On the Gribov Copy Problem for the Coulomb Gauge

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Abstract.

We consider the problem of gauge degeneracy. In particular, for connections on three-dimensional Euclidean space with the structure group SU(2), we show that a large class of spherically symmetric connections in the Coulomb gauge have distinct gauge copies in the Coulomb gauge.

1 Introduction

In abelian gauge theories such as electrodynamics, the Coulomb gauge condition is sufficient to fix all gauge degrees of freedom. It was first pointed out by Gribov in [Gr] that in non-abelian gauge theories one may be confronted with the problem of degeneracy of the Coulomb gauge, i.e. the gauge orbits can intersect the Coulomb gauge hypersurface at more than one point. Later Singer [Si] showed that this problem occurs whenever one tries to choose a global section of the configuration space of Yang-Mills potentials over a compact manifold with compact gauge group.

In order to discuss further the problem of gauge degeneracy, we fix some general notation; this will be made more precise in Section 2. We consider a smooth, n-dimensional Riemannian manifold M with structure group G and associated Lie algebra $\mathcal{G}$; we further consider connections $A \in \Lambda^1(M, \mathcal{G})$, the space of $\mathcal{G}$-valued 1-forms over M, and gauge copies of A:

$$g^*A = gAg^{-1} - dg \cdot g^{-1}$$

with $g \in \Lambda^0(M^n, G)$ a gauge transformation. There are two basic questions which arise, namely that of determining when gauge degeneracy occurs, and that of determining whether there is a reasonable way of choosing a distinguished representative in some "nice" gauge, in particular in the Coulomb gauge, for each gauge orbit. When the structure group is abelian, the gauge orbits are linear and the problem of finding a representative in the Coulomb gauge reduces to solving the linear Laplace equation. If, on the other hand, the structure group is nonabelian then the gauge orbits are complicated nonlinear objects and the question of finding Coulomb gauge representatives reduces to a non-linear equation of harmonic map type, for which no regularity theory is directly available.

One of the fundamental difficulties in dealing with these questions is to decide on an appropriate class of connections A and gauge transformations g. For example, Uhlenbeck considers in [Uh] Sobolev spaces of connections, and gauge transformations having one more weak derivative than the connections. In this setting, she is able to show ([Uh, Theorem 1.3]) that the orbit of any connection $A \in W^{1,p}(\mathbb{B}^n, \mathcal{G})$, $p \geq \frac{n}{2}$, defined on the unit ball $\mathbb{B}^n$ and whose curvature F_A has sufficiently small $L^{n/2}$-norm has a representative $g^*A \in W^{1,p}(\mathbb{B}^n, \mathcal{G})$ in the Coulomb gauge with $g \in W^{2,p}(\mathbb{B}^n, G)$.

On the other hand, by considering a weaker notion of connections and gauge transformations, Dell'Antonio and Zwanziger [DZ2] were able to show the existence of gauge copies in the Coulomb gauge for any $A \in L^2(\mathbb{R}^n)$ defined on the non-compact space $\mathbb{R}^n$. Note that this result will also apply to configurations on $\mathbb{R}^n$ which decay so fast at infinity that they induce regular connections on S^n, i.e. their result is also applicable in the noncompact setting. Due to the methods they employ, any additional regularity of the initial configuration is not a priori possessed by the gauge copy, in contrast to Uhlenbeck's result.

In the current paper we will produce smooth distinct gauge copies of smooth connections which can have arbitrarily large L^2-norms, for a restricted class of radially symmetric connections on $\mathbb{R}^3$.

Concerning the question of finding distinguished representatives of gauge orbits, the original hope of Gribov was that imposing the positivity of the so-called Faddeev-Popov operator $F(A) = -\partial_i(D_i(A)\cdot)$ (see Section 4) would be sufficient to uniquely specify the gauge orbit representative inside the Coulomb gauge. Although this hope turned out not to be justified –see [He], [DZ1, Appendix B]) – this set of potentials nevertheless provides a natural domain, as it can be characterized as the set of local minima of the L^2-norm along the gauge orbit:

$$I(g) = \frac{1}{2}\|g^*A\|^2_{L^2}$$

Minimizing this functional over the the gauge orbits was precisely the technique used by Dell'Antonio and Zwanziger in [DZ2]. As a consequence then, the copy they produce always lies inside the so-called Gribov horizon and in particular, they produce distinct gauge copies provided the initial connection lies outside the horizon. That one can find an initial connection outside the horizon follows from the fact that the horizon is bounded in L^2 in any given direction (see [Zw, Theorem 3]).

In our setting the variational methods we employ ensure that the gauge copy we produce is always distinct from the original. We are also able to show that, for a large subclass of the potentials considered, the copy produced does in fact lie inside the Gribov horizon (see Section 4).

The paper is organized as follows. In Section 2 we discuss notation, explain the symmetry being considered and show how the existence of a gauge copy reduces to the existence of a nontrivial solution to a particular second-order boundary value problem with degenerate coefficients. The symmetry considered is similar to that used in [Gr]. However, it is the topological triviality implied by our boundary conditions (see Section 3) $\alpha(0) = 0, \alpha(\infty) = 0$ (a property essential to the functioning of our minimizing arguments) which is a distinguishing aspect of the current paper.

Our main theorem, the existence of a distinct gauge copy in the Coulomb gauge for a large class of symmetric potentials can now be stated (the Pauli matrices $\{T_a\}_{a=1}^{a=3}$ are defined in (2.5)).

Theorem 1.1 *Consider connections on $\mathbb{R}^3$ with structure group $SU(2)$ with components $A_i = A_i^a T_a$ of the form*

$$A_i^a(x) = \frac{f(r) - 1}{r}\epsilon_{iab}\frac{x_b}{r} ,$$

where $r = |x|$ (such connections lie in the Coulomb gauge hypersurface). For a large class of potentials $f \in C^\infty((0,+\infty))$ we can find a nontrivial gauge transformation

$$g(x) = \exp(\theta(r)\frac{x^a}{r}T_a) \in C^\infty(\mathbb{R}^3, SU(2))$$

such that the gauge copy $g^*A = gAg^{-1} - dg \cdot g^{-1}$ is smooth, square-integrable, and also lies in the Coulomb gauge.

In Section 3 we prove the required existence result by employing the direct method of the calculus of variations. We are required to work in particular weighted Sobolev spaces to obtain existence. We also need to obtain a rather detailed picture of the qualitative behaviour of our minimizer in order to understand its behaviour at the origin (the point where the coefficients degenerate), and hence be able to show that the solution obtained is in fact smooth. As mentioned above, in Section 4 we discuss the relation of our solution to the so-called Gribov horizon.

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2 Spherically Symmetric Gauge Fields on $\mathbb{R}^3$

Let E denote a principal fiber bundle over an n-dimensional manifold M^n with structural group G, a compact semi-simple Lie Group. We denote by $\mathcal{G}$ the Lie algebra of G and by $[\cdot,\cdot]$ its Poisson bracket. We shall assume in what follows that G is a matrix group.

