Stabilization not for certain and the usefulness of bounds

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Abstract. Stabilization is still a somewhat controversial issue concerning its very existence and also the precise conditions for its occurrence. The key quantity to settle these questions is the ionization probability, for which hitherto no computational method exists which is entirely agreed upon. It is therefore very useful to provide various consistency criteria which have to be satisfied by this quantity, whose discussion is the main objective of this contribution. We show how the scaling behaviour of the space leads to a symmetry in the ionization probability, which can be exploited in the mentioned sense. Furthermore, we discuss how upper and lower bounds may be used for the same purpose. Rather than concentrating on particular analytical expressions we obtained elsewhere for these bounds, we focus in our discussion on the general principles of this method. We illustrate the precise working of this procedure, its advantages, shortcomings and range of applicability. We show that besides constraining possible values for the ionization probability these bounds, like the scaling behaviour, also lead to definite statements concerning the physical outcome. The pulse shape properties which have to be satitisfied for the existence of asymptotical stabilization is the vanishing of the total classical momentum transfer and the total classical displacement and not smoothly switched on and off pulses. Alternatively we support our results by general considerations in the Gordon-Volkov perturbation theory and explicit studies of various pulse shapes and potentials including in particular the Coulomb- and the delta potential.

INTRODUCTION

There is considerable interest in the high intensity regime (intensities larger than $3.5\times \mathrm{Wcm}^{-2}$ for typical frequencies), because since the early nineties it may be realized experimentally. The perturbative description, which was a very successful approach in the low intensity regime, breaks down for such high intensities. Thus, this regime constitutes a new challenge to theorists. Comparing the status of the understanding and clarity of the description of the two regimes one certainly

[1)] To appear in the Proceedings of the ICOMP8 (Monterey (USA), October 1999)

observes a clear mismatch and should probably conclude that the challenge has not been entirely met so far. One also observes a clear imbalance between numerical calculations and analytical descriptions.

In particular, the issue of stabilization has led to several controversies and there are still several recent computations which are in clear contradiction to each other. Since it is not very constructive simply to count the numbers of numerical results which agree and those which do not[2], our investigations aim at analytical descriptions which unravel the physical assumptions and might serve to pinpoint possible errors.

In view of the panel discussion at this meeting the main purpose of this contribution is to summarize our findings [1-6] and in particular explain the working and limitations of our method in the hope to dispel a few misunderstandings and misconceptions which have occurred.

FRAMEWORK AND PHYSICAL PROPERTIES

We start by stating our physical assumptions. We consider an atom with potential $V(\vec{x})$ in the presence of a sufficiently intense laser field, such that it may be described in the non-relativistic regime by the time-dependent Schrödinger equation in the dipole approximation

$$i\frac{\partial\psi(\vec{x},t)}{\partial t} = \left(-\frac{\Delta}{2} + V(\vec{x}) + \vec{x}\cdot\vec{E}(t)\right)\psi(\vec{x},t) = H(\vec{x},t)\,\psi(\vec{x},t). \tag{1}$$

We will use atomic units throughout this article. We take the pulse to be of the general form

$$\vec{E}(t) = \vec{E}_0 f(t) \tag{2}$$

where $f(t)$ is assumed to be a function whose integral over t is well behaved with $f(t) = 0$ unless $0 \leq t \leq \tau$. This means τ constitutes the pulse duration, $f(t)$ the pulse shape function and E_0 the amplitude of the pulse, which we take to be positive without loss of generality.

