Integrable geodesic flows on the suspensions of toric automorphisms

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Abstract

For any toric automorphism $A \in SL(n, \mathbb{Z})$ with only real eigenvalues a Riemannian metric with an integrable geodesic flow on the suspension of this automorphism is constructed. A qualitative analysis of such a flow on a three-solvmanifold constructed by the authors in the previous work is done. This flow is an example of the geodesic flow, which has vanishing Liouville entropy and, moreover, is integrable but has positive topological entropy. The authors also discuss some open problems on integrability of geodesic flows and related subjects.

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1 Introduction and main results

In this paper we resume our study of integrable geodesic flows on the suspensions of toric automorphisms which we started in [2].

A closed manifold $M_A = M^{n+1}$ is called the suspension of a toric automorphism $A : T^n \to T^n$ if there is a fibration

$$\pi : M^{n+1} \to S^1 \tag{1}$$

of this manifold over the circle S^1 with T^n-fibres such that the monodromy of this fibration is given by $A \in SL(n, \mathbb{Z})$.

The manifold M_A is constructed as the quotient of the free $\mathbb{Z}$-action

$$(X, z) \to (AX, z + 1)$$

on the cylinder $T^n \times \mathbb{R}$ where $X \in T^n = \mathbb{R}^n/\mathbb{Z}^n$, $z \in \mathbb{R}$.

Theorem 1 *If all eigenvalues of an automorphism $A \in SL(n, \mathbb{Z})$ are real or $n = 2$, then M_A, the suspension of A, admits a real-analytic Riemannian metric such that*

1) the geodesic flow of this metric is (Liouville) integrable in terms of C^∞ first integrals;

2) the measure entropy of the geodesic flow with respect to any smooth invariant measure vanishes;

3) the topological entropy of this flow meets the following inequality

$$h_{\text{top}} \geq \log \left(\max_{\lambda \in \operatorname{Sp} A} |\lambda| \right), \tag{2}$$

where $\operatorname{Sp} A$ is the spectrum of A, i.e. the set of its eigenvalues.

For A the identity, M_A is a torus and in this case the statement of the theorem is evident.

The first nontrivial case was found by Butler [3] who constructed an integrable geodesic flow on the manifold M_A with

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

He constructed the metric as a homogeneous metric on a nilmanifold and worked in terms of global coordinates on the corresponding nilpotent Lie group. In particular, Butler showed that some topological obstructions to integrability of geodesic flows in terms of real-analytic (or in some sense geometrically simple) first integrals found in [14, 15] do not obstruct integrability in terms of C^∞ functions.

The suspension construction was found in [2]. In this paper generalizing Butler's analytic trick for constructing C^∞ first integrals we constructed an integrable geodesic flow on the manifold M_A with

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \tag{3}$$

and had explained that this suspension construction is quite general. In [2] we discussed only one concrete example which appears to be the first example of Liouville integrable geodesic flow with positive topological entropy and also the first example of the geodesic flow for which the Liouville entropy vanishes but the topological entropy is positive.

We shall study the Lyapunov exponents of the flow from [2] and prove the following statement.

Theorem 2 *Given the Riemannian manifold M_A with A of the form (3) and the metric constructed in [2] (see Section 4), the unit cotangent bundle SM_A contains two four-dimensional invariant submanifolds N^u and N^v such that*

1) N^u and N^v are diffeomorphic to $M_A \times S^1$;

2) the intersection $N^u \cap N^v = V$ consists of two three-dimensional components V^+ and V^-, each of these components is diffeomorphic to M_A and consists in trajectories orthogonal to the fibers of the fibration (1);

3) the Lyapunov exponents vanish at points from $SM_A \setminus \{N^u \cup N^v\}$ and for any point from $N^u \cup N^v$ there are nonzero Lyapunov exponents;

4) all invariant (Borel) measures on N^u and N^v are supported on $V^+ \cup V^-$ and there are smooth invariant measures on V^+ and V^-;

5) N^u is a stable manifold for V^+ and an unstable manifold for V^-, i.e., any trajectory in $N^u \setminus V$ is asymptotic to a trajectory from V^+ as $t \to \infty$ and is asymptotic to a trajectory from V^- as $t \to -\infty$;

6) N^v is a stable manifold for V^- and an unstable manifold for V^+;

7) the complement to $N^u \cup N^v$ is fibered by invariant tori.

Now we derive from this theorem that

Corollary 1 *Given the Riemannian manifold M_A with A of the form (3) and the metric constructed in [2] (see Section 4), the topological entropy of this flow equals*

$$h_{\text{top}} = \log \frac{3+\sqrt{5}}{2}$$

and there are measures of maximal entropy supported on V^+ or V^-.

We would like to mention the following property of this integrable flow:

Corollary 2 *The restrictions of the geodesic flow on M_A onto V^+ or V^- are Anosov flows.*

One can see that easily: take a fiber of the fibration (1) and at each point q of the fiber take a covector $p = (p_u = p_v = 0, p_z = 1)$. Such points (q, p) form a two-torus T^2 embedded into SM_A. Then draw a geodesic in the direction of this covector. After the unit time it will return back to this fiber and therefore we have a recurrence mapping

$$T^2 \to T^2$$

given by the hyperbolic matrix (3).

2 Entropy and integrability

In this section we recall some well known definitions and facts from the theory of dynamical systems. For detailed explanation of different facts from this section we refer to [1, 6, 13].