A Yang-Mills potential A is a $\mathcal{G}$-valued 1-form on E compatible with the action of the structure group on E. We denote by $C^\infty(M, \mathrm{Ad}\ E \otimes T^*M)$ the set of all smooth connections on E and by $C^\infty(M, \mathrm{Aut}(E))$ the set of all smooth gauge transformations: these are sections of the bundle Aut (E) acting on the connections by conjugation. If $A \in C^\infty(M, \mathrm{Ad}\ E \otimes T^*M)$ and $g \in C^\infty(M, \mathrm{Aut}(E))$, then we define the gauge transformed connection g^*A as:

$$g^*A = gAg^{-1} - dg \cdot g^{-1}. \tag{2.1}$$

The curvature $F_A = D_A A \in C^\infty(M, \mathrm{Ad}\ E \otimes \Lambda^2 T^*M)$, the 2-form obtained by taking the covariant derivative $D_A = d + [A, \cdot]$ of the connection A, changes accordingly to:

$$F_{g^*A} = gF_A\, g^{-1}. \tag{2.2}$$

In this paper we shall consider the case when $M = \mathbb{R}^n$ and E is the trivial bundle $E = \mathbb{R}^n \times G$. In this case we can identify the spaces of connections and gauge transformations with the space $C^\infty(\mathbb{R}^n, \Lambda^1\mathcal{G})$ and $C^\infty(\mathbb{R}^n, G)$ of smooth $\mathcal{G}$-valued 1-forms $A = A_i dx^i$ and smooth mappings $g : \mathbb{R}^n \to G$ respectively. On $\mathbb{R}^n$, the curvature $F_A = D_A A$ is a $\mathcal{G}$-valued 2-form $F_A = F_{ij} dx^i dx^j$ given by the components

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]. \tag{2.3}$$

More generally, one can define the Sobolev spaces of connections and gauge transformations as:

$$\begin{aligned} W^{s,p}(\mathbb{R}^n, \Lambda^1\mathcal{G}) &= \{A : A_i \in W^{s,p}(\mathbb{R}^n)\} \\ W^{s+1,p}(\mathbb{R}^n, G) &= \{g : g \in W^{s+1,p}(\mathbb{R}^n),\ g(x) \in G, \text{a.e.}\}. \end{aligned} \tag{2.4}$$

If $p \geq n/2$, then the Sobolev embedding theorem shows that $F_A \in L^p$ in the case that $A \in W^{1,p}$, and that if $p > n/2$, then $g^*A \in W^{1,p}$ provided $A \in W^{1,p}$ and $g \in W^{2,p}$: see [Uh, Lemma 1.2] for details.

We consider the action of the rotation group $\mathrm{SO}(n)$ on the base manifold $\mathbb{R}^n$, and all possible lifts of this action to the bundle $E = \mathbb{R}^n \times G$. Fixing one of these liftings, it is possible to say that a connection $A = A_i dx^i$ is spherically symmetric if it is invariant under the action of the rotation group $\mathrm{SO}(n)$ on the bundle. It has been established that all possible lifts of the action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ to the bundle E are in 1-1 correspondence with all possible endomorphisms $\lambda : \mathrm{SO}(n) \to G$ (see e.g [KN, Theorem 11.5]). When $n = 3$ and $G = \mathrm{SU}(2)$, then, apart from the trivial map (which induces abelian connections), there is only one type of spherical symmetry. We fix a basis

$\{T_a\}_{a=1}^3$ of the Lie algebra su(2):

$$T_1 = \frac{1}{2i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad T_2 = \frac{1}{2i}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad T_3 = \frac{1}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.5}$$

and write the components of A as $A_i = A_i^a T_a$. One says that a connection $A \in C^\infty(\mathbb{R}^3, \Lambda^1 \mathrm{su}(2))$ is spherically symmetric when the Lie derivatives of A along the infinitesimal generators $\mathbb{O}_a = \epsilon_{abc} x_b \partial_c$ of SO(3) verify the relations:

$$\mathcal{L}_{\mathbb{O}_a} A + [T_a, A] = 0, \quad a = 1, 2, 3. \tag{2.6}$$

The general solution of equations (2.6) is a 3-parameter family of connections described by three radial functions f_1, f_2, f_3 of $r = |x|$:

$$A_i^a(x) = \frac{f_1(r) - 1}{r}\epsilon_{iab}\frac{x_b}{r} + \frac{f_2(r)}{r}(\delta_{ia} - \frac{x_i x_a}{r^2}) + f_3(r)\frac{x_i x_a}{r^2}. \tag{2.7}$$

The curvature $F_A = dA + [A, A]$ is given by:

$$\begin{aligned} F_{ij}^a(x) = \epsilon_{ijk}\Bigg[\frac{f_1 f_3 - f_2'}{r}\epsilon_{kab}\frac{x_b}{r} \quad &+ \quad \frac{f_1' + f_2 f_3}{r}(\delta_{ka} - \frac{x_k x_a}{r^2}) \\ &+ \quad \frac{1}{r^2}(f_1^2 + f_2^2 - 1)\frac{x_k x_a}{r^2}\Bigg]. \end{aligned} \tag{2.8}$$

The class of spherically symmetric connections (2.6) is preserved by a residual gauge group consisting of all gauge transformations $g \in C^\infty(\mathbb{R}^3, SU(2))$ of the kind:

$$g(x) = \exp(\theta(r)\frac{x^a}{r}T_a) \tag{2.9}$$

where θ is a smooth function on $\mathbb{R}^+$. The action of such an element g on the connections (2.6) is described by

$$\begin{aligned} (g^*A)_i^a = \frac{f_1\cos\theta + f_2\sin\theta - 1}{r}\epsilon_{iab}\frac{x_b}{r} \quad &- \quad \frac{f_1\sin\theta - f_2\cos\theta}{r}(\delta_{ia} - \frac{x_i x_a}{r^2}) \\ &+ \quad (f_3 - \theta'(r))\frac{x_i x_a}{r^2} \end{aligned} \tag{2.10}$$

From the expressions (2.7) and (2.8) it is easy to compute the L^2-norms of A and F_A:

$$\|A\|^2_{L^2(\mathbb{R}^3)} \quad = \quad 4\pi\int_0^\infty [2(f_1(r) - 1)^2 + 2f_2(r)^2 + f_3(r)^2]dr \tag{2.11}$$

$$\|F_A\|^2_{L^2(\mathbb{R}^3)} = 4\pi \int_0^\infty [2(f_1f_3 - f_2')^2 + 2(f_1' + f_2f_3)^2 + \frac{(f_1^2 + f_2^2 - 1)^2}{r^2}]\, dr. \tag{2.12}$$

In this work, we shall restrict ourselves to the subfamily of connections of the form

$$A_i^a(x) = \frac{f(r) - 1}{r} \epsilon_{iab} \frac{x_b}{r} \tag{2.13}$$

with $f \in C^\infty((0, +\infty))$. These connections already satisfy the Coulomb gauge condition $\partial_i A_i = 0$. If g is of the form (2.9) with $\theta \in C^\infty((0, +\infty))$, then $g^*A \in C^\infty((0, +\infty))$. It is natural to ask for assumptions on the gauge transformation g which will ensure that the connection g^*A will lie in $L^2(\mathbb{R}^3)$, provided $A \in L^2(\mathbb{R}^3)$ initially. We claim that this will be the case when $\theta \in H^1(\mathbb{R}^3)$ and f approaches 1 so fast that F_A has finite L^2-norm on the sphere S^3. Note that this means that, in particular,

$$\int_0^\infty [\frac{(f^2 - 1)^2}{r^2} + f'(r)^2] dr < +\infty \tag{2.14}$$

$$\int_0^\infty [(f^2 - 1)^2 + r^2 f'(r)^2] dr < +\infty. \tag{2.15}$$

These two inequalities imply that f is bounded on $(0, +\infty)$ and the claim then follows from considering (2.11) in the light of (2.10), (2.14) and (2.15). Such properties will play a role in fixing our class of admissable potentials $\mathcal{S}$ in Section 3.

