Berlin Sfb288 Preprint
physics/9911046

# Existence criteria for stabilization from the scaling behaviour of ionization probabilities

C. Figueira de Morisson Faria,[†] A. Fring[*] and R. Schrader[*] [1]

† *Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Str. 38, D-01187 Dresden, Germany*
∗ *Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, D-14195 Berlin, Germany*



**Abstract**

We provide a systematic derivation of the scaling behaviour of various quantities and establish in particular the scale invariance of the ionization probability. We discuss the gauge invariance of the scaling properties and the manner in which they can be exploited as a consistency check in explicit analytical expressions, in perturbation theory, in the Kramers-Henneberger and Floquet approximation, in upper and lower bound estimates and fully numerical solutions of the time dependent Schrödinger equation. The scaling invariance leads to a differential equation which has to be satisfied by the ionization probability and which yields an alternative criterium for the existence of atomic bound state stabilization.
PACS numbers: 32.80.Rm, 32.80.Fb, 33.80.Rv, 42.50.Hz, 03.65.Db

November 1999

[1] e-mail addresses: Faria@mpipks-dresden.mpg.de,
Fring@physik.fu-berlin.de,
Schrader@physik.fu-berlin.de

# 1 Introduction

It is up to now still not possible to carry out computations of ionization probabilities or ionization rates in the high intensity regime in a totally satisfactory manner. In particular analytical results are extremely rare. Especially concerning the issue of so-called atomic stabilization [1], numerous computations may be found in the literature which are contradictory in many cases. Alone for the relatively simple problem of the one-dimensional delta-potential there exist various recent computations which do [2, 3] or do not [4, 5] support the existence of stabilization. Roughly speaking, stabilization is the effect that the ionization probability (or ionization rate to some authors) as a function of the laser field intensity is decreasing or constant in some region. For further references and a more detailed terminology, that is, a distinction into transient, adiabatic, interference or resonance stabilization, see for instance [6].

It would be highly desirable to settle the controversial issue and pinpoint possible mistakes, erroneous physical or mathematical assumptions made in the course of the computations. The main intention of this note is to contribute to this debate and provide additional alternative consistency criteria. For this purpose we analyze the scaling behaviour of several quantities involved in the calculations which address the above-mentioned problem. This constitutes an adaptation of an idea which has been proved to be extremely powerful in the context of quantum field theory, for instance in form of the renormalization group flow (see almost any book on quantum field theory). In the context of atomic physics, scaling laws have been considered before [7] in a "semiempirical" fashion, as the authors refer themselves to their own analysis. In order to overcome the slightly ad hoc way of arguing we provide in this note a systematic derivation of various scaling laws, which are compatible with the ones proposed in [7]. In particular, we establish the scale invariance property of the ionization probability. As a consequence one may exploit these symmetry properties and check various analytical as well as numerical expressions for the ionization probability for consistency. In addition, when considering the ionization probability as a function of various parameters the scale invariance property allows to trade one particular variable for others. This permits to interpret and rigorously explain various types of behaviour which occurred before for more specific situations in the literature. For instance, for the Hydrogen atom it was found in [8] that for increasing principal quantum number the ionization probability decreased and in [9] the opposite behaviour was observed. Our analysis culminates in the formulation of a simple alternative criterium for the existence of stabilization.

Our manuscript is organized as follows: In section 2 we derive systematically the scaling properties of various quantities and establish in particular the invariance of the ionization probability under scaling. We show that this property is preserved under gauge invariance. Furthermore, the scale invariance can be exploited as a consistency check in various computations. We exhibit this for explicit analytical expressions, for perturbative calculations, for approximate evaluations in form of Kramers-Henneberger- and Floquet states and for upper and lower bound estimates. We demonstrate how the scaling properties can be exploited to trade some variables for others and use this feature to explain several types of physical behaviour. As a consequence of our analysis in section 2, we provide in section 3 a differential equation which has to be satisfied by the ionization probability and an alternative criterium for the existence of stabilization.

We state our conclusions in section 4.

## 2 Scaling behaviour of ionization probabilities

We consider an atom with potential $V(\vec{x})$ in the presence of a sufficiently intense laser field, such that it may be described in the non-relativistic regime by the time-dependent Schrödinger equation in the dipole approximation

$$i\hbar \frac{\partial \psi(\vec{x},t)}{\partial t} = \left( -\frac{\hbar^2}{2m_e}\Delta + V(\vec{x}) + e\,\vec{x}\cdot \vec{E}(t) \right) \psi(\vec{x},t) = H(\vec{x},t)\,\psi(\vec{x},t). \tag{1}$$

We further take the pulse to be of the general form

$$\vec{E}(t) = \vec{E}_0 f(t) \tag{2}$$

where $f(t)$ is a function whose integral over $t$ is assumed to be well behaved, with $f(t) = 0$ unless $0 \leq t \leq \tau$. This means $\tau$ is the pulse duration, $f(t)$ the pulse shape function and $E_0$ the amplitude of the pulse, which we take to be positive without loss of generality.

