Finite time blow-up for the Yang-Mills heat flow in higher dimensions

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Abstract

We consider the L^2-gradient flow associated with the Yang-Mills functional, the so-called Yang-Mills heat flow. In the setting of a trivial principal $SO(n)$-bundle over $\mathbb{R}^n$ in dimension n greater than 4, we show blow-up in finite time for a class of $SO(n)$-equivariant initial connections.

1 Introduction

We let E denote a principal fibre bundle over an n-dimensional Riemannian manifold M, with structure group G, a semi-simple Lie group, and canonical projection π. We denote by $\mathcal{G}$ the Lie algebra of G, and by $[\cdot,\cdot]$ its Poisson bracket. A (smooth) connection on E is a (smooth) map from M into $\operatorname{Ad} E \otimes T^*M$, where $\operatorname{Ad} E$ denotes the adjoint bundle to E. Locally, a connection can be considered as a $\mathcal{G}$-valued 1-form A defined on the coordinate patches U_α of M, $A = A_\mu(x)\, dx^\mu$, with $A_\mu : U_\alpha \to \mathcal{G}$, i.e. for $v \in T_xM$ we have $A(x,u) = A_\mu(x)v^\mu$. A (smooth) gauge transformation is a (smooth) section of the bundle $\operatorname{Aut}(E)$ of automorphisms of E acting on connections by conjugation: a gauge transformation $g : U_\alpha \cap U_\beta \to G$ changing coordinates on fibres in the intersection of the trivializations $\pi^{-1}(U_\alpha)$ and $\pi^{-1}(U_\beta)$ acts on A as defined above by

$$g^* A_\mu = g^{-1} A_\mu g + g^{-1}\partial_\mu g\,. \tag{1.1}$$

The curvature F_A of a connection A is defined by $F_A = D_A A$, where D_A is the covariant derivative associated with A, and is given locally by the $\mathcal{G}$-valued 2-form $F_{\mu\nu} dx^\mu dx^\nu$, where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\,. \tag{1.2}$$

The curvature transforms under g via

$$g^* F_{\mu\nu} = g^{-1} F_{\mu\nu} g\,. \tag{1.3}$$

This means that the Yang-Mills functional (or Yang-Mills action) $\mathcal{F}$, defined by

$$\mathcal{F}(A) = \int_M F_{\mu\nu} F^{\mu\nu} \, dvol_M \,, \tag{1.4}$$

is invariant under gauge transformations. Critical points of $\mathcal{F}$ are *Yang-Mills connections*; they solve the system

$$D^\mu F_{\mu\nu} = 0 \tag{1.5}$$

where $D_\mu = \partial_\mu + [A_\mu, \cdot]$.
One approach to finding Yang-Mills connections is to study the L^2- gradient flow associated with $\mathcal{F}$, the so-called Yang-Mills heat flow, i.e. the initial value problem

$$\left.\begin{array}{rl} \partial_t A_\mu(t,x) & = -D^\nu F_{\mu\nu}(t,x) \,, \\ A_\mu(0,x) & = A_{0\mu}(x) \end{array}\right\} \tag{1.6}$$

for some given initial connection A_0.
In this paper we wish to concentrate on one particular aspect of the study of (1.6), and refer the reader to [St1] and the references contained therein for a discussion of other results. Precisely, we wish to consider the phenomenon of *blow-up* for solutions of (1.6), i.e. the question of whether smooth initial data A_0 can be found such that the solution to (1.6) fails to be smooth after some finite, positive time T. Before discussing existing results concerning blow-up for solutions of (1.6), it is instructive to consider this phenomenon for a related problem, namely that of the harmonic map heat-flow.
The harmonic map heat flow between Riemannian manifolds X and N is the L^2-gradient flow associated with the energy functional

$$E(u) = \int_X |Du|^2 \, dvol_X \tag{1.7}$$

for $u : X \to N$ (where for convenience, N is taken as isometrically embedded in $\mathbb{R}^k$ for some k); critical points of E are called *harmonic maps*. We refer the reader to the reports [EL1], [EL2] and the article [St2] for general results concerning this evolution problem, and concentrate again on the aspect of blow-up; we note, however that the success of harmonic map heat-flow in producing harmonic maps in a variety of circumstances (meaning appropriate restrictions on X, N and the initial map), as well as the large number of interesting problems arising from the system (including existence and uniqueness for various types of weak solutions, rates of convergence or blow-up as applicable) was one of the original motivations for considering the evolution problem (1.6).
The first examples of finite-time blow-up for the harmonic map heat flow were given by Coron and Ghidaglia ([CG]), who showed that this occurred for certain symmetric initial data for the heat flow from $\mathbb{R}^n$ or S^n to S^n, for $n \geq 3$. It was also shown by Chen and Ding ([CD]) that, for $\dim X \geq 3$, finite-time blow-up is guaranteed if the initial map u_0 is homotopically nontrivial and has sufficiently small energy, provided that the homotopy class of u_0 contains maps of arbitrarily small energy. The situation