Computing the divergence of the connection g^*A, we have:

$$\partial_i(g^* A_i) = -(\theta_{rr} + \frac{2}{r}\theta_r - \frac{2\sin\theta}{r^2} f(r)) \frac{x^a}{r} T_a \tag{2.16}$$

and then, by setting $\theta(r) = 2\alpha(r)$, we see that g^*A will also be in the Coulomb gauge if $\alpha(r)$ verifies the ordinary differential equation:

$$\alpha_{rr} + \frac{2}{r}\alpha_r - \frac{\sin 2\alpha}{r^2} f(r) = 0$$

subject to the conditions $\alpha(0) = \alpha(+\infty) = 0$. We shall examine this problem next section by using the direct method of the calculus of variations.

3 The Variational Problem

In this section we are concerned with finding particular solutions of the ordinary differential equation

$$\alpha_{rr} + \frac{2\alpha_r}{r} - \frac{\sin 2\alpha}{r^2} f = 0 \tag{3.1}$$

on $[0, \infty)$. Here f is a given function in a subclass of $C^\infty[0,\infty)$ which we will describe shortly. Equation (3.1) is the Euler Lagrange equation associated with the functional

$$J_f(\alpha) = \int_0^\infty \left(\frac{1}{2}\alpha_r^2 + \frac{\sin^2 \alpha}{r^2} f\right) r^2 dr\,; \tag{3.2}$$

where there is no risk of confusion we will omit the subscript f. We will look for solutions of (3.1) in a particular weighted Sobolev space, which we now describe. We define $||u||_{L^{2;\delta}}$ by

$$||u||_{L^{2;\delta}} := [\int_{\mathbb{R}^3} (1 + |x|^2)^\delta |\beta(x)|^2 \, dx]^{\frac{1}{2}},$$

and $||u||_{H^{1;\delta}}$ by

$$||u||_{H^{1;\delta}} := [\int_{\mathbb{R}^3} (1 + |x|^2)^\delta |\beta(x)|^2 + (1 + |x|^2)^{1+\delta} |\nabla\beta(x)|^2] dx]^{\frac{1}{2}}.$$

The space $L^{2;\delta}$ is then defined to be the space of functions for which $||u||_{L^{2;\delta}}$ is finite, and $H^{1;\delta}$ is the space of functions for which $||u||_{H^{1;\delta}}$ is finite (the latter can of course be characterized as the subspace of functions in $L^{2;\delta}$ with weak derivatives in $L^{2;1+\delta}$); We refer the reader to [Ba, Section 1] for a description of such spaces and a summary of their properties.

We will look for solutions of (3.1) in the space $H^{1;\delta}_{\mathcal{R}}$, by which we denote radial functions in $H^{1;\delta}$. We will further consider a fixed δ satisfying $\frac{1}{2} < \delta < 1$ (as will be seen in the proof, any fixed δ in this range allows us to show the desired continuity properties of our functional J).

We now specify the class of potentials which we will consider. We denote by $\mathcal{S}$ those functions f in $C^\infty[0,\infty)$ for which

$$\int_0^\infty \Big[\frac{(f^2-1)^2}{r^2} + (f')^2\Big] dr \quad < \quad \infty \quad ; \tag{3.3}$$

$$\int_0^\infty \Big[(f^2-1)^2 + r^2(f')^2\Big]dr \;\;<\;\; \infty \quad ; \tag{3.4}$$

$$\int_0^\infty (f-1)^2(1+\tfrac{1}{r^2})\,dr \;\;<\;\; \infty \quad ; \tag{3.5}$$

$$\inf_{\beta\in H^{1;\delta}_{\mathcal{R}}} J_f(\beta) < 0. \tag{3.6}$$

In the first instance, these conditions should be regarded as analytic constraints on f_i, imposed in order to obtain our desired existence result. As we discussed in Section 2, they are at least partially geometrically motivated: (3.3) and (3.4) can be viewed as the condition that F_A be in $L^2(S^3)$. The condition (3.5) can be thought of additionally requiring that A be in $H^{1,2}_{loc}$ near the origin in $\mathbb{R}^3$. We of course need to verify that the class $\mathcal{S}$ is nonempty; this follows directly by a perturbation argument, once we establish

Lemma 3.1 *There exists f, uniformly Lipschitz continuous on $[0,\infty)$, which satisfies (3.3)-(3.6).*

Proof. We consider a potential $f(=f_{\varepsilon,\rho,d,\kappa})$ defined by

$$f(r) := \begin{cases} 1 & 0\le r<\rho-\varepsilon \\ -\frac{1}{\varepsilon}(d+1)(r-\rho+\varepsilon)+1 & \rho-\varepsilon\le r<\rho \\ -d & \rho\le r<\kappa\rho \\ \frac{1}{\varepsilon}(d+1)(r-\kappa\rho-\varepsilon)+1 & k\rho\le r<k\rho+\varepsilon \\ 1 & r\ge\kappa\rho+\varepsilon \end{cases}$$

for positive ε,ρ,d and κ with $\varepsilon<\rho$, $\kappa>1$. Direct calculation shows that f satisfies (3.3)-(3.5). We now consider $\beta(=\beta_{\mu,\rho,\varepsilon,\kappa})$ to be the $H^{1;\delta}_{\mathcal{R}}$ function defined by

$$\beta(r) := \begin{cases} 0 & 0\le r<\rho-\varepsilon \\ \frac{\mu}{\varepsilon}(r-\rho+\varepsilon) & \rho-\varepsilon\le r<\rho \\ \mu & \rho\le r<\kappa\rho \\ \frac{\mu}{\varepsilon}(\kappa\rho+\varepsilon-r) & \kappa\rho\le r<\kappa\rho+\varepsilon \\ 0 & r\ge\kappa\rho+\varepsilon \end{cases}$$

for positive μ. We calculate

$$
\begin{aligned}
J_f(\beta) &< \tfrac{1}{2}\int_{\rho-\varepsilon}^{\rho} \beta_r^2 r^2 dr + \tfrac{1}{2}\int_{\kappa p}^{\kappa\rho+\varepsilon} \beta_r^2 r^2 dr - d\int_{\rho}^{\kappa\rho} \sin^2\mu\, dr + \int_{\rho-\varepsilon}^{\rho} \sin^2\mu\, dr + \int_{\kappa\rho}^{\kappa\rho+\varepsilon} \sin^2\mu\, dr \\
&< \frac{\mu^2}{6\varepsilon^2}\Big[(\rho^3 - (\rho-\varepsilon)^3) + ((\kappa\rho+\varepsilon)^3 - \kappa^3\rho^3)\Big] + \sin^2\mu[2\varepsilon - d\rho(\kappa-1)] \\
&= \frac{\mu^2}{6\varepsilon}\Big[3(1+\kappa^2)\rho^2 + 3(k-1)\rho\varepsilon + \varepsilon^2\Big] + \sin^2\mu[2\varepsilon - d(\kappa-1)\rho]. \quad (3.7)
\end{aligned}
$$

From (3.7), we see that there are a number of choices of $\kappa, \rho, \varepsilon, d$ for which we can choose μ small enough that $J_f(\beta)$ is negative. For example, for fixed κ, ρ and ε, this will hold if d is sufficiently large. Alternatively for fixed $d > \frac{1}{2}$ and fixed κ, (3.6) will also hold for sufficiently small μ if ρ and ε are chosen small enough. □

We are now in a position to state the main theorem of this section.

Theorem 3.1 *For fixed $f \in \mathcal{S}$, equation (3.1) has a nontrivial solution $\alpha \in H^{1;\delta}_{\mathcal{R}}$ satisfying (in the trace sense) $\alpha(0) = 0$.*

The proof will follow in a number of steps. We begin by assembling a few facts about our potential f, and solutions of equation (3.1).