Denoting by $\lambda > 0$ the dilatation factor and by $\eta$ the scaling dimension of the eigenfunction $\varphi(\vec{x}) := \psi(\vec{x}, t=0)$ of the Hamiltonian $H(\vec{x}, t=0)$, we consider the following scale transformations[1]

$$\vec{x} \to \vec{x}' = \lambda \vec{x} \qquad \text{and} \qquad \varphi(\vec{x}) \to \varphi'(\vec{x}') = \lambda^{-\eta}\varphi(\vec{x}) \ . \tag{3}$$

Making the physical assumption that the Hilbert space norm remains invariant, i.e. $\|\varphi(\vec{x})\| = \|\varphi'(\vec{x}')\|$, we deduce immediately that the scaling dimension has to be $\eta = d/2$, with $d$ being the dimension of the space. Introducing now the scaling of the dimensional parameters $\hbar, m_e$ and $e$ as

$$\hbar \to \hbar' = \lambda^{\eta_\hbar}\hbar, \quad m_e \to m_e' = \lambda^{\eta_{m_e}} m_e \quad \text{and} \quad e \to e' = \lambda^{\eta_e} e \tag{4}$$

we can scale the whole problem to atomic units, i.e. $\hbar = e = m_e$, for instance by the choice $\lambda = \hbar$, $\eta_\hbar = -1$, $\eta_e = -\log_\hbar(e)$ and $\eta_{m_e} = -\log_\hbar(m_e)$. Staying for the time being in these units the scaling behaviour (3) may be realized by scaling the coupling constant. Considering for instance the wavefunction $\varphi(x) = \sqrt{\alpha}\exp(-\alpha|x|)$ of the only bound state when the potential in (1) is taken to be the one-dimensional delta-potential $V(x) = \alpha\delta(x)$, equation (3) imposes that the coupling constant has to scale as $\alpha \to \alpha' = \lambda^{-1}\alpha$. Choosing instead the Coulomb potential in the form $V(\vec{x}) = \alpha/r$ requires the same scaling behaviour of the coupling constant for (3) to be valid. This is exhibited directly by the explicit expressions of the corresponding wavefunctions $\varphi_{nlm}(\vec{x}) \sim \alpha^{3/2}(\alpha r)^l \exp(-\alpha r/n) L_{n+l}^{2l+1}(2\alpha r/n)$ (see e.g. [10]).

From a physical point of view it is natural to require further, that the scaling behaviour of the wavefunction is the same for all times

$$\psi(\vec{x},t) \to \psi'(\vec{x}',t') = U'(t',0)\varphi'(\vec{x}') = \lambda^{-d/2}\psi(\vec{x},t) = \lambda^{-d/2}U(t,0)\varphi(\vec{x}) \ . \tag{5}$$

[1] More formally we could also carry out all our computations by using unitary dilatation oparators $U(\lambda)$, such that the transformation of the eigenfunction is described by $U(\lambda)\varphi(\vec{x}) = \lambda^{\eta}\varphi'(\lambda\vec{x})$ and operators $\mathcal{O}$ acting on $\varphi(\vec{x})$ transform as $U(\lambda)\mathcal{O}U(\lambda)^{-1} = \mathcal{O}'$.

Consequently this means that the time evolution operator should be an invariant quantity under these transformations

$$U(t_1,t_0) = T\left(e^{-\frac{i}{\hbar}\int_{t_0}^{t_1} H(\vec{x},s)ds}\right) \rightarrow U'(t'_1,t'_0) = T\left(e^{-\frac{i}{\hbar}\int_{\lambda^2 t_0}^{\lambda^2 t_1} H'(\vec{x},s)ds}\right) = U(t_1,t_0)\ . \quad (6)$$

Here $T$ denotes the time ordering. Equation (6) then suggests that the scaling of the time has to be compensated by the scaling of the Hamiltonian and Planck's constant in order to achieve invariance. Scaling therefore the time as

$$t \rightarrow t' = \lambda^{\eta_t} t\ , \quad (7)$$

with $\eta_t$ being unknown for the moment, equation (6) only holds if the Stark Hamiltonian of equation (1) scales as

$$H(\vec{x},t) \rightarrow H'(\vec{x}',t') = \lambda^{\eta_H} H(\vec{x},t) \qquad \text{with} \qquad \eta_H = \eta_\hbar - \eta_t\ . \quad (8)$$

The properties (7) and (8) could also be obtained by demanding the invariance of the Schrödinger equation (1). The overall scaling behaviour of $H(\vec{x},t)$ is governed by the scaling of the Laplacian, the electron mass and Planck's constant, such that we obtain the further constraint

$$\eta_H = 2\eta_\hbar - \eta_{m_e} - 2\ . \quad (9)$$

As a consequence we can read off the scaling properties of the potential as

$$V(\vec{x}) \rightarrow V'(\vec{x}') = \lambda^{\eta_H} V(\vec{x}) \qquad . \quad (10)$$

