $\bar{V}_t$, hence the second assertion since $\bar{D}$ is Fredholm, by Corollary **??**. □

We choose $t_0 >> 0$ and introduce boundary conditions at t_0. To that end we write a section u over U_k as

$$u = \sum_j \left(u_{kj}^+ \Phi_{kj}^+ + u_{kj}^- \Phi_{kj}^- \right)\,, \tag{6.4}$$

where Φ_{kj}^+, Φ_{kj}^- is a frame of L_{kj} as in Lemma **??**. We recall that $P_{kj,>}(t_0)$ and $P_{kj,<}(t_0)$ denote the spectral projections in $L^2(S_{k,t_0})_{w_{kj}} \otimes \mathbb{C}^2$ corresponding to the eigenvalues of $A_{kj}(t_0)$ which are > 0 and < 0, respectively. Since A_{kj} decomposes into two operators according to the decomposition $u = (u^+, u^-)$, we have a corresponding decomposition $P_{kj,>} = P_{kj,>}^+ + P_{kj,>}^-$ and similarly for $P_{kj,<}$.

We also recall the orthogonal projection $P_{kj,0}(t_0)$ onto the kernel of $A_{kj}(t_0)$ in $L^2(S_{k,t_0})_{w_{kj}} \otimes \mathbb{C}^2$ and the decomposition $P_{kj,0} = P^+_{kj,0} + P^-_{kj,0}$ as in Lemma **??**.

In their discussion of the index problem for Dirac operators on compact manifolds with boundary, Atiyah, Patodi, and Singer used the projections

$$P_{kj,APS} = P_{kj,<} + P^+_{kj,0} \tag{6.5}$$

on the component S_{kt_0} of ∂M_{t_0}, assuming in addition that the metric is a product near the boundary, i.e., $f(s,t) = \bar{f}(t_0)$ for $s \in [0,1]$ and t near t_0, see [**?**]. In our discussion we need a variation of this. We set

$$P_{kj}(t_0) = \begin{cases} P_{kj,<}(t_0) + P^+_{kj,0}(t_0) & \text{if } 1/2 < c\,, \\ P_{kj,<}(t_0) + P^-_{kj,0}(t_0) & \text{if } 1/2 > c\,. \end{cases} \tag{6.6}$$

The identity

$$\gamma P_{kj}(t_0) = (1 - P_{kj}(t_0))\gamma \tag{6.7}$$

where γ is as in (**??**), implies that D with each of the following two domains

$$\begin{aligned} \mathcal{D}_{\text{int}} &= \{u \in C^1(M_{t_0}, E) \mid P_{kj}(t_0)(u_{kj}|S_{k,t_0}) = 0 \text{ for all } k, j\ \} \\ \mathcal{D}_{\text{ext}} &= \{u \in C^1_0(\bar{V}_{t_0}, E) \mid (1 - P_{kj}(t_0))(u_{kj}|S_{k,t_0}) = 0 \text{ for all } k, j\ \} \end{aligned}$$

is symmetric. We denote the corresponding operators by D_{int} and D_{ext}, respectively. Then the boundary conditions are elliptic. To see this, we invoke again [**?**, Theorem 4.1]. This requires a localization in the standard form used in [**?**, (4.2)]: the map

$$u \mapsto (f_{t_0}/f_t)^{1/2} u$$

transforms D_{kj} into the operator

$$D_{kj} = \gamma(\partial_t + \tilde{A}(t))\,, \tag{6.8}$$

on the Hilbert space $L^2\big((S \times (0,\infty), f_{t_0} ds dt\big)$. Here

$$\tilde{A}(t_0) = A(t_0) + \kappa(t_0)B =: \tilde{A} = \begin{pmatrix} \tilde{A}_+ & 0 \\ 0 & -\tilde{A}_+ \end{pmatrix}.$$

Next we need to determine a projection $P_+(\tilde{A})$ with the properties listed in (3.10) of [BL], and we have to show that $(P_+(\tilde{A}), P(t_0))$ forms a Fredholm pair. Now it is clear that the projection

$$P_+(\tilde{A}) = \begin{pmatrix} P_{\geq 0}(\tilde{A}_+) & 0 \\ 0 & P_{<0}(\tilde{A}_+) \end{pmatrix}$$

of Atiyah, Patodi, and Singer is an admissible choice. Since B is bounded and $P(t_0)$ differs by a finite rank projection from the analogous projection $P_+(A(t_0))$, the Fredholm property follows from standard perturbation theory. It follows also from Theorem 4.1 in [**?**] (with an obvious modifiction for D_{ext}) that both D_{int} and D_{ext} are essentially self–adjoint. By Lemma **??**, they are both Fredholm.

The boundary projections respect the decomposition $E = E^+ \oplus E^-$, hence we have the subdomains $\mathcal{D}^+_{\text{int}}$ and $\mathcal{D}^+_{\text{ext}}$ and the corresponding restrictions

$$\begin{aligned} D^+_{\text{int}} &: \mathcal{D}^+_{\text{int}} \to L^2(M_{t_0}, E^-)\,, \\ D^+_{\text{ext}} &: \mathcal{D}^+_{\text{ext}} \to L^2(\bar{V}_{t_0}, E^-)\,. \end{aligned}$$

The proof of our index formula below rests on the following result.

6.9. THEOREM (BRÜNING–LESCH, [?]). $\operatorname{ind} D^+ = \operatorname{ind} D^+_{\text{int}} + \operatorname{ind} D^+_{\text{ext}}$.

The proof of this formula consists in showing that the transmission condition at ∂M_{t_0} for sections in $H^1(M, E^+)$ can be deformed into the above boundary conditions in the definition of D_{int} and D_{ext} without affecting the index. The important point is that the interior boundary condition above is independent of the exterior one, so that they can be handled separately. Another important feature of our choice of boundary condition is that the exterior contribution to the index can be computed explicitly: it is zero if t_0 is sufficiently large.

6.10. LEMMA. *For t_0 sufficiently large,* $\ker D_{\text{ext}} = 0$. *In particular,* $\operatorname{ind} D^+_{\text{ext}} = 0$.

Proof. It suffices to consider the (self–adjoint) closure of the operator $D_{kj,\infty} =: D_{kj}$ on $\mathcal{D}_{kj,\text{ext}}$, that is, on $C^1_0\big(S \times [t_0, \infty)\big)_{w_{kj}} \otimes \mathbb{C}^2$ with the boundary condition $(1 - P_{kj}(t_0))$. Let $C_3 > 0$ be a lower bound for the absolute value of the nonzero eigenvalues of the operator $A_{kj}(1)$. Then $C_3\bar{f}(1)/\bar{f}(t_0)$ is a lower bound for the nonzero eigenvalues of $A_{kj}(t)$ for all $t \geq t_0$, by Proposition **??**, hence for any $u \in C^1_0\big([t_0, \infty)\big)_{w_{kj}} \otimes \mathbb{C}^2$ satisfying the boundary condition we have

$$\begin{aligned} C_3 \frac{\bar{f}(1)}{\bar{f}(t_0)} \|(1 - P_0)u\|_{[t_0,\infty)} &\leq \|Au\|_{[t_0,\infty)} \\ &\leq C_4\big(\|D_{kj}u\|_{[t_0,\infty)} + \|(1 - P_0)u\|_{[t_0,\infty)}\big)\,, \end{aligned}$$

by Lemmas **??** and **??**, where C_4 is a constant independent of k, j, and t_0. Since $\mathcal{D}_{kj,\text{ext}}$ is dense in the domain of the closure of D_{kj}, the above inequality and the inequality in Lemma **??** persist to hold in that domain. On the other hand, $\bar{f}(t) \to 0$ as $t \to \infty$ so

$$C_3\bar{f}(1) > C_4\bar{f}(t_0)$$

if t_0 is sufficiently large. Thus if $u \in \ker D_{kj}$, then $(1 - P_0)u = 0$ and so, by Lemma **??**, also $u = 0$. ☐