Important quantities for our discussion are the total classical momentum transfer $\vec{b}(\tau)$, the classical displacement $\vec{c}(\tau)$ and the classical energy transfer $a(\tau)$ defined through the relations

$$\vec{b}(t) = \int_0^t ds\vec{E}(s), \qquad \vec{c}(t) = \int_0^t ds\vec{b}(s), \qquad a(t) = \frac{1}{2}\int_0^t ds\, b^2(s) \ . \tag{3}$$

The quantity of interest, which one aims to compute, is the ionization probability $\mathcal{P}(\varphi)$ defined as

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2) Panel discussion at this meeting.

$$\mathcal{P}(\varphi) = \|(1-P)\,U(\tau,0)\,\varphi\|^2 = 1 - \|PU(\tau,0)\,\varphi\|^2 . \tag{4}$$

Here P denotes the orthogonal projection in the space $L^2(\mathbb{R}^3)$ of square integrable wave functions onto the subspace spanned by the bound states φ of $H(\vec{x}, t=0)$, $\|\cdot\|$ is the usual Hilbert space norm and the time evolution operator is defined by

$$U\left(t,t^{'}\right) \equiv T[\mathrm{Exp}(-i\int_{t'}^{t} H(\vec{x},s)ds)] \;, \tag{5}$$

with T denoting the time ordering. The question one is interested in is: How does $\mathcal{P}(\varphi)$ behave as a function of E_0? In particular is it possible that $\mathcal{P}(\varphi)$ decreases when the field amplitude E_0 increases, in other words does stabilization exist? Quantitatively this means we should find a behaviour of the form

$$\frac{d\mathcal{P}(\varphi)(E_0)}{dE_0} \leq 0 \qquad \text{for} \quad \mathcal{P}(\varphi) \neq 1 \tag{6}$$

with $0 \leq E_0 \leq \infty$ on a finite interval for E_0. We refer to a behaviour in (6) for the equal sign as weak stabilization and for strict inequality we call this strong stabilization.

We stress once more that this description is entirely non-relativistic. The relativistic regime surely poses a new challenge and a full quantum field theoretical treatment is desirable, but it should be possible to settle the question just raised within the framework outlined above, since stabilization is not claimed to be a relativistic effect. In particular it is not clear which consequences on the physics in this regime one expects from a description in the form of the Klein-Gordon equation[3]. Furthermore, appealing to a more formal description[4] in terms of scattering matrices[5] instead of the time evolution operator $U\left(t,t^{'}\right)$ will not shed any new light on the question raised, unless one deals with non-trivial asymptotics.

The time-ordering in (5) poses the main obstacle for the explicit computations of $\mathcal{P}(\varphi)$. To get a handle on the issue, one can first resort to general arguments which provide analytical expressions constraining the outcome. The least such arguments are good for is to serve as consistency checks for results obtained by other means. This is especially useful when one has a controversy as in the case at hand. In addition we will demonstrate that they also allow some definite statements and explain several types of physical behaviour without knowing the exact expression of the quantities which describe them.

CONSTRAINTS FROM SCALING PROPERTIES

More details concerning the arguments of this section may be found in [5]. Denoting by $\lambda > 0$ the dilatation factor and by η the scaling dimension of the eigenfunc-

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3) See contribution to the panel discussion at this meeting by F.H.M. Faisal.

4) See contributions to the panel discussion at this meeting by F.H.M. Faisal and H. Reiss.

5) For pulses of the form (2) the scattering matrix $S = \lim_{t_\pm \to \pm\infty} \exp(it_+ H_+) \cdot U(t_+, t_-) \cdot \exp(-it_- H_-)$ and $U(\tau, 0)$ coincide in the weak sense. (see e.g. [1] for a more detailed discussion)

tion $\varphi(\vec{x}) := \psi(\vec{x}, t=0)$ of the Hamiltonian $H(\vec{x}, t=0)$, we consider the following scale transformations[6]

$$\vec{x} \to \vec{x}' = \lambda \vec{x} \qquad \text{and} \qquad \varphi(\vec{x}) \to \varphi'(\vec{x}') = \lambda^{-\eta} \varphi(\vec{x}) \ . \tag{7}$$

As the only two physical assumptions we now demand that the Hilbert space norm, i.e. $\|\varphi(\vec{x})\| = \|\varphi'(\vec{x}')\|$, remains invariant and that the scaling of the wavefunction is preserved for all times. From the first assumption we deduce immediately that the scaling dimension has to be $\eta = d/2$ with d being the dimension of the space.