$\dim X = 2$ is particularly interesting in the study of harmonic maps and the harmonic map heat flow, due to the fact that the energy functional is conformally invariant in this case. It was shown by Chang, Ding and Ye ([CDY]) that blow-up can occur in this setting.
The Yang-Mills functional $\mathcal{F}$ is conformally invariant if $\dim M = 4$. Råde showed in [R] that, in dimensions 2 and 3, the system (1.6) will not blow up. In dimension 4 the situation appears to be even more delicate than the analogous case of the harmonic map heat flow from a 2-dimensional manifold, and the general question of whether blow-up can occur remains open; however Schlatter, Struwe and Tahvildah-Zadeh have recently shown in [SST] that, under a symmetry Ansatz which is the natural analogue of that that considered in [CDY], solutions to (1.6) will not blow up in finite time.
For $\dim M \geq 5$, Naito gave the first proof of finite-time blow-up, with an approach analogous to that of [CD]: he showed

Theorem 1.1 ([N, Theorem 1.3]) *Let $G \subset SO(n)$, for $n \geq 5$. For E a nontrivial principal G-bundle over S^n there exists $\varepsilon_1 > 0$ such that $||F(A_0)||_{L^2(S^n)} < \varepsilon_1$ implies that the solution of (1.6) blows up in finite time.*

The purpose of the current paper is to give another approach to showing the occurence of finite-time blow-up in dimensions 5 and above. In contrast to [N] we show this in the case of a trivial bundle (over $\mathbb{R}^n$); i.e. our blow-up occurs as a result of the geometric nature of the symmetry and the analytic properties of the initial data, rather than being induced by topology. In addition our arguments are simpler than those in [N].
Our basic strategy is to find an analogue of the procedure used in [CG] for harmonic maps in order to produce finite-time blow-up in the setting of a trivial principal $SO(n)$-bundle over $\mathbb{R}^n$ for a suitable $SO(n)$-equivariant initial connection; see Theorem 2.1 for a precise statement.
The remainder of the paper is organized as follows: in Section 2 we describe the symmetry we use, and state our main theorem, Theorem 2.1; in Section 3 we show how (1.6) reduces to a scalar-valued initial value problem; and in Section 4 we complete the proof of the main theorem.

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2 Equivariant connections

The situation we consider is similar to that given in [CST, Section 1.2], cf. also [SST, Section 2], [W]; in all cases (including the current paper) the authors draw on earlier results of [Du] and [I].

We assume that a symmetry group $\mathcal{S}$ acts on M, the action $\gamma : \mathcal{S} \times M \to M$ being given by $(s,x) \mapsto sx$. We further suppose that γ lifts to an action $\overline{\gamma}$ on the bundle E, i.e. $\overline{\gamma} : \mathcal{S} \times E \to E$, such that: $\gamma \circ \pi = \pi \circ \overline{\gamma}$, i.e.

$$\gamma(s, \pi(z)) = \pi(\overline{\gamma}(s,z)) \qquad \text{for all } s \in \mathcal{S},\ z \in E\,;$$

and such that $\overline{\gamma}$ commutes with the right action of G on E. In this situation we have from [I, Section 2] (cf. [KN, Theorem 11.5]) the existence of a homomorphism $\lambda : \mathcal{S} \to G$ such that, for U a neighbourhood of $x \in M$, on the trivialization $\pi^{-1}(U)$ we have

$$\overline{\gamma}(s, (y,g)) = (sy, \lambda(s)g)) \qquad \text{for all } s \in \mathcal{S},\ y \in U \text{ and } g \in G.$$

The action $\overline{\gamma}$ induces an action on connections. If this action has the effect of a (global) gauge transformation on the local $\mathcal{G}$-valued 1-forms A defined in the introduction, i.e. if

$$A(x,v) = (\lambda(s))^* A(sx, s_* v) \qquad \text{for all } s \in \mathcal{S},\ x \in M \text{ and } v \in T_x M,$$

