Nonlinear elliptic systems with Dini continuous coefficients

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0 Introduction

This paper will be concerned with the interior regularity theory of weak solutions to nonlinear elliptic systems of second order in divergence form of the following type:

$$\operatorname{div} A(x,u,Du) = 0 \qquad \text{in } \Omega. \tag{0-1}$$

Here Ω is a bounded domain in $\mathbb{R}^n$, u taking values in $\mathbb{R}^N$, and the coefficients $A(x,\xi,p)$ are in $\operatorname{Hom}(\mathbb{R}^n,\mathbb{R}^N)$.

Under natural hypotheses concerning the regularity and the growth of A with respect to p it has been proved by Giaquinta & Modica [GM] that weak solutions of (0-1) admit Hölder continuous first derivatives outside a **singular set** of Lebesgue measure 0 provided $(1+|p|)^{-1}A(x,\xi,p)$ is Hölder continuous in the variables (x,ξ) uniformly with respect to p. In [DG] the first author and Grotowski gave a simplified (direct) proof of this result avoiding the use of L^p-L^2-estimates for the derivative Du of u. The method of proof also gives the optimal result in one step, i.e. if $(1+|p|)^{-1}A(x,\xi,p)$ is of class $C^{0,\alpha}$ for some $0<\alpha<1$ in (x,ξ) then u is of class $C^{1,\alpha}$ outside the singular set. The essential new feature is the use of the **harmonic approximation lemma** (cf. [DS, Lemma 3.3]; see also Lemma 4.1).

The aim of this paper is to weaken the assumptions on A with respect to continuity in the variables (x,ξ) and to show a partial regularity result with optimal estimates for the modulus of continuity for the derivative Du under this weaker assumption. To be more precise, we assume for the continuity of A with respect to the variables (x,ξ) that

$$(1+|p|)^{-1}|A(x,\xi,p)-A(\tilde{x},\tilde{\xi},p)| \le \kappa(|\xi|)\mu(|x-\tilde{x}|+|\xi-\tilde{\xi}|)\,, \tag{0-2}$$

for all x, $\tilde{x}\in\Omega$, ξ, $\tilde{\xi}\in\mathbb{R}^N$, $p\in\operatorname{Hom}(\mathbb{R}^n,\mathbb{R}^N)$, where $\kappa\colon[0,\infty)\to[1,\infty)$ is nondecreasing, and $\mu\colon(0,\infty)\to[0,\infty)$ is nondecreasing and concave with $\mu(0+)=0$. We also have to require that $r\mapsto r^{-\alpha}\mu(r)$ is nonincreasing for some $0<\alpha<1$, and that

$$\mathsf{M}(r) = \int_0^r \frac{\mu(\rho)}{\rho}\,d\rho < \infty \quad \text{for some } r>0. \tag{0-3}$$

Then our **main result** can roughly be stated as follows: *Assume additionally to the standard hypotheses (i.e. conditions concerning the ellipticity of A, the regularity and the growth of A with respect to the p variable) that $(1+|p|)^{-1}A(x,\xi,p)$ satisfies (0-2) and (0-3). Let $u\in H^{1,2}(\Omega,\mathbb{R}^N)$ be a weak solution to (0-1). Then u is of class C^1 outside a (relatively) closed singular set $\operatorname{Sing}(u)\subset\Omega$ of Lebesgue measure 0. Moreover, for $x_0\in\Omega\setminus\operatorname{Sing}(u)$ the derivative Du of u has the modulus of continuity $r\mapsto\mathsf{M}(r)$ in a neighborhood of x_0.*

Our result result is optimal in the sense that in the case $\mu(r)=r^\alpha$, $0<\alpha<1$, we have $\mathsf{M}(r)=\alpha^{-1}r^\alpha$ and $C^{1,\alpha}$ regularity is known to optimal in that case.

A condition like (0-3) seems to be natural in elliptic regularity theory with Dini continuous coefficients. It is called **Dini condition** in the literature, although Dini himself [Di] used a slightly weaker condition a century ago. It had some significance for the theory of linear elliptic partial differential equations in the first half of the century, cf. [HW]. Condition (0-3) (in its reformulation (η3); see Section 2) also plays a crucial role in the regularity theory for *almost minimizers* to elliptic variational problems, a notion which was first introduced by Almgren [Alm] (see also [Bo], [Ta], [DS], and [DGG]). In [DS] the first author and Steffen prove an optimal boundary regularity result for almost minimizing rectifiable currents of general elliptic integrands. The assumptions on the regularity of the prescribed boundary and the function appearing in the definition of an almost minimizing current are similar to those imposed above for the modulus of continuity μ. Finally, in a recent paper, Kovats [Ko] studies solutions of fully nonlinear elliptic equations of the form $F(D^2u) = f$. It is shown that solutions are of class C^2 with modulus of continuity $r \mapsto \mathsf{M}(r)$ for the second derivative D^2u provided f is continuous with modulus of continuity μ satisfying (0-3).

It is our impression that the generalization of earlier partial regularity proofs to the case of Dini continuous coefficients should involve technical difficulties in handling the higher integrability of the derivatives of weak solutions to (0-1). Moreover, the optimal modulus of continuity of Du outside the singular can not be obtained by the use of this method in one fairly elementary step.