The scaling behaviour (7) may usually be realized by scaling the coupling constant. Considering for instance the wavefunction $\varphi(x) = \sqrt{\alpha} \exp(-\alpha|x|)$ of the only bound state when the potential in (1) is taken to be the one-dimensional delta-potential $V(x) = \alpha\delta(x)$, equation (7) imposes that the coupling constant has to scale as $\alpha \to \alpha' = \lambda^{-1}\alpha$. Choosing instead the Coulomb potential in the form $V(\vec{x}) = \alpha/r$ requires the same scaling behaviour of the coupling constant for (7) to be valid. This is exhibited directly by the explicit expressions of the corresponding wavefunctions $\varphi_{nlm}(\vec{x}) \sim \alpha^{3/2}(\alpha r)^l \exp(-\alpha r/n) L_{n+l}^{2l+1}(2\alpha r/n)$.

From the second assumption we conclude

$$\psi(\vec{x}, t) \to \psi'(\vec{x}', t') = U'(t', 0)\varphi'(\vec{x}') = \lambda^{-d/2}\psi(\vec{x}, t) = \lambda^{-d/2} U(t, 0)\varphi(\vec{x}) \ . \tag{8}$$

Consequently this means that the time evolution operator should be an invariant quantity under these transformations

$$U(t_1, t_0) = T\left(e^{-i\int_{t_0}^{t_1} H(\vec{x}, s)ds}\right) \to U'(t_1', t_0') = T\left(e^{-i\int_{\lambda^2 t_0}^{\lambda^2 t_1} H'(\vec{x}, s)ds}\right) = U(t_1, t_0) \ . \tag{9}$$

Equation (9) then suggests that the scaling of the time has to be compensated by the scaling of the Hamiltonian in order to achieve invariance. Scaling therefore the time as

$$t \to t' = \lambda^{\eta_t} t \ , \tag{10}$$

equation (9) only holds if the Stark Hamiltonian of equation (1) scales as

$$H(\vec{x}, t) \to H'(\vec{x}', t') = \lambda^{\eta_H} H(\vec{x}, t) \qquad \text{with} \qquad \eta_H = -\eta_t \ . \tag{11}$$

The properties (10) and (11) could also be obtained by demanding the invariance of the Schrödinger equation (1). The overall scaling behaviour of $H(\vec{x}, t)$ is governed by the scaling of the Laplacian, such that we obtain the further constraint

$$\eta_H = -2 \ . \tag{12}$$

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[6] More formally we could also carry out all our computations by using unitary dilatation operators $U(\lambda)$, such that the transformation of the eigenfunction is described by $U(\lambda)\varphi(\vec{x}) = \lambda^{\eta}\varphi'(\lambda\vec{x})$ and operators $\mathcal{O}$ acting on $\varphi(\vec{x})$ transform as $U(\lambda)\mathcal{O}U(\lambda)^{-1} = \mathcal{O}'$.

As a consequence we can read off the scaling properties of the potential as

$$V\left(\vec{x}\right) \rightarrow V'\left(\vec{x}'\right) = \lambda^{-2} V\left(\vec{x}\right) \quad . \tag{13}$$

Considering for instance the one-dimensional delta-potential and the Coulomb potential in the forms specified above, equation (13) imposes that the coupling constant has to scale as $\alpha \rightarrow \alpha' = \lambda^{-1}\alpha$ in both cases. This behaviour of the coupling constant is in agreement with our earlier observations for the corresponding wave-functions.
We will now discuss the constraint resulting from equation (11) on the scaling behaviour of the pulse. We directly observe that

$$\vec{E}\left(t\right) \rightarrow \vec{E}'\left(t'\right) = \lambda^{-3}\vec{E}\left(t\right) \quad . \tag{14}$$