(where here $s_* : T_x M \to T_{sx} M$ is the push-forward $s_* v = d\gamma(s,v)$), then the connection is called *equivariant* with respect to the $\mathcal{S}$-action γ.
In particular in this paper we consider the situation $M = \mathbb{R}^n$, $n \geq 5$, $\mathcal{S} = G = SO(n)$ and E is the trivial bundle $\mathbb{R}^n \times SO(n)$. In this case the homomorphism λ is simply id_G, and leads, as in [Du], to A being given by

$$A_\mu(x) = \frac{-h(r)}{r^2} \sigma_\mu(x) \tag{2.1}$$

for $r = |x|$, and h a real-valued function on $[0,\infty)$. Here the $\{\sigma_\mu\}_{\mu=1}^{n(n-1)/2}$ are a basis for the Lie-algebra $\mathfrak{so}(n)$, given by

$$\sigma_\mu^{ij}(x) = \delta_\mu^i x^j - \delta_\mu^j x^i \qquad \text{for } 1 \leq i,j, \leq n. \tag{2.2}$$

We will refer to connections satisfying (2.1) simply as $SO(n)$*-equivariant connections.*
We are now in a position to state the main result of the paper.

Theorem 2.1 *Fix $n \geq 5$, and consider the trivial $SO(n)$-bundle over $\mathbb{R}^n$. Consider an $SO(n)$-equivariant connection given by*

$$A_0 = A_{0\mu}\, dx^\mu\,, \tag{2.3}$$

with

$$A_{0\mu}(x) = \frac{-h_0(r)}{r^2} \sigma_\mu(x) \tag{2.4}$$

for $r = |x|$, $h_0 : [0,\infty) \to \mathbb{R}$ and $\{\sigma_\mu\}_{i=1}^{n(n-1)/2}$ a basis for the Lie algebra $\mathfrak{so}(n)$ as given by (2.2). Then there exists smooth h_0 such that the Yang-Mills action of the curvature of A_0 is finite, and such that the Yang-Mills heat flow with initial data A_0 blows up in finite time.

3 The reduced equation

We first note that if we have a smooth $SO(n)$-equivariant solution of the Yang-Mills heat flow (1.6) on $[0,T)$ then a straightforward calculation (cf. [Du, Section 3], [W, Chapter 2]), shows that the system can be rewritten as

$$\left.\begin{aligned} -\sigma_\mu(x) r^{-2} h_t &= -\sigma_\mu(x) r^{-2} h_{rr} + (n-3)\frac{h_r}{r} - (n-2)\frac{h(h-1)(h-2)}{r^2}\,, \\ -\sigma_\mu(x) r^{-2} h(0,r) &= -\sigma_\mu(x) r^{-2} h_0(r) \end{aligned}\right\} \tag{3.1}$$

for some smooth $h : [0,T) \times [0,\infty) \to \mathbb{R}$ and $h_0 : [0,\infty) \to \mathbb{R}$. Setting

$$\ell h = h_{rr} + (n-3)\frac{h_r}{r} - (n-2)\frac{h(h-1)(h-2)}{r^2}\,,$$

the system (3.1) (and hence (1.6)) will be satisfied on $[0,T)$ if we can find a smooth h satisfying

$$h_t = \ell h \tag{3.2}$$

subject to the initial condition

$$h(0,\cdot) = h_0(\cdot)\,. \tag{3.3}$$

In order to justify restricting our attention to (3.2) (3.3), we need to show that $SO(n)$-equivariant initial data leads to an $SO(n)$-equivariant flow, i.e. we need to show that $SO(n)$-equivariance is preserved by the flow (1.6). The approach we take is comparable to that of [Sch, Section 9]; there the author was concerned with symmetry preservation for the hyperbolic Yang-Mills flow.
We begin by giving an alternative characterization of $SO(n)$-equivariance; for a proof of the first part, see [FM], and cf. [Sch, Section 4]. The second part follows directly from the linearity of the infinitesimal generators of $\mathfrak{so}(n)$, again see [FM].

