# Partial regularity for almost minimizers of quasiconvex integrals

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October 1999

**Abstract**

We consider almost minimizers of variational integrals whose integrands are quasiconvex. Under suitable growth conditions on the integrand and on the function determining the almost minimality, we establish almost everywhere regularity for almost minimizers, and obtain results on the regularity of the gradient away from the singular set. We give examples of problems from the calculus of variations whose solutions can be viewed as such almost minimizers.

## 1 Introduction

One of the most basic questions in the calculus of variations is that of existence and regularity of minimizers of regular functionals subject to some sort of boundary conditions. To fix ideas we consider a functional

$$\mathcal{F}(u) = \int_U f(x, u, Du)\, dx\,, \tag{1-1}$$

for $x \in U$, a domain in $\mathbb{R}^n$, $u$ mapping $U$ into $\mathbb{R}^N$; then $\mathcal{F}$ is *regular* if $f(x, u, p)$ is convex in $p$. Appropriate growth conditions on $f$ can be imposed to ensure that the Euler equation corresponding to $\mathcal{F}$ is elliptic, or at least degenerate elliptic; however even under reasonable assumptions on $f$, in the case of systems of equations (i.e. $N>1$) one cannot, in general, expect that minimizers of $\mathcal{F}$ will be classical, i.e. $C^2$-solutions. This was first shown by De Giorgi [DeG]; we refer the reader to [G1, Chapter 2.3] for further discussion. It is thus of interest to consider questions of partial regularity. The *regular set* of a solution $u$ is defined by

$$\operatorname{Reg} u = \{x \in \Omega \mid u \text{ is continuous on a neighbourhood of } x\},$$

and the *singular set* by

$$\operatorname{Sing} u = \Omega \setminus \operatorname{Reg} u.$$

Partial regularity theory involves estimating the size of $\operatorname{Sing} u$ (i.e. showing that $\operatorname{Sing} u$ has zero $n$-dimensional Lebesgue measure or better, controlling the Hausdorff dimension of $\operatorname{Sing} u$), and showing higher regularity on $\operatorname{Reg} u$. There is a wealth of literature covering the existence and regularity of minimizers (and, more generally, of stationary points) of regular functionals; we refer the reader to the monographs [G1] [G2] and the literature contained therein.

The condition (for $\mathcal{F}$ to be regular) that the integrand be convex in the gradient is quite restrictive. There are a number of interesting and important problems in the calculus of variations which are not regular; in addition, weak lower semicontinuity, an essential notion for showing the existence of minimizers, is implied by convexity (in appropriate Sobolev spaces), but not vice versa. This lead Morrey to introduce the notion of *quasiconvexity* in the paper [M1]; we postpone giving a precise definition until Section 2, and simply note here that Morrey showed that, in many circumstances, quasiconvexity and weak lower semicontinuity were equivalent, and refer the reader additionally to [Da], [Ba] and [AF] for discussion, literature and further references.

The first results on partial regularity for minimizers of general quasiconvex integrands were obtained by Evans [Ev]. He considered integrals of the form $\mathcal{F}(u) = \int_U f(Du)\, dx$ and showed, under the principle assumption of uniform strict quasiconvexity (see (H2) of the current paper), that a minimizer $u$ of such an $\mathcal{F}$ satisfies $\mathcal{L}^n(\operatorname{Sing} u) = 0$, and that $Du$ is Hölder continuous for all exponents between 0 and 1; see [Ev, Section 2] for precise statements. These results were extended independently by Fusco-Hutchinson [FH] and Giaquinta-Modica [GM] to more general functionals of the form (1-1) under assumptions comparable to our (H1)–(H4), and an additional assumption concerning the Hölder continuity of the integrand $f(x, u, p)$ in $x$ and $u$; see [FH, Section 2] and [GM, Theorem 1.1]. Note in particular that in these results $Du$ is shown to be Hölder continuous for some exponent depending on the Hölder continuity of the integrand $f$.

In the current paper we wish to consider a more general class of functions than minimizers, namely *almost minimizers*. Writing $\mathcal{F}(u; D)$ for $\int_D f(x, u, Du)\, dx$, an almost minimizer (at $x_0$) for $\mathcal{F}$ is a function $u$ for which

$$\mathcal{F}(u; B_\rho(x_0)) \leq \mathcal{F}(u + \varphi; B_\rho(x_0)) + \omega(\rho) \int_{B_\rho(x_0)} (1 + |Du|^2 + |D\varphi|^2) dx \tag{1-2}$$

for all suitable test functions $\varphi$ with $\operatorname{supp} \varphi \subset B_\rho(x_0)$; see Definition 2.1 for a precise statement. Here $\omega$ is a real-valued function. Obviously $\omega$ identically vanishing corresponds to the case of $\mathcal{F}$-minimizers, and minimal conditions on $\omega$ (continuous and nondecreasing at 0 with $\omega(0) = 0$) ensure that the the term almost minimizer makes sense. In the next section we impose some additional (mild) conditions on $\omega$ and give examples that show that solutions of a number of problems in the calculus of variations (precisely, minimizers subject to certain constraints) are almost minimizers of suitable functionals, so the notion of an almost minimizer is in fact useful.

A comparable but more restrictive definition of an almost minimizer was given by Anzellotti [An]. In that paper the author shows partial regularity for almost minimizers of the (regular) functional with integrand given by $a^{\alpha\beta}(x) D_\alpha u D_\beta u + g(x)$ for suitably regular $a^{\alpha\beta}$ and $g$; see [An, Theorem 1.5]. Anzellotti's definition was more restrictive in two respects; he required Hölder continuity for the function $\omega$, and required a sharper inequality than (1-2). We also mention that there is another related concept for regular integrands, viz. that of a quasi-minimizer (or Q-minimizer); here the right-hand side of (1-2) is replaced by $Q\,\mathcal{F}(u+\varphi; B_\rho(x_0))$ for some constant $Q \geq 1$; see [G1, Chapter IX] for details and further references.

We also note here that there are close ties between the current setting and the study of elliptic parametric variational problems in geometric measure theory. In particular, our notion of an almost minimizer is analogous to Almgren's definition of an $(\mathbf{F}, \varepsilon, \delta)$-minimizer; see [Al, Chapter III]. Indeed our regularity result, Theorem 2.2, is the analogue of Almgren's regularity theorem ([Al, Theorem III.3.7]) in the current setting; of course [Al, Theorem III.3.7] is broader in scope, and the proof is considerably more involved than the proof of our regularity result. We refer the reader to [Ev, Section 1] for more comments on the connections to geometric measure theory, and restrict ourselves here to noting the abovementioned work of Almgren [Al] as well as the paper of Bombieri [Bo]. The closest analogue of the current paper in the setting of geometric measure theory is the paper [DS], where the authors prove optimal regularity results for almost minimizing rectifiable currents of general elliptic integrands.

The main regularity result of this paper is given in Theorem 2.2. We consider integrals of the form $F(u) = \int_U f(Du)\,dx$ and show, under reasonable conditions on $f$ (the main one being uniform strict quasiconvexity) and the function $\omega$, that $(F, \omega)$-minimizers are regular away from a set of zero-measure. In addition we obtain an optimal local modulus of continuity for $Du$ on $\operatorname{Reg} u$. The structure of the proof and the nature of our definition of an almost minimizer enable us to extend this result to families of such integrals. This allows us to obtain, as an easy corollary, partial regularity for *minimizers* of integrals of the form $F(u) = \int_U f(x, Du)\,dx$ where $f$ is quasiconvex, but where we only require a Dini condition (cf. [HW, Section 1]) on the continuity of the coefficients in $x$. In particular, we do not need to assume that the coefficients are Hölder continuous with respect to $x$, in contrast to the results of [FH] and [GM] (of course, the results there admit $u$-dependency, in contrast to the current paper). Indeed even for minimizers of regular integrals of the form $F(u) = \int_U f(x, Du)\,dx$, in the case of systems (i.e. $N > 1$) this appears to be the first time that partial regularity results have been obtained for coefficients which are not Hölder continuous (there are a number of results for scalar valued problems; we mention here specifically [HW] and the recent paper [Ko]).