We now come to the proof of the asserted index formula. By Theorem **??** and Lemma **??** we have

$$\operatorname{ind} D^+ = \operatorname{ind} D^+_{\text{int}}\,, \tag{6.11}$$

and hence it remains to compute $\operatorname{ind} D^+_{\text{int}}$. Now in the standard versions of the index theorem for manifolds with boundary, it is assumed that near the boundary the manifold is a product. To arrive at such a situation, we deform the given metric in a small neighborhood of t_0 by replacing f with

$$f^\alpha = (1-\alpha)f + \alpha(\phi \bar{f}(t_0) + (1-\phi)f)\,, \quad \alpha \in [0,1]\,, \tag{6.12}$$

where ϕ is a suitable cut–off function near t_0. Using the representation (**??**), we see that the family D^α_{int} is graph continuous on $\mathcal{D}_{\text{int}}$, hence has constant index. Hence we can, and will, assume from now on that near the boundary, the metric of M_{t_0} is a product. Thus we are finally in the situation considered in [**?**], except for a variation in the boundary condition. This is taken care of by a result of Agranovich-Dynin, see Theorem 23.1 in [**?**] or Theorem 4.2 in [**?**].

6.13. Theorem. *Denote by* ω_{ind} *the index form of* D. *Then*

$$\begin{aligned} \operatorname{ind} D^+_{\text{int}} = \int_{M_{t_0}} \omega_{\text{ind}} - \frac{1}{2}\sum_{k,j} \dim\ker A^+_{kj}(t_0) + \frac{1}{2}\sum_{k,j} \eta_{kj}(t_0) \\ + \sum_{k,j} \operatorname{ind}(P^+_{kj}(t_0) : \operatorname{im} P^+_{APS}(t_0) \to \operatorname{im} P^+_{kj}(t_0))\,, \end{aligned}$$

where $\eta_{kj}(t_0)$ *denotes the* η*–invariant of* $A^+_{kj}(t_0)$ *and where* P_{APS} *denotes the Atiyah–Patodi–Singer spectral projection.*

It remains to explain and evaluate the different terms on the right hand side of this formula.

We start with the index form ω_{ind}. We could compute ω_{ind} by using the local structure of graded geometric Dirac bundles as in Section 2. However, there is also the following way: By the Local Index Theorem, we have

$$\omega_{\text{ind}}(p) = \lim_{t\to 0} \operatorname{tr}_E[\omega_{\mathbb{C}} e^{-tD^2}(p,p)]\,,$$

hence the computation of ω_{ind} is a local problem. Therefore we may consider an open contractible subset $W \subset M$, for which we choose an orientation. If M is oriented, we assume that the orientations of W and M coincide. The field C is parallel, hence the subbundles $E^\pm$ split over W as a direct sum of the pairwise orthogonal and parallel subbundles

$$E^\pm(c,\varepsilon) = \{u \in E^\pm \mid Cu = cu\,,\, \omega_{\mathbb{C}} u = \varepsilon u\}\,,$$

where $c \in \mathbb{R}$ and $\varepsilon \in \{-1,+1\}$. Now the subbundles

$$E(c,\varepsilon) = E^+(c,\varepsilon) \oplus E^-(1-c,-\varepsilon)\,.$$

are graded Dirac subbundles of E, hence we may assume that $E = E(c,\varepsilon)$. We discuss the case $\varepsilon = +1$ first. Then

$$E^+ = \{u \in E \mid \omega_{\mathbb{C}} u = u\}\,.$$

Let Σ be the spinor bundle associated to the spin structure of W. Then we have

$$E = \Sigma \otimes F \quad \text{with} \quad F = \operatorname{Hom}_{\text{Cliff}}(\Sigma, E)\,,$$

where $\mathrm{Hom}_{\mathrm{Cliff}}(\Sigma, E)$ is the bundle of homomorphisms from Σ to E which are linear over Clifford multiplication, a parallel subbundle of $\mathrm{Hom}(\Sigma, E)$ with the canonical connection. Now the curvature endomorphism C_Σ on $\Sigma^\pm$ is multiplication by $K/2$. By our assumption on E, the curvature endomorphism C_E on $E^\pm$ is multiplication by Kc and $K(1-c)$, respectively. It follows easily that for any oriented orthonormal frame X, Y of W, $R_F(X,Y)$ is multiplication by $Ki(c-1/2)$. Hence

$$\omega_{\mathrm{ind}} = \frac{1}{2\pi}(1/2 - c)\dim F\, KdA = \frac{1}{2\pi}(1/2-c)\dim E^+ KdA\,,$$

by the formula for the index form of twisted Dirac operators as explained in [?].

In the case where $\varepsilon = -1$ we reverse the roles of E^+ and E^-, then we are back in the previous case but ω_{ind} changes sign. We get

$$-\omega_{\mathrm{ind}} = \frac{1}{2\pi}(1/2 - (1-c))\dim E^- KdA = -\frac{1}{2\pi}(1/2-c)\dim E^+ KdA\,.$$

In conclusion,

$$\omega_{\mathrm{ind}} = \frac{1}{2\pi}\{m/2 - \mathrm{tr}(C^+)\}KdA\,,$$

and this gives the first term of the claimed index formula.

Now the second term in the index formula **??** is obviously equal to $-1/2$ times $\#\{(k,j) \mid w_{kj} = 1\}$. Furthermore, note that for each pair (k,j) the projection $P^+_{kj} : \mathrm{im}\, P^+_{APS} \to \mathrm{im}\, P^+_{kj}$ is surjective, by (**??**) and (**??**). It has trivial kernel if $w_{kj} \neq 1$ or if $w_{kj} = 1$ and $c_{kj} > 1/2$. The kernel is of dimension 1 if $w_{kj} = 1$ and $c_{kj} < 1/2$. This gives the second term of the claimed index formula.

If $w_{kj} = 1$, then the spectrum of $A^+_{kj}(t_0)$ is symmetric about 0 and hence $\eta_{kj}(t_0) = 0$ in this case. If $w_{kj} \neq 1$, we write $w_{kj} = \exp(2\pi i\rho)$ with $0 < \rho = \rho_{kj} < 1$. Then if $\varepsilon_{kj} = 1$, we get from Proposition **??**

$$\eta_{kj}(s) = -\rho^{-s} - \sum_{k\geq 1}\{(k+\rho)^{-s} - (k-\rho)^{-s}\}\,.$$

If $\varepsilon_{kj} = -1$, we get the corresponding negative of the right hand side. Now

$$\lim_{s\to 0}\sum_{k\geq 1}\{(k+\rho)^{-s} - (k-\rho)^{-s}\} = -2\rho\,.$$

This concludes the proof of Theorem **??**.

6.14. REMARK. The proof of Theorem **??** gives a somewhat more general result. We only need to assume that the given graded Dirac bundle E is geometric along the ends of M. Then the same index formula holds, except that the first term on the right hand side has to be replaced by the integral over the corresponding index form.

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Irregular Spectral Asymptotics

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Dedicated to Sergio Albeverio

Abstract

In this note we derive the resolvent expansion for elliptic operators with irregular singularities in the coefficients, like Laplace type operators on manifolds with metric horns. The main technical tool is a generalization of the "Singular Asymptotics Lemma" of Brüning and Seeley.

1 Introduction

The resolvent trace expansion for elliptic operators is a well established tool with many important applications in Geometric Analysis, see e.g. [Gi] and its bibliography. It is known for a long time in the case of elliptic differential operators on compact manifolds [MP], [S1] and differential boundary value problems [S2], [Gre]. Recently, this expansion has been extended fully to pseudodifferential elliptic operators and boundary value problems [GruS], [Gru].