A) *Geodesic flows as Hamiltonian systems.*

Let M^n be a Riemannian manifold with the metric g_{ij}. Denote local coordinates on the cotangent bundle T^*M^n as $(x^1, \dots, x^n, p_1, \dots, p_n)$ where $(x_1, \dots, x^n)$ are (local) coordinates on M^n and the momenta $p_1, \dots, p_n$ are defined from tangent vectors (velocities of curves on M^n) by the Legendre transformation:

$$p_i = g_{ij}\dot{x}^j.$$

There is a symplectic form

$$\omega = \sum_{i=1}^{n} dx^i \wedge dp_i$$

on T^*M^n which is correctly defined globally and in its turn defines the Poisson brackets on the space of smooth functions on T^*M^n or on open domains in T^*M^n:

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i} \right). \tag{4}$$

The geodesic flow is a Hamiltonian system on T^*M^n with the Hamiltonian function

$$H(x,p) = \frac{1}{2} g^{ij}(x) p_i p_j.$$

This means that the evolution of any function f along trajectories of the system is given by the Hamiltonian equations

$$\frac{df}{dt} = \{f, H\}.$$

If a function f is preserved by the flow, i.e.

$$\frac{df}{dt} = \{f, H\} = 0,$$

it is said that this function is a first integral of the system.

Since the Poisson brackets are skew-symmetric, the function H is a first integral. This implies that the set of unit momenta vectors SM^n is invariant under the flow:

$$SM^n = \{(x,p) : |p| = \sqrt{g^{ij}(x) p_i p_j} = 1\} = \left\{ H = \frac{1}{2} \right\}.$$

The restrictions of the geodesic flow onto different level sets $H = \mathrm{const} \neq 0$ are smoothly trajectory equivalent and this equivalence is established by constant reparametrization depended only on the values of H. Therefore it is enough to consider the flow only on SM^n.

Take the Liouville measure on SM^n. This means that the measure of a set $U \subset SM^n$ is defined as

$$\mu(U) = \int_x \mu(U \cap S_x) \sqrt{\det g}\, dx^1 \wedge \ldots \wedge dx^n$$

where S_x is the $(n-1)$-dimensional sphere of unit covectors at the point $x \in M^n$. In this event the measure on S_x coincides with the measure on the unit sphere in $\mathbb{R}^n$ and this coincidence is established by an orthogonal map $T_x^* M^n \to \mathbb{R}^n$.

B) *Integrability of geodesic flows* [2, 14].

The geodesic flow is called Liouville integrable if in addition to $I_n = H$ there are $(n-1)$ first integrals $I_1, \ldots, I_{n-1}$ defined on SM^n such that

1) the integrals $I_1, \ldots, I_n$ are in involution: $\{I_j, I_k\} = 0$,

2) the integrals $I_1, \ldots, I_{n-1}$ are functionally independent on the full measure subset $W \subset SM^n$.

To define the Poisson brackets correctly we extend $I_1, \dots, I_{n-1}$ onto a neighborhood of $SM^n \subset T^*M^n$ as follows

$$I_j(x,p) = I_j(x, p/|p|), \quad j = 1, \dots, n-1.$$

Vanishing of the Poisson brackets of these functions on SM^n does not depend on the choice of f.

If the metric and the first integrals $I_1, \dots, I_{n-1}$ are real-analytic we say that the flow is analytically integrable.

If the geodesic flow is integrable, then a full measure subset $\widetilde{W}$ of $W \subset SM^n$ is foliated by invariant n-dimensional tori and moreover for any such a torus there is its neighborhood $U \subset W \subset SM^n$ such that

1) there are coordinates $\varphi_1, \dots, \varphi_n$ defined modulo $\mathbb{Z}$ and $I_1, \dots, I_{n-1}$ in U;

2) every level set $\{I_1 = c_1, \dots, I_{n-1} = c_{n-1}\}$ is an invariant (Liouville) torus;

3) the flow is linearized in these coordinates as follows:

$$\dot{\varphi}_1 = \omega_1(I_1, \dots, I_{n-1}), \ \dots \ , \dot{\varphi}_n = \omega_n(I_1, \dots, I_{n-1}), \tag{5}$$

$$I_1 = \text{const}, \ \dots \ , I_{n-1} = \text{const}.$$

This subset $\widetilde{W}$ is distinguished as the preimage of the set of regular values of the momentum map $SM^n \to \mathbb{R}^{n-1}$:

$$x \to (I_1(x), \dots, I_{n-1}(x)).$$

C) *Entropy.*

Let X be a compact space and $T : X \to X$ be its homeomorphism.

Take an invariant Borel measure μ on X such that $\mu(X) < \infty$. For any a disjoint measurable countable decomposition

$$X = \sqcup U_i$$

the entropy of the decomposition is defined by the following formula

$$h(U) = -\sum \mu(U_i) \log \mu(U_i)$$

assuming that $\mu(U_j) \log \mu(U_j) = 0$ for $\mu(U_j) = 0$. Let $\{U_i\}$ be such a decomposition. For any $k \in \mathbb{N}$ define the decomposition $\wedge^k U$ as follows:

$$X = \sqcup U_{i_0 \dots i_{k-1}}$$

where

$$x \in U_{i_0 \dots i_{k-1}} \quad \text{iff} \quad x \in U_{i_0}, Tx \in U_{i_1}, \dots, T^{k-1}x \in U_{i_{k-1}}.$$

Now put

$$h_\mu(U, T) = \limsup_{k \to \infty} \frac{h(\wedge^k U)}{k}$$

and define the measure entropy of T with respect to μ (the Kolmogorov–Sinai entropy) as

$$h_\mu(T) = \sup_{U \text{ with } h_\mu(U,T) < \infty} h_\mu(U, T).$$

To any open covering

$$X \subset \cup V_j$$

of X corresponds the series of coverings $\wedge^k V$ defined as follows:

$$X \subset \cup V_{j_0 \dots j_{k-1}}$$

where

$$x \in V_{j_0 \dots j_{k-1}} \quad \text{iff} \quad x \in V_{j_0}, Tx \in V_{j_1}, \dots, T^{k-1}x \in U_{j_{k-1}}.$$

Usually $\wedge^k V$ contains subsets which still form coverings of X and for any $k \in \mathbb{N}$ put $C(k, V, T)$ to be a minimal cardinality of such a subset. Now put

$$h(V, T) = \limsup_{k \to \infty} \frac{\log C(k, V, T)}{k}$$

and define the topological entropy of T as

$$h_{\text{top}}(T) = \sup_V h(V, T).$$

By the Bowen theorem, $h_{\text{top}}(T)$ equals the supremum of the measure entropies with respect to invariant ergodic Borel measures μ such that $\mu(X) = 1$.