Lemma 3.1 *A connection A characterized by the $\mathfrak{so}(n)$-valued 1-forms A_μ given in Section 1 is $SO(n)$-equivariant if and only if the Lie derivatives of A along the infinitesimal generators $\Omega_{\alpha\beta} = x_\alpha\partial_\beta - x_\beta\partial_\alpha$ of $\mathfrak{so}(n)$ satisfy*

$$\mathcal{L}_{\Omega_{\alpha\beta}} A_\mu + [\sigma_{\alpha\beta}, A_\mu] = 0\,; \tag{3.4}$$

here $\sigma_{\alpha\beta} = \delta^i_\alpha\delta^j_\beta - \delta^i_\beta\delta^j_\alpha$. Further we have, for the curvature 2-forms $F_{\mu\nu}$:

$$\mathcal{L}_{\Omega_{\alpha\beta}}(\partial^\nu F_{\mu\nu}) = \partial^\nu(\mathcal{L}_{\Omega_{\alpha\beta}} F_{\mu\nu})\,. \tag{3.5}$$

We denote the operator $\mathcal{L}_{\Omega_{\alpha\beta}} + [\sigma_{\alpha\beta},\cdot]$ by $\widehat{\mathcal{L}}_{\Omega_{\alpha\beta}}$. The next lemma gives two useful properties of $\widehat{\mathcal{L}}_{\Omega_{\alpha\beta}}$.

Lemma 3.2 *With A a solution of (1.6) and notation as above, we have that each $\widehat{\mathcal{L}}_{\Omega_{\alpha\beta}}$ commutes with ∂_t, in particular*

$$\partial_t(\widehat{\mathcal{L}}_{\Omega_{\alpha\beta}} A_\mu) = \widehat{\mathcal{L}}_{\Omega_{\alpha\beta}}(\partial_t A_\mu)\,. \tag{3.6}$$

Further there holds

$$\widehat{\mathcal{L}}_{\Omega_{\alpha\beta}} F_{\mu\nu} = D_\mu(\widehat{\mathcal{L}}_{\Omega_{\alpha\beta}} A_\nu) - D_\nu(\widehat{\mathcal{L}}_{\Omega_{\alpha\beta}} A_\mu)\,. \tag{3.7}$$

Proof. The commutation relationship (3.6) is immediate from the definition of $\widehat{\mathcal{L}}_{\Omega_{\alpha\beta}}$. For ease of notation we set $\Omega = \Omega_{\alpha\beta}$, $\sigma = \sigma_{\alpha\beta}$ and write B_μ for $\widehat{\mathcal{L}}_\Omega A_\mu$. Recalling that $\mathcal{L}_\Omega$ commutes with the exterior derivative, and using (1.2) and the Jacobi identity we calculate

$$\begin{aligned}
\widehat{\mathcal{L}}_\Omega F_{\mu\nu} &= \mathcal{L}_\Omega F_{\mu\nu} + [\sigma, F_{\mu\nu}] \\
&= \mathcal{L}_\Omega(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]) + [\sigma, F_{\mu\nu}] \\
&= \partial_\mu(\mathcal{L}_\Omega A_\nu) - \partial_\nu(\mathcal{L}_\Omega A_\mu) + [\mathcal{L}_\Omega A_\mu, A_\nu] + [A_\mu, \mathcal{L}_\Omega A_\nu] + [\sigma, F_{\mu\nu}] \\
&= D_\mu(B_\nu - [\sigma, A_\nu]) - D_\nu(B_\mu - [\sigma, A_\mu]) + [\sigma, F_{\mu\nu}] \\
&= D_\mu B_\nu - D_\nu B_\mu - \partial_\mu[\sigma, A_\nu] - [A_\mu, [\sigma, A_\nu]] + \partial_\nu[\sigma, A_\mu] \\
&\qquad\qquad + [A_\nu, [\sigma, A_\mu]] + [\sigma, F_{\mu\nu}] \\
&= D_\mu B_\nu - D_\nu B_\mu - [\sigma, \partial_\mu A_\nu - \partial_\nu A_\mu] - [\sigma, [A_\mu, A_\nu]] + [\sigma, [A_\mu, A_\nu]] \\
&\qquad\qquad + [A_\mu, [A_\nu, \sigma]] + [A_\nu, [\sigma, A_\mu]] \\
&= D_\mu B_\nu - D_\nu B_\mu\,,
\end{aligned}$$

showing (3.7). □

With this preparation, we are now able to show the desired result concerning the preservation of the equivariance.