This equation is not quite as restrictive as the one for the potential, since in the latter case we could determine the behaviour of the coupling whereas now a certain ambiguity remains in the sense that we can only deduce

$$\vec{E}_0 \rightarrow \vec{E}_0' = \lambda^{\eta_{E_0}} \vec{E}_0 \,, \quad f\left(t\right) \rightarrow f'\left(t'\right) = \lambda^{\eta_f} f\left(t\right) \,, \quad \text{with } \eta_{E_0} + \eta_f = -3 \,. \tag{15}$$

Thus, under the assumptions we have made, it is not possible to disentangle the contribution coming from the scaling of the amplitude or the pulse shape function. However, there might be pulse shape functions for which η_f has to be 0, since no suitable parameter, analogously to the coupling constant for the potential, is available in its explicit form to achieve the expected scaling.

Finally, we come to the scaling behaviour of the ionization probability. Noting that the projection operator has to be a scale invariant quantity, i.e. $P \rightarrow P' = P$, we obtain together with (7) and (9) that the ionization probability remains an invariant quantity under the scaling transformation

$$\mathcal{P}\left(\varphi\right) = \left\| \left(1 - P\right) U\left(\tau, 0\right) \varphi \right\|^2 \rightarrow \mathcal{P}'\left(\varphi'\right) = \mathcal{P}\left(\varphi\right) . \tag{16}$$

We have therefore established that transforming the length scale corresponds to a symmetry in the ionization probability $\mathcal{P}\left(\varphi\right)$. This symmetry can be exploited as a consistency check in various approximation methods in numerical or analytical form as outlined in [5]. In this sense the arguments of this section are similar in spirit to those of the next section. Nonetheless, scaling properties may also be used to explain directly certain types of physical behaviour, as for instance the behaviour of $\mathcal{P}\left(\varphi\right)$ as a function of the coupling constant (see [5]).

CONSTRAINTS FROM BOUNDS

In this section we wish to comment on the method of computing bounds which is alternative to computing $\mathcal{P}(\varphi)$ exactly. This means we estimate expressions of the form

$$\|(1-P)U(\tau,0)\varphi\|^2 \leq \mathcal{P}_u(\varphi) \quad \text{and} \quad \|PU(\tau,0)\varphi\|^2 \leq 1-\mathcal{P}_l(\varphi) \tag{17}$$

such that

$$\mathcal{P}_l(\varphi) \leq \mathcal{P}(\varphi) \leq \mathcal{P}_u(\varphi) \ . \tag{18}$$

How does this work? We can not go into all the technical details, but we would like to illustrate the general principle of the computational steps involved. First one should note that from a mathematical point of view there are seldom general principles for deriving such inequalities, except for a few elementary theorems (see e.g. [7]). Therefore the steps in the derivations very often do not always appear entirely compelling. In mathematics, absolute inequalities, i.e. those which hold for all real numbers, are important in analysis especially in connection with techniques to prove convergence or error estimates, and in physics they have turned out to be extremely powerful for instance in proving the stability of matter [8] or to establish properties of the entropy [9].

The basic ingredients which are always exploited are the Minkowski and Hölder inequalities

$$\|\psi+\psi'\| \leq \|\psi\|+\|\psi'\|, \qquad \|\psi\psi'\| \leq \|\psi\|\cdot\|\psi'\|, \tag{19}$$

used in the form

$$\|\psi-\psi'+X-X\| \leq \|\psi-X\|+\|X-\psi'\|, \tag{20}$$

$$\|\psi\, X\, X^{-1}\,\psi'\| \leq \|\psi X\|\cdot\|X^{-1}\psi'\| \ , \tag{21}$$

where ψ and ψ' are meant to be formal objects. The aim and sometimes the art of all considerations is now to choose X such that the loss in accuracy is minimized. One should resort here to as much physical inspiration as possible, for instance if there is a conjecture or a result from other sources which suggests a dynamics one can compare with. There exist also more sophisticated possibilities to estimate the norm, as for instance to relate the Hilbert space norm to different types of norms, e.g. the operator norm[7] or the Hilbert-Schmidt norm[8]

$$\|A\psi\| \leq \|A\|_{op}\|\psi\| \leq \|A\|_{H.S.}\|\psi\| \ . \tag{22}$$

Where do we start? In fact, the starting point is identical to the one of perturbation theory, that is the Du Hamel formula involving the time evolution operator associated to two different Hamiltonians $H_1(t)$ and $H_2(t)$

$$U_1(t,t^{'}) = U_2(t,t^{'}) - i\int_{t'}^{t} ds\; U_1\,(t,s)\,(H_1(s)-H_2(s))\,U_2(s,t^{'}) \quad . \tag{23}$$