We wish to briefly comment on our technique. The central idea in our proof is that of $A$-harmonic approximation, as expressed in Lemma 4.2. This idea, too, has its origins in the field of geometric measure theory, specifically in Simon's proof of the regularity theorem of Allard ([A]), see [S1, Section 23], and cf. [Bo]. The point here is to show that, for $A \in \operatorname{Bil}(\operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N))$ which is rank-one elliptic, a function which is "approximately $A$-harmonic", i.e. a function $g$ for which $\int_U A(Dg, D\varphi)\,dx$ is sufficiently small for all test functions $\varphi$, lies $L^2$–close to some $A$-harmonic function. Lemma 4.2 is due to Duzaar-Steffen (see [DS, Lemma 3.3]). The lemma is also vital to the paper [DG], where the authors give an elementary, self-contained approach to partial regularity for nonlinear elliptic systems of divergence type.

Many of the advantages of the approach of [DG] are relevant in the current paper. In particular we note that the arguments in both papers avoid the technical complications associated with using Gehring's Lemma ([Ge]); as noted above, in the current setting this is essential to obtaining the optimal modulus of continuity. Furthermore the $A$-harmonic approximation lemma is the only time where we argue indirectly, so we keep some control on the sensitivity to the structure constants in our proof.

In Section 2 we discuss our assumptions on the integrand $f$ and the function $\omega$, give a number of examples (as discussed above, these are concerned with applications of the partial regularity theorem and with showing that the notion is in fact useful; we also show how the result is optimal in a certain sense). The remainder of the paper is concerned with the proof of the regularity theorem.

We close this section by briefly summarizing the notation we use in this paper. As noted above, we consider a domain $U \subset \mathbb{R}^n$, and maps from $U$ to $\mathbb{R}^N$, where we take $n \geq 2$, $N \geq 1$. For a given set $X$ we denote by $\mathcal{L}^n(X)$ its $n$–dimensional Lebesgue measure. We write $B_\rho(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$, and further $B_\rho = B_\rho(0)$, $B = B_1$. For bounded $X \subset \mathbb{R}^n$ we denote the average of a given $g \in L^1(X)$ by $⨍_X g\,dx$, i.e. $⨍_X g\,dx = \frac{1}{\mathcal{L}^n(X)} \int_X g\,dx$. In particular, we write $g_{x_0,\rho} = ⨍_{B_\rho(x_0)} g\,dx$. We let $\alpha_n$ denote the volume of the unit–ball in $\mathbb{R}^n$, i.e. $\alpha_n = \mathcal{L}^n(B)$. We write $\mathrm{Bil}(\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N))$ for the space of bilinear forms on the space $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^N$.

# 2 Assumptions, examples and the partial regularity theorem

We consider a function $\omega : [0, \infty) \to [0, \infty)$, and define

$$\Omega(r) := \Big( \int_0^r \frac{\sqrt{\omega(\rho)}}{\rho}\, d\rho \Big)^2 .$$

We impose the conditions:

**($\boldsymbol{\omega}$0)** $\omega$ is nondecreasing;

**($\boldsymbol{\omega}$1)** $r \mapsto \omega(r)/r^{2\alpha}$ is nonincreasing for some $\alpha \in (0,1)$;

**($\boldsymbol{\omega}$2)** $\omega(r) \leq 1$ for all $r$; and

**($\boldsymbol{\omega}$3)** $\Omega(r)$ is finite for some $r > 0$.

Note that all the arguments involving $\omega$ in this paper are local in nature, so ($\omega$2) is always realizable. In addition ($\omega$3) shows that $\Omega(r)$ is in fact finite for all positive $r$. Before we discuss some of the consequences of ($\omega$0)–($\omega$3) we define the central concept of the paper, that of an almost minimizer.

**Definition 2.1** Consider a functional $\mathcal{F}$ defined on $H^{1,2}_{loc}(U, \mathbb{R}^N)$ and $\omega : [0, \infty) \to [0, \infty)$. A function $u \in H^{1,2}_{loc}(U, \mathbb{R}^N)$ is called $(\mathcal{F}, \omega)$*-minimizing at* $x_0 \in U$ if, for all $\rho > 0$ with $B_\rho(x_0) \subset\subset U$, there holds:

$$\mathcal{F}(u; B_\rho(x_0)) \leq \mathcal{F}(u + \varphi; B_\rho(x_0)) + \omega(\rho) \int_{B_\rho(x_0)} (1 + |Du|^2 + |D\varphi|^2)\, dx \qquad \text{(2-1)}$$

for all $\varphi \in H_0^{1,2}(B_\rho(x_0), \mathbb{R}^N)$.

A function $u$ is $(\mathcal{F}, \omega)$-*minimizing* if $u$ is $(\mathcal{F}, \omega)$-minimizing for each $x_0 \in U$.

We now note some less immediate consequences of the above conditions, which we will need in Section 5. From $(\omega 1)$ we immediately see

$$\omega(tr) \leq t^{2\alpha} \omega(r) \qquad \text{for } t \geq 1, \tag{2-2}$$

and from the definition of $\Omega$ we thus have

$$\Omega(tr) \leq t^{2\alpha} \Omega(r) \qquad \text{for } t \geq 1. \tag{2-3}$$

We further have, for $0 < \tau < 1$, $r > 0$, $j \in \mathbb{N} \cup \{0\}$

$$\frac{1}{\alpha}(1 - \tau^\alpha)\sqrt{\omega(\tau^j r)} = \frac{\sqrt{\omega(\tau^j r)}}{(\tau^j r)^\alpha} \int_{\tau^{j+1} r}^{\tau^j r} \rho^{\alpha - 1}\, d\rho \leq \int_{\tau^{j+1} r}^{\tau^j r} \frac{\sqrt{\omega(\rho)}}{\rho}\, d\rho. \tag{2-4}$$

This estimate has two useful consequences. We firstly note

$$\sum_{j=0}^{\infty} \sqrt{\omega(\tau^j r)} \leq \frac{\alpha}{1 - \tau^\alpha} \sqrt{\Omega(r)}. \tag{2-5}$$

In addition we see

$$\omega(r) \leq \Omega(r) \tag{2-6}$$

for all $r > 0$. We note further that $(\omega 0)$ and $(\omega 1)$ imply continuity of $\omega$ at 0, as well as $\omega(0) = 0$.

We now discuss our assumptions on the functional in question. We consider functionals of the form

$$F(u) := \int_U f(Du)\, dx,$$

where $U$ is a domain in $\mathbb{R}^n$, and $f : \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N) \to \mathbb{R}$ satisfies the following conditions:

**(H1)** there exist positive constants $c_1$ and $c_2$ such that, for all $p \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$

$$c_1^{-1}|p|^2 - c_2 \leq f(p) \leq c_1 |p|^2 + c_2;$$

**(H2)** the function $f$ is *(uniformly) strictly quasiconvex*, i.e. there exists $\lambda > 0$ such that for all $B_\rho(x_0) \subset\subset U$, $p \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$, $\varphi \in C_0^1(B_\rho(x_0), \mathbb{R}^n)$ there holds:

$$\int_{B_\rho(x_0)} \Big( f(p + D\varphi) - f(p) \Big)\, dx \geq \lambda \int_{B_\rho(x_0)} |D\varphi|^2\, dx;$$

**(H3)** the function $f$ is $C^2$ and there exists $L \geq 0$ such that for all $p \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ there holds $|D^2 f(p)| \leq L$.

Note that the upper bound in (H1) follows from (H3), and the lower bound is only useful for questions of existence, cf. [M2, 4.4.7], [Ev, p. 228]. We include the condition here largely for completeness in the examples which follow.

Condition (H2) implies the *Legendre-Hadamard condition*, see [M2, 4.4.3, 4.4.1] or [Fe, 5.1.10], i.e.

$$\sum_{i,j=1}^{N}\sum_{\alpha,\beta=1}^{n}\frac{\partial^2 f}{\partial p_\alpha^i \partial p_\beta^j}(p)\xi^i\xi^j\eta_\alpha\eta_\beta \geq \lambda|\xi|^2|\eta|^2 \tag{2-7}$$

for all $p \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$, $\xi \in \mathbb{R}^N$ and $\eta \in \mathbb{R}^n$.