Example. Let A be an automorphism of a torus $T^n = \mathbb{R}^n / \mathbb{Z}^n$ given by a matrix $A \in SL(n, \mathbb{Z})$. Take a coordinates $x^1, \dots, x^n$ on T^n such that these coordinates are defined modulo $\mathbb{Z}$, the automorphism A is linear in terms of $x^1, \dots, x^n$ and

$$\int_{T^n} dx^1 \wedge \dots \wedge dx^n = 1.$$

Then the topological entropy of A and the measure entropy with respect to $d\mu = dx^1 \wedge \ldots \wedge dx^n$ coincide and equal

$$h_{\mathrm{top}}(A) = h_\mu(A) = \log\left(\max_{\lambda \in \mathrm{Sp}\, A} |\lambda|\right).$$

Therefore, $h_{\mathrm{top}}(A)$ vanishes if and only if all eigenvalues of A lies on the unit circle in $\mathbb{C}$.

D) *The entropies of geodesic flows.*

Let

$$F_t : SM^n \to SM^n$$

be a translation along trajectories per the time t.

By the definition, the entropy of the geodesic flow is the entropy of the map

$$T : SM^n \to SM^n$$

which is the translation along trajectories per unit time: $T = F_1$.

Recall the definition of Lyapunov exponents. Let v be a tangent vector to SM^n. For any such a vector its norm $|v|$ is defined as follows. Let $v \in T_q SM^n$ and decompose it into the sum $v = v_M + v_S$, where v_M is the component tangent to M^n and v_S is the component tangent to S_x where $q = (x,p) \in SM^n$. As in the definition of the Liouville measure, S_x is endowed with a metric by an orthogonal map $T_x M^n \to \mathbb{R}^n$. Now put

$$|v|^2 = |v_M|^2 + |v_S|^2$$

where norms of v_M and v_S are defined by the metrics on M^n and S_x.

On the full measure subset U of SM^n there is a correctly defined map from nonzero tangent vectors at the points of U to $\mathbb{R}$:

$$v \longrightarrow \limsup_{t \to \infty} \frac{\log |F_t^*(v)|}{t}.$$

At any point $q \in U \subset SM^n$ such a map takes $2n-1$ values

$$l_1 \leq l_2 \leq \ldots \leq l_k \leq 0 \leq l_{k+1} \leq \ldots \leq l_{2n-2}$$

where the zero value is attained on the vector tangent to the trajectory of the flow. Another values $l_1, \ldots, l_{2n-2}$ are called Lyapunov exponents and some of them may coincide with another. The number of negative Lyapunov exponents depends on q.

The Pesin formula for the measure entropy of the geodesic flow with respect to any smooth invariant measure μ on SM^n reads

$$h_\mu = -\int_{SM^n} \sum_{j=1}^{k(q)} l_j(q) d\mu.$$

It is evident that for the flow (5) its Lyapunov exponents vanish. Since an integrable geodesic flow has such a behavior on a full measure set, the Pesin formula implies that the entropy of an integrable flow vanish for any smooth invariant measure on SM^n and, in particular, for the Liouville measure.

This already follows from the inequality

$$h_\mu \leq -\int_{SM^n} \sum_{j=1}^{k(q)} l_j(q) d\mu.$$

first established by Margulis in the middle of the 1960s.

3 The construction of the metric and the lower estimate for the entropy

The construction of the metric on M_A is as follows.

Take linear coordinates $x^1, \dots, x^n$ on T^n for which the map A is linear and take a coordinate z on $\mathbb{R}/\mathbb{Z}$. These are coordinates on an infinite cylinder $\mathcal{C} = T^n \times \mathbb{R}$ which descend to coordinates on M_A, the quotient of $\mathcal{C}$ with respect to the $\mathbb{Z}$-action generated by

$$(X, z) \to (AX, z), \quad X = (x^1, \dots, x^n)^\top. \tag{6}$$

The symplectic form takes the form

$$\omega = \sum_{i=1}^{n} dx^i \wedge dp_i + dz \wedge dp_z. \tag{7}$$

Define the metric

$$ds^2 = g_{jk}(z) dx^j dx^k + dz^2$$

where

$$G(z) = (g_{jk}(z)) = \gamma(z)^\top \widehat{G} \gamma(z). \tag{8}$$

where $\widehat{G}$ is an arbitrary positive symmetric $n \times n$-matrix and $\gamma(z)$ is an analytic curve in $SL(n, \mathbb{R})$ satisfying the two following properties:

$$\gamma(z+1) = \gamma(z) A^{-1} \quad \text{and} \quad \gamma(0) = E.$$

It is easily seen that such a curve always exists. Indeed, if all the eigenvalues of A are positive, then it suffices just to put $\gamma(z) = e^{-zG_0}$, where $e^{G_0} = A$.