Lemma 3.2 *(i) Solutions of (3.1) are smooth on $(0, \infty)$.*

(ii) Given $c_1, c_2 \in \mathbb{R}$, $r_0 \in (0, \infty)$, there exists a unique solution of (3.1) with $\alpha(r_0) = c_1$, $\alpha_r(r_0) = c_2$.

(iii) A function $f \in C^\infty[0, \infty)$ satisfying (3.3) – (3.5) is bounded on $[0, \infty)$, and further there exist r_1, r_2 such that:

$$f(r) > 0 \quad \textit{for} \quad r \in [0, r_1) \cup (r_2, \infty) \tag{3.8}$$

Proof. The coefficients of (3.1) are smooth and uniformly bounded on compact neighbourhoods of $(0, \infty)$, establishing (i). The second claim is a consequence of the existence and uniqueness theorem for first order systems (simply consider (3.1) as the system $\alpha_r = \gamma$, $\gamma_r = -\frac{\gamma}{r} + \frac{\sin 2\alpha}{r^2} f$).

To show the third claim we first note that, by (3.3), (3.5) and the Sobolev embedding theorem, $\lim\limits_{r\to 0} f(r)$ must exist, and by (3.5) we in fact must have that $\lim\limits_{r\to 0} f(r) = 1$. It remains to study the behaviour as r approaches infinity. We consider a Cauchy sequence $\{r_k\} \to \infty$, and estimate for $r_m > r_n$

$$
\begin{aligned}
|f^2(r_n) - f^2(r_m)| &\leq 2\int\limits_{r_n}^{r_m} |ff'|dr \\
&\leq 2\Big[\int\limits_0^{\infty} (f')^2 r^2 dr\Big]^{1/2}\Big[\int\limits_{r_n}^{r_m} \frac{f^2-1}{r} \cdot \frac{1}{r}dr + \int\limits_{r_m}^{r_n} \frac{dr}{r^2}\Big]^{1/2} \\
&\leq 2\Big[\int\limits_0^{\infty} (f')^2 r^2 dr\Big]^{1/2}\Big[\Big(\int\limits_0^{\infty} \frac{(f^2-1)^2}{r^2}dr\Big)^{1/4}\Big(\frac{1}{r_n} - \frac{1}{r_m}\Big)^{1/4} \\
&\quad +\Big(\frac{1}{r_n} - \frac{1}{r_n}\Big)^{1/2}\Big].
\end{aligned}
$$

Allowing first r_m then r_n to tend to ∞ and noting (3.3) and (3.4), we conclude that $\lim\limits_{r\to\infty} f^2(r)$ exists and by (3.4) this must be equal to 1. By (3.5) we in fact have $\lim\limits_{r\to\infty} f(r) = 1$. □

For convenience, we take $r_1 = \inf\{r|f(r) < 0\}$, $r_2 = \sup\{r|f(r) < 0\}$ in (3.8).
We now describe the properties of the functional J, in order to be able to realize our desired solution as a minimizer of J over $H^{1;\delta}_{\mathcal{R}}$.

Lemma 3.3 *J is a C^1-functional on $H^{1;\delta}_{\mathcal{R}}$, bounded from below.*

Proof. We begin by establishing that J is a continuous functional on $H^{1;\delta}_{\mathcal{R}}$. The Dirichlet functional $\alpha \mapsto \frac{1}{2}\int \alpha_r^2 r^2 dr$ is continuous on $H^{1;\delta}_{\mathcal{R}}$; hence we only need to show continuity of the nonlinear functional $N(\alpha) = \int\limits_0^{\infty} f(r)\sin^2\alpha(r)dr$. We have, for β and γ in $H^{1;\delta}_{\mathcal{R}}$ and for arbitrary positive ε:

$$
|N(\beta) - N(\gamma)| = \int\limits_0^{\infty} |\sin^2\beta - \sin^2\gamma|\ |f|dr
$$

$$\leq \quad 2\|f\|_\infty \Big[\int_0^\varepsilon |\sin\beta - \sin\gamma| dr + \int_\varepsilon^\infty |\sin\beta - \sin\gamma| dr \Big].$$

We have

$$\int_0^\varepsilon |\sin\beta - \sin\gamma| dr < 2\varepsilon, \tag{3.9}$$

and we estimate, using Hölder's inequality

$$\begin{aligned}
\int_\varepsilon^\infty |\sin\beta - \sin\gamma| dr &\leq \int_\varepsilon^\infty |\beta - \gamma| dr \\
&\leq \Big[\int_\varepsilon^\infty |\beta - \gamma|^2 r^{2\delta} dr \Big]^{1/2} \Big[\int_\varepsilon^\infty r^{-2\delta} dr \Big]^{1/2} \\
&= \frac{2\sqrt{\pi}\varepsilon^{1/2-\delta}}{\sqrt{1-2\delta}} \Big[\int_{\{x\in\mathbb{R}^3 \mid \ |x|>\varepsilon\}} \frac{|\beta-\gamma|^2}{|x|^2} |x|^{2\delta} dx \Big]^{1/2} \\
&\leq \frac{2\sqrt{\pi}}{\varepsilon^{1/2+\delta}\sqrt{1-2\delta}} \Big[\int_{\mathbb{R}^3} |\beta-\gamma|^2 (1+|x|^2)^\delta dx \Big]^{1/2} \\
&\leq \frac{1}{\varepsilon^{1/2+\delta}\sqrt{1-2\delta}} \|\beta - \gamma\|_{H^{1;\delta}}.
\end{aligned} \tag{3.10}$$

From (3.9) and (3.10) we see that, for $\|\beta-\gamma\|_{H^{1;\delta}}$ sufficiently small, we can ensure that

$$|N(\beta) - N(\gamma)| \leq 6\varepsilon\|f\|_\infty;$$

since ε is arbitrary, we have established the continuity of J.

Formally, for $\varphi \in C_0^\infty(\mathbb{R}^3)$ radial we have

$$J'(\alpha)\varphi = \int_{\mathbb{R}^3} (\alpha_{rr} + \frac{2\alpha_r}{r} - \frac{\sin 2\alpha}{r^2} f)\varphi\, dx.$$

Since $\alpha \mapsto \Delta\,\alpha$ is continuous, we need to check the continuity of $N'(\alpha)$, where

$$N'(\alpha)\varphi = \int_{\mathbb{R}^3} \frac{\sin 2\alpha(|x|)}{|x|^2} f(|x|)\varphi(|x|)\, dx.$$

For $\beta, \gamma \in H^{1;\delta}_{\mathcal{R}}$ we have

$$\|N'(\beta) - N'(\gamma)\| = \sup_{\varphi \in H^{1;\delta}_{\mathcal{R}}} \frac{|\langle N'(\beta) - N'(\gamma), \varphi\rangle|}{\|\varphi\|_{H^{1;\delta}}}, \tag{3.11}$$

and hence from Hölder and Sobolev's inequalities

$$\begin{aligned} |\langle N'(\beta) - N'(\gamma), \varphi\rangle| &\leq \|f\|_\infty \|\varphi\|_{L^6(\mathbb{R}^3)} \Big\| \frac{\sin 2\beta - \sin 2\gamma}{r^2} \Big\|_{L^{6/5}(\mathbb{R}^3)} \\ &\leq \|f\|_\infty \|\varphi\|_{H^{1;\delta}} \Big\| \frac{\sin 2\beta - \sin 2\gamma}{r^2} \Big\|_{L^{6/5}(\mathbb{R}^3)}. \end{aligned} \tag{3.12}$$