From condition (H3) we have

$$|Df(p) - Df(\tilde{p})| \leq L|p - \tilde{p}|\,; \tag{2-8}$$

this condition also implies the existence of a modulus of continuity of $D^2 f$, more precisely of a family of monotone nondecreasing, concave functions $\nu(M,\cdot) : [0,\infty) \to [0,\infty)$ for $M > 0$ satisfying $\nu(M,0) = 0$ and

$$|D^2 f(p) - D^2 f(\tilde{p})| \leq \nu(M, |p - \tilde{p}|^2) \tag{2-9}$$

for all $p, \tilde{p} \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ with $|p| \leq M$ .

For the proof of our main theorem we will initially strengthen (H3) by further imposing

**(H4)** $D^2 f$ is uniformly continuous.

In conjunction with (H3) this leads to the existence of a monotone nondecreasing, concave function $\nu : [0,\infty) \to [0,\infty)$ satisfying $\nu(0) = 0$ and

$$|D^2 f(p) - D^2 f(\tilde{p})| \leq \nu(|p - \tilde{p}|^2) \tag{2-10}$$

for all $p, \tilde{p} \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$. At the end of the paper (Corollary (5.3)) we show how the arguments can be modified to remove (H4).

We are also interested in being able to consider a different functional at each point, i.e. functionals of the form

$$F_{x_0}(u) := \int_U f_{x_0}(Du)\,dx,$$

for $x_0 \in U$. Given a family of such functionals, the analogues of (H2) and (H3) are:

**(h2)** the functions $f_{x_0}$ are *uniformly strictly quasiconvex*, i.e. there exists $\lambda > 0$ such that for all $B_\rho(x_0) \subset\subset U$, $p \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$, $\varphi \in C_0^1(B_\rho(x_0), \mathbb{R}^n)$ there holds:

$$\int_{B_\rho(x_0)} \Big(f_{x_0}(p + D\varphi) - f_{x_0}(p)\Big)\,dx \geq \lambda \int_{B_\rho(x_0)} |D\varphi|^2\,dx;$$

**(h3)** the functions $f_{x_0}$ are $C^2$ and there exists $L \geq 0$ such that for all $p \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ and $x_0 \in U$ there holds $|D^2 f_{x_0}(p)| \leq L$.

Just as we imposed the additional condition (H4) to obtain a uniform modulus of continuity above, we will have occasion to require:

**(h4)** The second derivatives $D^2 f_{x_0}$ admit a *uniform* modulus of continuity, i.e. there exists a monotone nondecreasing, concave function $\nu : [0,\infty) \to [0,\infty)$ satisfying $\nu(0) = 0$ and

$$|D^2 f_{x_0}(p) - D^2 f_{x_0}(\tilde{p})| \leq \nu(|p - \tilde{p}|^2) \tag{2-11}$$

for all $p, \tilde{p} \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ and $x_0 \in U$.

We are now in a position to state our main result.

**Theorem 2.2** *(i) On a domain $U \subseteq \mathbb{R}^n$ consider a function $\omega$ satisfying ($\omega 0$)–($\omega 3$), and a function $f$ which satisfies (H2) and (H3). Let $F$ be the functional on $H^{1,2}(U, \mathbb{R}^N)$ given by $F(u) = \int_U f(Du)\, dx$. Let $u \in H^{1,2}(U, \mathbb{R}^N)$ be $(F, \omega)$-minimizing on $U$. Then there exists a relatively closed subset of $U$, $\mathrm{Sing}\, u$, such that*

$$u \in C^1(U \setminus \mathrm{Sing}\, u)\,.$$

*Further $\mathrm{Sing}\, u \subseteq \Sigma_1 \cup \Sigma_2$, where here*

$$\begin{aligned} \Sigma_1 &= \{x_0 \in U \ : \ \liminf_{\rho \to 0^+} \fint_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2\, dx > 0\}, \quad \textit{and} \\ \Sigma_2 &= \{x_0 \in U \ : \ \sup_{\rho > 0} |(Du)_{x_0,\rho}| = \infty\}\ ; \end{aligned}$$

*in particular $\mathcal{L}^n(\mathrm{Sing}\, u) = 0$.*

*In addition in a neighbourhood of any $x_0 \in U \setminus \mathrm{Sing}\, u$ and for any $\beta$ with $\alpha < \beta < 1$, $Du$ has a modulus of continuity given by*

$$\mu(r) = c(r^\beta + \sqrt{\Omega(r)})\,,$$

*where $c$ is a constant depending only on $\limsup_{\rho \to 0} |(Du)_{x_0,\rho}|$, on $\beta$, on the dimensions $n$ and $N$, on the structural parameters $\lambda$, $L$ and $\alpha$, and on the functions $\omega(\cdot)$ and $\nu(\cdot)$.*

*(ii) The conclusion also holds when $\{F_{x_0}\}_{x_0 \in U}$ is a family of functionals arising from functions $\{f_{x_0}\}_{x_0 \in U}$ satisfying (h2),(h3) and (h4), $\omega$ is as above, and $u \in H^{1,2}(U, \mathbb{R}^N)$ is $(F_{x_0}, \omega)$-minimizing for each $x_0 \in U$.*

We now give a few examples of almost minima, and applications of the partial regularity result.

**Example 1:** Consider $u$ minimizing a functional of the form

$$G(u) := \int_U g(x, Du)\, dx\,,$$

where here the frozen coefficients

$$g_{x_0}(p) := g(x_0, p)$$

satisfy (h2), (h3) and (h4), and in addition

$$|g(x,p) - g(\tilde{x}, p)| \leq \omega(|x - \tilde{x}|)(1 + |p|^2) \tag{2-12}$$

for all $x, \tilde{x} \in U$ and all $p \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ for some $\omega$ satisfying ($\omega 1$)–($\omega 3$). Writing

$$G_{x_0}(u) := \int_U g_{x_0}(Du)\, dx = \int_U g(x_0, Du)\, dx,$$

we have that $u$ is $(G_{x_0},\omega)$–minimizing for each $x_0 \in U$.

**Example 2:** Consider $v$ in $C^{1,\alpha}$. For every $x_0 \in U$, consider

$$G_{x_0}(u) := \int_U |Du(x) - Dv(x_0)|^2\, dx\,;$$

this functional satisfies (h2), (h3) and (h4) with $\lambda = 1$ and $L = 2$. For $\rho$ so small that $B_\rho(x_0) \subset\subset U$, we have:

$$\begin{aligned}\int_{B_\rho(x_0)} |Dv(x) - Dv(x_0)|^2\, dx &\leq \int_0^\rho \omega_{n-1} t^{n-1} t^{2\alpha}\, dt = \frac{\rho^{2\alpha}}{n+2\alpha}\int_{B_\rho(x_0)} 1\, dx \\ &\leq \; G_{x_0}(v+\varphi; B_\rho(x_0)) + c_3 \rho^{2\alpha} \int_{B_\rho(x_0)} (1 + |Dv|^2 + |D\varphi|^2)\, dx \qquad (2\text{-}13)\end{aligned}$$

for all $\varphi \in H_0^{1,2}(B_\rho(x_0), \mathbb{R}^N)$. Hence $v$ is $(G_{x_0},\omega)$–minimizing at each $x_0 \in U$, with $\omega(r) = c_3 r^{2\alpha}$. This shows that our regularity theorem is optimal.

There are a number of additional examples given in [An, Section 3], in particular that of an obstacle problem and of a volume-constrained problem, which are of course also valid with our more general notion of an almost minimizer. We also note here that comparable examples exist in the setting of geometric measure theory, see e.g. [Al] and [Ta].

# 3 The Caccioppoli inequality

We begin by stating an elementary technical lemma from Fusco-Hutchinson, [FH, Lemma 3.2] (cf. [G1, Chapter V, Lemma 3.1]); for completeness we include the result here.

**Lemma 3.1** *Let $h$ be nonnegative and bounded on $[\rho/2, \rho]$, and satisfy*

$$h(t) \leq \theta h(s) + A(s-t)^{-2} + B$$

*for positive constants $A$, $B$ and $\theta$ with $0 < \theta < 1$, for all $s$ and $t$ with $\rho/2 \leq s < t < \rho$. Then there exists a constant $c$ depending only on $\theta$ such that*

$$h(\rho/2) \leq c(A\rho^{-2} + B)\,.$$

□

We now prove a suitable version of the Caccioppoli inequality. The proof is close to that of [Ev, Lemma 5.1] and [GM, Proposition 4.1].