What does this behaviour imply for some concrete potentials? Having scaled everything to atomic units, relation (9) suggests that $\eta_H = -2$. Considering for this situation the one-dimensional delta-potential and the Coulomb potential in the forms specified above, equation (10) imposes that the coupling constant has to scale as $\alpha \rightarrow \alpha' = \lambda^{-1}\alpha$ in both cases. This behaviour of the coupling constant is in agreement with our earlier observations for the corresponding wavefunctions. We also observe immediately that the behaviour (10) may be realized for the general class of Kato small potentials. We recall that if for each constant $\beta$ with $0 < \beta < 1$ there exists a constant $\gamma$, such that $\|V\psi\| \leq \beta \|-\Delta\psi\| + \gamma \|\psi\|$ holds for all $\psi$ in the domain $\mathcal{D}(-\Delta/2)$, the potential $V$ is called Kato small. We see immediately that the scaling of the first term is entirely governed by the Laplacian such that $\beta \rightarrow \beta' = \beta$ is scale invariant and that $\gamma$ has to scale as $\gamma \rightarrow \gamma' = \lambda^{-2}\gamma$ due to the scale invariance of the norm.

It is intriguing to note that there exists an interesting class of potentials which scale alone via their dependence on $\vec{r}$ and which do not contain any energy scale $\alpha$ at all, as for instance $V(\vec{x}) = 1/r^2$ or the two-dimensional delta potential.

In [7] the interesting proposal was made to exploit the scaling behaviour in order to use known properties of the Hydrogen atom to understand the behaviour of Helium. For this purpose the Schrödinger equation describing Helium, i.e. (1), for the potential $V_{He}(\vec{x}) = -Ze^2/r$ and the mass $m_e$ replaced by the reduced mass $\mu$, is scaled to the one which describes Hydrogen. Translating the quantities of [7] into our conventions this transformation is realized by $\lambda = (\mu/m_e)Z$, $\eta_t = \log_\lambda(Z^2\mu/m_e)$, $\eta_\mu = \log_\lambda(m_e/\mu)$,

$\eta_Z = -\log_\lambda Z$ and $\eta_\hbar = \eta_e = 0$. These quantities are consistent with the additional constraint $\eta_H = 2\eta_\hbar - \eta_{m_e} - 2$, which results for the potential $V_{He}(\vec{x})$ from the scaling arguments. We would like to point out that this is only one of many possible choices. It might be more convenient to use for instance $\lambda = Z$, $\eta_t = 2$, $\eta_\mu = \eta_\hbar = \eta_Z + 1 = \log_Z(m_e/\mu)$ and $\eta_e = 0$ instead.
We will now consider the constraint resulting from equation (8) on the scaling behaviour of the pulse. We have

$$\vec{E}(t) \to \vec{E}'(t') = \lambda^{\eta_E} \vec{E}(t) \qquad \text{with} \quad \eta_E = \eta_H - \eta_e - 1. \tag{11}$$

This equation is not quite as restrictive as for the potential, since in the latter case we could determine the behaviour of the coupling whereas now a certain ambiguity remains in the sense that we can only deduce

$$\vec{E}_0 \to \vec{E}_0' = \lambda^{\eta_{E_o}} \vec{E}_0\,, \quad f(t) \to f'(t') = \lambda^{\eta_f} f(t)\,, \quad \text{with } \eta_{E_0} + \eta_f = \eta_E\,. \tag{12}$$

Thus, under the assumptions we have made, it is not possible to disentangle the contribution coming from the scaling of the amplitude or the pulse shape function. However, there might be pulse shape functions for which $h_f$ has to be 0, since no suitable parameter is available in its explicit form to achieve the scaling.

Finally, we come to the scaling behaviour of the ionization probability. Denoting by $P$ the orthogonal projection in $L^2(I\!R^3)$ onto the subspace spanned by the bound states of $H(\vec{x}, t=0)$, the ionization probability turns out to be a scale invariant quantity

$$\mathcal{P}(\varphi) = \|(1-P)\,U(\tau,0)\,\varphi\|^2 \to \mathcal{P}'(\varphi') = \mathcal{P}(\varphi)\,. \tag{13}$$

This follows by means of (3), (6) and by noting that the projection operator has to be a scale invariant quantity, i.e. $P \to P' = P$. From a physical point of view this is clear unless we were able to scale bound states into the continuum, which is impossible, since negative energies will remain always negative even after being scaled. Mathematically this means we have to demand that $P'$ and $P$ are related to each other by a unitary transformation.