We have, for arbitrary real ε,

$$\begin{aligned} \Big\| \frac{\sin 2\beta - \sin 2\gamma}{r^2} \Big\|^{6/5}_{L^{6/5}(\mathbb{R}^3)} &= \int_0^\infty \frac{|\sin 2\beta - \sin 2\gamma|^{6/5}}{r^{2/5}} dr \\ &= \int_0^\varepsilon \frac{|\sin 2\beta - \sin 2\gamma|^{6/5}}{r^{2/5}} dr + \int_\varepsilon^\infty \frac{|\sin 2\beta - \sin 2\gamma|^{6/5}}{r^{2/5}} dr. \end{aligned}$$

The first of these integrals is bounded above by $5\varepsilon^{3/5}$. We estimate the second, using Hölder's inequality, by

$$\begin{aligned} \int_\varepsilon^\infty \frac{|\sin 2\beta - \sin 2\gamma|^{6/5}}{r^{2/5}} dr &\leq 2 \int_\varepsilon^\infty \frac{|\beta - \gamma|^{6/5} r^{6\delta/5}}{r^{(2+6\delta)/5}} dr \\ &\leq 2 \Big[\int_\varepsilon^\infty |\beta - \gamma|^2 r^{2\delta} dr \Big]^{3/5} \Big[\int_\varepsilon^\infty \frac{dr}{r^{1+3\delta}} \Big]^{2/5} \\ &\leq \frac{1}{(3\delta)^{2/5} \varepsilon^{6(1-\delta)/5}} \|\beta - \gamma\|^{6/5}_{H^{1;\delta}}. \end{aligned}$$

Since ε is arbitrary, we conclude from (3.11) and (3.12) the desired continuity of $N'(\alpha)$.

To see that J is bounded from below on $H^{1;\delta}_{\mathcal{R}}$, we use (3.8) to estimate

$$J(\alpha) \geq \int_{\{r \mid f(r) < 0\}} f \sin^2 \alpha \, dr = \int_{r_1}^{r_2} f \sin^2 \alpha dr \geq -\|f\|_\infty (r_2 - r_1),$$

which gives the required lower bound. $\square$

Using (3.9) and (3.10) we observe

$$|N(\beta)| \leq 2||f||_\infty(2\varepsilon + \frac{1}{\varepsilon^{\frac{1}{2}+\delta}\sqrt{1-2\delta}}||\beta||_{H^{1;\delta}}),$$

whence

$$J(\beta) \geq \frac{1}{2}||\beta||^2_{H^{1;\delta}} - c_1 - c_2||\beta||_{H^{1;\delta}}.$$

Thus for a sequence $\{\alpha_n\}$ minimizing J in $H^{1;\delta}_{\mathcal{R}}$ we see that $\{\alpha_n\}$ is bounded in $H^{1;\delta}$, and hence possesses a subsequence (which we shall also denote by $\{\alpha_n\}$) which is weakly convergent in $H^{1;\delta}$, strongly convergent in $L^{2;\delta}$ and pointwise convergent almost everywhere to a function $\alpha \in H^{1;\delta}_{\mathcal{R}}$. By the pointwise a.e. convergence we see that

$$N(\alpha_n) \to N(\alpha)$$

and hence, due to the weak lower semicontinuity of the norm on $H^{1;\delta}$, we have

$$J(\alpha) \leq \lim_{n\to\infty} J(\alpha_n)\,;$$

this implies equality, since $\{\alpha_n\}$ is a minimizing sequence. Thus the convergence is in fact strong in $H^{1;\delta}$, and α is a minimizer.

Since (3.1) is the Euler-Lagrange equation associated with J, by Lemma 3.2 (i), α satisfies (3.1) on $(0,\infty)$. Further by (3.6) we see that α is not identically zero.

Lemma 3.4 *A minimizer α of J over $H^{1;\delta}_{\mathcal{R}}$ satisfies either $0 < \alpha < \pi/2$ on $(0,\infty)$ or $-\pi/2 < \alpha < 0$ on $(0,\infty)$.*

Proof. We first establish:

$$\liminf_{r\to 0^+} \alpha(r), \liminf_{r\to\infty} \alpha(r) \quad \text{are both finite.} \tag{3.13}$$

To see this, we note that if the first of these quantities is not finite, we can find $r^* \in (0, r_1)$ with $\alpha(r^*) = \ell\pi$ for some $\ell \in \mathbb{Z}$. Defining α_1 by

$$\alpha_1(r) = \begin{cases} \ell\pi & r < r^* \\ \alpha(r) & r \geq r^* \end{cases}$$

we see $\alpha_1 \in H^{1;\delta}_{\mathcal{R}}$, $J(\alpha_1) < J(\alpha)$ (recall $f > 0$ on $[0, r^*]$) which is a contradiction. Arguing analogously we also show the finiteness of $\liminf_{r\to\infty} \alpha(r)$, and hence establish (3.13).
We next show that α is bounded on $[0, \infty)$. From (3.13) and the regularity of f, α is bounded on the compact set $[0, r_2]$. We define

$$\alpha_2(r) = \begin{cases} \alpha(r) & r \le r_2 \\ p\pi & r > r_2, \quad \alpha(r) < p\pi \\ \alpha(r) & r > r_2, \quad p\pi \le \alpha(r) \le (p+1)\pi \\ (p+1)\pi & r > r_2, \quad \alpha(r) > (p+1)\pi, \end{cases}$$

where $p \in \mathbb{Z}$ is specified by the condition $p\pi \le \alpha(r_2) < (p+1)\pi$. We see $\alpha_2 \in H^{1;\delta}_{\mathcal{R}}$, and $J(\alpha_2) < J(\alpha)$ with strict inequality unless $\alpha_2 \equiv \alpha$. Thus $p\pi \le \alpha < (p+1)\pi$ on $[r_2, \infty)$ i.e. α is bounded on $[0, \infty)$. Note that we must have $p = 0$ or $p = -1$ since $\alpha \in H^{1;\delta}_{\mathcal{R}}$.

We argue analogously to above to show the existence of $q \in \mathbb{Z}$ such that $q\pi \le \alpha(r) < (q+1)\pi$ on $(0, r_1]$. The next step is to show

$$p = q. \tag{3.14}$$

We consider the case $p = 0$, and define

$$\alpha_3(r) = \begin{cases} \alpha(r) & \alpha(r) \le \pi \\ 2\pi - \alpha(r) & \alpha(r) > \pi. \end{cases}$$

Then $\alpha_3 \in H^{1;\delta}_{\mathcal{R}}$ (note $\alpha_3 \equiv \alpha$ on $[r_2, \infty)$), $J(\alpha_3) = J(\alpha)$. Thus α_3 is a minimizer, and so satisfies (3.1) on $(0, \infty)$. Since $q > 0$, $p = 0$ there exists $\hat{r} \in [r_1, r_2)$ with $\alpha_3(\hat{r}) = \pi$, and hence α_3 achieves a local maximum at $\hat{r}$. From Lemma 3.2 (ii) we conclude $\alpha_3 \equiv \pi$ on $(0, \infty)$, contradicting (3.7). For $q < 0$ we argue analogously, using

$$\alpha_4(r) = \begin{cases} -2\pi - \alpha(r) & \alpha(r) < -\pi \\ \alpha(r) & \alpha(r) \ge -\pi. \end{cases}$$

This shows (3.14) for the case $p = 0$, and similar constructions establish (3.14) for $p = -1$.
For $p = 0$ we consider

$$\alpha_5(r) = \begin{cases} -\alpha(r) & \alpha(r) < 0 \\ \alpha(r) & \alpha(r) \ge 0. \end{cases}$$

We have $\alpha_5 \in H^{1;\delta}_{\mathcal{R}}$, $J(\alpha_5) = J(\alpha)$ and, by Lemma 3.2 (ii), $\alpha_5 \equiv 0$ if there exists $r_5 \in (0,\infty)$ with $\alpha(r_5) = 0$. Hence $\alpha_5 > 0$ on $(0,\infty)$ and since α is continuous, $\alpha > 0$ on $(0,\infty)$. Arguing similarly with

$$\alpha_6(r) = \begin{cases} \alpha(r) & \alpha(r) < \pi \\ 2\pi - \alpha(r) & \alpha(r) \geq \pi \end{cases}$$

we conclude that $\alpha < \pi$ on $(0,\infty)$ and hence $0 < \alpha(r) < \pi$ for $r \in (0,\infty)$.