**Lemma 3.2** *Let $f$ satisfy (H2) and (H3), and $\omega$ satisfy ($\omega$0), ($\omega$1) and ($\omega$2). Let $F$ be the functional on $H^{1,2}(U,\mathbb{R}^N)$ given by $F(u) = \int_U f(Du)\, dx$. Then there exist* *positive constants $\rho_1 = \rho_1(\lambda, \omega(\cdot))$ and $c_4 = c_4(\lambda, L)$, (without loss of generality we take $c_4 \geq 1$) such that for every $B_\rho(x_0) \subset\subset U$ with $\rho \leq \rho_1$, $p_0 \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ and every $u \in H^{1,2}(B_\rho(x_0), \mathbb{R}^N)$ which is $(F, \omega)$–minimizing at $x_0$ there holds:*

$$\begin{aligned}
&\int_{B_{\rho/2}(x_0)} |Du - p_0|^2 \, dx \\
&\leq \; c_4 \Big[ \rho^{-2} \int_{B_\rho(x_0)} |u - p_0(x - x_0)|^2 \, dx + \alpha_n \omega(\rho) \rho^n (1 + |p_0|^2) \Big]. \qquad (3\text{-}1)
\end{aligned}$$

**Proof:** For $\frac{\rho}{2} \leq t < s \leq \rho$ choose $\eta \in C_0^\infty(B_\rho(x_0), [0,1])$, $\eta \equiv 1$ on $B_t(x_0)$, $\eta \equiv 0$ outside $B_s(x_0)$ and $|\nabla \eta| \leq 2/(s-t)$. We set

$$\begin{aligned}
\varphi &:= \eta(u - p_0(x - x_0)), \\
\psi &:= (1 - \eta)(u - p_0(x - x_0)).
\end{aligned}$$

Then

$$D\varphi + D\psi = Du - p_0 \qquad (3\text{-}2)$$

and, with $v(x) := u(x) - p_0(x - x_0)$,

$$\begin{aligned}
|D\varphi|^2 &\leq 2|Du - p_0|^2 + \frac{8}{(s-t)^2}|v|^2, \qquad (3\text{-}3) \\
|D\psi|^2 &\leq 2|Du - p_0|^2 + \frac{8}{(s-t)^2}|v|^2. \qquad (3\text{-}4)
\end{aligned}$$

From (H2) and (3-2) we have

$$\lambda \int_{B_s(x_0)} |D\varphi|^2 \, dx \leq \int_{B_s(x_0)} [f(p_0 + D\varphi) - f(p_0)] \, dx = I + II + III \,, \qquad (3\text{-}5)$$

where

$$\begin{aligned}
I &= \int_{B_s(x_0)} [f(Du - D\psi) - f(Du)] \, dx \,, \\
II &= \int_{B_s(x_0)} [f(Du) - f(Du - D\varphi)] \, dx \qquad \text{and} \\
III &= \int_{B_s(x_0)} [f(p_0 + D\psi) - f(p_0)] \, dx \,.
\end{aligned}$$

The $(F, \omega)$-minimality and (3-3), along with $(\omega 2)$, imply

$$\begin{aligned}
II &\leq \omega(s) \int_{B_s(x_0)} \Big(1 + |Du|^2 + |D\varphi|^2\Big) \, dx \\
&\leq \omega(s) \int_{B_s(x_0)} \Big(1 + 2|p_0|^2 + 4|Du - p_0|^2 + \frac{8}{(s-t)^2}|v|^2\Big) \, dx \\
&\leq \frac{\lambda}{2} \int_{B_s(x_0)} |Du - p_0|^2 \, dx + \frac{8}{(s-t)^2} \int_{B_s(x_0)} |v|^2 \, dx \\
&\quad + 2\alpha_n \omega(\rho) \rho^n (1 + |p_0|^2), \qquad (3\text{-}6)
\end{aligned}$$

as long as $\rho$ is sufficiently small that $8\omega(\rho) \le \lambda$; by $(\omega 0)$ and $(\omega 1)$ we can choose $\rho_1 > 0$ such that this holds for all $\rho \in (0, \rho_1]$. For the other terms we have (via (2-8) and (3-2), as well as (3-4))

$$\begin{aligned} I + III \le L \int_{B_s(x_0)} \Big(|Du - p_0| + |D\psi|\Big)|D\psi|\, dx \\ = \;\; L \int_{B_s(x_0)\setminus B_t(x_0)} \Big(|Du - p_0| + |D\psi|\Big)|D\psi|\, dx \\ \le \;\; \frac{L}{2} \int_{B_s(x_0)\setminus B_t(x_0)} |Du - p_0|^2\, dx + \frac{3L}{2} \int_{B_s(x_0)\setminus B_t(x_0)} |D\psi|^2\, dx \\ \le \;\; \frac{7L}{2} \int_{B_s(x_0)\setminus B_t(x_0)} |Du - p_0|^2\, dx + \frac{12L}{(s-t)^2} \int_{B_s(x_0)} |v|^2\, dx. \end{aligned} \tag{3-7}$$

Combining (3-6) and (3-7) in (3-5) and noting $D\varphi = Du - p_0$ on $B_t(x_0)$ we see

$$\begin{aligned} \frac{\lambda}{2} \int_{B_t(x_0)} |Du - p_0|^2\, dx \le \frac{7L + \lambda}{2} \int_{B_s(x_0)\setminus B_t(x_0)} |Du - p_0|^2\, dx \\ + \frac{12L + 8}{(s-t)^2} \int_{B_s(x_0)} |v|^2\, dx + 2\alpha_n \omega(\rho)\rho^n (1 + |p_0|^2). \end{aligned} \tag{3-8}$$

Thus we have

$$\begin{aligned} \int_{B_t(x_0)} |Du - p_0|^2\, dx \le \frac{7L + \lambda}{7L + 2\lambda} \int_{B_s(x_0)} |Du - p_0|^2\, dx \\ + \frac{24L + 16}{7L(s-t)^2} \int_{B_s(x_0)} |v|^2\, dx + \frac{4}{7L} \alpha_n \omega(\rho)\rho^n (1 + |p_0|^2). \end{aligned} \tag{3-9}$$

Since $\frac{7L+\lambda}{7L+2\lambda} < 1$ we can apply Lemma 3.1 to conclude (3-1). $\square$

# 4 Approximate $A$-harmonicity and $A$-harmonic approximation

The next lemma is a prerequisite for applying the $A$-harmonic approximation technique.

**Lemma 4.1** *Let $\omega$ satisfy $(\omega 2)$, and $f$ satisfy (H2), (H3) and (H4). Then there exists $c_5 = c_5(n, L)$ such that for every $u \in H^{1,2}(U, \mathbb{R}^N)$ that is $(F, \omega)$–minimizing at $x_0$, every ball $B_\rho(x_0) \subset\subset U$ and every $p_0 \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ we have:*

$$\begin{aligned} \Big|\rho^{-n} \int_{B_\rho(x_0)} D^2 f(p_0)(Du - p_0, D\varphi)\, dx\Big| \\ \le \;\; c_5 \Big[\omega^{1/2}(\rho)(1 + \Phi + |p_0|^2) + \nu^{1/2}(\Phi)\Phi^{1/2}\Big] \sup_{B_\rho(x_0)} |D\varphi| \end{aligned} \tag{4-1}$$

*for all $\varphi \in C_0^1(B_\rho(x_0), \mathbb{R}^N)$. Here we write*

$$\Phi = \Phi(x_0, \rho, p_0) := \fint_{B_\rho(x_0)} |Du - p_0|^2\, dx. \tag{4-2}$$

**Proof:** Without loss of generality we take $x_0 = 0$. We firstly note

$$\begin{aligned}\int_{B_\rho} Df(Du)\cdot D\varphi\, dx &= \int_{B_\rho} Df(Du)\cdot D\varphi\, dx - \int_{B_\rho} Df(p_0)\cdot D\varphi\, dx \\ &= \int_{B_\rho}\int_0^1 D^2 f(p_0+\tau(Du-p_0))\,(Du-p_0, D\varphi)d\tau dx. \qquad (4\text{-}3)\end{aligned}$$