We recapitulate that our sole assumptions in this derivation were to demand the invariance of the Hilbert space norm, i.e. $\|\varphi(\vec{x})\| = \|\varphi'(\vec{x}')\|$, and that the scaling of the wavefunction is preserved for all times.

We shall now utilize this symmetry property in various approaches, which can be carried out either numerically or analytically. At this point we scale everything to atomic units which we will use from now onwards.

## 2.1 Gauge invariance

First of all we would like to establish that these scaling properties hold in every gauge, as one naturally expects. We recall that different gauges are related by a time-dependent unitary operator $A_{g_2\leftarrow g_1}(t)$. For instance the wavefunction in gauge $g_1$ and gauge $g_2$ are related as $\Psi_{g_2}(\vec{x},t) = A_{g_2\leftarrow g_1}(t)\Psi_{g_1}(\vec{x},t)$. The velocity gauge is obtained from the length gauge by

$$A_{v\leftarrow l}(t) = e^{i\vec{b}(t)\cdot\vec{x}} \to A'_{v\leftarrow l}(t') = A_{v\leftarrow l}(t) \tag{14}$$

the velocity gauge from the Kramers-Henneberger gauge by

$$A_{v\leftarrow KH}(t) = e^{-ia(t)}e^{i\vec{c}(t)\cdot\vec{p}} \rightarrow A'_{v\leftarrow KH}(t') = A_{v\leftarrow KH}(t) \tag{15}$$

and the length gauge from the Kramers-Henneberger gauge by

$$A_{l\leftarrow KH}(t) = e^{-ia(t)}e^{-i\vec{b}(t)\cdot\vec{x}}e^{i\vec{c}(t)\cdot\vec{p}} \rightarrow A'_{l\leftarrow KH}(t') = A_{l\leftarrow KH}(t)\ . \tag{16}$$

The defining relations for the classical momentum transfer $\vec{b}(t)$, the classical displacement $\vec{c}(t)$ and the classical energy transfer $\vec{a}(t)$ then yield

$$\vec{b}(t) \;=\; \vec{E}_0 b_0(t) = \int_0^t ds\vec{E}(s) \rightarrow \vec{b}'(t') = \int_0^{t\lambda^2} ds\lambda^{-3}\vec{E}(s\lambda^{-2}) = \lambda^{-1}\vec{b}(t) \tag{17}$$

$$\vec{c}(t) \;=\; \vec{E}_0 c_0(t) = \int_0^t ds\vec{b}(s) \rightarrow \vec{c}'(t') = \int_0^{t\lambda^2} ds\lambda^{-1}\vec{b}(s\lambda^{-2}) = \lambda\vec{c}(t) \tag{18}$$

$$\vec{a}\,(t) \;=\; \vec{E}_0 a_0(t) = \frac{1}{2}\int_0^t ds\, b^2\,(s) \rightarrow \vec{a}'(t') = \int_0^{t\lambda^2} ds\lambda^{-2}b^2(s\lambda^{-2}) = \vec{a}(t) \quad . \tag{19}$$

These quantities scale in the expected manner, that is $\vec{b}(t)$ scales as a momentum, $\vec{c}(t)$ as the space and $\vec{a}(t)$ remains invariant. Taking these properties into account, we observe easily that the operator $A_{g_2\leftarrow g_1}(t)$ is an invariant quantity under scaling

$$A_{g_2\leftarrow g_1}(t) \rightarrow A'_{g_2\leftarrow g_1}(t) = A_{g_2\leftarrow g_1}(t) \tag{20}$$

for all cases $g_1$ and $g_2$ mentioned. Hence the scaling behaviour is preserved in all gauges. It is interesting to note that one may reverse the logic here and deduce from a broken scale invariance onto a broken gauge invariance. However, in general gauge invariance is not broken in such a crude manner, e.g. in [11] (see eqn. (22) therein) the gauge invariance is broken in a scale invariant fashion.

### 2.2 Symmetry properties for analytical expressions of $\mathcal{P}$

Keeping the pulse shape function invariant under the scaling transformations we incorporate now the explicit functional dependence into the ionization probability. The fundamental parameters are the field amplitude, the pulse length and the coupling constant. The previous observations then suggest that

$$\mathcal{P}(E_0, \tau, \alpha) = \mathcal{P}(E_0', \tau', \alpha')\ . \tag{21}$$

Assuming from now on that the coupling constant scales as for the one-dimensional delta- and the Coulomb potential, the meaning of equation (21) is that the ionization probability remains invariant under the transformations

$$E_0 \rightarrow E_0' = \lambda^{-3}E_0, \qquad \tau \rightarrow \tau' = \lambda^2\tau, \qquad \alpha \rightarrow \alpha' = \lambda^{-1}\alpha\ . \tag{22}$$

We can exploit the symmetry property (21) most easily when we have an explicit analytical expression for $\mathcal{P}\,(\varphi)$ at hand. Considering for example the $\delta$-potential and taking the pulse to be the $\delta$-kick, i.e. $E(t) = E_0\delta(t)$, $b(t) = E_0 0^+$, $c(t) = 0$, the ionization probability of the bound state was computed to be [12]

$$\mathcal{P}(\varphi) = 1 - \frac{4}{\pi^2}\left| \int_{-\infty}^{\infty} dp \frac{\exp\left(-i\tau\alpha^2\frac{p^2}{2}\right)}{\left(1+\left(p+b(\tau)/\alpha\right)^2\right)(1+p^2)} \right|^2 . \tag{23}$$

Obviously the r.h.s. of (23) passes the test and remains invariant under the scale transformation in form of (17) and (22).