For $p = -1$ we analogously obtain $-\pi < \alpha < 0$.

For $p = 0$ (i.e., as we have established, the case of nonnegative α) we now consider

$$\alpha_7(r) = \begin{cases} \alpha(r) & \alpha(r) \leq \pi/2 \\ \pi - \alpha(r) & \alpha(r) > \pi/2. \end{cases}$$

We see $\alpha_7 \in H^{1;\delta}_{\mathcal{R}}$, $J(\alpha_7) = J(\alpha)$ and, using Lemma 3.2 (ii) as above, we conclude $0 < \alpha < \pi/2$ on $(0,\infty)$. The same reflection argument holds for negative α, completing the proof of the lemma. $\square$

We can use Lemma 3.4 to give a precise description of the qualitative behaviour of our minimizer α; this result, and the subsequent comparison argument (Lemma 3.6) are needed to establish the desired boundary condition $\alpha(0) = 0$.

Lemma 3.5 *Let α be a minimizer of J over $H^{1;\delta}_{\mathcal{R}}$. Then α is monotone increasing on $[0, r_1)$ and monotone decreasing on (r_2, ∞) or vice versa.*

Proof. We consider the case $\alpha > 0$ on $(0,\infty)$: the case $\alpha < 0$ follows analogously. Straight-forward maximum principle arguments applied to (3.1) show that the only possible local extrema of α are minima in $(0, r_1)$ and (r_2, ∞), and maxima in (r_1, r_2) (note in particular that we can apply the Hopf maximum principle to rule out critical points at r_1 and r_2). This also establishes the existence of $\lim\limits_{r\to 0} \alpha(r)$.

If we have a local minimum at $\bar{r} \in (0, r_1)$ we consider α_8 defined by

$$\alpha_8(r) = \begin{cases} \alpha(\bar{r}) & 0 \leq r < \bar{r} \\ \alpha(r) & r \geq \bar{r}, \end{cases}$$

and see $\alpha_8 \in H^{1;\delta}_{\mathcal{R}}$; $J(\alpha_8) < J(\alpha)$ (note that there can be at most one such $\bar{r}$), contradicting the minimality of α. Hence α is monotone on $(0, r_1]$. If α is monotone decreasing on this interval, define

$$\alpha_9(r) = \begin{cases} \alpha(r_1) & 0 \leq r < r_1 \\ \alpha(r) & r > r_1 \end{cases}$$

to obtain a contradiction to the minimality of α. Thus α is monotone increasing on $(0, r_1)$. $\square$

Since α is smooth on $[r_2, \infty)$, in $H^{1;\delta}_{\mathcal{R}}$ and only permitted minima in this region, we see immediately that α is in fact monotone decreasing to 0 as r tends to infinity.

We are now in a position to prove the desired behaviour of α at zero (obviously, we define $\alpha(0) := \lim\limits_{r\to 0} \alpha(r)$).

Lemma 3.6 *For α a minimizer of J over $H^{1;\delta}_{\mathcal{R}}$, we have $\alpha(0) = 0$.*

Proof. As usual, we restrict to the case $\alpha > 0$, the case of negative α following analogously. We consider $\alpha(0) \neq 0$; by Lemma 3.5, we see that $0 < \alpha(0) < \pi/2$.
We will argue by contradiction, constructing $\beta \in H^{1;\delta}_{\mathcal{R}}$ with $J(\beta) < J(\alpha)$ and $0 < \beta(0) < \alpha(0)$, $J(\beta) < J(\alpha)$.
We set $\alpha_0 := \alpha(0)$.
Noting

$$\begin{aligned} \sin 2\alpha_0 - \sin(2\alpha_0 - 2\gamma) &= \sin 2\alpha_0 - \sin 2\alpha_0 \cos 2\gamma + \cos 2\alpha_0 \sin 2\gamma \\ &= 2\gamma \cos 2\alpha_0 + 2\gamma^2 \sin 2\alpha_0 + \mathcal{O}(\gamma^3), \end{aligned}$$

we deduce the existence of $\gamma_0 (= \gamma_0(\alpha_0))$ such that $\sin 2\alpha_0 - \sin(2\alpha_0 - 2\gamma) > 2\gamma \cos 2\alpha_0$ for all $\gamma \in (0, \gamma_0)$. We fix

$$\gamma < \min\{\gamma_0, \frac{1}{2} \sin 2\alpha_0, 1\}. \tag{3.15}$$

Now given any $\varepsilon > 0$, the continuity of f and α at 0 ensures the existence of $\tilde{\varepsilon} > 0$ such that

$$|f(r) - 1| < \varepsilon, \quad 0 \le \alpha(r) - \alpha_0 < \varepsilon$$

for $r \in [0, \tilde{\varepsilon}]$.

We write $\alpha(\tilde{\varepsilon}) = \alpha_0 + \lambda$, where $\lambda := \alpha(\tilde{\varepsilon}) - \alpha_0$; note that $0 < \lambda < \varepsilon$. We now choose ε sufficiently small such that:

$$|\sin 2\alpha(\tilde{\varepsilon}) - \sin 2\alpha_0| < 4\varepsilon; \text{ and} \tag{3.16}$$

$$\varepsilon < \frac{\gamma^2}{30} \sin 2\alpha_0. \tag{3.17}$$

For a comparison function we define

$$\beta(r) = \begin{cases} \alpha_0 - \gamma + \frac{1}{\tilde{\varepsilon}}(\alpha(\tilde{\varepsilon}) - \alpha_0 + \gamma)r & 0 \le r \le \tilde{\varepsilon} \\ \alpha(r) & r > \tilde{\varepsilon}; \end{cases}$$

note that $\beta \in H^{1;\delta}$. In order to compare $J(\beta)$ with $J(\alpha)$, we calculate

$$\begin{aligned} \frac{1}{2}\int_0^{\tilde{\varepsilon}} \beta_r^2 r^2 dr = \frac{\tilde{\varepsilon}}{6}(\alpha(\tilde{\varepsilon}) - \alpha_0 + \gamma)^2 &= \frac{\tilde{\varepsilon}}{6}(\lambda + \gamma)^2 \\ &< \frac{\tilde{\varepsilon}}{6}(\gamma^2 + 2\varepsilon\gamma + \varepsilon^2), \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} \int_0^{\tilde{\varepsilon}} f \sin^2 \beta dr &< (1+\varepsilon) \int_0^{\tilde{\varepsilon}} \sin^2(\alpha_0 - \gamma + \frac{\lambda+\gamma}{\tilde{\varepsilon}} r) dr \\ &= (1+\varepsilon)\Big[\frac{\tilde{\varepsilon}}{2} - \frac{\tilde{\varepsilon}}{4(\lambda+\ \gamma)}\Big(\sin 2\alpha(\tilde{\varepsilon}) - \sin(2\alpha_0 - 2\gamma)\Big)\Big] \\ &= (1+\varepsilon)\tilde{\varepsilon}\Big[\frac{1}{2} - \frac{1}{4(\lambda+\gamma)}(\sin 2\alpha(\tilde{\varepsilon}) - \sin 2\alpha_0) \\ &\quad - \frac{1}{4(\lambda+\gamma)}(\sin 2\alpha_0 - \sin(2\alpha_0 - 2\gamma))\Big] \\ &< (1+\varepsilon)\tilde{\varepsilon}\Big[\frac{1}{2} + \frac{\varepsilon}{\gamma} - \frac{\gamma\cos 2\alpha_0}{2(\lambda+\gamma)} - \frac{\gamma^2 \sin 2\alpha_0}{4(\lambda+\gamma)}\Big] \text{ via (3.15), (3.16)} \end{aligned}$$