Lemma 3.3 *Let $A_\mu(t,\cdot)$ be a smooth solution to (1.6) on $(0,T]$. If the initial data $A_{0\mu}$ is $SO(n)$-equivariant, then the solution $A_\mu(t,\cdot)$ is also $SO(n)$-equivariant on $(0,T]$.*

Proof. For the given solution $A_\mu(t,\cdot)$ we retain the notation of Lemma 3.2, and further abbreviate $\widehat{\mathcal{L}}_\Omega F_{\mu\nu}$ by $G_{\mu\nu}$. We will show that each B_μ (initially zero by assumption) stays zero on $(0,T]$. We then have

$$\begin{aligned}
\partial_t B_\mu &= \widehat{\mathcal{L}}_\Omega(\partial_t A_\mu) && \text{via (3.6)} \\
&= \widehat{\mathcal{L}}_\Omega(-D^\nu F_{\mu\nu}) && \text{via (1.6)} \\
&= \mathcal{L}_\Omega(-D^\nu F_{\mu\nu}) + [\sigma, -D^\nu F_{\mu\nu}] \\
&= \mathcal{L}_\Omega(-\partial^\nu F_{\mu\nu} - [A^\nu, F_{\mu\nu}]) + [\sigma, -\partial^\nu F_{\mu\nu} - [A^\nu, F_{\mu\nu}]] \\
&= -\partial^\nu \mathcal{L}_\Omega F_{\mu\nu} - [\mathcal{L}_\Omega A^\nu, F_{\mu\nu}] - [A^\nu, \mathcal{L}_\Omega F_{\mu\nu}] - [\sigma, \partial^\nu F_{\mu\nu}] \\
&\qquad - [\sigma, [A^\nu, F_{\mu\nu}]] && \text{via (3.5)} \\
&= -D^\nu \mathcal{L}_\Omega F_{\mu\nu} - [\mathcal{L}_\Omega A^\nu, F_{\mu\nu}] - D^\nu[\sigma, F_{\mu\nu}] - [\sigma, [F_{\mu\nu}, A^\nu]] - [[\sigma, A^\nu], F_{\mu\nu}] \\
&\qquad - [\sigma, [A^\nu, F_{\mu\nu}]] && \text{via the Jacobi identity} \\
&= -D^\nu G_{\mu\nu} - [B^\nu, F_{\mu\nu}]\,.
\end{aligned} \tag{3.8}$$

Using (3.7) and integrating by parts we have (with the norms being the norm in $L^2(\mathbb{R}^n)$ unless otherwise noted, and denoting the Killing product of g, h in $\mathfrak{so}(n)$ by $g \cdot h$):

$$\begin{aligned}
\int_{\mathbb{R}^n} D^\nu G_{\mu\nu} \cdot B^\mu \, dx &= -\int_{\mathbb{R}^n} G_{\mu\nu} \cdot D^\nu B^\mu \, dx \\
&= ||G||^2 - \int_{\mathbb{R}^n} G_{\mu\nu} \cdot D^\mu B^\nu \, dx \\
&= ||G||^2 + \int_{\mathbb{R}^n} D^\mu G_{\mu\nu} \cdot B^\nu \, dx = ||G||^2 + \int_{\mathbb{R}^n} D^\nu G_{\nu\mu} \cdot B^\mu \, dx \\
&= ||G||^2 - \int_{\mathbb{R}^n} D^\nu G_{\mu\nu} \cdot B^\mu \, dx \,,
\end{aligned}$$

i.e.

$$\int_{\mathbb{R}^n} D^\nu G_{\mu\nu} \cdot B^\mu \, dx = \tfrac{1}{2}||G||^2 \,. \tag{3.9}$$

Thus we have

$$\begin{aligned}
\tfrac{1}{2}\partial_t ||B||^2 &= -\int_{\mathbb{R}^n} D^\nu G_{\mu\nu} \cdot B^\mu \, dx - \int_{\mathbb{R}^n} [B^\nu, F_{\mu\nu}] \cdot B^\mu \, dx && \text{via (3.8)} \\
&= -\tfrac{1}{2}||G||^2 - \int_{\mathbb{R}^n} [B^\nu, F_{\mu\nu}] \cdot B^\mu \, dx && \text{via (3.9)} \\
&\leq ||F(t,\cdot)||_\infty \, ||B(t,\cdot)||^2 \,,
\end{aligned}$$

i.e. from Gronwall's inequality

$$||B(t,\cdot)|| \leq ||B(0,\cdot)|| \exp\Big(\int_0^T ||F(s,\cdot)||_\infty \, ds \Big).$$

Since by assumption the integral on the right-hand side is bounded and $||B(0,\cdot)|| = 0$, this shows the desired result. □

4 Finite time blow-up for the reduced equation

We begin by considering some properties of solutions of the system (3.1), (3.2). Note firstly that finiteness of the action (1.4) for a connection of the form (2.4) is equivalent to finiteness of $F(h_0)$, where the functional F (which we will term the *energy*) is given by

$$F(h) = \int_0^\infty \Big(h_r^2 + \frac{h^2(h-2)^2}{r^2} \Big) r^{n-3} \, dr \,. \tag{4.1}$$