If the matrix $G_0 = \log A \in sl(n, \mathbb{R})$ does not exist, then we can use the following simple construction. Decompose A into product of matrices A_1 and A_2 such that

1) $A = A_1 A_2$;

2) there are $G_i \in sl(n, \mathbb{R})$ such that $e^{G_i} = A_i$, $i = 1, 2$;

3) A_2 commute with e^{zG_1} for any z (in particular, A_1 and A_2 commute).

To prove that such a decomposition exists take a Jordan form of A, which is a block matrix. Take now a diagonal matrix A_2, whose entires equal ± 1 and such that all eigenvalues of $AA_2 = AA_2^{-1}$ has positive eigenvalues. Since $\det A = 1$, the matrix A_2 has an even number of diagonal elements, which equal -1, and therefore there is a matrix $G_2 \in so(n) \subset sl(n, \mathbb{R})$ such that $A_2 = e^{G_2}$. Now it remains to put $A_1 = AA_2^{-1}$.

Given A_1 and A_2, put $\gamma(z) = e^{-zG_2} e^{-zG_1}$.

It is clear that (8) defines a metric on an infinite cylinder $\mathcal{C}$ and the metric is invariant with respect to the action (6). Therefore, this metric descends to a metric on the quotient space $M_A = \mathcal{C}/\mathbb{Z}$.

Lemma 1 *The geodesic flow of the metric (8) on the cylinder $\mathcal{C}$ is integrable, i.e., it admits $n+1$ first integrals*

$$I_1 = p_1, \ \ldots, \ I_n = p_n, \ I_{n+1} = H = \frac{1}{2} \left(g^{ij}(z) p_i p_j + p_z^2 \right)$$

which are in involution and for any open subset $U \subset T^*\mathcal{C}$ these integrals are functionally independent on a full measure subset of U with respect to the Liouville measure.

Proof. It is clear that these integrals are functionally independent at least on the set where $p_z \neq 0$. By (4) and (7), the momenta variables are in involution:

$$\{p_i, p_j\} = 0, \quad i, j = 1, \ldots, n,$$

and, moreover, since H does not depend on $x^1, \ldots, x^n$, we have

$$\{p_i, H\} = 0, \quad i = 1, \ldots, n.$$

This proves the lemma.

Now take a torus $T^n \subset SM^n$ formed by the points with $z = 0$ and $p_1 = \ldots = p_n = 0$. Since $p_1, \ldots, p_n$ are preserved by the flow, the translation $T = F_1$ along trajectories of the geodesic flow per unit time maps its torus into itself:

$$(X, 0) \to (X, 1) \sim (AX, 0)$$

and we see that the dynamical system $T : SM_A \to SM_A$ contains a subsystem isomorphic to the torus automorphism $A : T^n \to T^n$. It is known that the topological entropy of a system is not less than the topological entropy of any of its subsystems. Therefore we conclude that

$$h_{\mathrm{top}}(T) \geq h_{\mathrm{top}}(A) = \log \left(\max_{\lambda \in \mathrm{Sp}\, A} |\lambda| \right).$$

For proving integrability of the flow we are left to descend the first integrals $p_1, \ldots, p_n$ to SM_A. We can not do that straightforwardly but may substitute them by some functions of $p_1, \ldots, p_n$ which are invariant under the action of A and functionally independent almost everywhere.

4 Proof of Theorem 1 for A with real eigenvalues

The action of A on M_A generates the natural action on tangent vectors, the differential. We expand the action of A onto T^*M_A by assuming that A preserves the form ω. This action is also linear in terms of $p_1, \ldots, p_n$. Denote this action by $\widetilde{A}$. It is uniquely define by the equation

$$\begin{pmatrix} A^\top & 0 \\ 0 & \widetilde{A}^\top \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & \widetilde{A} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which means that ω is preserved and reads

$$A^\top \widetilde{A} = 1.$$

Let all eigenvalues of A be real. Then all eigenvalues of $\widetilde{A}$ are real. Take linear coordinates $p_1, \ldots, p_n$ such that $\widetilde{A}$ attains its Jordan form:

$$\widetilde{A} = \begin{pmatrix} B_0 & 0 & \ldots & 0 \\ 0 & B_1 & \ldots & 0 \\ 0 & \ldots & \ldots & 0 \\ 0 & \ldots & 0 & B_k \end{pmatrix}$$

where B_0 is a diagonal matrix

$$B_0 = \mathrm{diag}(\mu_1, \ldots, \mu_l)$$

and for $j \geq 1$ each matrix B_j is an $n_j \times n_j$-matrix of the form

$$B_j = \begin{pmatrix} \lambda_j & 1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda_j & 1 & \ldots & 0 & 0 \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & \lambda_j & 1 \\ 0 & 0 & 0 & \ldots & 0 & \lambda_j \end{pmatrix}$$

where $n_1 + \ldots + n_k + l = n$. Hence redenote the variables as follows

$$p_1, \ldots, p_n \longrightarrow q_1, \ldots, q_l, p_{11}, \ldots, p_{1n_1}, \ldots, p_{k1}, \ldots, p_{kn_k}.$$

Introduce the following polynomial

$$Q = q_1 \ldots q_l p_{11}^{n_1} \ldots p_{k1}^{n_k}.$$

Since $A \in SL(n, \mathbb{Z})$, we have $A^\top \in SL(n, \mathbb{Z})$ and, therefore, $\widetilde{A} = (A^\top)^{-1} \in SL(n, \mathbb{Z})$. This implies

$$\det \widetilde{A} = \mu_1 \ldots \mu_l \lambda_1^{n_1} \ldots \lambda_k^{n_k} = 1.$$

Since

$$Q \to (\mu_1 q_1) \ldots (\mu_l q_l)(\lambda_1 p_{11})^{n_1} \ldots (\lambda_k k_{k1})^{n_k} = (\mu_1 \ldots \mu_l \lambda_1^{n_1} \ldots \lambda_k^{n_k}) Q$$

this results in the following lemma.