### 2.3 Perturbation theory

Usually one is not in the fortunate situation to have explicit expressions for the ionization probability available as in the previous subsection. However, the symmetry property may also be utilized when computing $\mathcal{P}(\varphi)$ approximately either in a numerical or analytical fashion. We recall that the essential ingredient of perturbation theory is to expand the time evolution operator as a series in $E_0$ or $\alpha$ for small or large field intensities, respectively. We can formally write

$$U(t_1, t_0) = \sum_{n=0}^{\infty} U(n|t_1, t_0) . \tag{24}$$

Since $U(t_1, t_0)$ is a scale invariant quantity, the invariance property (6) must hold order by order, that is for $0 \leq n \leq \infty$ we have

$$U(n|t_1, t_0) \to U'(n|t_1', t_0') = U(n|t_1, t_0) . \tag{25}$$

Considering now for instance the high intensity regime and performing Gordon-Volkov perturbation theory (e.g. [13, 14]), the first terms in (24) read

$$\begin{aligned} U(0|t_1, t_0) &= \exp(i(t_1 - t_0)\Delta/2) = \exp(i(t_1 - t_0)\lambda^2\lambda^{-2}\Delta/2) = U'(0|t_1', t_0') && (26) \\ U(1|t_1, t_0) &= i\int_{t_0}^{t_1} ds U(0|t_1, s) V U(0|s, t_0) = U'(1|t_1', t_0') \; . && (27) \end{aligned}$$

Whilst it was fairly obvious that the general expressions (26) and (27) remain invariant under scaling, this consistency check might be less trivial when carried out after the expressions have been evaluated explicitly either numerically or analytically.

### 2.4 Expansions in terms of Kramers-Henneberger states or Floquet states

The essence of the Kramers-Henneberger approximation (e.g. [15]) is to exploit the fact that when the gauge transformation (16) is carried out on the Stark Hamiltonian, the term involving the laser pulse disappears and instead the potential is shifted by the total classical displacement, i.e. $V(\vec{x}) \to V(\vec{x} - \vec{c}(t))$. Having chosen the pulse in such a way that $\vec{c}(t)$ is a periodic function in time, one can expand the shifted potential in a Fourier series

$$V(\vec{x} - \vec{c}(t)) = \sum_{n=-\infty}^{\infty} V_n e^{in\omega t} . \tag{28}$$

In the Kramers-Henneberger approximation one assumes now that the zero mode is dominant such that the remaining terms may be treated as perturbations. From the scaling behaviour of the l.h.s. of (28) we deduce immediately that the frequency has to scale inverse to the time, i.e. $\omega \to \omega' = \lambda^{-2}\omega$, and that each mode in the series scales in the same way as the potential, i.e.

$$V_n \to V_n' = \lambda^{-2} V_n \ . \tag{29}$$

As an example let us consider the expansion for a square-well potential of depth $\alpha V_0$ and of width $d$ subjected to a pulse of linearly polarized monochromatic light. The modes are of the general form (first reference in [15])

$$V_n = |\alpha V_0| \, g[(d/2 - x)\omega^2/E_0] \ , \tag{30}$$

where the explicit formula of the function $g$ is given in term of Chebyshev-polynomials which, however, is not important for our purpose. We only need to know that it scales by means of its argument alone. Since the argument is scale invariant, we observe with the help of (10) for $\eta_H = 2$ in atomic units that (29) holds for each coefficient in (30).

The analysis of the scaling behaviour of the Floquet expansion is very similar. Instead of exploiting the periodicity of the potential one makes additional use of the periodicity of the field and expands $\psi(\vec{x}, t) = \sum_{n=-\infty}^{\infty} \psi_n(\vec{x}) e^{in\omega t}$. It is then obvious by the same reasoning as before that the scaling of the individual modes has to be the same as for the field itself, i.e. $\psi_n(\vec{x}) \to \psi_n'(\vec{x}') = \lambda^{-d/2}\psi_n(\vec{x})$.