Lemma 4.1 *Let h be a solution of (3.2) on $[0,T] \times [0,\infty)$ with $F(h_0)$ finite, and such that $h_\infty(0) = \lim_{r\to\infty} h_0(r)$ exists. Then:*

(i) $F(h(t,\cdot)$ *is finite for all* $t \in [0,T]$*;*

(ii) $h_\infty(t) = \lim_{r\to\infty} h(t,r)$ *exists for all* $t \in [0,T]$*;*

(iii) there exists positive C *depending only on* T*,* h_0 *and* n *such that* $|h(t,r)-h(s,\rho)| \le C|r-\rho|^{1/4}$ *for all* $t \in [0,T]$ *and* $r,\rho \in [0,\infty)$*;*

(iv) $h_\infty(t) = h_\infty(0)$ *for all* $t \in (0,T]$.

Proof: (i) By multiplying (3.2) by h_t and integrating by parts (alternatively, by working directly with (1.6), cf. [St1, Section 3]) we obtain the *energy inequality*

$$\int_0^t \int_0^\infty |h_t(s,r)|^2 r^{n-3} dr ds + F(h(t,\cdot)) \le F(h_0)\,, \tag{4.2}$$

which shows (i).
(ii) For fixed $t \in (0,T]$ we write $h(r)$ for $h(t,r)$ where no confusion arises, and define $\Sigma = \{r : |h(r)| > 4\}$. Considering $0 < r_\ell < r_m$ and noting that $|h-2| > 1$ on Σ we have

$$\begin{aligned}
\int_{r_m}^{r_\ell} h^2 r^{3-n}\,dr &= \int\limits_{[r_m,r_\ell]\cap\Sigma} h^2 r^{3-n}\,dr + \int\limits_{[r_m,r_\ell]\cap\Sigma^c} h^2 r^{3-n}\,dr \\
&\le \int\limits_{[r_m,r_\ell]\cap\Sigma} h^2(h-2)^2 r^{3-n}\,dr + 16\int_{r_m}^{r_\ell} r^{3-n}\,dr \\
&\le r_m^{8-2n}\int_0^\infty h^2(h-2)^2 r^{3-n}\,dr + \tfrac{16}{n-4}(r_m^{4-n} - r_\ell^{4-n}) \\
&\le r_m^{8-2n} F(h_0) + \tfrac{16}{n-4}(r_m^{4-n} - r_\ell^{4-n}) \qquad \text{via (4.2).}
\end{aligned} \tag{4.3}$$

Thus we have, from Cauchy-Schwarz, (4.2) and (4.3),

$$\begin{aligned}
|h^2(r_\ell)-h^2(r_m)|^2 &\le 4\Big[\int_{r_m}^{r_\ell} |hh_r|\,dr\Big]^2 \\
&\le 4\Big[\int_{r_m}^{r_\ell} h_r^2 r^{n-3}\,dr\Big]\Big[\int_{r_m}^{r_\ell} h^2 r^{3-n}\,dr\Big] \\
&\le 4F(h_0)\Big[r_m^{8-2n}F(h_0) + \tfrac{16}{n-4}(r_m^{4-n} - r_\ell^{4-n})\Big]\,.
\end{aligned} \tag{4.4}$$

Allowing first r_ℓ and then r_m to tend to ∞, we have the desired conclusion from (4.4) and the continuity of h.
(iii) Note firstly that finite energy means that $h_\infty(t) = 0$ or 2 for all $t \in [0,T]$. Further by letting $r_\ell \to \infty$ in (4.4) we see that $h(t,r) \to h_\infty(t)$ as $r \to \infty$ uniformly for $t \in [0,T]$. Given this and the energy inequality (4.2) we have that $||h(t,\cdot)-h_\infty(t)||_{H^{1,2}([0,\infty))}$ is uniformly bounded for $t \in [0,T]$, yielding the desired conclusion by the Sobolev embedding theorem.
(iv) For fixed $t \in [0,T]$ and fixed $\rho > 1$ we have from (4.2)

$$\int_0^t \int_\rho^{\rho+1} |h_t^2(s,r)|^2 r^{n-3} drds \le F(h_0)\,,$$

and hence

$$\int_0^t \int_\rho^{\rho+1} |h_t^2(s,r)|^2 r^{n-3} drds \le (\rho+1)^{n+3} F(h_0)\,.$$