Lemma 2 *The polynomial Q is an invariant of the action $\widetilde{A}$.*

Before constructing the full family of first integrals let us prove the technical lemma which we shall need.

Lemma 3 *Let L be an operator acting on the ring $\mathbb{R}[p_1, \ldots, p_n]$ of polynomials in $p_1, \ldots, p_n$ as follows:*

$$L \cdot f(p_1, \ldots, p_n) = f(L \cdot p_1, \ldots, L \cdot p_n), \qquad f \in \mathbb{R}[p_1, \ldots, p_n], \tag{9}$$

where

$$L \cdot p_1 = \lambda p_1, \quad L \cdot p_k = \lambda p_k + p_{k-1} \quad \text{for } k = 2, \ldots, n \tag{10}$$

and λ is a constant.

Then for any $k = 1, \dots, n$ there is a polynomial $G_k \in \mathbb{R}[p_1, \dots, p_n]$ of degree k such that

1) G_k depends only on $p_1, \dots, p_{k+1}$ and has the form

$$p_{k+1} H_{k1}(p_1, \dots, p_k) + H_{k2}(p_1, \dots, p_k)$$

where $H_{k1}, H_{k2} \in \mathbb{R}[p_1, \dots, p_k]$;

2) the operator L acts on G_k as follows

$$L \cdot G_k = \lambda^k G_k + p_1^k.$$

Proof. Let V_k^l be the space of homogeneous polynomials in $p_1, \dots, p_l$ of degree k. It is clear from (9) and (10) that $L(V_k^l) \subset V_k^l$.

Notice that the linear operator

$$(L - \lambda^k) : V_k^l \to V_k^l \tag{11}$$

is nilpotent. Indeed, let us introduce the following order on monomials from V_k^l:

$$p_1^{\alpha_1} \dots p_l^{\alpha_l} \prec p_1^{\beta_1} \dots p_l^{\beta_l} \quad \text{if} \quad \alpha_r = \beta_r \ \text{ for } l > m \text{ and } \ \alpha_m < \beta_m.$$

Then L acts on any monomial $F = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ as follows

$$L \cdot F = \lambda^k F + \sum_j D_j$$

where D_j are monomials such that $D_j \prec F$.

It is also clear that the kernel of the action (11) is generated by p_1^k.

This implies that in some basis $e_1, \dots, e_N$ for V_k^l L takes the form

$$\begin{pmatrix} \lambda^k & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda^k & 1 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & \lambda^k & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda^k \end{pmatrix}, \tag{12}$$

where $e_1 = p_1^k$.

Put $F_k = p_{k+1} p_1^{k-1}$. Then we have

$$L \cdot F_k = \lambda^k F_k + \lambda^{k-1} p_k p_1^{k-1}.$$

Look for solutions H_k and c_k to the equation

$$(L - \lambda^k) \cdot H_k = c_k p_1^k - \lambda^{k-1} p_k p_1^{k-1} \tag{13}$$

where $H \in V_k^k$ and $c_k \in \mathbb{R}$. In some basis $e_1, \ldots, e_N$ for V_k^k the operator L has the form (12) and, since the monomial $p_k p_1^{k-1}$ is not maximal in V_k^k,

$$p_k p_1^{k-1} = \sum_{j \le (N-1)} a_j e_j.$$

The vectors $e_2, \ldots, e_{N-1}$ lies in the image of $(L - \lambda^k)$ and therefore the equation (13) is solvable in H_k for $c_k = a_1 \lambda^{k-1}$. Take a solution H_k to it. We see that $F_k + H_k = p_{k+1} p_1^{k-1}$ satisfy the equation

$$L \cdot (F_k + H_k) = \lambda^k (F_k + H_k) + c_k p_1^k.$$

If $C_k = 0$ then $(F_k + H_k)$ lies in the kernel of $(L - \lambda^k)$ but $(F_k + H_k)$ is not proportional to p_1^k. Hence $c_k \neq 0$ and we are left to put

$$G_k = \frac{1}{c_k} (F_k + H_k).$$

This proves the lemma.

These are some simplest examples of the polynomials G_k:

$$G_1 = p_2, \quad G_2 = p_2^2 - 2p_1 p_3, \quad G_3 = p_2^3 + 3p_1^2 p_4 - 3p_1 p_2 p_3,$$

$$G_4 = p_2^4 - 4p_1^3 p_5 - 4p_1 p_2^2 p_3 + 2p_1^2 p_3^2 + 4p_1^2 p_2 p_4.$$

Corollary 3 *Given an action L on $\mathbb{R}[p_1, \ldots, p_n]$ such that it satisfies (9), this action admits $(n-1)$ almost invariant rational functions functionally independent outside an algebraic subvariety of positive codimension. These are*

$$J_k = \frac{G_k}{p_1^k}, \quad k = 1, \ldots, n-1$$

which are transformed by A as follows

$$J_k \xrightarrow{L} J_k + \frac{1}{\lambda^k}.$$

Notice that the functional independence statement follows from the fact that each polynomial G_k depends only on $p_1, \ldots, p_{k+1}$ and is linear in p_{k+1}.

Now we are ready to finish the proof of Theorem 1 for A with real eigenvalues.