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[7] The operator norm is defined as $\|A\|_{op} = \sup_{\varphi:\|\varphi\|=1}\|A\varphi\|$.

[8] Denoting by $\alpha_1 \geq \alpha_2 \geq \ldots$ the positive eigenvalues of the operator $T=(A^*A)^{1/2}$ the Hilbert-Schmidt norm of the operator A is defined as $\|A\|_{H.S.} = (\sum_{n=1}^{\infty}\alpha_n^2)^{1/2}$.

For instance, identifying the Stark Hamiltonian in (1) with $H_1(s)$, one chooses $H_2(s) = -\Delta/2 + \vec{x} \cdot \vec{E}\,(t)$ or $H_2(s) = -\Delta/2 + V\,(\vec{x})$ in the high- or low intensity regime, respectively. Instead of iterating (23) and ending up with a power series in V in the former or a power series in E_0 in the second case one inserts (23) into (17) and commences with the estimation of the norm in the way just outlined. Most conveniently these considerations are carried out in a different gauge, for the high intensity regime in the Kramers-Henneberger gauge.

Where do we stop? The whole procedure may be terminated when one arrives at expressions which may be computed explicitly.

When can we apply bounds? In general in all circumstances. In particular problems occurring in the context of perturbative considerations, like the convergence, are avoided completely. Especially when the strength of the potential and the field are comparable, e.g. in the turn-on and off region, this method is not limited in its applicability, as is for instance the case for the Gordon-Volkov series.

What can we deduce? Ideally $P_l(\varphi)$ and $P_u(\varphi)$ are very close to each other, in which case we are in the position of someone solving the problem numerically with $P_l(\varphi)$ and $P_u(\varphi)$ related to the numerical errors. If the lower bound tends to 1 for an increasing finite realistic value of E_0 there will be little room left for $\mathcal{P}\,(\varphi)$ to decrease and one may deduce that stabilization is absent (see figure 9 in [2]). Furthermore, we can always make statements about the extreme limits. For instance for the extreme frequency limit we obtain

$$\frac{d}{dE_0}\left(\lim_{\omega\to\infty} \mathcal{P}\,(\varphi)\right) = 0 \quad . \tag{24}$$

This relates our discussion to the seminal paper on the stabilization issue by Gavrila and Kaminski [10]. For the extreme field amplitude limit we found

$$\lim_{E_0\to\infty} \mathcal{P}\,(\varphi) = 1 - |\langle\varphi, \psi_{GV}(\tau)\rangle|^2 \quad \text{for } b(\tau) = c(\tau) = 0 \tag{25}$$

$$\lim_{E_0\to\infty} \mathcal{P}\,(\varphi) = 1 \qquad\qquad \text{otherwise} , \tag{26}$$

where $\psi_{GV}(\tau) = U_{GV}(\tau, 0)\varphi$ is the Gordon-Volkov wave function. For the definition of U_{GV} see (39). We would like to stress that this limit is not merely of mathematical interest[9]. The result (25) is a clear indication of weak stabilization, though it is still desirable to find the precise onset of this behaviour. It is also clear that as a consequence of (25) a value of $\mathcal{P}\,(\varphi)$ which is equal or larger than the r.h.s. of (25) for **any finite** and experimentally realisable value of E_0 immediately implies the existence of strong stabilization.