By Fubini's theorem this means that $\int_0^t h_t^2(s,r)ds$ is finite for almost all $r \in [\rho,\rho+1]$. Since h is continuous there exists c depending only on ρ and T such that $|h(t,r)| \le c$ for $t \in [0,T]$ and $r \in [\rho,\rho+1]$. Thus $h(\cdot,r) \in H^{1,2}([0,T])$ for almost all $r \in [\rho,\rho+1]$ and in particular is absolutely continuous on $[0,T]$ for such r. Hence, for ε positive to be fixed later and $\rho > 1$, we have

$$\begin{aligned}
\int_\rho^{\rho+\varepsilon} |h^2(t,r) - h_0^2(r)|\, dr &\le 2\int_\rho^{\rho+\varepsilon}\int_0^t |h(s,r)|\,|h_r(s,r)|\, dsdr \\
&\le 2\int_0^t\int_\rho^{\rho+\varepsilon} |h(s,r)|\,|h_r(s,r)|\, drds \\
&\le \int_0^t\int_\rho^{\rho+\varepsilon} h^2(r,s) r^{\frac{7}{2}-n}\, drds + \frac{1}{\sqrt{\rho}}\int_0^t\int_\rho^\infty h_t^2 r^{n-3}\, drds \\
&\le \rho^{9-2n} t + \frac{F(h_0)}{\sqrt{\rho}}\,, \qquad (4.5)
\end{aligned}$$

the estimate for the first term in the second-last line following from analolgous arguments to those in the proof of part (ii).
If $h_\infty(t) \ne h_\infty(0)$ then by finiteness of F and by the remarks at the start of the proof of part (iii) we have, for almost all ρ sufficiently large:

$$|h^2(t,\rho) - h_0^2(\rho)| > 1\,.$$

By part (iii), this means

$$|h^2(t,r) - h_0^2(r)| > 1 - 2C^2\varepsilon^{1/2}$$

for $r \in [\rho,\rho+\varepsilon]$, i.e. the left-hand side of (4.5) is bounded below by $\varepsilon - 2C^2\varepsilon^{3/2}$, which is strictly positive for ε sufficiently small, contradicting (4.5) when ρ is chosen sufficiently large. This yields the desired conclusion. □

We next construct a family of self-similar subsolutions to (3.2), which will be used to show finite time blow-up. For fixed, positive $\lambda > 0$ and T we consider $\varphi(r) = \frac{2r^2}{\lambda^2+r^2}$, and define

$$\xi(t,r) = \varphi(\rho) \qquad \text{for} \qquad \rho = \frac{r}{\sqrt{T-t}}\,. \qquad (4.6)$$

We have

$$\begin{aligned}
\ell\xi &= \xi_{rr} + \frac{n-3}{r}\xi_r - \frac{n-2}{r^2}\xi(\xi-1)(\xi-2) \\
&= \frac{1}{T-t}\Big[\varphi_{\rho\rho} + \frac{n-3}{\rho}\varphi_\rho - \frac{n-2}{\rho^2}\varphi(\varphi-1)(\varphi-2)\Big] \\
&= \frac{8(n-4)\lambda^2\rho^2}{(T-t)(\rho^2+\lambda^2)^2},
\end{aligned}$$

and

$$\xi_t = \frac{2\lambda^2\rho^2}{(T-t)(\rho^2+\lambda^2)^2}$$

i.e. we see that $\xi_t \le \ell\xi$. Further we see that $\xi(t, r_0)$ increases monotonicly to 2 as t approaches T for all $r_0 > 0$; since $\xi(t, 0)$ is identically zero, this means that the subsolution ξ blows up at $t = T$.

We now establish a suitable maximum principle for equation (3.2).

Lemma 4.2 *Let h be a solution of (3.2), (3.3) on $[0,T) \times [0,\infty)$ with $h_0(0) = 0$ and $lim_{r\to\infty} h_0(r) = 2$ such that h_0 has finite energy, and let ξ be a subsolution of (3.2) given by (4.6) for some $\lambda > 0$, such that $\xi(0,r) < h(0,r)$ for $0 < r < \infty$. Then $\xi(t,r) < h(t,r)$ for all $t \in (0,T)$, $r \in (0,\infty)$.*

Proof: We consider $\eta = h - \xi$. From (3.2) and the inequality $\xi_t \le \ell\xi$ we see

$$\begin{aligned} \eta_t &\ge \eta_{rr} + (n-3)\frac{\eta_r}{r} - \frac{(n-2)}{r^2}\Big[h(h-1)(h-2) - \xi(\xi-1)(\xi-2)\Big] \\ &= \eta_{rr} + (n-3)\frac{\eta_n}{r} - \frac{n-2}{r^2}a\eta, \end{aligned} \tag{4.7}$$

where $a = [(h-1)(h-2) + (\xi-1)(\xi-2) + h\xi - 2]$.