The method has been developed further to cover spaces with singularities, notably of conic type, including families of cones [Ch1], [BS2], [BS3]. Here, operators of Laplace type are considered which are most interesting in applications. Typically, near a conic singularity such an operator takes the form

$$\triangle^\circ = -\left(\frac{\partial}{\partial x}\right)^2 + x^{-2}A(x), \quad x \in (0, \varepsilon_0], \tag{1.1}$$

where A is a family of Laplace type operators on the base of the cone (a compact smooth manifold), and A varies smoothly with $x \in [0, \varepsilon_0]$.

The next step towards more complicated (isolated) singularities will introduce operators of the form

$$\triangle^\circ = -\left(\frac{\partial}{\partial x}\right)^2 + x^{-2\alpha_0}A(x), \quad x \in (0, \varepsilon_0], \tag{1.2}$$

where $\alpha_0 \geq 1$ and A is no longer smooth but admits an asymptotic expansion in powers of x, as $x \to 0$. This comprises the case of "metric horns" which refers to Riemannian manifolds with metric singularities of the form $dx^2 + x^{2\alpha_0}g_N$ on $(0, \varepsilon_0] \times N$, with (N, g_N) a compact Riemannian manifold; in this case we have

$$A(x) = x^{-2\alpha_0} A_0 + x^{-\alpha_0 - 1} A_1 + x^{-2} A_2. \tag{1.3}$$

Various aspects of this class of singular spaces have been studied, like Hodge theory [Ch2] and index theorems [LPe]. In [B1], we have announced the full asymptotic expansion of the heat kernel for the Laplacian (Thm 1.2) and have used it to derive a Signature Theorem (Thm.1.3). The proof of the expansion result was based on an extension of the Singular Asymptotics Lemma (SAL) (Thm.2.2) which was introduced in [BS1], a very convenient tool to produce asymptotic expansions in singular situations which admit scaling.

In this note we supply the proofs of [B1, Thms.1.2 and 2.2] and, at the same time, provide substantial generalizations. The main observation concerns the structure of the SAL. Recall that it deals with a function, $\sigma(x, \xi)$, of two variables $x, \xi > 0$, and derives an asymptotic expansion of the function

$$I(\sigma; z) := \int_0^\infty \sigma(x, xz) dx, \tag{1.4}$$

as $x \to \infty$. To do so one assumes an asymptotic expansion of σ as $\xi \to \infty$, with coefficients in $\mathcal{S}(\mathbb{R}_+)$ (as functions of x), and smoothness in $x \geq 0$, in order to use the Taylor expansion at $x = 0$; both expansions are matched by a special "integrability condition". The emphasis on smoothness at $x = 0$ links the SAL in this form closely to the special structure of the Taylor expansion, whereas the expansion at ∞ is unrestricted.

The main generalization we introduce here is that we put the expansions at 0 and at ∞ on an equal footing. This leads us to different remainder estimates which replace the "Integrability Condition" in [BS1]. It seems that they are more easy to verify, but for the time being we do not see the precise relationship between both types of remainder estimates. The price we have to pay is the exclusion of "small eigenvalues" which can be easily treated, however, by ode methods. It should be noted that our expansion includes a new term which is not present if we assume smoothness at $x = 0$.

In Section 2, we prove Theorem 1, the generalization of the SAL, which was announced (in a more restrictive form) in [B1, Thm.1.2]. In Section 3, we use this result to deduce Theorem 2, as announced in [B1, Thm.2.2].

2 The Generalized Singular Asymptotics Lemma

We begin with some terminology which is essentially taken from [L, Ch. II]. By Γ we denote a discrete closed subset of $\mathbb{C}$ with

$$\underline{\gamma} := \inf \mathrm{Re}\ \gamma > -\infty, \tag{2.1}$$

and

$$\Gamma_N := \{\gamma \in \Gamma; \mathrm{Re}\ \gamma \leq N\} \tag{2.2}$$

finite for all $N \in N$, and equipped with a “weight function” $\nu : \Gamma \to \mathbb{Z}_+$. In addition, we write $\Gamma^*_{(N)} := \Gamma_{(N)} \cap \{\mathrm{Re}\, z > 0\}$. Then we consider the following types of asymptotic expansions.

(1) For $f \in L^1_{\mathrm{loc}}(0,1]$, we want the expansion

$$f(x) = \sum_{\substack{\alpha \in \Gamma_N \\ k \leq \nu(\alpha)}} f^0_{\alpha k} x^\alpha \log^k x + R^0_N(f;x), \tag{2.3}$$

with certain numbers $f^0_{\alpha k} \in \mathbb{C}$ and remainder estimate

$$|R^0_N(f;x)| \leq C_{N,\varepsilon} x^{N-\varepsilon}, \tag{2.4}$$

uniformly in $x \in (0,1]$, for every $\varepsilon > 0$. This class of functions we denote by $\mathcal{F}^0_{\Gamma,\nu}$.

(2) For $f \in L^1_{\mathrm{loc}}[1,\infty)$, we want the expansion

$$f(x) = \sum_{\substack{\beta \in \Gamma_N \\ l \leq \nu(\beta)}} f^\infty_{\beta l} x^{-\beta} \log^l x + R^\infty_N(f;x), \tag{2.5}$$

with certain numbers $f^\infty_{\beta l} \in \mathbb{C}$ and remainder estimate

$$|R^\infty_N(f;x)| \leq C_{N,\varepsilon} x^{\varepsilon-N}, \tag{2.6}$$

uniformly in $x \in [1,\infty)$, for every $\varepsilon > 0$. This class of functions we denote by $\mathcal{F}^\infty_{\Gamma,\nu}$.

Then we put

$$\mathcal{F}_{\Gamma,\nu} := \{f \in L^1_{\mathrm{loc}}(0,\infty); f_0 := f \mid (0,1] \in \mathcal{F}^0_{\Gamma,\nu}, f_\infty := f \mid [1,\infty) \in \mathcal{F}^\infty_{\Gamma,\nu}\}; \tag{2.7}$$

this is the function space we are interested in. We can define the Mellin transform on $\mathcal{F}_{\Gamma,\nu}$ by

$$\mathcal{M}f(z) := \mathcal{M}f_0(z) + \mathcal{M}f_1(z). \tag{2.8}$$

$\mathcal{M}f$ is meromorphic in $\mathbb{C}$, hence we can define the “regularized integral” by

$$\fint_0^\infty f(x)dx := \mathrm{Res}_0 \mathcal{M}f(1), \tag{2.9}$$

where “Res_0” denotes the constant term in the Laurent expansion. This is the appropriate notion for our purposes.

We also employ the notion of “two-variable asymptotic expansion”, associated with Γ and ν, as explained e.g. in [B2]. By this we mean an asymptotic expansion for functions, f, in $L^1_{\mathrm{loc}}((0,1] \times (0,1])$ of the form

$$f(\varepsilon, w) = \sum_{\substack{\alpha,\beta\in\Gamma_N \\ k\le\nu(\alpha), l\le\nu(\beta)}} f^{kl}_{\alpha\beta}\varepsilon^\alpha \log^k\varepsilon \;\; w^\beta \log^l w + R_N(f;\varepsilon, w), \tag{2.10}$$

with certain $f^{kl}_{\alpha\beta} \in \mathbb{C}$ and remainder estimate

$$|R_N(f;\varepsilon, w)| \le C_{N,\varepsilon}(\varepsilon + w)^{N-\varepsilon}, \tag{2.11}$$

uniformly in $0 < \varepsilon, w \le 1$, for every $\varepsilon > 0$. We will mainly use the fact that (2.10) determines the coefficients $f^{kl}_{\alpha\beta}$ uniquely.

We are interested in the asymptotic expansion of expressions of the form

$$I(\sigma; z) := ⨍_0^\infty \sigma(x, xz)dx, \quad 1 \le z \to \infty, \tag{2.12}$$

where σ is in $C((0,\infty) \times (0,\infty), \mathbb{C})$. To ensure the existence of the integral and its asymptotic expansion, we introduce two assumptions.