Initially we assume $|D\varphi| \le 1$ on $B_\rho$. For positive $s$ we have from the $(F,\omega)$–minimality of $u$:

$$\begin{aligned}&\int_{B_\rho} D^2 f(p_0)(Du-p_0, D\varphi)\, dx \\
&\geq \frac{1}{s}\Big[\int_{B_\rho}(f(Du)-f(Du+sD\varphi))\, dx - \omega(\rho)\int_{B_\rho}\Big(1+|Du|^2+s^2|D\varphi|^2\Big)\, dx\Big] \\
&\qquad + \int_{B_\rho} D^2 f(p_0)(Du-p_0, D\varphi)\, dx \\
&\geq \frac{1}{s}\Big[-\int_{B_\rho}\int_0^s \frac{d}{dt} f(Du+tD\varphi)\, dt\, dx + s\int_{B_\rho} D^2 f(p_0)(Du-p_0, D\varphi)\, dx \\
&\qquad - \omega(\rho)\int_{B_\rho}\Big(1+s^2+|Du|^2\Big)\, dx\Big] \qquad \text{since } |D\varphi|\le 1 \\
&= \frac{1}{s}\Big[\int_{B_\rho}\int_0^s \Big(Df(Du)-Df(Du+tD\varphi)\Big)\cdot D\varphi\, dt\, dx \\
&\qquad + s\int_{B_\rho}\int_0^1 \Big(D^2 f(p_0) - D^2 f(p_0+\tau(Du-p_0))\Big)\, d\tau(Du-p_0, D\varphi)\, dx \\
&\qquad - \omega(\rho)\int_{B_\rho}\Big(1+s^2+|Du|^2\Big)\, dx\Big] \qquad \text{via (4-3)} \\
&\geq -\frac{1}{s}\Big[\frac{L}{2}s^2\alpha_n\rho^n + s\sqrt{2L}\int_{B_\rho}\nu^{1/2}(|Du-p_0|^2)|Du-p_0|\, dx \\
&\qquad + \omega(\rho)\int_{B_\rho}\Big(1+s^2+|Du|^2\Big)\, dx\Big] \qquad \text{via (2-8),(2-10),(H3)} \\
&\geq -\frac{L}{2}s\alpha_n\rho^n\, dx - \sqrt{2L}\,\alpha_n\rho^n\nu^{1/2}(\fint_{B_\rho}|Du-p_0|^2\, dx)\Big(\fint_{B_\rho}|Du-p_0|^2\, dx\Big)^{1/2} \\
&\qquad - \frac{\omega(\rho)}{s}\int_{B_\rho}\Big(1+s^2+2|Du-p_0|^2+2|p_0|^2\Big)\, dx \\
&\geq -\alpha_n\rho^n\Big[\frac{L}{2}s + \sqrt{2L}\,\nu^{1/2}(\Phi)\Phi^{1/2} + \frac{2\omega(\rho)}{s}(1+s^2+\Phi+|p_0|^2)\Big]; \qquad (4\text{-}4)\end{aligned}$$

we have used Jensen's and Hölder's inequalities to obtain the second-last inequality. Completely analogously we see

$$\begin{aligned}&\int_{B_\rho} D^2 f(p_0)(Du-p_0, D\varphi)\, dx \\ &\quad\le \alpha_n\rho^n\Big[\frac{L}{2}s + \sqrt{2L}\,\nu^{1/2}(\Phi)\Phi^{1/2} + \frac{2\omega(\rho)}{s}(1+s^2+\Phi+|p_0|^2)\Big]. \qquad (4\text{-}5)\end{aligned}$$

By choosing $s := \omega^{1/2}(\rho)$ and using $(\omega 2)$ we have the desired conclusion for $\varphi$ such that $|D\varphi| \le 1$ with $c_5 = \alpha_n(4+L)$. By a simple scaling argument this yields the result for general $\varphi$. $\square$

We close this section by giving a result which is central to our technique, the $A$-harmonic approximation lemma. The lemma was first proven in [DS, Lemma 3.3], cf. [S2, Section 1.6] for the case $A = id$ (i.e., the harmonic approximation lemma); for completeness, we quote it here.

**Lemma 4.2** *Consider fixed positive $\lambda$ and $L$, and $n$, $N \in \mathbb{N}$ with $n \geq 2$. Then for any given $\varepsilon > 0$ there exists $\delta = \delta(n, N, \lambda, L, \varepsilon) \in (0, 1]$ with the following property: if $A \in \mathrm{Bil}(\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N))$ is rank-one elliptic with ellipticity constant $\lambda > 0$ and upper bound $L$, then for any $u \in H^{1,2}(B_\rho(x_0), \mathbb{R}^N)$ (for some $\rho > 0$, $x_0 \in \mathbb{R}^n$) satisfying*

$$\rho^{-n} \int_{B_\rho(x_0)} |Du|^2 \, dx \leq 1, \qquad \textit{and}$$

$$\left| \rho^{-n} \int_{B_\rho(x_0)} A(Du, D\varphi) \, dx \right| \leq \delta(n, N, \lambda, L, \varepsilon) \sup |D\varphi|$$

*for all $\varphi \in C^1(B_\rho(x_0), \mathbb{R}^N)$, there exists an $A$–harmonic function $h \in H^{1,2}(B_\rho(x_0), \mathbb{R}^N)$ such that*

$$\rho^{-n} \int_{B_\rho} |Dh|^2 \, dx \leq 1 \qquad \textit{and} \quad \rho^{-n-2} \int_{B_\rho} |h - u|^2 \, dx \leq \varepsilon.$$

$\square$

Here $h$ is called *$A$–harmonic* if

$$\int_{B_\rho} A(Dh, D\varphi) \, dx = 0$$

for all $\varphi \in C^\infty(B_\rho(x_0), \mathbb{R}^N)$.

# 5 Proof of the main theorem

To prove the result we follow the general lines of [DG, Section 3]. We firstly establish appropriate smallness conditions sufficient to deduce growth estimates on $\Phi$.

**Proposition 5.1** *Consider $u$ satisfying the conditions of Theorem 2.2, and $\beta$ fixed, $\alpha < \beta < 1$. We write $\Phi(x_0, r)$ for $\Phi(x_0, r, (Du)_{x_0,r})$. Then we can find positive constants $c_6$, $c_7$ and $\delta$, and $\theta \in (0, 1)$ (with $c_6$ depending only on $n$, $N$, $\lambda$ and $L$, and with $c_7$, $\theta$ and $\delta$ depending only on these quantities as well as $\beta$) such that the smallness conditions $\rho \leq \rho_1$,*

$$\nu(\Phi(x_0, \rho)) + \Phi(x_0, \rho) \leq \delta^2/2\,, \tag{5-1}$$

*and*

$$c_6 \omega(\rho)(1 + |(Du)_{x_0,\rho}|^4) \leq \delta^2 \tag{5-2}$$

*together imply the growth condition*

$$\Phi(x_0, \theta\rho) \leq \theta^{2\beta} \Phi(x_0, \rho) + c_7 \omega(\rho)(1 + |(Du)_{x_0,\rho}|^4). \tag{5-3}$$

*Here $\rho_1$ depending on $\lambda$ and $\omega(\cdot)$ is given by Lemma 3.2.*

**Proof:** From Lemma 4.1 we have (with $c_8 := 1 + \sqrt{2}c_5$)

$$\Big|\rho^{-n}\int_{B_\rho(x_0)} D^2f(p_0)(Du - p_0, D\varphi)\,dx\Big| \le c_8\Big[\Phi(x_0,\rho,p_0)$$
$$+\nu^{1/2}(\Phi(x_0,\rho,p_0))\Phi^{1/2}(x_0,\rho,p_0) + (\omega(\rho)/2)^{1/2}(1+|p_0|^2)\Big] \sup_{B_\rho(x_0)}|D\varphi|. \quad \text{(5-4)}$$