If the lemma were false, we could find $t' \in (0,T)$ such that $\inf\limits_{\{t'\}\times[0,\infty)} \eta < 0$; then δ defined by $\delta = \inf\limits_{[0,t']\times[0,\infty)} \eta$ is negative. By Lemma 4.1, $h(t,r)$ tends to 2 as r tends to ∞ uniformly for $t \in [0,t']$. The same holds for $\xi(t,r)$ by the above remarks, and hence $\eta(t,r)$ tends to 0 as r tends to ∞ uniformly for $t \in [0,t']$. This means that η must take on the value δ, a negative minimum, at some point in $(0,t'] \times (0,\infty)$. Now fix $\varepsilon \in (0,1/3)$. By continuity of the solution and the definition of ξ we can find $\rho > 0$ such that $|\xi(t,r)|, |h(t,r)| < \varepsilon$ for $0 \le t \le t'$, $0 \le r \le \rho$. This means

$$a \ge 2(1-\varepsilon)(2-\varepsilon) - \varepsilon^2 - 2 > 0 \text{ on } [0,t'] \times [0,\rho].$$

Further (after again appealing to Lemma 4.1 to obtain uniform L^∞-bounds on h) there exists $M > 0$ such that $|a| < M$ on $[0,M] \times [0,\infty)$.
Consider $\zeta = e^{-Mt/2\rho^2}\eta$: by (4.7) we have

$$\zeta_t \ge \zeta_{rr} + (n-2)\frac{\zeta_r}{r} - (n-2)\zeta\Big(\frac{a}{2r^2} + \frac{M}{2\rho^2}\Big). \tag{4.8}$$

Since the function η achieves a negative minimum on $[0,t'] \times [0,\infty)$ and the scaling factor $e^{-Mt/2\rho^2}$ is bounded above and below away from 0, the function ζ also achieves a negative minimum at some point $(\tau,\sigma) \in (0,t'] \times (0,\infty)$. We consider two cases:

(i) $\sigma \leq \rho$. At (τ, σ) we have $\zeta_t \leq 0$, $\zeta_{rr} \geq 0$, $\zeta_r = 0$, $\zeta < 0$ and $a > 0$, which contradicts (4.8).

(ii) $\sigma > \rho$. At (τ, σ) we have $\zeta_t \leq 0$, $\zeta_{rr} \geq 0$, $\zeta_r = 0$, $\zeta < 0$ and $\frac{|a|}{2\sigma^2} > \frac{-M}{2\rho^2}$, so that $\zeta(\frac{a}{2r^2} + \frac{M}{2\rho^2}) < 0$, again contradicting (4.8).

This verifies the claim. □

Proof of Theorem 2.1 Given smooth initial data, the system (1.6) will have a smooth solution for some positive time interval, see [DK, Section 6.3.1], and cf. [St1, Section 4.4]. By Lemma 3.3, if we take $SO(n)$-equivariant initial data, this symmetry is preserved. Given $T > 0$, we will construct finite-action inital data such that the solution to (1.6) must blow up before $t = T$. To do this we fix $\lambda > 0$, and consider $\xi(t, r)$ as defined by (4.6). For an equivariant initial connection given by $A_{0\mu}(x) = -h_0(r)r^{-2}\sigma_\mu(x)$, It is easy to construct finite-energy initial data h_0 satisfying the conditions of Lemma 4.2 with respect to the chosen subsolution $\xi(t, r)$ (for example, we could take h_0 to be identically equal to 2 for all r sufficiently large). With such initial data, we can apply Lemma 4.2 to conclude that the solution $h(t, \cdot)$ to (3.2), (3.3) must blow up at or before time $t = T$. This means that the same conclusion holds for the corresponding solution of (3.1), which is the desired result. □

We close by remarking that the approach we use to show blow-up avoided the sometimes delicate question of determining whether the connections arising are generic/ irreducible (see [BF], and cf. [St1, Section 5], [DK, Section 6.3.1]), since we only used results concerning existence of a short-term solution to (1.6), as opposed to uniqueness results.

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