### 2.5 Upper and lower bounds

In [13, 17, 16, 14] analytical expressions for upper and lower bounds, $\mathcal{P}_u(\varphi)$ and $\mathcal{P}_l(\varphi)$, respectively, were derived and analyzed. Depending on the particular parameters these expressions put more or less severe constraints on the actual value of $\mathcal{P}(\varphi)$, in the sense that $\mathcal{P}_l(\varphi) \leq \mathcal{P}(\varphi) \leq \mathcal{P}_u(\varphi)$. Since $\mathcal{P}(\varphi)$ is a scale invariant quantity, also the bounds have to respect this symmetry. Otherwise they could be scaled to any desired value. We present just one example for one particular upper bound (the arguments carry through equally for lower bounds) to convince ourselves that this is indeed the case. For instance under the condition $b(\tau)^2/2 > -E \equiv$ binding energy of $\varphi$, the following upper bound was derived in [13]

$$\mathcal{P}_u(\varphi) = \left\{ \int_0^\tau \|(V(\vec{x} - c(t)e_z) - V(\vec{x}))\varphi\| dt + |c(\tau)| \, \|p_z\varphi\| + \frac{2|b(\tau)| \|p_z\varphi\|}{2E + b(\tau)^2} \right\}^2 \ . \tag{31}$$

It is easy to see term by term that the r.h.s. of (31) scales invariantly. In [16] we have already exploited this property. In fact, we found that the bound (31) is only considerably below 1 for very small values of the pulse length $\tau$. Since the binding energy has to scale in the same manner as the Hamiltonian $H(\vec{x}, t = 0)$, that is $E \to E' = \lambda^{-2}E$, we could also, due to the scale invariance property, enlarge the pulse durations by considering higher Rydberg states. In this way we could study pulses which are physically more conceivable, at the cost of having to deal with higher principal quantum numbers.

## 2.6 Trading some variables for others

Of course the principle mentioned at the end of the last subsection is very general and we may always trade some variables for others, simply by bringing the relevant $\lambda$'s in (22) to the other side of the equation. For instance from $\mathcal{P}(\lambda^3 E_0, \tau, \alpha) = \mathcal{P}(E_0, \lambda^2\tau, \lambda^{-1}\alpha)$ it follows that instead of varying the field amplitude and keeping $\tau$ and $\alpha$ fixed, we could equivalently keep $E_0$ fixed and vary simultaneously $\tau$ and $\alpha$ in the described fashion. As a consequence we can give some alternative physical interpretation to the extreme intensity limit considered in [17, 12]

$$\lim_{E_0\to\infty} \mathcal{P}(\varphi) = \lim_{\substack{\tau\to\infty \\ \alpha\to 0}} \mathcal{P}(\varphi) \; . \tag{32}$$

This means switching off the potential and exposing the atom to an infinitely long pulse with some finite field amplitude is equivalent to keeping the pulse length and the coupling constant finite and taking the field amplitude to infinity.

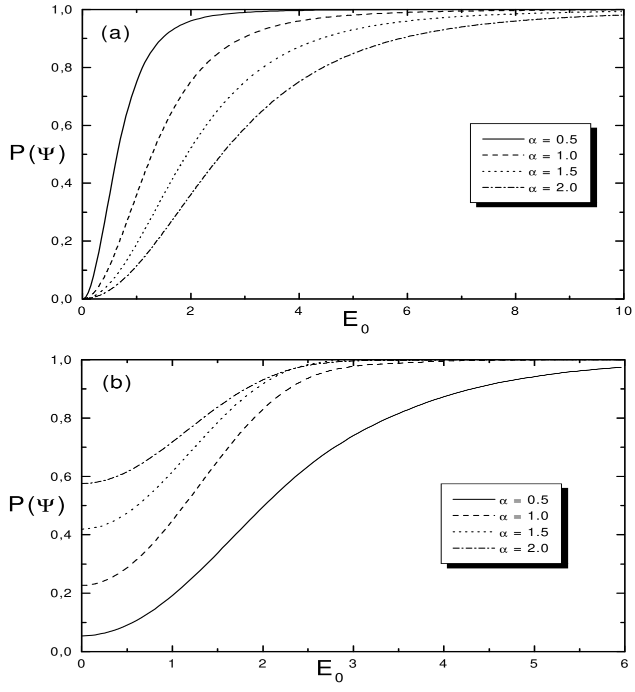


Figure 1: Part (a) shows the ionization probability as a function of the field amplitude $E_0$ for a $\delta$-potential atom subjected to a $\delta$-kick pulse (23) for $\tau = 0.001$ and various coupling constants. Part (b) shows the ionization probability to zeroth order Gordon-Volkov perturbation theory as a function of the field amplitude $E_0$ for a $\delta$-potential atom subjected to a double $\delta$-kick pulse of the form $E(t) = E_0(\delta(t) - 2\delta(t-\tau))$ for $\tau = 1.1$ and various coupling constants. Notice that for this pulse the conditions $b(\tau) = 0$ and $c(\tau) \neq 0$ hold. For a detailed derivation see [12].