We set

$$w = \frac{u - p_0(x - x_0)}{2c_8\sqrt{\Phi(x_0,\rho,p_0) + \delta^{-2}\omega(\rho)(1+|p_0|^2)^2}} \quad \text{(5-5)}$$

and deduce from (5-4) that for all $\varphi \in C_c^\infty(B_\rho(x_0), \mathrm{I\!R}^n)$ there holds

$$\begin{aligned}
&\Big|\rho^{-n}\int_{B_\rho(x_0)} D^2f(p_0)(Dw, D\varphi)\,dx\Big| \\
&\le \tfrac{1}{2}\Big[\Phi^{1/2}(x_0,\rho,p_0) + \nu^{1/2}(\Phi(x_0,\rho,p_0)) + \delta/\sqrt{2}\Big] \sup_{B_\rho(x_0)}|D\varphi| \\
&\le [\nu(\Phi(x_0,\rho,p_0)) + \Phi(x_0,\rho,p_0) + \delta^2/2]^{1/2} \sup_{B_\rho(x_0)}|D\varphi| \qquad \text{(5-6)}
\end{aligned}$$

and (since $c_8 \ge \max\{\alpha_n, 1\}$),

$$\rho^{-n}\int_{B_\rho(x_0)} |Dw|^2\,dx \le \frac{\alpha_n}{4c_8^2} \le 1. \quad \text{(5-7)}$$

We further set

$$A(\xi,\eta) := D^2f(p_0)(\xi,\eta). \quad \text{(5-8)}$$

From (2-7) we see that the bilinear form $A$ satisfies the conditions of Lemma 4.2. For positive $\varepsilon$ to be determined later, we denote by $\delta = \delta(n, N, \lambda, L, \varepsilon) \in (0, 1]$ the corresponding constant from Lemma 4.2; via this lemma the *smallness condition*

$$\nu(\Phi(x_0,\rho,p_0)) + \Phi(x_0,\rho,p_0) \le \delta^2/2 \quad \text{(5-9)}$$

guarantees the existence of an $A$-harmonic $h \in H^{1,2}(B_\rho(x_0), \mathrm{I\!R}^N)$ satisfying

$$\rho^{-n}\int_{B_\rho(x_0)} |Dh|^2\,dx \le 1 \qquad \text{and} \quad \text{(5-10)}$$

$$\rho^{-n-2}\int_{B_\rho(x_0)} |w - h|^2\,dx \le \varepsilon. \quad \text{(5-11)}$$

We also note that $h$ satisfies the estimate

$$\rho^{-2}\sup_{B_{\rho/2}(x_0)}|Dh|^2 + \sup_{B_{\rho/2}(x_0)}|D^2h|^2 \le c_9\rho^{-n-2}\int_{B_\rho(x_0)}|Dh|^2dx \le \frac{c_9}{\rho^2}\,, \quad \text{(5-12)}$$

with $c_9 = c_9(n, N, \lambda, L)$ (without loss of generality we take $c_9 \ge 1$). For elliptic $A$ the first inequality follows from a standard argument due to Campanato (see [Ca, Teorema 9.2]) combined with Sobolev's and Poincaré's inequalities; the same arguments are valid in the current setting because the Legendre-Hadamard condition is satisfied, cf.

[Ev, p. 236]. The second inequality follows from (5-10). For $\theta \in (0, 1/4]$ we can thus apply Taylor's theorem to $h$ at $x_0$ to deduce

$$\sup_{x \in B_{2\theta\rho}(x_0)} |h(x) - h(x_0) - Dh(x_0)(x - x_0)|^2 \le \frac{c_9}{\rho^2}(2\theta\rho)^4 = 16c_9\theta^4\rho^2. \tag{5-13}$$

Thus we have, using also (5-11) and (5-13),

$$\begin{aligned}
&(2\theta\rho)^{-n-2}\int_{B_{2\theta\rho}(x_0)} |w - h(x_0) - Dh(x_0)(x - x_0)|^2 dx \\
&\le 2(2\theta\rho)^{-n-2}\Big(\int_{B_{2\theta\rho}(x_0)} |w - h|^2 dx + \int_{B_{2\theta\rho}(x_0)} |h - h(x_0) - Dh(x_0)(x - x_0)|^2 dx\Big) \\
&\le 2(2\theta\rho)^{-n-2}(\rho^{n+2}\varepsilon + 16c_9\alpha_n(2\theta\rho)^n\theta^4\rho^2) \\
&= 2^{-n-1}\theta^{-n-2}\varepsilon + 8c_9\alpha_n\theta^2. && (5\text{-}14)
\end{aligned}$$

We now set $\gamma = 2c_8\sqrt{\Phi(x_0, \rho, p_0) + \delta^{-2}\omega(\rho)(1 + |p_0|^2)^2}$. Taking advantage of the fact that $u$ and $u - (p_0 + \gamma Dh(x_0))(x - x_0)$ have the same mean value on balls centred at $x_0$ we have

$$\begin{aligned}
&(2\theta\rho)^{-n-2}\int_{B_{2\theta\rho}(x_0)} |u - u_{x_0,2\theta\rho} - (p_0 + \gamma Dh(x_0))(x - x_0)|^2 dx \\
&\le (2\theta\rho)^{-n-2}\int_{B_{2\theta\rho}(x_0)} |u - p_0(x - x_0) - \gamma(h(x_0) + Dh(x_0)(x - x_0))|^2 dx \\
&= \gamma^2(2\theta\rho)^{-n-2}\int_{B_{2\theta\rho}(x_0)} |w - h(x_0) - Dh(x_0)(x - x_0)|^2 dx \\
&\le 4c_8^2\left(2^{-n-1}\theta^{-n-2}\varepsilon + 8c_9\alpha_n\theta^2\right)\left(\Phi(x_0, \rho, p_0) + \delta^{-2}\omega(\rho)(1 + |p_0|^2)^2\right) \\
&\le c_{10}\left(\theta^{-n-2}\varepsilon + \theta^2\right)\left(\Phi(x_0, \rho, p_0) + \delta^{-2}\omega(\rho)(1 + |p_0|^2)^2\right), && (5\text{-}15)
\end{aligned}$$

where we have used (5-14) in the second-last line; here we have set $c_{10} = (2^{1-n} + 32\alpha_n c_9)c_8^2 + 1$, which depends only on $n$, $N$, $\lambda$ and $L$. We now fix $p_0 = (Du)_{x_0,\rho}$. With $P = (Du)_{x_0,\rho} + \gamma Dh(x_0)$ we deduce from (5-15), assuming $\rho \le \rho_1$,

$$\begin{aligned}
\Phi(x_0, \theta\rho) &= \alpha_n^{-1}(\theta\rho)^{-n}\int_{B_{\theta\rho}(x_0)} |Du - (Du)_{x_0,\theta\rho}|^2\, dx \\
&\le \alpha_n^{-1}(\theta\rho)^{-n}\int_{B_{\theta\rho}(x_0)} |Du - P|^2\, dx \\
&\le 2^n c_4\alpha_n^{-1}(2\theta\rho)^{-n-2}\int_{B_{2\theta\rho}(x_0)} |u - u_{x_0,2\theta\rho} - P(x - x_0)|^2\, dx \\
&\quad + 2^n c_4\omega(2\theta\rho)(1 + |P|^2) \\
&\le 2^n c_4 c_{10}\alpha_n^{-1}\left(\theta^{-n-2}\varepsilon + \theta^2\right)\left(\Phi(x_0, \rho) + \delta^{-2}\omega(\rho)(1 + |(Du)_{x_0,\rho}|^2)^2\right) \\
&\quad + 2^n c_4\omega(\rho)(1 + |P|^2)\,; && (5\text{-}16)
\end{aligned}$$

here the second-last inequality follows from Lemma 3.2, the last from (5-15). Under the additional smallness condition

$$2c_9\gamma^2 \le 1 \tag{5-17}$$

we have, using (5-10) and (5-12),

$$
\begin{aligned}
1+|P|^2 &\le 1+2|(Du)_{x_0,\rho}|^2+2\gamma^2|Dh(x_0)|^2\\
&\le 1+2|(Du)_{x_0,\rho}|^2+2c_9\gamma^2\rho^{-n}\int_{B_\rho(x_0)}|Dh|^2\,dx\\
&\le 1+2|(Du)_{x_0,\rho}|^2+2c_9\gamma^2\\
&\le 2(1+|(Du)_{x_0,\rho}|^2)\,. \qquad (5\text{-}18)
\end{aligned}
$$

We now fix $\theta$ sufficiently small that

$$2^{n+1}c_4c_{10}\alpha_n^{-1}\theta^2 \le \theta^{2\beta}, \qquad (5\text{-}19)$$

and then set $\varepsilon := \theta^{n+4}$, which also fixes $\delta$; without loss of generality we assume that $\delta$ is sufficiently small that we have $8c_8^2c_9\delta^2 < 1$. Note that $\theta$, $\varepsilon$ and $\delta$ depend on $n$, $N$, $\lambda$, $L$, $\alpha$ and $\beta$.