(A_0) For any $C_0 > 0$ and $0 < x \le C_0$, $\xi \in (0,\infty)$, $N \in \mathbb{N}$, we have the expansion

$$\sigma(x,\xi) = \sum_{\substack{\alpha\in\Gamma_N \\ k\le\nu(\alpha)}} \sigma^0_{\alpha k}(\xi)x^\alpha \log^k x + R^0_N(x,\xi),$$

where $\sigma^0_{\alpha k} \in \mathcal{F}_{\Gamma,\nu}$ and, for $0 < \varepsilon \le 1$ and certain numbers $\alpha^0_0 \le 0, \alpha^0_\infty \ge 0$,

$$|R^0_N(x,\xi)| \le C(N,\varepsilon,C_0)x^{N-\varepsilon}\xi^{\alpha^0_0}(1+\xi)^{\alpha^0_\infty}.$$

(A_∞) For any $C_0 > 0$ and $\xi \ge 1/C_0$, $x \in (0,\infty)$, $N \in \mathbb{N}$ we have the expansion

$$\sigma(x,\xi) = \sum_{\substack{\beta\in\Gamma_N \\ l\le\nu(\beta)}} \sigma^\infty_{\beta l}(x)\xi^{-\beta} \log^l \xi + R^\infty_N(x,\xi),$$

where $\sigma^\infty_{\beta l} \in \mathcal{F}_{\Gamma,\nu}$ and, for $0 < \varepsilon \le 1$ and certain numbers $\alpha^\infty_0 \le 0$, $\alpha^\infty_\infty \ge 0$,

$$|R^\infty_N(x,\xi)| \le C(N,\varepsilon,C_0)\xi^{\varepsilon-N}x^{\alpha^\infty_0}(1+x)^{\alpha^\infty_\infty}.$$

We point out the following consequence of (A_0), (A_∞): as $x \to 0$, $\xi \to \infty$ we obtain the expansion, from (A_0),

$$\sigma(x,\xi) \sim \sum_{\substack{\alpha,\beta\in\Gamma \\ k\le\nu(\alpha), l\le\nu(\beta)}} [\sigma^0_{\alpha k}]^\infty_{\beta l} \;\; x^\alpha \log^k x \;\; \xi^{-\beta} \log^l \xi,$$

and from (A_∞),

$$\sigma(x,\xi) \sim \sum_{\substack{\alpha,\beta\in\Gamma \\ k\le\nu(\alpha),l\le\nu(\beta)}} [\sigma^{\infty}_{\beta l}]^{0}_{\alpha k}\ x^{\alpha}\log^{k} x\ \ \xi^{-\beta}\log^{l}\xi.$$

By uniqueness of the expansion coeffcients, we conclude

$$[\sigma^{0}_{\alpha k}]^{\infty}_{\beta l} = [\sigma^{\infty}_{\beta l}]^{0}_{\alpha k} =: \sigma_{\alpha k,\beta l}. \tag{2.13}$$

Now we can formulate the first result of this paper.

THEOREM 1. *Let* $\sigma \in C((0,\infty)\times(0,\infty),\mathbb{C})$. *Under the assumptions* (A_0) *and* (A_∞) *we have the following asymptotic expansion, as* $1 \le z \to \infty$*:*

$$I(\sigma;z) \sim \sum_{\substack{\alpha\in\Gamma \\ k\le\nu(\alpha)}} {\int\!\!\!\!\!-}_{0}^{\infty} (x/z)^{\alpha}\log^{k}(x/z)\sigma^{0}_{\alpha k}(x)dx/z \tag{2.14}$$

$$+ \sum_{\substack{\beta\in\Gamma \\ l\le\nu(\beta)}} {\int\!\!\!\!\!-}_{0}^{\infty} (xz)^{-\beta}\log^{l}(xz)\sigma^{\infty}_{\beta l}(x)dx \tag{2.15}$$

$$+ \sum_{\substack{\alpha\in\Gamma \\ k\le\nu(\alpha) \\ 0\le k''\le k, 0\le k'\le\nu(\alpha+1)}} z^{-\alpha-1}\log^{k+k'+1} z \binom{k}{k''}(-1)^{k-k'+1}(k+k'+1)^{-1}[\sigma^{0}_{\alpha k}]^{0}_{-\alpha-1,k'} \tag{2.16}$$

$$+ \sum_{\substack{\beta\in\Gamma \\ k\le\nu(\beta+1),l\le\nu(\beta) \\ 0\le k'\le k}} z^{-\beta}\log^{k+l+1} z \binom{k}{k'}(-1)^{k-k'}(k'+l+1)^{-1}\sigma_{\beta l;\beta-1,k}. \tag{2.17}$$

In particular, $I(\sigma;z) \in \mathcal{F}^{\infty}_{\Gamma,\tilde\nu}$ *with* $\tilde\nu(\alpha) = \nu(\alpha)+\nu(\alpha+1)+1$.

REMARKS. Comparing with [BS1] we see that the terms (Ia) and (Ib) there - which equal (2.14) and (2.15) here - are now seen in a more symmetric way; indeed, our condition (A_0) replaces the Taylor expansion used before. Also, the fairly subtle integrability condition [BS1, (1.2b)] and the remainder estimate [BS1, (1.2a)] are now replaced by the rather straightforward estimates in (A_0) and (A_∞). They are enough for the application we have in mind here but it is not clear at the moment how they relate to the previous assumptions in general. In particular, it is not obvious how they imply the vanishing of (2.16).

The systematic use of the regularized integral throughout also adds some transparency. In particular, we see that the "singular" contributions (2.17) - which corresponds to [BS1, (Ic)] - and the new contribution (2.16) arise simply from the formula for the change of variables [L, Lemma 2.1.4]; this was pointed out in [L,Ch.II].

Proof. We clearly need to split the integral into a part where x is small, $x \leq C_0$, and one where x is large, $xz \geq 1/C_0$, in order to use the different asymptotics (A_0), (A_∞). This will be the case if we split at $x = 1/C_0 z$ where the choice of C_0 is arbitrary. We take advantage of this by introducing two new expansion variables:

$$\varepsilon \in (0,1] \ \text{ and } \ w := \varepsilon z \geq 1. \tag{2.18}$$

Then we write

$$\begin{aligned} I(\sigma; z) &= \fint_0^\infty (\phi(xw) + (1-\phi(xw)))\sigma(x, xz)dx \\ &=: I_0(\sigma; \varepsilon, w) + I_\infty(\sigma; \varepsilon, w), \end{aligned} \tag{2.19}$$

with $0 \leq \phi \in C_o^\infty(-2,2)$ and $\phi \mid [-1,1] = 1$. We will show that I_0 and I_∞ admit asymptotic expansions as $\varepsilon^2 + w^{-2} \to 0$, which will imply the theorem.

Beginning with I_0, we obtain from (A_0) and [L, Lemma 2.1.4.], since $x \leq 2$ on the support of the integrand,

$$\begin{aligned} I_0(\sigma; \varepsilon, w) &= \fint_0^\infty \phi(xw)\sigma(x, xz)dx \\ &= \fint_0^\infty \phi(xw) \left[\sum_{\substack{\alpha \in \Gamma_N \\ k \leq \nu(\alpha)}} x^\alpha \log^k x \ \ \sigma^0_{\alpha k}(xz) + R^0_N(x, xz) \right] dx \end{aligned}$$

$$\begin{aligned} =: \quad & \sum_{\substack{\alpha \in \Gamma \\ k \leq \nu(\alpha)}} \{ \fint_0^\infty \phi(x\varepsilon)(x/z)^\alpha \log^k(x/z)\sigma^0_{\alpha k}(x)dx/z \\ + \quad & \sum_{\substack{\alpha \in \Gamma \\ k \leq \nu(\alpha) \\ 0 \leq k'' \leq k, 0 \leq k' \leq \nu(\alpha+1)}} z^{-\alpha-1} \log^{k+k'+1} z \binom{k}{k''} (-1)^{k-k'+1}(k+k'+1)^{-1}[\sigma^0_{\alpha k}]^0_{-\alpha-1,k'} \\ + \quad & \tilde{R}^0_N(\sigma; \varepsilon, \omega). \end{aligned} \tag{2.20}$$

Now, (2.21) already gives the first singular contribution, (2.16). (2.20) we rewrite as

$$\sum_{\substack{\alpha \in \Gamma \\ k \leq \nu(\alpha)}} [\fint_0^\infty (x/z)^\alpha \log^k(x/z)\sigma^0_{\alpha k}(x)dx/z \tag{2.21}$$

$$- \fint_0^\infty (1-\phi(x\varepsilon))(x/z)^\alpha \log^k(x/z)\sigma^0_{\alpha k}(x)dx/z], \tag{2.22}$$

such that (2.23) gives (2.14), while (2.24) will be dealt with later.