We can also use the scale invariance property to give a simple explanation to a behaviour, which at first sight appears somewhat puzzling. In [12, 8] it was observed that the ionization probability is sometimes a decreasing and sometimes an increasing function of the coupling constant when the other parameters are kept fixed, refer to figure 1.

Important for the explanation of this feature is that in the former case $b(\tau) = 0, c(\tau) \neq 0$ and in the latter $b(\tau) \neq 0, c(\tau) = 0$. Assuming now that the dependence of the ionization probability on the field amplitude enters only through the quantities $b(\tau)$ and $c(\tau)$ and in addition that the dependence on the pulse length is very weak in comparison with the one on $b(\tau), c(\tau)$ and $\alpha$, according to the scale invariance property we can write

$$\mathcal{P}(b(\tau), c(\tau), \alpha) \approx \mathcal{P}(\lambda^{-1} b(\tau), \lambda c(\tau), \lambda^{-1}\alpha) \ . \tag{33}$$

Thus, in case the functional dependence on $c(\tau)$ is much weaker than the one on $b(\tau)$, we have to increase the coupling constant when the total classical momentum transfer is increased in order to keep the ionization probability fixed. Noting that $E \sim \alpha^{-2}$, this is expected from the classical point of view, since to free a more deeply bound state with the same probability requires a larger momentum transfer. In the reversed case, in which the functional dependence on $c(\tau)$ is much stronger than the one on $b(\tau)$ we have to decrease the coupling constant when the total classical displacement is increased in order to keep the ionization probability at the same value. Also this behaviour is expected from a classical point of view, since when a less deeply bound state is freed with the same probability, it will be further displaced.

The behaviour in figure 1 is therefore explained by relation (33). Note that in figure 1(b) the value of $\mathcal{P}(E_0 = 0)$, which of course has to be zero, is a measure for the poor quality of the zeroth order Gordon-Volkov perturbation theory, at least in this low intensity regime. Finally it is worth to note that the crossover which takes place for the curves of $\alpha = 1.5$ and $\alpha = 2$ indicates that in fact (33) is not exact and the pulse length has to be scaled also. It is not an indication that the higher order terms need to be taken into account, since, as we discussed in subsection 2.3, scale invariance holds order by order in perturbation theory.

# 3 Existence criteria for stabilization

As a consequence of (21) it is elementary to derive a differential equation which has to be satisfied by the ionization probability

$$\lambda \frac{d\mathcal{P}}{d\lambda} = 2\tau \frac{\partial \mathcal{P}}{\partial \tau} - \alpha \frac{\partial \mathcal{P}}{\partial \alpha} - 3E_0 \frac{\partial \mathcal{P}}{\partial E_0} + \lambda \frac{\partial \mathcal{P}}{\partial \lambda} \ . \tag{34}$$

As an example one may easily convince oneself that (23) indeed satisfies (34).

One way to speak of stabilization is when the ionization probability as a function of the field amplitude satisfies

$$\frac{\partial \mathcal{P}}{\partial E_0} \leq 0 \tag{35}$$

for $E_0 \in [0, \infty)$ on a finite interval. Noting now that the transformation of the length scale is a symmetry for the ionization probability, i.e. relation (13), we have $\partial \mathcal{P}/\partial \lambda =$

$d\mathcal{P}/d\lambda = 0$. Then, according to the differential equation (34), the criterium (35) for the existence of stabilization may be written alternatively as

$$2\tau \frac{\partial \mathcal{P}}{\partial \tau} \leq \alpha \frac{\partial \mathcal{P}}{\partial \alpha} \; . \tag{36}$$

Once again it will be instructive to verify this statement for an explicit example. We believe that hitherto no analytical expression for the ionization probability is known which obeys the strict inequality in (35). However, it was shown [17, 12] that in the extreme intensity limit $E_0 \to \infty$ the equal sign holds. In particular when $b(\tau) = c(\tau) = 0$ one obtains non-trivial expressions in this case. Taking for instance the potential to be the $\delta$-potential in three dimensions, the ionization probability of the only bound state was computed to [17]

$$\mathcal{P}(\varphi) = 1 - \frac{1}{\pi} \left| U\left(\frac{3}{2}, \frac{1}{2}; \frac{i\tau\alpha^2}{2}\right) \right|^2 \quad , \tag{37}$$

with $U$ being the confluent hypergeometric function. Obviously (37) satisfies the criterium (36) for the equal sign.

It is interesting to note that for potentials which do not possess an energy scale, like the ones mentioned after (10), relation (36) reduces to $\partial\mathcal{P}/\partial\tau \leq 0$ for $\tau \in [0, \infty)$ on a finite interval. This means that for increasing pulse length the ionization probability should decrease, which is as counterintuitive as the statement (35).