What are the shortcomings? For realistic values of the parameters involved the expressions sometimes yield

$$P_l(\varphi) = 0 \qquad \text{or} \qquad P_u(\varphi) = 1 \tag{27}$$

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[9] See contribution to the panel discussion at this meeting by F.H.M. Faisal.

in which case the constraint is of course not very powerful. In that situation it simply means that we have lost too much accuracy in the derivation for that particular parameter setting. One should note, however, there is no need to give up in that situation since as is evident **the expressions for the bounds are by no means unique**. It should then be quite clear that one can not deduce[10] that **the** bound is useless if one encounters the situation (27). Even more such a conclusion seems very much astray in the light of [1,2,4,6], where we presented numerous examples for which the bounds are well beyond the values in (27). Sometimes this could, however, only be achieved for extremely short pulses. As we pointed out in [2] this can be overcome at the cost of having to deal with higher Rydberg states[11], which is a direct consequence of the scaling behaviour outlined in the previous section.

How do typical expressions look like? In [1] we derived for instance the expression

$$\mathcal{P}_l(\varphi) = 1 - \left\{ \int_0^\tau \|(V(\vec{x} - c(t)e_z) - V(\vec{x}))\varphi\| dt + \right.$$
$$\left. \frac{2}{2E + b(\tau)^2} \|(V(\vec{x} - c(\tau)e_z) - V(\vec{x}))\varphi\| + \frac{2|b(\tau)|}{2E + b(\tau)^2} \|p_z \varphi\| \right\}^2 \tag{28}$$

for a lower bound. For given potentials and pulse shapes terms involved in (28) may be computed at ease. As stated in [1], it is important to pay attention to the fact that (28) is derived for the condition $b(\tau)^2/2 > -E \equiv$ binding energy[12]. Such restrictions which at first emerge as technical requirements in the derivations usually indicate at some physical implications. In this case it points at the different physical situation we encounter when the total momentum transfer is vanishing (see also (25)).

What still needs to be done? Probably it is unrealistic to expect to find a bound which is universally applicable and restrictive at the same time, rather one should optimize the bounds for particular situations. For instance it would be highly desirable to find more powerful bounds for the situations $b(\tau) = 0, c(\tau) \neq 0$ and $b(\tau) = c(\tau) = 0$. For the latter case we expect in hindsight from (25) that the loss in the estimations may be minimized if in (23) we chose to compare the Stark Hamiltonian with the free Hamiltonian $-\Delta/2$ instead of $H = -\Delta/2 + \vec{x} \cdot \vec{E}\,(t)$ as was done in [1].

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10) As was done by J.H. Eberly at the panel discussion at this meeting.

11) This should not lead to the conclusion that bounds in general are exclusively applicable to higher Rydberg states, see contribution to the panel discussion at this meeting by M. Gavrila.

12) During the panel discussion at this meeting J.H. Eberly exhibited a plot of our result for $\mathcal{P}_l(\varphi)$ involving a pulse which did not satisfy this condition. As he confirmed to a question from the audience his pulse satisfied $b(\tau) = 0$. The conclusions drawn by J.H. Eberly concerning the usefulness of bounds based on this plot are therefore meaningless. (See also footnote 9.)

IMPORTANCE OF PULSE SHAPES

From our previous discussion it is evident that the physical outcome differs for different pulse shapes. However, the fact that a pulse is adiabatically switched on or off is not very important, rather the precise values of $b(\tau)$ and $c(\tau)$ are the determining quantities. In particular the case