Using (A_0) again, we estimate the remainder (2.22):

$$
\begin{aligned}
|\tilde{R}_N^0(\sigma;\varepsilon,w)| &\leq C_N \int_0^\infty \phi(xw)x^{N-1/2}(xz)^{\alpha_0^0}(1+xz)^{\alpha_\infty^0}dx \\
&\leq C_N \ z^{\alpha_0^0+\alpha_\infty^0} \ w^{-N-1/2-\alpha_0^0}. \qquad (2.23)
\end{aligned}
$$

I_∞ is now treated analogously. Since $xz \geq \varepsilon^{-1} \geq 1$ on the support of the integrand, we can use (A_∞) to get

$$
\begin{aligned}
I_\infty(\sigma;\varepsilon,w) &= \fint_0^\infty (1-\phi(xw)) \left[\sum_{\substack{\beta\in\Gamma_N \\ l\leq\nu(\beta)}} \sigma_{\beta l}^\infty(x)(xz)^{-\beta}\log^l xz + R_N^\infty(x,xz) \right] dx \\
&=: \sum_{\substack{\beta\in\Gamma_n \\ l\leq\nu(\beta)}} \{ \fint_0^\infty (xz)^{-\beta}\log^l xz \ \ \sigma_{\beta l}^\infty(x)dx \qquad (2.24) \\
&- \fint_0^\infty \phi(xw)(xz)^{-\beta}\log^l xz \ \ \sigma_{\beta l}^\infty(x)dx \} \qquad (2.25) \\
&+ \tilde{R}_N^\infty(\sigma;\varepsilon,w). \qquad (2.26)
\end{aligned}
$$

Again, (2.27) gives the desired term (2.15); and the remainder we estimate using (A_∞):

$$
\begin{aligned}
|\tilde{R}_N^\infty(\sigma;\varepsilon,w)| &\leq C_N \int_{w^{-1}}^\infty (xz)^{1/2-N}x^{\alpha_0^\infty}(1+x)^{\alpha_\infty^\infty}dx \\
&= C_N \int_{\varepsilon^{-1}}^\infty z^{-1-\alpha_0^\infty}x^{1/2-N+\alpha_0^\infty}(1+x/z)^{\alpha_\infty^\infty}dx \\
&\leq C_N \ z^{-1-\alpha_0^\infty} \ \varepsilon^{N-3/2-\alpha_0^\infty-\alpha_\infty^\infty} \\
&\leq C_N \ z^{-\alpha_0^\infty} \ \varepsilon^{N-(3/2+\alpha_0^\infty+\alpha_\infty^\infty)}. \qquad (2.27)
\end{aligned}
$$

It remains to deal with the terms (2.27) and (2.24). In the integrand of (2.24) we have $x \geq \varepsilon^{-1}$, so we plug in the expansion of $\sigma_{\alpha k}^0$ as $x \to \infty$; the resulting remainder of order $L > N$ can be estimated by

$$
C_L \int_{\varepsilon^{-1}}^\infty \ z^{1/2-\alpha} \ x^{\alpha-L}dx = C_L \ z^{1/2-\alpha} \ \varepsilon^{L-\alpha-1} \leq C_L \ z^{1/2-\gamma} \ \varepsilon^{L-N-1}.
$$

Using [L, II (1.12)], this leads to the expansion

$$
\begin{aligned}
& - \sum_{\substack{\alpha\in\Gamma_N\\ k\le\nu(\alpha)}} \fint_0^\infty (1-\phi(x\varepsilon))(x/z)^\alpha \log^k x/z\ \ \sigma^0_{\alpha k}(x)dx/z \\
= & \sum_{\substack{\alpha\in\Gamma_N,\beta\in\Gamma_L\\ k\le\nu(\alpha),l\le\nu(\beta)}} [\sigma^0_{\alpha k}]^\infty_{\beta l} \fint_0^\infty \phi(x\varepsilon)(x/z)^\alpha \log^k (x/z)^{x-\beta} \log^l x\ \ dx/z \\
+ & \ O_{N,L}(z^{1/2-\gamma}\varepsilon^{L-N-1}).
\end{aligned}
\tag{2.28}
$$

In the integrand of (2.27) we have $x \le 2w^{-1}$, so we can use the expansion of $\sigma^\infty_{\beta l}$ as $x \to 0$. Then we obtain quite similarly:

$$
\begin{aligned}
& - \sum_{\substack{\beta\in\Gamma_N\\ l\le\nu(\beta)}} \fint_0^\infty \phi(xw)(xz)^{-\beta}\ \ \log^l(xz)\sigma^\infty_{\beta l}(x)dx \\
= & - \sum_{\substack{\beta\in\Gamma_N,\alpha\in\Gamma_L\\ l\le\nu(\beta),k\le\nu(\alpha)}} [\sigma^\infty_{\beta l}]^0_{\alpha k} \fint_0^\infty \phi(x\varepsilon z)(xz)^{-\beta} \log^l(xz) x^\alpha \log^k x dx \\
+ & \ O_{N,L}(z^{-\gamma}w^{-(L+1/2-N)}).
\end{aligned}
\tag{2.29}
$$

But now we see from (2.13) and [L, Lemma 2.1.4] again that the terms in (2.30) and (2.31) with $\alpha, \beta \in \Gamma_N$ add up to the term

$$
\sum_{\substack{\beta\in\Gamma_N\\ k\in\nu(\beta+1),l\le\nu(\beta)}} \sigma_{\beta l;\beta-1,k}\ \ z^{-\beta}\ \ \log^{k+l+1} z \sum_{k'=0}^{k} (-1)^{k-k'} \binom{k}{k'} (k'+l+1)^{-1},
\tag{2.30}
$$

which coincides with the second singular contribution, (2.17). It now remains to estimate the terms in (2.30) with $N \le \beta < L$, and in (2.31) with $N \le \alpha < L$. For definiteness, we choose $L = 2N$. Now, each such term from (2.30) or (2.31) is a finite linear combination of the functions $w^{-\alpha-1}\varepsilon^\beta \log^{k'}\varepsilon \log^{l'} w$ with $0 \le k', l' \le \nu(\alpha)+\nu(\beta)+1$. Thus it suffices to observe that we can estimate

$$
w^{-\alpha}\varepsilon^\beta \le \begin{cases} 1/2(z^{-N/2} + z^{-\gamma}\varepsilon^{N/2}), & \beta \ge \max\{\alpha, N\}; \\ 1/2(z^{-N/2} + z^{-\gamma}w^{-N/2}), & \alpha \ge \max\{\beta, N\}. \end{cases}
\tag{2.31}
$$

Collecting all the remainder terms, (2.25), (2.29), (2.30), (2.31), and (2.33), we see that there is a problem only if one of the quantities

$$
\delta_1 := \alpha^0_0 + \alpha^0_\infty,\ \ \delta_2 := -\alpha^\infty_0,\ \ \delta_3 := 1/2-\gamma,\ \ \delta_4 := -\gamma,
$$

is positive. Thus, we put

$$
\delta := \sum_{i=1}^{4} |\delta_i|
$$

and consider the expansion of $\tilde{I}(\sigma;z) := z^{-\delta}I(\sigma;z)$ resulting from our considerations. Then all remainder terms are small, so we obtain an asymptotic expansion of $\tilde{I}$ hence also of I. The proof is complete. □