## 4 Conclusions

We have shown that transforming the length scale corresponds to a symmetry in the ionization probability $\mathcal{P}(\varphi)$. We demonstrated that this symmetry property may be used as a consistency check in various approximation methods in numerical or analytical form. One should also note that every numerical code which fully solves the Schrödinger equation can be tested for consistency by appropriately scaling all variables. Moreover one can employ the scale invariance to avoid certain problems which sometimes plague numerical calculations as for instance the occurrence of very small numbers near machine precision or of very large numbers. By re-scaling all parameters one might be able to avoid such difficulties and still describe exactly the same physical situation.

We have further shown, in section 2.6, that certain types of behaviour for which one has very often intuitive physical explanations may be confirmed by means of scaling arguments.

We like to stress that none of the above considerations is restricted to a particular intensity regime of the pulse in comparison with the potential and they hold for low as well as ultra high intensities, although the latter regime is of course currently of more interest. The above considerations may of course be carried out also for other quantities of interest like ionization rates $\mathcal{I}$, harmonic spectra etc. It follows for instance immediately that the ionization rate has to scale inverse to the time, i.e. $\mathcal{I} \to \mathcal{I} = \lambda^{-\eta_t}\mathcal{I}$. Fermi's golden rule scales for instance in this way.

As an outlook one should keep in mind that like in numerous other situations the physics becomes more interesting when the symmetry is broken. For instance for the two-dimensional delta potential we noted already that there is a priori no energy scale available. However, these potentials suffer from ultraviolet divergencies at the origin which have to be renormalised. Through this procedure one then introduces an additional scale, which is a situation reminiscent of relativistic quantum field theory. Another interesting situation arises when we have more than one intrinsical physical scale in our system. In many situations one scale is dominating the other and the problem is reducible to one with only one parameter. However, there might intriguing situations in which the scales combine in an arbitrary complicated manner as for instance in a statistical physics problem where we have a microscopic length scale which specifies the typical distance between fluctuating magnetic degrees of freedom and the correlation length.

**Acknowledgment:** A.F. and R.S. are grateful to the Deutsche Forschungsgemeinschaft (Sfb288) for partial support. We would like to thank M. Dörr for bringing the existence of the second reference in [7] to our attention.

## References

[1] M. Gavrila and J.Z. Kaminski, *Phys. Rev. Lett.* **52** (1984) 613.

[2] Q. Su, B.P. Irving, C.W. Johnson and J.H. Eberly, *J. of Phys. B: At. Mol. Opt. Phys.* **29** (1996) 5755.

[3] S. Geltman, *Jour. of Phys. B: At. Mol. Opt. Phys.* **32** (1999) 853.

[4] S. Geltman, *Phys. Rev. A* **45** (1992) 5293; *J. of Phys. B: At. Mol. Opt. Phys.* **27** (1994) 257; *J. of Phys. B: At. Mol. Opt. Phys.* **27** (1994) 1497.

[5] T. Mercouris and C.A. Nicolaides; *J. of Phys. B: At. Mol. Opt. Phys.* **32** (1999) 2371.

[6] H.G. Muller, *Proceedings of Super-Intense-Laser-Atom Physics IV* , eds. H.G.Muller and M.V. Fedorov (Kluwer Acad. Publ., Amsterdam, 1996*).*

[7] P. Lambropoulos and X. Tang, *J. Opt. Soc. Am. B* **4** (1987) 821;

L.B. Madsen and P. Lambropoulos, *Phys. Rev. A* **59** (1999) 4574.

[8] M. Pont and R. Shakeshaft, *Phys. Rev. A* 44 (1991) R4110.

[9] W.E. Cooke and R. Shakeshaft, *Phys. Rev.* A **43** (1991) 251.

[10] H.A. Bethe and E.E. Salpeter, "*Quantum Mechanics of One and Two-Electron Atoms*" (Springer, Berlin, 1957).

[11] P.W. Milonni, *Phys. Rev. A* **38** (1988) 2682.

[12] C. Figueira de Morisson Faria, A. Fring and R. Schrader "*Momentum Transfer, Displacement and Stabilization*", in preparation.

[13] A. Fring, V. Kostrykin and R. Schrader, *J. of Phys. B: At. Mol. Opt. Phys.* **29** (1996) 5651.

[14] C. Figueira de Morisson Faria, A. Fring and R. Schrader "*Analytical Treatment of Stabilization*", *Laser Physics* **9** (1999) 379.

[15] E.A. Volkova, A.M. Popov and O.V. Smirnova, *JETP* **79** (1994) 736; *JETP* **82** (1996) 72;
E.A. Volkova, A.M. Popov, O.V. Smirnova and O.V. Tikhonova, *JETP* **84** (1997) 658.

[16] C. Figueira de Morisson Faria, A. Fring and R. Schrader, *J. of Phys. B: At. Mol. Opt. Phys.* **31** (1998) 449.

[17] A. Fring, V. Kostrykin and R. Schrader, *J. of Phys. A: Math. Gen.* **30** (1997) 8599.