Put

$$I_1 = \exp\left(-\frac{1}{Q^2}\right)\sin\left(2\pi\frac{\log q_1}{\log\mu_1}\right), \ \ldots, I_l = \exp\left(-\frac{1}{Q^2}\right)\sin\left(2\pi\frac{\log q_l}{\log\mu_l}\right).$$

To each series of variables $p_{j1}, \ldots, p_{jn_j}$ we apply Lemma 3 and construct the polynomials $G_1, \ldots, G_{n_j-1}$. Now put

$$I_{j1} = \exp\left(-\frac{1}{Q^2}\right)\sin\left(2\pi\frac{\log p_{j1}}{\log\lambda_j}\right), \quad I_{j2} = \exp\left(-\frac{1}{Q^2}\right)\sin\left(2\pi\lambda_j\frac{G_1}{p_{j1}}\right),$$

$$\ldots, \quad I_{jm} = \exp\left(-\frac{1}{Q^2}\right)\sin\left(2\pi\lambda_j^{m-1}\frac{G_{m-1}}{p_{j1}^{m-1}}\right), \quad \ldots,$$

$$\ldots, \quad I_{jn_j} = \exp\left(-\frac{1}{Q^2}\right)\sin\left(2\pi\lambda_j^{n_j-1}\frac{G_{n_j-1}}{p_{j1}^{n_j-1}}\right).$$

These functions are smooth, invariant under the action of $\widetilde{A}$ and functionally independent at any fiber $S_x\mathcal{C}$ outside an algebraic subset of positive codimension. In fact, outside this singular set where they functionally dependent these functions substitute $p_{j1}, \ldots, p_{jn_j}$.

The functions $I_1, \ldots, I_l, I_{11}, \ldots, I_{kn_k}$ are functionally independent at any fiber $S_x\mathcal{C}$, invariant under $\widetilde{A}$ and, therefore, descend to functions on SM_A. Since these functions depend only on the momenta variables, they are in involution and are first integrals of the geodesic flow on M_A.

We conclude that this family gives us a complete family of first integrals and therefore the geodesic flow on M_A is integrable.

The case of Theorem 1 concerning automorphisms A with real eigenvalues is established.

5 Proof of Theorem 1 for $n = 2$

The case when all eigenvalues are real is already considered. In fact, the case when A is not diagonalized and therefore in a convenient coordinates equals

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

was the initial one discovered by Butler [3] and the case when A is diagonalized with real eigenvalues was considered by us in [2].

Hence we assume that λ and $\bar{\lambda}$ are eigenvalues of A and, since $A \in SL(2,\mathbb{Z})$, we have

$$\lambda + \bar{\lambda} \in \mathbb{Z}, \quad |\lambda| = 1.$$

This means that $\lambda = \cos\varphi + i\sin\varphi$ and $2\cos\varphi \in \mathbb{Z}$. The latter inclusion implies $\cos\varphi \in \{\pm 1, \pm 1/2, 0\}$. If $\cos\varphi = \pm 1$ then $\lambda = \pm 1$ and hence λ is real. Therefore we are left with the following cases: in the momenta coordinates p_1, p_2 the action A is a rotation by

$$\varphi = \pm\frac{\pi}{2},\ \pm\frac{2\pi}{3},\ \pm\frac{\pi}{3}.$$

It is clear that this action preserves

$$I_1(p_1,p_2) = p_1^2 + p_2^2.$$

Put

$$\psi = \arcsin\frac{p_2}{\sqrt{p_1^2+p_2^2}}$$

and notice that A acts as

$$\psi \to \psi + \varphi.$$

Now we put

$$I_2(p_1,p_2) = \mathrm{Re}\,(p_1 + ip_2)^k,$$

where $\varphi = \pm 2\pi/k$.

It is easy to notice that the functions I_1 and I_2 are functionally independent almost everywhere.

This proves Theorem 1 for $n = 2$.

6 Proof of Theorem 2

Take linear coordinates u and v on T^2 such that A of the form (3) acts as

$$u \to \lambda^{-1}u, \quad v \to \lambda v \tag{14}$$

with

$$\lambda = \frac{3+\sqrt{5}}{2},$$

and also take a linear coordinate z on S^1 which is lifted to a coordinate on M_A defined modulo $\mathbb{Z}$. These coordinates are completed by (p_u, p_v, p_z) to coordinates on T^*M_A such that the symplectic form on the cotangent bundle is

$$\omega = du \wedge dp_u + dv \wedge dp_v + dz \wedge dp_z$$

and A acts on the momenta as

$$p_u \to \lambda p_u, \quad p_v \to \lambda^{-1} p_v, \quad p_z \to p_z. \tag{15}$$

Now the metric on M_A is

$$ds^2 = dz^2 + e^{2z\log\lambda} du^2 + e^{-2z\log\lambda} dv^2$$

and the Hamiltonian function on $T^* M_A$ is

$$H = \frac{1}{2}\left(p_z^2 + e^{-2z\log\lambda} p_u^2 + e^{2z\log\lambda} p_v^2\right).$$

There are three functionally independent almost everywhere first integrals of the geodesic flow on the universal covering:

$$I_1 = p_u, \quad I_2 = p_v, \quad I_3 = H.$$

Consider the restriction of the flow on the compact level surface SM_A defined as

$$SM_A = \left\{H = \frac{1}{2}\right\}.$$

1) If $p_u p_v \neq 0$, then the lift of a trajectory on the universal covering is trapped in the layer

$$c_1 e^{-2z\log\lambda} + c_2 e^{2z\log\lambda} \leq 1$$

with the constants $c_1 = p_u^2$ and $c_2 = p_v^2$. This layer is invariant under $\mathbb{Z}^2$ actions under translations by vectors of the lattice Λ. Here $T^2 = \mathbb{R}/\Lambda$ and in the coordinates u and v the vectors from Λ have irrational coefficients. There are two different kinds of such trajectories:

1a) A trajectory, for which

$$p_z \neq 0 \quad \text{or} \quad p_u^2 \neq p_v^2,$$

lies on an invariant torus in SM_A and its Lyapunov exponent vanish. These inequalities describe the set on which the first integrals I_1, I_2, and I_s are functionally independent;

1b) Trajectories with

$$p_z = p_u^2 - p_v^2 = 0$$

form a submanifold which is evidently diffeomorphic to two copies of M_A corresponding to two possibilities: $p_u = \pm p_v$. Each of this copies is fibered over S^1 and this fibration is induced by (1). Since $\dot{z} = p_z = 0$ on such a trajectory, it lies on the level $z = \text{const}$ which is a torus with linear coordinates u and v. The flow is linear in these coordinates and has constant velocities. Therefore the Lyapunov exponents for such a trajectory are zero.