3 Irregular Singular Equations

We want to apply Theorem 1 to obtain the resolvent expansion of certain elliptic equations with irregular singularities. The prototypical example is given by the differential operator

$$\triangle^\circ := -\left(\frac{\partial}{\partial x}\right)^2 + \frac{a}{x^{2\alpha_0}}, \quad x > 0, \tag{3.1}$$

with domain $C_0^2(0,\infty)$, here $a > 0$ and $\alpha_0 \geq 1$. $\triangle^\circ$ is symmetric in $L^2(\mathbb{R}_+)$ and bounded from below, hence we can consider its Friedrichs extension, $\triangle$, which is self-adjoint and bounded from below with the same lower bound. The expansion of the resolvent trace (or, almost equivalently, of the heat trace) for $\triangle$ has been announced by Callias [Ca]. In [B1], we have announced a different proof, in the spirit of 'regular singular analysis', and have applied it to derive the Signature Theorem for metric horns. Here we supply the missing details, in a more general and probably also more accessible form then studied before.

In order to apply the results to singular spaces like metric horns, we generalize (3.1) to an operator valued setting. Thus we consider a Hilbert space, H, a dense subspace, H_1, and a family of self-adjoint operators in H, $A(x)$, defined for $x > 0$, with common domain H_1. We assume in addition the following properties

$$(0,\infty) \ni x \mapsto A(x) \in \mathcal{L}(H_1, H) \text{ is smooth}, \tag{3.2}$$

$$A(x) \geq a > 0 \text{ for all } x > 0. \tag{3.3}$$

The smoothness of A in $\mathbb{R}_+$ - which holds in the conic case - is replaced by assuming an asymptotic expansion

$$A(x) = A_0 + \sum_{\gamma \in \Gamma_N^*} A_\gamma x^\gamma + R_N(x), \tag{3.4}$$

where all A_γ are defined and symmetric on $\mathcal{D}(\triangle)$, and

$$|R_N(x)| \leq C_N x^{\mu_n} A_0^{1-\delta_N}, \tag{3.5}$$

for some $\delta_N \in (0,1]$ and some sequence $\mu_N \nearrow \infty$ (note that A_0 is self-adjoint with domain H, and $A_0 \geq a$). We also assusme, with $\mathcal{C}_p$ the Schatten-von Neumann class of order p, that

$$A_0^{-1} \in \mathcal{C}_p(H) \tag{3.6}$$

for some $p > 0$.

Then we can consider the analogue of (3.1),

$$\triangle^\circ := -\left(\frac{\partial}{\partial x}\right)^2 + \frac{A(x)}{x^{2\alpha_0}}, \tag{3.7}$$

with domain $C_0^2((0,\infty), H_1)$, and also its Friedrichs extension, $\triangle$, in the Hilbert space $\mathcal{H} := L^2(\mathbb{R}_+, H)$. Because $\mathbb{R}_+$ is not compact, the resolvent may not be trace class. To remedy this, we choose $\phi \in C_0^\infty(-2,2)$ with $\phi = 1$ in a neighbourhood of 0 and consider for $z \geq 1$ and $l \in \mathbb{N}$ the function

$$r(z) := \mathrm{tr}_{\mathcal{H}}[\phi(\triangle + z^2)^{-l}]. \tag{3.8}$$

We will show that r is well defined and admits an asymptotic expansion in powers of z and $\log z$, as $z \to \infty$; this will be a consequence of Theorem 1.

To put r into the framework of the SAL we use the natural scaling, as in [BS 2,3], and the Trace Lemma from [BS 2]. Starting with the latter, we have the following result.

LEMMA 3.1 *For $l \geq p + 1/2$, $(\triangle + z^2)^{-l}$ has an operator kernel, $(\triangle + z^2)^{-l}(x,y)$, on $\mathbb{R}_+$ such that*

$$(\triangle + z^2)^{-l}(x,x) \in \mathcal{C}_1(H), \;\; x > 0. \tag{3.9}$$

Moreover,

$$r(z) = \int_0^\infty \phi(x)\mathrm{tr}_H[(\triangle + z^2)^{-l}(x,x)]dx. \tag{3.10}$$

PROOF. We write $\mathcal{S} := C_0^\infty(\mathbb{R}_+)$ and note that, by construction,

$$\psi\mathcal{D}(\triangle) = \mathcal{D}(\triangle) \text{ for } \psi \in \mathcal{S}. \tag{3.11}$$

We also write $\psi_2 > \psi_1$, for $\psi_1, \psi_2 \in \mathcal{S}$, if $\psi_2 = 1$ in a neighbourhood of supp ψ_1. Then, by the Trace Lemma, it is enough to prove that

$$\psi_2(\triangle + z^2)^{-l}\psi_1 \in \mathcal{C}_1(\mathcal{H}), \tag{3.12}$$

for all such ψ_1 and ψ_2 in $\mathcal{S}$ if l is large enough. Thus, we fix $\psi_i \in \mathcal{S}$, $1 \leq i \leq 3$, with $\psi_3 > \psi_2 > \psi_1$ and supp $\psi_3 \subset [0,T)$, for some $T > 0$. We want to prove that

$$\psi_2(\triangle + z^2)^{-l} \;\; \psi_1 \in \mathcal{C}_{(p+1/2)/l}(\mathcal{H}), \tag{3.13}$$

with p from (3.6). To show this for $l = 1$, we denote by $\triangle_T$ the Friedrichs extension of $\triangle^0$ in $L^2([0,T], H)$ with domain $C_0^2((0,T), H_1)$. Then we obtain from the max-min principle and (3.4), (3.5) the estimate

$$\begin{aligned} \| (\triangle_T + C_1)^{-1} \|^{p+1/2}_{\mathcal{C}_{p+1/2}(\mathcal{H}_T)} &\leq C_2 \sum_{i,j\geq 1} \left(\frac{4\pi^2}{T^2} i^2 + \lambda_j \right)^{-(p+1/2)} \\ &\leq C_2 \sum_{j\geq 1} \int_0^\infty \left(\frac{4\pi^2}{T^2} x^2 + \lambda_j \right)^{-(p+1/2)} dx \\ &\leq C_3 \| A_0^{-1} \|_{\mathcal{C}_p(H)}, \end{aligned} \tag{3.14}$$

where we have written $\mathcal{H}_T = L^2((0,T),H)$, $(\lambda_j)_{j\geq 1} = \text{spec A}_0$, and the constants C_i depend only on A_0 and p. Now we find by standard arguments

$$\begin{aligned} \psi_2(\triangle + z^2)^{-1}\psi_1 &= \psi_2(\triangle_T + z^2)^{-1}\psi_1 \\ &+ \psi_2(\triangle_T + z^2)^{-1}[\triangle_T, \psi_3](1-\psi_2)(\triangle + z^2)^{-1}\psi_1. \end{aligned} \tag{3.15}$$

Next we observe the easy a priori estimate

$$\| [\triangle_{(T)}, \psi]u \|_{\mathcal{H}_{(T)}} \leq C(\psi)((\triangle_{(T)}u, u) + \| u \|^2_{\mathcal{H}_{(T)}}), \tag{3.16}$$

which holds for any $\psi \in \mathcal{S}_{(T)}$ and $u \in C_0^2((0,\infty), H_1)$, $(C_0^2((0,T), H_1))$, where $\mathcal{S}_T = C_0^\infty[0,T]$. But then it is readily seen that $(\triangle_{(T)} + I, \mathcal{S}_{(T)})$ satisfy the axioms (Co1) through (Co5) in [B3, Sec.4]; in particular, from (Co2) we see that

$$(\triangle_T + I)^{-1/2}[\triangle_T, \psi_3] \text{ is bounded in } \mathcal{H}_T. \tag{3.17}$$