$$b(\tau) = c(\tau) = 0 \tag{29}$$

is very special, since then asymptotically weak stabilization is certain to exist. An adiabatically switched on or off pulse sometimes satisfies (29), but this condition is by no means identical to it. We found no evidence for stabilization for an adiabatically switched on field when $b(\tau) \neq 0$. To our knowledge the importance of (29) was first pointed out by Grobe and Fedorov [11], using intuitive arguments, who employed a trapezoidal enveloping function with symmetrical turn-on and turn-off time T, which has the nice feature that for T and τ being integer cycles $b(\tau) = c(\tau) = 0$ and for T half τ being integer cycles $b(\tau) = 0$, $c(\tau) \neq 0$. Thereafter, this observation seems to have been widely ignored in the literature since many authors still employ pulses which do not have this property, trading (29) for the condition of an adiabatic smooth turn-on or/and turn-off[13]. For instance a sine-squared switch on and off with T and τ being integer cycles has $b(\tau) = 0$, $c(\tau) \neq 0$, an entire sine-squared envelope for τ being integer cycles satisfies $b(\tau) = 0$, $c(\tau) \neq 0$. Using gaußian envelopes or gaußian switch on and no switch off usually yields $b(\tau) \neq 0$, $c(\tau) \neq 0$. A pulse which has the nice features that it allows a theoretical investigation of all possible cases for the values of $b(\tau)$ and $c(\tau)$ is the triple δ-kick in the form

$$f(t) = \delta(t) + \beta_1 \delta(t - \tau/2) + \beta_2 \delta(t - \tau) \ , \tag{30}$$

which we employed in [6]. This pulse obviously satisfies

$$b(\tau) = E_0(1 + \beta_1 + \beta_2/2) \qquad \text{and} \qquad c(\tau) = E_0(1 + \beta_1/2) \tag{31}$$

such that by tuning the constants β_1, β_2 we may realise any desired value of $b(\tau)$ and $c(\tau)$.

How do real pulses look like[14]? The quantity which is experimentally accessible is the Fourier transform of the pulse (2)

$$\widetilde{E}(\omega) = \int_{-\infty}^{\infty} E(t) e^{i\omega t} dt \ = \sum_{n=0}^{\infty} \alpha_n \omega^n \ . \tag{32}$$

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13) As may be supported by numerous publications, this observation appears not to have become common knowledge as claimed by M. Gavrila in the introduction to the panel discussion at this meeting.

14) We acknowledge that the following argument was initiated, though not agreed upon in this form, by an e-mail communication with H.G. Muller.

with α_n being constants. For finite pulses this quantity coincides with the total momentum transfer for vanishing frequency ω

$$\widetilde{E}(\omega = 0) = \int_{-\infty}^{\infty} E(t)dt = \int_{0}^{\tau} E(t)dt = b(\tau) \; . \tag{33}$$

Provided that $\alpha_0 = b(\tau) = 0$, the Fourier transform of the momentum transfer

$$\widetilde{b}(\omega) = \int_{-\infty}^{\infty} b(t)e^{i\omega t}dt \tag{34}$$

is on the other hand related to the total displacement for vanishing frequency

$$\widetilde{b}(\omega = 0) = \int_{-\infty}^{\infty} b(t)dt = \int_{0}^{\tau} b(t)dt = c(\tau) \tag{35}$$

such that

$$\widetilde{E}(\omega) = b(t)e^{i\omega t}|_{-\infty}^{\infty} - i\omega\widetilde{b}(\omega) \sim -i\omega c(\tau) + \mathcal{O}(\omega^2) \; . \tag{36}$$

This means that when the experimental outcome is

$$\widetilde{E}(\omega) = \alpha_2\omega^2 + \alpha_3\omega^3 + \alpha_4\omega^4 + \ldots \tag{37}$$

the total momentum transfer and the total displacement are zero. Experimentally, the observed fall off is expected to be even stronger [12].

COMPARISON WITH GV-PERTURBATION THEORY

It is instructive to compare our findings with other standard methods as for instance the Gordon-Volkov (GV) perturbation theory. Using now in (23) for H_2 the Hamiltonian just involving the field and the free particle Hamiltonian in the Kramers-Henneberger frame subsequent iteration yields

$$\begin{aligned} U_1(t,t') &= U_{GV}(t,t') - i\int_{t'}^{t} ds\; U_{GV}(t,s)\,VU_{GV}(s,t') \\ &- \int_{t'}^{t} ds \int_{s}^{t} ds'\; U_{GV}(t,s')\,VU_{GV}(s',s)VU_{GV}(s,t') + \ldots \end{aligned} \tag{38}$$

where U_{GV} corresponds to the free-particle evolution operator in the KH frame

$$U_{GV}(t,t') = e^{-ia(t)}e^{-ib(t)z}e^{ic(t)p_z}e^{-i(t-t')\frac{p^2}{2}}e^{-ic(t')p_z}e^{ib(t')z}e^{ia(t')}. \tag{39}$$