We now set $c_6 = 32c_8^2c_9$ and $c_7 = 2^{n+2}c_4(\delta^{-2}+1)$. Combining these choices with conditions (5-9) and (5-17), with (5-16) and with (5-18) yields the desired result. □

For a given $M>0$ we can find $\Phi_0(M)>0$ (dependent also on $n$, $N$, $\lambda$, $L$, $\beta$ and $\nu(\,\cdot\,)$) sufficiently small that

$$\nu(2\Phi_0(M))+2\Phi_0(M)\le \delta^2/2, \qquad \text{and} \qquad (5\text{-}20)$$

$$\Phi_0(M)\le \frac{1}{4}M^2\theta^n(1-\theta^\beta)^2. \qquad (5\text{-}21)$$

Given this, we can also find $\rho_0(M)\in(0,\rho_1]$ (dependent also on $n$, $N$, $\lambda$, $L$, $\beta$, $\nu(\,\cdot\,)$ and $\omega(\,\cdot\,)$) so small that, writing $c_{11}(M)$ for $\frac{c_6+c_7}{\theta^{2\alpha}-\theta^{2\beta}}(1+16M^4)$ (with $c_{11}$ thus depending also on $n$, $N$, $\lambda$, $L$, $\alpha$ and $\beta$), we have

$$c_{11}(M)\omega(\rho_0(M))\le \min\{\delta^2,\Phi_0(M)\}, \qquad \text{and} \qquad (5\text{-}22)$$

$$c_{11}(M)\Omega(\rho_0(M))\le \tfrac{1}{4}M^2\theta^n(1-\theta^\alpha)^2. \qquad (5\text{-}23)$$

If the quantities $\Phi(x_0,\rho)$ and $\rho$ are sufficiently small for some $B_\rho(x_0)$, the next lemma shows that we can iterate Proposition 5.1:

**Lemma 5.2** *For $M_0>0$ and $B_\rho(x_0)\subset\subset U$, suppose that the conditions*

$$
\begin{array}{lrcl}
(i) & |(Du)_{x_0,\rho}| & \le & M_0,\\
(ii) & \rho & \le & \rho_0(M_0), \qquad \textit{and}\\
(iii) & \Phi(x_0,\rho) & \le & \Phi_0(M_0)
\end{array}
$$

*are satisfied. Then the smallness conditions (5-1) and (5-2) are fulfilled on $B_{\theta^j\rho}(x_0)$ for all $j\in\mathbb{N}$. Furthermore there exists*

$$\Upsilon_{x_0} := \lim_{j\to\infty}(Du)_{x_0,\theta^j\rho},$$

*and there exists $c_{12}$ depending only on $n$, $N$, $\lambda$, $L$, $\alpha$, $\beta$ and $M_0$ such that for all $r<\rho$ there holds:*

$$⨍_{B_r(x_0)}|Du-\Upsilon_{x_0}|^2\,dx \le c_{12}\Big(\Big(\frac{r}{\rho}\Big)^{2\beta}\Phi(x_0,\rho)+\Omega(r)\Big). \qquad (5\text{-}24)$$

**Proof:** In order to show the first part of the lemma we prove two statements by induction. Precisely, for $j \in \mathbb{N} \cup \{0\}$ we shall show:

$$\begin{array}{rrcll} \textbf{(I)}_{\mathrm{j}} & \Phi(x_0, \theta^j \rho) & \leq & \theta^{2\beta j}\Phi(x_0,\rho) + c_{11}(M_0)\omega(\theta^j\rho)\,; & \text{and} \\ \textbf{(II)}_{\mathrm{j}} & |(Du)_{x_0,\theta^j\rho}| & \leq & 2M_0\,. & \end{array}$$

Note firstly that $(\mathrm{II})_{\mathrm{j}}$ combined with (5-22) and (iii) yields

$$\textbf{(I}'\textbf{)}_{\mathrm{j}} \qquad \Phi(x_0,\theta^j\rho) \leq 2\Phi_0(M_0)\,.$$

We now proceed to the proof by induction. The case $j = 0$ follows immediately from (5-20), (5-22) and the monotonicity of $\nu$ and of $\omega$. We assume $(\mathrm{I})_\ell$ and $(\mathrm{II})_\ell$ for $\ell = 0, \ldots, j-1$. We first calculate, using (5-3), $(\mathrm{II})_\ell$ for $\ell = 0, \ldots, j-1$ and $(\omega 1)$,

$$\begin{aligned} \Phi(x_0,\theta^j\rho) &\leq \theta^{2\beta j}\Phi(x_0,\rho) + c_7\sum_{\ell=0}^{j-1}\theta^{2\beta\ell}\omega(\theta^{j-\ell-1}\rho)(1+|(Du)_{x_0,\theta^{j-\ell-1}\rho}|^4) \\ &\leq \theta^{2\beta j}\Phi(x_0,\rho) + c_7\theta^{-2\alpha}\Big(\sum_{\ell=0}^{j-1}\theta^{2(\beta-\alpha)\ell}\Big)\omega(\theta^j\rho)(1+16M_0^4) \\ &\leq \theta^{2\beta j}\Phi(x_0,\rho) + \frac{c_7(1+16M_0^4)}{\theta^{2\alpha}-\theta^{2\beta}}\omega(\theta^j\rho) \\ &\leq \theta^{2\beta j}\Phi(x_0,\rho) + c_{11}(M_0)\omega(\theta^j\rho)\,, \end{aligned}$$

showing $(\mathrm{I})_{\mathrm{j}}$. To show $(\mathrm{II})_{\mathrm{j}}$ we estimate

$$\begin{aligned} |(Du)_{x_0,\theta^j\rho}| &\leq M_0 + \sum_{\ell=1}^{j}|(Du)_{x_0,\theta^\ell\rho} - (Du)_{x_0,\theta^{\ell-1}\rho}| && \text{via (iii)} \\ &\leq M_0 + \sum_{\ell=1}^{j}\Big[\fint_{B_{\theta^\ell\rho}(x_0)}|Du-(Du)_{x_0,\theta^{\ell-1}\rho}|^2\,dx\Big]^{1/2} \\ &\leq M_0 + \theta^{-n/2}\sum_{\ell=1}^{j}\Big[\fint_{B_{\theta^{\ell-1}\rho}(x_0)}|Du-(Du)_{x_0,\theta^{\ell-1}\rho}|^2\,dx\Big]^{1/2} \\ &\leq M_0 + \theta^{-n/2}\sum_{\ell=0}^{j-1}\sqrt{\theta^{2\beta\ell}\Phi(x_0,\rho)+c_{11}(M_0)\omega(\theta^\ell\rho)} && \text{via } (\mathrm{I})_\ell,\ \ell=0,\ldots,j-1 \\ &\leq M_0 + \theta^{-n/2}\Big(\frac{\sqrt{\Phi(x_0,\rho)}}{1-\theta^\beta} + \frac{\sqrt{c_{11}(M_0)}}{1-\theta^\alpha}\sqrt{\Omega(\rho)}\Big) && \text{via (2-5)} \\ &\leq M_0 + \theta^{-n/2}\Big(\frac{\sqrt{\Phi_0(M_0)}}{1-\theta^\beta} + \frac{\sqrt{c_{11}(M_0)\Omega(\rho_0(M_0))}}{1-\theta^\alpha}\Big) && \text{via (iii), (ii)} \\ &\leq 2M_0 && \text{via (5-21), (5-23).} \end{aligned}$$

The conclusion of the lemma then follows from $(\mathrm{I}')_{\mathrm{j}}$ and $(\mathrm{II})_{\mathrm{j}}$ after taking into account (5-20) and (5-22).