Thus, we derive from (3.14), (3.15), and (3.17) that

$$\psi_1(\triangle + z^2)^{-1}\psi_1 \in \mathcal{C}_{2p+1}(\mathcal{H}),$$

hence

$$\psi_1(\triangle + z^2)^{-1/2} \in \mathcal{C}_{4p+2}(\mathcal{H}).$$

But then it follows from [B3, Lemma 4.1] that

$$(1-\psi_2)(\triangle + z^2)^{-1}\psi_1 \in \mathcal{C}_q(\mathcal{H}), \text{ for all } q > 0,$$

which gives, with (3.15), (3.13) for $l = 1$. If we have proved this for some $l \geq 1$, then we write

$$\begin{aligned} \psi_2(\triangle + z^2)^{-l-1}\psi_1 &= \psi_2(\triangle + z^2)^{-l}\psi_3(\triangle + z^2)^{-1}\psi_1 \\ &+ \psi_2(\triangle + z^2)^{-l}(1-\psi_3)(\triangle + z^2)^{-1}\psi_1. \end{aligned}$$

Using [B3, Lemma 4.1] again and the Hölder inequality for $\mathcal{C}_p$-norms, we complete the induction; hence the assertion. □

The scaling mentioned above is effected through the unitary family

$$U_\varepsilon u(X) := \varepsilon^{1/2} u(\varepsilon x), \;\; \varepsilon > 0, \;\; u \in \mathcal{H}. \tag{3.18}$$

Clearly, with $\alpha_0 =: 1 + \beta_0$,

$$\begin{aligned} U_\varepsilon \triangle^\circ U_\varepsilon^\star &=: \varepsilon^{-2} \left(-\partial_x^2 + \varepsilon^{-2\beta_0} \frac{A(\varepsilon x)}{x^{2\alpha_0}} \right) \\ &= \varepsilon^{-2} \triangle_\varepsilon^\circ. \end{aligned} \tag{3.19}$$

It follows that for the Friedrichs extension, $\triangle_\varepsilon$, we have

$$\varepsilon^2 U_\varepsilon \triangle U_\varepsilon^* = \triangle_\varepsilon. \tag{3.20}$$

It follows also that

$$\varepsilon^{-2l} U_\varepsilon (\triangle + z^2)^{-l} U_\varepsilon^\star = (\triangle_\varepsilon + (\varepsilon z)^2)^{-l}, \tag{3.21}$$

and for the kernels

$$\varepsilon^{1-2l} (\triangle + z^2)^{-l} (\varepsilon x, \varepsilon y) = (\triangle_\varepsilon + (\varepsilon z)^2)^{-l} (x, y). \tag{3.22}$$

Thus, we obtain with Lemma 3.1 (and $x = y = 1, \varepsilon = y$)

$$r(z) = \int_0^\infty y^{2l-1} \phi(y) \mathrm{tr}_H [(\triangle_y + (yz)^2)^{-l} (1,1)] dy. \tag{3.23}$$

In the conic case (when $\alpha_0 = 1$) we have applied the SAL directly to (3.9), cf. [BS2, Sec.7]. To be able to do this also in this case, we rewrite our operator as

$$\triangle_y + (yz)^2 = - \left(\frac{\partial}{\partial x} \right)^2 + y^{-2\beta_0} \left(\frac{A(yx)}{x^{2\alpha_0}} + (y z^{1/2\alpha_0})^{2\alpha_0} \right). \tag{3.24}$$

This suggests that we introduce $\tilde{z} := z^{1/2\alpha_0}$ and write

$$\tilde{r}(\tilde{z}) = r(z) =: \int_0^\infty \sigma(y, y\tilde{z}) dy, \tag{3.25}$$

where

$$\sigma(y, \zeta) = y^{2l-1} \phi(y) \mathrm{tr}_H [(-(\frac{\partial}{\partial x})^2 + y^{-2\beta_0} (\frac{A(yx)}{x^{2\alpha_0}} + \zeta^{2\alpha_0}))^{-l} (1,1)]. \tag{3.26}$$

In what follows, it is convenient to write

$$\tilde{A}(x) := \frac{A(x)}{x^{2\alpha_0}}, \tag{3.27}$$

such that

$$y^{-2\beta_0}\frac{A(yx)}{x^{2\alpha_0}} = y^2\tilde{A}(yx) =: \tilde{A}_y(x). \tag{3.28}$$

We also write

$$P(y,\xi;\zeta) := (\xi^2 + y^2\tilde{A}(y) + \zeta^{2\alpha_0}/y^{2\beta_0})^{-1}. \tag{3.29}$$

Now, for any fixed $y > 0$, we have to determine the resolvent kernel expansion of the elliptic operator with operator coefficients, (3.11b), at a *regular* point, in order to verify the conditions of Theorem 1.

To do so, we have to add some assumptions on resolvent expansions for the operator family $\tilde{A}$. First of all, we assume now that the expansion (3.2c) can be differentiated any number of times, i.e. that for $j \in \mathbb{Z}_+$ we have

$$A^{(j)}(x) = \sum_{\gamma\in\Gamma_N^*} A_\gamma j!\binom{\gamma}{j}x^{\gamma-j} + R_{Nj}(x),$$

where, with μ_N and δ_N as above,

$$|R_{Nj}(x)| \le C_N x^{\mu_N - j}A_0^{1-\delta_N}. \tag{3.30}$$

Moreover, we will have to consider expressions of the form

$$I_\alpha(A, B_1, \dots, B_{j-1}; x, \eta)$$

$$:= \mathrm{tr}_H[(A(x)+\eta^2)^{-\alpha_1}B_1(A(x)+\eta^2)^{-\alpha_2}B_2\cdots B_{j-1}(A(x)+\eta^2)^{-\alpha_j}]. \tag{3.31}$$

Here, $\alpha \in \mathbb{N}^j$ is a multiindex satisfying

$$|\alpha| \ge j + p, \tag{3.32}$$

and the operators B_i are closed with domain containing H_1; in addition, we assume that

$$A_0^{-1}B_i \in \mathcal{L}(H), \tag{3.33}$$

and that there is a locally convex space, $\mathcal{B}$, such that

$$B_i \in \mathcal{B}, \ \ 1 \le i \le j. \tag{3.34}$$

We observe that (3.30), (3.32), and (3.33) imply that (3.31) is, indeed, well defined.

If we now add the assumption that

$$A_\gamma \in \mathcal{B} \text{ and } R_N(x) \in \mathcal{B}, \tag{3.35}$$

for all $\gamma \in \Gamma^*$, $x > 0$, $N \in \mathbb{N}$, then we see from (3.4,5) that, for x sufficiently small, the Neumann series

$$\begin{aligned}(A(x)+\eta^2)^{-1} &= (A_0+\eta^2)^{-1}(I+R_1(x)(A_0+\eta^2)^{-1})^{-1}\\ &= \sum_{j\ge 0}(-1)^j(A_0+\eta^2)^{-1}(R_1(x)(A_0+\eta^2)^{-1})^j\end{aligned} \tag{3.36}$$

is asymptotic with respect to η in $\mathcal{C}_1(H)$. Hence the η-expansion of I_α is reduced to the expansion of

$$I^0_\alpha(B_1,\cdots,B_{j-1};\eta) :=$$

$$\mathrm{tr}_H[(A_0+\eta^2)^{-\alpha_1}B_1(A_0+\eta^2)^{-\alpha_2}B_2\dots B_{j-1}(A_0+\eta^2)^{-\alpha_j}], \tag{3.37}$$

for $\alpha \in \mathbb{N}^j$, $|\alpha| \ge j+p$, and $B_i \in \mathcal{B}$ (possibly also depending on x). Thus, we are lead to the following assumption:

> There is an asymptotic expansion of the form
>
> $$\begin{aligned}I^0_\alpha(B_1,\dots,B_{j-1};\eta) &= \sum_{\substack{\gamma\in\Gamma\\ m\le\nu(\gamma)}} I^0_{\alpha,\gamma m}(B_1,\dots,B_{j-1})\eta^\gamma\log^m\eta\\ &+ R_N(B_1,\dots,B_{j-1};\eta),\end{aligned} \tag{3.38}$$
>
> where the multilinear forms $I^0_{\alpha,\gamma m}$ and $\eta^{\mu_N}R_N(B_1,\dots,B_{j-1};\eta)$ are bounded on bounded sets in $\mathcal{B}$, uniformly in $\eta \ge 0$.