As was explained in [4] we may use these expressions together with the Riemann-Lebesgue theorem in order to obtain the extreme frequency and intensity limit, finding (24), (25) and (26). For these arguments to be valid we have to assume that the Gordon-Volkov series makes sense, so in particular we have to assume its convergence.

The latter assumption may be made more rigorous when considering the one-dimensional delta potential $V(x) = -\alpha\,\delta(x)$ which is well known to possess only one bound state. In that case the problem of computing ionization probabilities is reduced to the evaluation of

$$\mathcal{P}(\varphi) = 1 - \left|\langle\varphi, \psi_{GV}(\tau)\rangle + \langle\varphi, \Psi(\tau)\rangle\right|^2 \tag{40}$$

with

$$\langle\varphi, \psi_{GV}(\tau)\rangle = \frac{2}{\pi} e^{-ia(\tau)} \int_{-\infty}^{\infty} dp \frac{\exp\left(-i\tau\alpha^2 \frac{p^2}{2} - ic(\tau)\,\alpha p\right)}{\left(1 + (p + b(\tau)/\alpha)^2\right)(1 + p^2)} \tag{41}$$

$$\langle\varphi, \Psi(\tau)\rangle = ie^{-ia(\tau)} \sqrt{\frac{\alpha^5}{2\pi^3}} \int_0^{\tau} \int_{-\infty}^{\infty} \psi_I(s) \frac{e^{i(c(\tau)-c(s))p} e^{-\frac{i}{2}p^2(\tau-s)} ds dp}{(\alpha^2 + (p + b(\tau))^2)} \,. \tag{42}$$

Here the only unknown is the function $\psi_I(t)$ which can be obtained as a solution of the Volterra equation

$$\psi_I(t) = \int_{-\infty}^{\infty} dp\, \psi_{GV}(p,t) + \alpha\sqrt{\frac{i}{2\pi}} \int_0^t ds \psi_I(s) \frac{e^{i\frac{(c(t)-c(s))^2}{2(t-s)}}}{\sqrt{t-s}} \,. \tag{43}$$

Iteration of this equation is a well controllable procedure and in [6] we found that the series converges for all values of α. The results obtained from the analysis of this equation match the results obtained from bounds.

CONCLUSIONS

The main outcome of our investigations is that the **classical momentum transfer** and **displacement** caused by a laser pulse on an electron are the essential parameters determining the existence of weak asymptotic stabilization. In fact, we obtained evidence for stabilization only for pulses for which these two quantities vanish at the end of the pulse, i.e., with $b(\tau) = 0$ and $c(\tau) = 0$.

Using purely analytical methods, we have shown that, for a wide range of potentials, namely Kato and one- and three-dimensional delta potentials, we always have $\lim_{E_0 \to \infty} \mathcal{P}(\psi) = 1$ unless $b(\tau) = 0$ and $c(\tau) = 0$, in which case the ionization probability tends to the lowest order in GV-perturbation theory, which corresponds simply to the free particle Green's function (39). Furthermore, for infinite frequencies, the high-frequency condition of [10] is a way to obtain $b(t) = 0$ and $c(t) = 0$ for *all* times.

Clearly, smooth pulses in general do not necessarily fullfil the above conditions, and therefore will not provide a mechanism for stabilization, but just prolong the onset of ionization. In fact, we have observed no stabilization for adiabatically switched on and off pulses of several shapes, for which analytic expressions for lower bounds of ionization probabilities lead to **conclusive** statements concerning the existence or absence of stabilization.

Therefore, as an overall conclusion: **Bounds are useful indeed, also in the context of high intensity laser physics!**

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