2) Trajectories with $p_v = 0$ form a submanifold N^u. Since M_A is parallelizable, we see that N^u is diffeomorphic to $M_A \times S^1$ and the flow on it is described by the equations

$$\dot{p}_u = 0, \quad \dot{p}_z = \log\lambda e^{-2z\log\lambda} p_u^2, \quad \dot{u} = e^{2z\log\lambda} p_u, \quad \dot{z} = p_z. \tag{16}$$

There are two invariant submanifolds of N^u, which are

$$V^+ = \{p_u = p_v = 0, p_z = 1\}, \quad V^- = \{p_u = p_v = 0, p_z = -1\}.$$

Any trajectory with $p_v = 0$ satisfies the inequality

$$e^{-2z\log\lambda} \leq \frac{2}{p_u^2}$$

and we see that the lift of such a trajectory onto the universal covering is not trapped into any layer but just bounded in z from below. Hence

any trajectory on SM_A with $p_u \neq 0$ and $p_v = 0$ is asymptotic to a trajectory from V^+ as $t \to \infty$ and asymptotic to a trajectory from V^- as $t \to -\infty$.

Since the metric is invariant with respect to the A-action and the action of (16) on the tangent vector field

$$\xi = \frac{\partial}{\partial p_u}$$

is trivial: $F_t^*(\xi) = \xi$, we derive from (14) that the Lyapunov exponent corresponding to this vector is positive:

$$\limsup_{t\to\infty} \frac{\log|F_t^*(\xi)|}{|\xi|} > 0.$$

3) The submanifold N^v of SM_A is defined by the equation $p_u = 0$. It is analyzed in completely the similar manner as N^u and we derive that

any trajectory from N^v with $p_v \neq 0$ is asymptotic to a trajectory from V^- as $t \to \infty$ and asymptotic to a trajectory from V^+ as $t \to -\infty$.

We see that all trajectories in $\{N^u \cup N^v\} \setminus \{V^+ \cup V^-\}$ are not closed which implies that all invariant Borel measures on N^u and N^v are supported by $V^+ \cup V^-$. Otherwise it would contradict to the Katok theorem [5], which reads that given a compact manifold with an invariant Borel measure with nonzero Lyapunov exponents the support of the measure lies in the closure of periodic trajectories.

This finishes the proof of Theorem 2.

There is a natural invariant measure on V^+, which is

$$d\mu = du \wedge dv \wedge dz, \tag{17}$$

and the measure entropy with respect to $d\mu$ equals the topological entropy of the automorphism A of the torus, which is $\log \lambda$.

By the Bowen theorem, the topological entropy of a flow equals the supremum of the measure entropies of the flow taken over all invariant ergodic Borel measures. For an integrable flow with first integrals $I_1, \dots, I_n$ it is easy to derive from this ergodicity restriction for measures that there are constants $C_1, \dots, C_n$ such that this supremum may be taken over all measures supported on the level $\{I_1 = C_1, \dots, I_n = C_n\}$ (see, for instance, [16]). Knowing the first integrals of the geodesic flow on M_A and the behavior of it trajectories, we see that the topological entropy of this flow is the supremum of the measure entropies supported by V^+ or V^-. But the restrictions of the flow onto these sets the topological entropy equals $\log \lambda$ and this establishes Corollary 1.

In fact, Theorem 2 describes the geodesic flow on the universal covering of M_A, which is the solvable Lie group SOL. This manifold is a model for one of Thurston's canonical three-geometries. Asymptotic properties of its geodesic flow were studied in [10] where some general results on solvable groups were proved, which imply that the Martin boundary of SOL consists in a single point, and in [17] where a rather complex "horison" of the group SOL defined via the asymptotics of geodesics was described.

Speaking about the geodesic flow on M_A we would like to remind the first integrals of it, which were found in [2]:

$$I_1 = p_u p_v, \quad I_2 = \exp\left(-\frac{1}{p_u^2 p_v^2}\right) \sin\left(2\pi \frac{\log p_u}{\log \lambda}\right), \quad I_3 = H.$$

It is easy to check from (15) that these functions are invariants of A and therefore descend to SM_A. They are the first integrals of the geodesic flow on SM_A which are functionally independent on a full measure subset of SM_A.

7 Some remarks and open problems

The problem of topological obstructions to integrability was posed by Kozlov who also found the first known obstruction: he proves that if there is an analytically integrable geodesic flow on an oriented closed two-dimensional manifold then this manifold is homeomorphic to the two-sphere S^2 or the two-torus T^2 [8, 9]. As shown by Kolokol'tsov [7] this also true for geodesic flows on two-manifolds, which are integrable in terms of smooth first integrals, which are real-analytic functions of the momenta. But the following problem remains unsolved

Problem 1 *Can the Kozlov theorem be generalized for C^∞ metrics on two-manifolds with geodesic flows integrable in terms of C^∞ first integrals ?*

Speaking not about integrability but on existence of metrics whose geodesic flows have zero Liouville entropy we would like to remind the problem posed by Katok:

Problem 2 *Does there exist a smooth (at least C^2) geodesic flow with zero Liouville entropy on a two-sphere with $g \geq 2$ handles ? Or more general, do there exists such a flow on a closed manifold admitting negatively curved metric ?*

There is a similar question for mappings which also belongs to Katok.