Note that in view of (3.3) we can in fact allow $\eta \ge 0$ for the resolvent estimates.

With these preparations we can turn to the expansion theorem we are after. We begin with the following lemma which follows e.g. from the material in [BS2, Sec.2]; a detailed treatment will be given in a forthcoming publication.

LEMMA 3.2 *For $l \ge p$, $j \ge 1$, and $\alpha \in \mathbb{N}^j_+$, $\beta \in \mathbb{Z}^{j-1}_+$ there are homogeneous polynomials $p_{\alpha\beta}$ in $\mathbb{R}$, of degree $d_{\alpha\beta}$, such that for $y,\zeta > 0$*

$$\begin{aligned}\sigma(y,\zeta) &= \sum_{\substack{1\le j\le N\\ \alpha,\beta\in\mathcal{M}_j}} y^{2l-1}\phi(y)\int_{\mathbb{R}}\phi_{\alpha\beta}(\xi)\mathrm{tr}_H[P(y,\xi;\zeta)^{\alpha_1}y^{2+\beta_1}\tilde{A}^{(\beta_1)}(y)..\\ &\quad ..\ y^{2+\beta_{j-1}}\tilde{A}^{(\beta_{j-1})}(y)P(y,\xi;\zeta)^{\alpha_j}]d\xi\\ &+ R_N(y,\zeta).\end{aligned} \tag{3.39}$$

Here, the index set is

$$\mathcal{M}_j := \{(\alpha,\beta) \in \mathbb{N}_+^j \times \mathbb{Z}_+^{j-1};\, |\alpha| \geq l + j - 1, d_{\alpha\beta} - 2|\alpha| = -2(l+j-1) - |\beta|\}, \tag{3.40}$$

and we have the uniform estimate

$$|R_N(y,\zeta)| \leq C_N(\| \tilde{A}(y)^{-1} A_{2\alpha} \|_H)(a/2 + \zeta/y^{2\beta})^{-H}, \tag{3.41}$$

with some continuous function $C_N : \mathbb{R}_+ \to \mathbb{R}_+$.

Now we are ready for the proof of our main result.

THEOREM 2. *Assume that the operator family,* $A(x)$, *satisfies the assumptions* (3.2) *through* (3.6) *and* (3.30), (3.35), *and* (3.38). *Then the function* σ *introduced in* (3.26) *satisfies the assumptions of Theorem* 1.

This implies a complete asymptotic expansion for

$$r(z) = \mathrm{tr}_{\mathcal{H}}[\phi(\triangle + z^2)^{-l}] \text{ as } z \to \infty,$$

whenever $l \geq p + 1/2$. *In particular,* $r \in \mathcal{F}^{\infty}_{\Gamma,\tilde{\nu}}$, *with* $\tilde{\nu}$ *as in Theorem* 1.

PROOF. We have to analyze the expansion of σ given in Lemma 3.2. Note first that, since supp $\phi \subset [-2, 2]$, we derive from (3.41) the remainder estimate

$$|R_N(y,\zeta)| \leq C_N(C_0) y^{2N\beta_0} (a/2 + \zeta)^{-N}, \tag{3.42}$$

uniformly in $\zeta > 0$. This estimate is good for (A_0) and (A_∞) since $\sigma(y,\zeta) = 0$ for $y > 2$.

Hence it only remains to verify that the individual terms in the sum (3.39) satisfy the assumptions (A_0) and (A_∞).

We begin with (A_0). Using (3.3), and (3.40), we obtain for a typical term in (3.20) the form

$$y^{2l-1+2\beta_0|\alpha|}\phi(y) \int_{-\infty}^{+\infty} p_{\alpha\beta}(\xi) \tilde{I}_\alpha(y, y^{\beta_0}\xi, \zeta)\, d\xi$$

$$= y^{2l-1+\beta_0(2|\alpha| - d_{\alpha\beta} - 1)}\phi(y) \int_{-\infty}^{+\infty} p_{\alpha\beta}(\xi) \tilde{I}_\alpha(y, \xi, \zeta)\, d\xi, \tag{3.43}$$

where

$$\tilde{I}_\alpha(y, \tilde{\xi}, \zeta)$$

$$:= I_\alpha(A(y) - a/2, y^{2+\beta_1}\tilde{A}^{(\beta_1)}, \ldots, y^{2+\beta_{j-1}}\tilde{A}^{(\beta_{j-1})}(y); y, (\tilde{\xi}^2 + \zeta^{2\alpha_0} + a/2)^{1/2}).$$

Now, we observe that, from (3.27) and (3.30),

$$y^{2+i}\tilde{A}^{(i)}(y) = O_i(y^{-2\beta_0}), \tag{3.44}$$

and admits a full asymptotic expansion with coefficients in $\mathcal{B}$. Plugging this into (3.43) and using (3.36), we obtain an asymptotic expansion as $y \to 0+$, with typical term

$$y^\delta \phi(y) \int\limits_{\mathbb{R}} p(\xi) I_\alpha^0(B_1, \dots, B_{k-1}; (\xi^2 + \zeta^{2\alpha_0} + a/2)^{1/2}) d\xi, \tag{3.45}$$

for some δ and some polynomial p, and with A_0 replaced by $A_0 - a/2$. Using (3.38) we see that each term of the form (3.45) is in $\mathcal{F}_{\Gamma,\nu}$, as required.

For (A_∞), we use (3.36) and (3.38) first, as $\zeta \to \infty$, and plug in the asymptotic expansions, as $y \to 0+$, for the operators (3.28) in each term. This completes the proof. □

We remark in conclusion that that condition (3.5) is unnecessarily restrictive, even though it suffices to deal with Laplace type operators in metric horns, as explained in the Introduction. It is fairly obvious how to modify the assumptions on the functions I_α in (3.38) in such a way that they are satisfied if $A(y)$ is a family of elliptic differential operators on a compact manifold, and still imply Theorem 2. It would be desirable, however, to find a more general and less clumsy set of conditions also implying this result.

Let us finally spell out the result for the "metric horns".

Corollary 3. *Denote by $\triangle$ the Friedrichs extension of the Laplacian on forms on a manifold with a metric horn, as in (1.3). Then we have for $l > m/2$*

$$\mathrm{tr}(\triangle + z^2)^{-l} \in \mathcal{F}^\infty_{\Gamma,\nu},$$

where, with $\alpha_0 =: 1 + \beta_0$,

$$\Gamma := \{k + l\beta_0; k, l \in \mathbb{Z}, k \geq 2l - m, l \geq 0\},$$

and $\nu(\alpha) \equiv 1$.

Proof: Since the conic case has been dealt with in [Ch1] and [BS2], we assume $\beta_0 > 0$. In view of (1.3), Theorem 2 applies if we can prove that $A(x) \geq a > 0$ for x near 0. The explicit form of the coefficients in (1.3), given in [B1, (2.4)] shows that this holds if ker $A_2 = 0$. Otherwise, the full operator function $A(x)$ is reduced by $\ker A_2 \cap \ker A_0$ which splits the problem as a sum of a "small eigenvalue part" and a "large eigenvalue part". To the latter, Theorem 2 applies, and the former is dealt with by standard ode methods. □

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