Analogously we calculate, for $k > j$,

$$|(Du)_{x_0,\theta^j\rho} - (Du)_{x_0,\theta^k\rho}| \leq \sum_{\ell=j+1}^{k}|(Du)_{x_0,\theta^\ell\rho} - (Du)_{x_0,\theta^{\ell-1}\rho}|$$

$$\leq \theta^{-n/2}\Big(\frac{\sqrt{\Phi(x_0,\rho)}}{1-\theta^\beta}\theta^{\beta j}+\frac{\sqrt{c_{11}(M_0)}}{1-\theta^\alpha}\sqrt{\Omega(\theta^j\rho)}\Big);$$

this shows that $\{(Du)_{x_0,\theta^j\rho}\}$ is a Cauchy sequence. For

$$\Upsilon_{x_0} := \lim_{j\to\infty}(Du)_{x_0,\theta^j\rho}$$

we thus have, with $c_{13}=\sqrt{2}\theta^{-n/2}\big(\frac{1+\sqrt{c_{11}(M_0)}}{1-\theta^\alpha}\big)$ depending only on $n$, $N$, $\lambda$, $L$, $\alpha$, $\beta$ and $M_0$

$$|(Du)_{x_0,\theta^j\rho}-\Upsilon_{x_0}|\leq c_{13}\Big[\theta^{2\beta j}\Phi(x_0,\rho)+\Omega(\theta^j\rho)\Big]^{1/2}$$

for all $j$. Combining this with $(\mathrm{I})_\mathrm{j}$ and setting $c_{14}=2(c_{11}(M_0)+c_{13}^2)$ (note that $c_{14}$ has the same dependencies as $c_{13}$) we have, using also (2-6),

$$\begin{aligned}
&\fint_{B_{\theta^j\rho}(x_0)}|Du-\Upsilon_{x_0}|^2\,dx \leq 2\Phi(x_0,\theta^j\rho)+2|(Du)_{x_0,\theta^j\rho}-\Upsilon_{x_0}|^2\\
&\quad\leq 2\theta^{2\beta j}\Phi(x_0,\rho)+2c_{11}(M_0)\omega(\theta^j\rho)+2c_{13}^2\Big(\theta^{2\beta j}\Phi(x_0,\rho)+\Omega(\theta^j\rho)\Big)\\
&\quad\leq c_{14}\Big(\theta^{2\beta j}\Phi(x_0,\rho)+\Omega(\theta^j\rho)\Big).
\end{aligned}$$

For $0<r\leq\rho$ we can find $j\in\mathbb{N}\cup\{0\}$ with $\theta^{j+1}\rho<r\leq\theta^j\rho$. For this $j$ we have

$$\begin{aligned}
&\fint_{B_r(x_0)}|Du-\Upsilon_{x_0}|^2\,dx \leq \theta^{-n}\fint_{B_{\theta^j\rho}(x_0)}|Du-\Upsilon_{x_0}|^2\,dx\\
&\quad\leq c_{14}\theta^{-n}\Big(\theta^{2j\beta}\Phi(x_0,\rho)+\Omega(\theta^j\rho)\Big)\\
&\quad= c_{14}\theta^{-n}\Big(\frac{\theta^{2(j+1)\beta}}{\theta^{2\beta}}\Phi(x_0,\rho)+\Omega\Big(\frac{\theta^{j+1}\rho}{\theta}\Big)\Big)\\
&\quad\leq c_{14}\theta^{-n}\Big(\Big(\frac{r}{\rho}\Big)^{2\beta}\theta^{-2\beta}\Phi(x_0,\rho)+\theta^{-2\alpha}\Omega(\theta^{j+1}\rho)\Big)\\
&\quad\leq c_{14}\theta^{-n-2\beta}\Big(\Big(\frac{r}{\rho}\Big)^{2\beta}\Phi(x_0,\rho)+\Omega(r)\Big)\,;
\end{aligned}\tag{5-25}$$

here we have used (2-3) to obtain the second-last inequality. This shows (5-24) with $c_{12}=c_{14}\theta^{-n-2\beta}$ (note that $c_{12}$ has the correct dependencies). □

We are now in a position to complete the partial-regularity proof.

**Proof of Theorem 2.2:** We give the proof of (i); the proof of (ii) is completely analogous. We assume that for some $x_0\in U$ and $M_0>0$ we have

$$|(Du)_{x_0,\rho}|<M_0 \quad\text{and}\quad \Phi(x_0,\rho)<\Phi_0(M_0)$$

on $B_\rho(x_0)$, where $B_{2\rho}(x_0)\subset\subset U$ with $0<\rho\leq\rho_0(M_0)$. Such a $\rho$ can always be found for each $x_0$ belonging to neither $\Sigma_1$ nor $\Sigma_2$. Since the functions $z\mapsto(Du)_{z,\rho}$

and $z \mapsto \Phi(z,\rho)$ are continuous there exists a ball $B_\sigma(x_0) \subset\subset U$, such that for all $z \in B_\sigma(x_0)$ we have $B_\rho(z) \subset\subset U$, and further there holds:

$$|(Du)_{z,\rho}| < M_0 \quad \text{and} \quad \Phi(z,\rho) < \Phi_0(M_0) \quad \text{for all } z \in B_\sigma(x_0). \tag{5-26}$$

We can thus apply Lemma 5.2 on $B_r(z)$ for any $z \in B_\sigma(x_0)$ and $r$ with $0 < r \le \rho$ to deduce

$$⨍_{B_r(z)} |Du - \Upsilon_z|^2\, dx \le c_{12}\Big(\Big(\frac{r}{\rho}\Big)^{2\beta}\Phi(z,\rho) + \Omega(r)\Big). \tag{5-27}$$

For $z, \tilde{z} \in B_\sigma(x_0)$ with $r = |z - \tilde{z}| < 2\sigma$ and $a = (z+\tilde{z})/2$ we obtain

$$\begin{aligned}
|\Upsilon_z - \Upsilon_{\tilde{z}}|^2 &= \frac{1}{\alpha_n (r/2)^n}\int_{B_{r/2}(a)} |\Upsilon_z - \Upsilon_{\tilde{z}}|^2\, dx \\
&\le \frac{2^n}{\alpha_n r^n}\int_{B_r(z)\cap B_r(\tilde{z})} |\Upsilon_z - \Upsilon_{\tilde{z}}|^2\, dx \\
&\le 2^{n+1}\Big[⨍_{B_r(z)} |Du - \Upsilon_z|^2\, dx + ⨍_{B_r(\tilde{z})} |Du - \Upsilon_{\tilde{z}}|^2\, dx\Big] \\
&\le 2^{n+1}c_{12}\Big[\Big(\frac{r}{\rho}\Big)^{2\beta}\Big(\Phi(z,\rho) + \Phi(\tilde{z},\rho)\Big) + 2\Omega(r)\Big] \\
&\le 2^{2n+2}c_{12}\Big[\Big(\frac{|z-\tilde{z}|}{\rho}\Big)^{2\beta}\Phi(x_0, 2\rho) + \Omega(|z-\tilde{z}|)\Big].
\end{aligned}$$

here we have used (5-27) in the third-last inequality, and the fact that $\Phi(z,\rho)+\Phi(\tilde{z},\rho) \le 2^{n+1}\Phi(x_0,2\rho)$ in obtaining the final inequality. Since $\Upsilon_z$ is the Lebesgue-representative of $Du(z)$, we can conclude the desired continuity. □

As noted in Section 2 we can weaken the hypotheses of the theorem by omitting (H4). This entails essentially only notational changes in the proof: in (5-1), (5-9) and (5-19) we need to replace $\nu(\,\cdot\,)$ by $\nu(M+1,\,\cdot\,)$ for $|(Du)_{x_0,\rho}|$ (respectively $|p_0|$) less than $M$, and check that this is preserved in the iteration. Analogous changes also need to be made in Lemma 4.1. We thus have

**Corollary 5.3** *The conclusion of Theorem 2.2 also follows if we omit the hypothesis (H4).* □

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