$$
\begin{aligned}
&< \ (1+\varepsilon)\tilde{\varepsilon}\Big[\frac{\varepsilon}{\gamma}+\frac{1}{2}(1-\cos 2\alpha_0)+\frac{\varepsilon}{\gamma}|\cos 2\alpha_0|-\frac{\gamma}{4}\sin 2\alpha_0 \\
&\quad +\frac{\varepsilon}{2}\sin 2\alpha_0\Big] \\
&< \ (1+\varepsilon)\tilde{\varepsilon}\Big[\sin^2\alpha_0-\frac{\gamma}{4}\sin 2\alpha_0+\frac{2\varepsilon}{\gamma}\Big]. \qquad (3.19)
\end{aligned}
$$

Finally we note

$$
\frac{1}{2}\int_0^{\tilde{\varepsilon}} \alpha_r^2 r^2 dr + \int_0^{\tilde{\varepsilon}} f\sin^2\alpha dr > (1-\varepsilon)\tilde{\varepsilon}\sin^2\alpha_0. \qquad (3.20)
$$

Combining (3.18), (3.19) and (3.20), we have

$$
\begin{aligned}
J(\beta)-J(\alpha) &< \ (1+\varepsilon)\tilde{\varepsilon}\Big[\sin^2\alpha_0-\frac{\gamma}{4}\sin^2\alpha_0+\frac{2\varepsilon}{\gamma}+\frac{1}{6}(\gamma^2+2\varepsilon\gamma+\varepsilon^2)\Big] \\
&\quad -(1-\varepsilon)\tilde{\varepsilon}\sin^2\alpha_0 \\
&< \ \tilde{\varepsilon}\Big[-(1+\varepsilon)\frac{\gamma}{4}\sin 2\alpha_0+2\varepsilon\sin^2\alpha_0+\frac{2\varepsilon}{\gamma}+\frac{(1+\varepsilon)}{6}(\gamma^2+2\varepsilon\gamma+\varepsilon^2)\Big] \\
&< \ \tilde{\varepsilon}\Big[-\frac{\gamma}{4}\sin 2\alpha_0+\frac{\gamma^2}{6}+\varepsilon(2\sin^2\alpha_0+\frac{2}{\gamma}+\frac{1}{6}(\gamma^2+2\varepsilon\gamma+\varepsilon^2+2\gamma+\varepsilon))\Big] \\
&< \ \tilde{\varepsilon}\Big[-\frac{\gamma}{4}\sin 2\alpha_0+\frac{\gamma}{12}\sin 2\alpha_0+\frac{\gamma}{12}\sin 2\alpha_0\Big] \quad \text{by (3.15), (3.17).} \\
&< \ -\frac{\gamma\tilde{\varepsilon}}{12}\sin 2\alpha_0 < 0.
\end{aligned}
$$

This provides the desired contradiction. ☐

The final step in the proof of theorem (3.1) is to establish the smoothness of the minimizer α at 0. In order to establish C^1 regularity we use arguments similar to those of the proof of [Sh, Lemma 4.1].

Lemma 3.7 *Any minimizer of J over $H^{1;\delta}_{\mathcal{R}}$ is smooth at* 0.

Proof. From Lemmas (3.5) and (3.6) we know that, for such a minimizer α, we have $\alpha(0) = 0$, and we can restrict to the case that α is monotone increasing on $[0, r_1)$. Multiplying (3.1) by α and integrating by parts we have, for $0 < \varepsilon' < \varepsilon$ small,

$$\int_{\varepsilon'}^{\varepsilon} \Big(\alpha_r^2 - \frac{2\alpha\alpha_r}{r} + \frac{\alpha \sin 2\alpha}{r^2}\Big) dr = \frac{1}{2}\partial_r(\alpha^2)|_{\varepsilon'}^{\varepsilon}. \tag{3.21}$$

Noting that $\liminf\limits_{\varepsilon' \to 0} \Big[\partial_r(\alpha^2)|_{\varepsilon'}\Big] \geq 0$ from Lemma (3.6), we have that (3.21) is uniformly bounded for ε small.

Noting $0 \leq 2\frac{\alpha_r \alpha}{r} \leq \alpha_r^2 + \frac{\alpha^2}{r^2}$, we have

$$\begin{aligned} 0 \leq \alpha_r^2 + \frac{\alpha^2}{r^2} &= 2\Big(\alpha_r^2 + \frac{\alpha^2}{r^2}\Big) - \Big(\alpha_r^2 + \frac{\alpha^2}{r^2}\Big) \\ &\leq 2\Big(\alpha_r^2 - \frac{\alpha_r \alpha}{r} + \frac{\alpha^2}{r^2}\Big) \\ &\leq 2\Big(\alpha_r^2 - \frac{\alpha_r \alpha}{r} + \frac{\alpha \sin 2\alpha}{r^2}\Big) \end{aligned}$$

on $(0, \varepsilon)$ for ε sufficiently small, using Lemma 3.7. Integrating this from 0 to ε we see

$$0 \leq \int_0^{\varepsilon} \Big(\alpha_r^2 + \frac{\alpha^2}{r^2}\Big) dr \leq 2\int_0^{\varepsilon} \Big(\alpha r^2 - \frac{\alpha_r \alpha}{r} + \frac{\alpha \sin 2\alpha}{r^2}\Big) dr, \tag{3.22}$$

which is uniformly bounded by the uniform boundedness of (3.21).
We now rewrite (3.1) as

$$\alpha_{rr} + \frac{2}{r}\alpha_r - \frac{2\alpha}{r^2} = \frac{\sin 2\alpha}{r^2} - \frac{2\alpha}{r^2}. \tag{3.23}$$

Since $\alpha(0) = 0$ and $\alpha \in C^\rho$ for any $\rho < \frac{1}{2}$, we have that the right-hand side of (3.23) is integrable on $(0, \varepsilon)$; from (3.23) then, so is $\partial_r(\alpha_r + \frac{2\alpha}{r})$. Thus $\alpha_r + \frac{2\alpha}{r}$ is absolutely continuous on $[0, \varepsilon)$, and in particular, there exists a finite limit

$$\lim_{r \to 0^+} \Big(\alpha_r + \frac{2\alpha}{r}\Big).$$

Since both these terms are nonnegative, α_r must also have a finite limit as $r \to 0^+$, i.e. $\alpha \in C^1[0, \infty)$. Higher regularity then follows from the fact that solutions to (3.1) solve the equation

$$\Delta \alpha = \frac{\sin 2\alpha}{r^2} f$$

in $\mathbb{R}^3$ (considering $r := |x|$). Since the right-hand side of this equation belongs to $L^p_{\mathrm{loc}}(\mathbb{R}^3)$ for any $p > 1$, we can apply standard elliptic regularity (see e.g. [GT, Theorem 8.15]) to conclude that $\alpha \in W^{2,p}(\mathbb{R}^3)$, and hence in $C^{1,\rho'}$ for some ρ'; a standard bootstrap argument then yields that α is everywhere smooth. □

By the above remarks, this completes the proof of Theorem 3.1.