Problem 3 *Does there exist a smooth (at least $C^{1+\alpha}$) diffeomorphism f of an n-dimensional torus T^n with $n \geq 3$ such that it induces an Anosov automorphism $f_* : \mathbb{Z}^n \to \mathbb{Z}^n$ in homologies (and therefore, its topological entropy is positive) and its measure entropy with respect to some invariant smooth measure on T^n vanishes ?*

A generalization of the Kozlov theorem for higher-dimensional manifolds was found in [14, 15] where it was shown that if the geodesic flow on a closed manifold M^n is analytically integrable then the unit cotangent bundle SM^n contains an invariant torus T^n such that its projection onto the base

$$\pi : T^n \subset SM^n \to M^n$$

induces a homomorphism of the fundamental groups $\pi_* : \pi_1(T^n) \to \pi_1(M^n)$ whose image $\pi_*(\pi_1(T^n))$ has a finite index in $\pi_1(M^n)$:

$$[\pi_1(M^n) : \pi_*(\pi_1(T^n))] < \infty.$$

This implies that

1) the fundamental group of M^n is almost commutative;

2) if the first Betti number $b_1(M^n)$ of M^n equals k: $b_1(M^n) = k$, then the real cohomology ring $H^*(M^n; \mathbb{R})$ of M^n contains a subring isomorphic to the real cohomology ring of the k-dimensional torus:

$$H^*(T^k; \mathbb{R}) \subset H^*(M^n; \mathbb{R}).$$

In particular, this implies that

$$b_1(M^n) \leq n = \dim M^n; \tag{18}$$

3) if $b_1(M^n) = \dim M^n$, then $H^*(T^n; \mathbb{R}) = H^*(M^n; \mathbb{R})$.

This result is valid for more general case when the flow is not analytically integrable but so-called geometrically simple and also is immediately generalized for superintegrable cases when there are more than n functionally independent real analytic first integrals and generic tori are l-dimensional with $l < n$ (in this case the "maximal" torus whose fundamental group projects into a group with finite index is l-dimensional).

As shown by Butler [3] some of these topological properties do not obstruct C^∞ integrability: for Butler's manifold we have $b_1 = 2$ and the fundamental group is not almost commutative and H^* contains no subring isomorphic to $H^*(T^2; \mathbb{R})$ but the inequality (18) is valid. In fact this is true also for the geodesic flows on M_A where A is not of finite order.

We would like to introduce the following

Conjecture 1 *Let the geodesic flow on a Riemannian manifold M^n is integrable in terms of C^∞ first integrals. Then the inequalities*

$$b_k(M^n) \leq b_k(T^n) = \frac{n!}{k!(n-k)!}, \tag{19}$$

hold.

These inequalities mean that homologically M^n is dominated by the n-dimensional torus. They were already mentioned in talks of the second author (I.A.T.) in the early 90s. It was derived by Paternain from results of Gromov and Yomdin that if the topological entropy of the geodesic flow of a C^∞ metric on a simply connected manifold vanishes, then this manifold is rationally elliptic (in the sense of Sullivan) [11] and he also mentioned that, by results Friedlander and Halperin, rational ellipiticity implies the inequalities (19).

Actually, it was Paternain who proposed the entropy approach to finding topological obstructions to integrability. He proposed to split this problem into two ones: proving the vanishing of the topological entropy of an integrable geodesic flow and finding topological obstructions to vanishing of the topological entropy of a flow. The second problem was already studied and in addition to the results of Gromov and Yomdin, which we already mentioned above, we would like to remind the theorem of Dinaburg who proved that if the fundamental group of the manifold has an exponential growth, then the topological entropy of the geodesic flow of any smooth metric on the manifold is positive [4].

Paternain found some conditions mainly concerning existence of rather good action-angle variables on the set, where the first integrals are functionally dependent, which in addition to integrability imply the vanishing of the topological entropy [11, 12] (after that some other similar conditions were exposed in [16]).

He also conjectured that the topological entropy of an integrable geodesic flow vanishes and that the fundamental group of a manifold with an integrable geodesic flow has a subexponential growth.

In [2] we disproved both these conjectures in the C^∞ case. Since it is proved in [14], that if the geodesic flow is analytically integrable, then the fundamental group of the manifold has a polynomial growth, we are left with the following real-analytic version of Paternain's conjecture:

Conjecture 2 *If the geodesic flow on a closed manifold is analytically integrable, then the topological entropy of the flow vanishes.*

We already mentioned about eight Thurston's canonical three-geometries, which are the homogeneous geometries of $S^3, \mathbb{R}^3, H^3, S^2 \times \mathbb{R}, H^2 \times \mathbb{R}, \mathrm{NIL}$, SOL, and $SL(2,\mathbb{R})$. Here we denote by H^n the n-th dimensional Lobachevsky space. Since the Lyapunov exponents does not vanish at any point, there are no compact quotients of H^3 and $H^2 \times \mathbb{R}$ with integrable geodesic flows. There are well-known examples of compact quotients of $\mathbb{R}^3$ and $S^2 \times \mathbb{R}$ with integrable geodesic flows, which are, for instance, flat tori T^3 and $S^2 \times S^1$. The geodesic flow of the Killing metric on $SU(2) = S^3$ is also integrable. As shown in [3] and [2] there are compact quotients of NIL and SOL with integrable geodesic flows. Hence it remains to answer the following question:

Problem 4 *Do there exist compact quotients of $SL(2,\mathbb{R})$ with integrable geodesic flows ?*

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