4 The Faddeev-Popov Operator

Given a connection $A \in C^\infty(\mathbb{R}^n, \Lambda^1\mathcal{G})$, the so-called Faddeev-Popov operator is defined by

$$\mathcal{D}_A = -\partial(D_i(A)\cdot)\,; \tag{4.1}$$

here $\mathcal{D}_A : \mathrm{dom}(\mathcal{D}_A) \subseteq L^2(\mathbb{R}^n, \mathcal{G}) \to L^2(\mathbb{R}^n, \mathcal{G})$. This operator is the second variation of the functional

$$\mathcal{F}(g) = \frac{1}{2}\|g^* A\|^2_{L^2(\mathbb{R}^n)}\,; \tag{4.2}$$

that is to say, if we compute the Hessian matrix of $\mathcal{F}$ at the identity, along two directions $g_i = \exp(\psi_i)$, with $\psi_i \in \mathcal{G}$, then:

$$D^2\mathcal{F}(id) \cdot (\psi_1, \psi_2) = \int_{\mathbb{R}^n} \mathcal{D}_A \psi_1(x) \cdot \psi_2(x)\, dx \tag{4.3}$$

and therefore, if $\mathcal{F}$ attains its minimum along a gauge orbit, then $\mathcal{D}_A$ is a positive-definite operator. When A is in the Coulomb gauge, $\mathcal{D}_A$ takes the form

$$\mathcal{D}_A = -\Delta - [A_i, \partial_i \cdot] \tag{4.4}$$

Restricting to configurations satisfying (2.9) - (2.11), it is not difficult to see that (4.4) becomes

$$\mathcal{D}_A = -\Delta + \frac{2f}{r^2}, \tag{4.5}$$

when acting on vectors of the form:

$$\psi(x) = \psi(r)\frac{x^a}{r}T_a, \tag{4.6}$$

corresponding to the allowable gauge transformations (2.11). Noting that

$$\mathcal{F}(g) = \frac{1}{2}\|g^*A\|_{L^2}^2 = \frac{1}{2}\|A\|_{L^2}^2 + 2J(\alpha), \tag{4.7}$$

we conclude that the operator (4.5) will be positive definite when acting on radial vectors of the form (4.6). It is our goal to show that our minima are not only minima with respect radial variations, but in fact global minima in $H^2(\mathbb{R}^3, \mathcal{G})$. This can be seen by demonstrating that the terms with higher angular momentum will not effect the positivity of the restricted Faddeev-Popov operator. More precisely, we will prove:

Proposition 4.1 *For $f \in \mathcal{S}$ satisfying $-1 \leq f \leq 3$, the full Faddeev-Popov operator (4.4) is a positive-definite operator on $H^2(\mathbb{R}^3, \mathcal{G})$.*

Proof. We shall consider an expansion of $\psi \in H^2(\mathbb{R}^3, su(2))$ in vector spherical harmonics (see e.g. [DM]):

$$\psi(x) = \sum_{\substack{j,\ell,m: \\ j\geq 0 \\ |j-\ell|\leq 1 \\ |m|\leq \ell}} \psi_{j\ell m}(r) Y^a_{j\ell m}(\omega) T_a \tag{4.8}$$

where $\{Y^a_{j\ell m}(\omega)\}_{a=1}^3$ form an orthonormal basis of $L^2(S^2, su(2))$ (we will assume that the restrictions on j,ℓ and m given in (4.8) hold in any summations considered for the remainder of this section). The radial vectors (4.6) correspond to the vector spherical harmonic $j = 0$ for which the only surviving modes are:

$$Y^a_{010} = \frac{x^a}{r} \tag{4.9}$$

Using the standard properties of the vector spherical harmonics we find:

$$-\Delta\psi(x) = \sum_{j,\ell,m} \Big[-\Delta + \frac{\ell(\ell+1)}{r^2} \Big] \psi_{j\ell m}(r) Y^a_{j\ell m}(\omega) T_a \tag{4.10}$$

The non-abelian term $[A_i, \partial_i \psi]$ can be calculated using the fact that A is radial and that $[T_a, T_b] = \epsilon_{abc} T_c$:

$$\begin{aligned}
[A_i, \partial_i \psi] &= \epsilon_{abc} A_i^b \partial_i \psi^c T_a = \epsilon_{abc}\epsilon_{ib\ell} \frac{f-1}{r} \cdot \frac{x^\ell}{r} \partial_i \psi^c T_A \\
&= \frac{f(r)-1}{r^2} \, (T \cdot \mathbb{O})\psi \qquad (4.11)
\end{aligned}$$

where $T \cdot \mathbb{O} = T_a \mathbb{O}_a$ is the so-called spin-orbit coupling operator (see [DM]). Therefore

$$[A_i, \partial_i \psi] = \frac{f(r)-1}{2r^2} \sum_{j,\ell,m} (j(j+1) - \ell(\ell+1) - 2)\psi_{j\ell m}(r) Y^a_{j\ell m}(\omega) T_a \qquad (4.12)$$

and so, using the orthogonality on S^2, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} \mathcal{D}_A \psi \cdot \psi dx &= \sum_{j,\ell,m} 4\pi \Big\{ \int_0^\infty (-\Delta + \frac{\ell(\ell+1)}{r^2}) \psi_{j\ell m}(r) \psi_{j\ell m}(r) r^2 dr \\
&\quad - \int_0^\infty \frac{f(r)-1}{2} (j(j+1) - \ell(\ell+1) - 2) \psi^2_{j\ell m}(r) dr \Big\} \\
&= I + II + III + IV
\end{aligned}$$

Where I is the term corresponding to $j = 0$, and II , III and IV correspond to the terms with $j \geq 1$ for $\ell = j$, $\ell = j-1$ and $\ell = j+1$ respectively.

We calculate (keeping in mind (4.10))

$$I = 4\pi \int_0^\infty (-\Delta + \frac{2f}{r^2}) \psi_{010}(r) \psi_{010}(r) r^2 \, dr = \int_{\mathbb{R}^3} \mathcal{D}_A (\psi_{010}(r) \frac{x^a}{r} T_a) \cdot (\psi_{010}(r) \frac{x^a}{r} T_a) dx$$

which is nonnegative by the positivity of $\mathcal{D}_A$ when acting on radial vectors. The second term, where $j = \ell$, is given by

$$II = 4\pi \sum_{\ell=j,m} \int_0^\infty \Big[r^2 (\partial_r \psi_{\ell\ell m})^2 + [\ell(\ell+1) + (f(r)-1)] \psi_{\ell\ell m}(r)^2 \Big] \, dr \, ;$$

since $\ell \geq 1$ and $f(r) \geq -1$, this term is nonnegative. The third term, where $\ell = j-1$, is

$$III = 4\pi \sum_{\ell=j-1,m} \int_0^\infty \Big[r^2 (\partial_r \psi_{j\ell m})^2 + (\ell)[\ell - f(r) + 2)] \psi^2_{j\ell m} \Big] dr \, ;$$

Since $f \leq 3$, for any $j \geq 2$ the summands are nonnegative, and for $j = 1$ the last term is zero. Finally, observe that the last term is even easier to estimate:

$$IV = 4\pi \sum_{\ell=j+1,m} \int_0^\infty r^2(\partial_r \psi_{j\ell m})^2 + (\ell+1)(f(r)+\ell-1)\psi_{j\ell m}^2 \, dr$$

which is nonnegative by inspection; this concludes the proof. □

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