Our methods also allow us to handle the case of weak solutions to the inhomogeneous system, i.e.

$$-\operatorname{div} A(x,u,Du) = f(x,u,Du) \quad \text{in } \Omega.$$

The reasoning is exactly the same as in the case of homogeneous systems. For this reason we restrict to the homogeneous case for brevity and state the result for the inhomogeneous case without proof.

1 Hypotheses and statement of results

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We consider second-order elliptic equations of divergence form, i.e.

$$\operatorname{div} A(x,u,Du) = 0 \tag{1-1}$$

in Ω, where $A : \Omega \times \mathbb{R}^n \times \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N) \to \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N)$.

We assume the following structure-conditions on A. The major novel feature is the assumption (A3).

(A1) $A(x,\xi,p)$ is differentiable in p with bounded and continuous derivatives

$$\Big|\frac{\partial A}{\partial p}(x,\xi,p)\Big| \le L \quad \text{for all } (x,\xi,p) \in \Omega \times \mathbb{R}^N \times \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N);$$

(A2) A is **uniformly strongly elliptic**, i.e. for some $\lambda > 0$ we have

$$\Big(\frac{\partial A}{\partial p}(x,\xi,p)\nu\Big)\nu \ge \lambda|\nu|^2 \quad \text{for all } x \in \Omega,\ \xi \in \mathbb{R}^N,\ p,\nu \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N);$$

(A3) There exists a **modulus of continuity** $\mu\colon (0,\infty) \to [0,\infty)$, and a nondecreasing function $\kappa : [0,\infty) \to [1,\infty)$ such that

$$|A(x,\xi,p) - A(\tilde{x},\tilde{\xi},p)| \le \kappa(|\xi|)\mu(|x-\tilde{x}| + |\xi - \tilde{\xi}|)(1+|p|)$$

for all $x, \tilde{x} \in \Omega$, $\xi, \tilde{\xi} \in \mathbb{R}^N$, and $p \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N)$.

Without loss of generality we can assume that

(μ1) μ is nondecreasing with $\mu(0+) = 0$;

(μ2) μ is concave; in the proof of the regularity theorem we have to require that $r \mapsto r^{-\alpha}\mu(r)$ is nonincreasing for some exponent $\alpha \in (0,1)$.

We also require **Dini's condition** which was already mentioned in the introduction.

(μ3) $\mathsf{M}(r) := \int_0^r \frac{\mu(\rho)}{\rho}\, d\rho < \infty$ for some $r > 0$.

We will prove the following theorem concerning **weak solutions** of (1-1), i.e. $u \in H^{1,2}(\Omega, \mathbb{R}^N)$ satisfying

$$\int_\Omega A(x,u,Du) \cdot D\varphi \, dx = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega, \mathbb{R}^N). \tag{1-2}$$

Theorem 1.1 *On $\Omega \times \mathbb{R}^N \times \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ consider coefficients $A(x,\xi,p)$ satisfying (A1)–(A3), (μ1)–(μ3) and let $u \in H^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution of (1-2). Then there exists a relatively closed set $\mathrm{Sing}\, u \subset \Omega$ such that $u \in C^1(\Omega \setminus \mathrm{Sing}\, u, \mathbb{R}^N)$. Furthermore $\mathrm{Sing}\, u \subset \Sigma_1 \cup \Sigma_2$ where*

$$\begin{aligned} \Sigma_1 &= \{x_0 \in \Omega \ : \ \liminf_{\rho \to 0^+} \fint_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \, dx > 0\}, \quad \textit{and} \\ \Sigma_2 &= \{x_0 \in \Omega \ : \ \sup_{\rho > 0} \Big(|u_{x_0,\rho}| + |(Du)_{x_0,\rho}| \Big) = \infty\}, \end{aligned}$$

and in particular, $\mathcal{L}^n(\mathrm{Sing}\, u) = 0$. In addition, for $\beta \in [\alpha, 1)$ and $x_0 \in \Omega \backslash \mathrm{Sing}\, u$ the derivative Du has the modulus of continuity $r \mapsto r^\beta + \mathsf{M}(r)$ in a neighborhood of x_0.

As stated in the introduction, the same methods allow us to handle also the regularity problem for weak solutions to **inhomogeneous** nonlinear elliptic systems, i.e. $u \in H^{1,2}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ satisfying

$$\int_\Omega A(x,u,Du) \cdot D\varphi \, dx = \int_\Omega f(x,u,Du) \cdot \varphi \, dx \tag{1-3}$$

for all test functions $\varphi \in C_c^\infty(\Omega, \mathbb{R}^N)$, where $f : \Omega \times \mathbb{R}^N \times \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N) \to \mathbb{R}^N$ satisfies the **natural growth condition**:

(f) There exist constants a and b, with a possibly depending on $M > 0$, such that

$$|f(x,\xi,p)| \leq a(M)|p|^2 + b$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^N$ with $|\xi| \leq M$, and $p \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$.

We state the theorem without giving the proof.

Theorem 1.2 *Let $u \in H^{1,2}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ be a weak solution to (1-3) under the structure conditions (A1)–(A3), (μ1)–(μ3), (f), satisfying $||u||_{L^\infty} \leq M$ and $2a(M)M < \lambda$. Then there is a relatively closed set $\mathrm{Sing}\, u \subset \Omega$, such that $u \in C^1(\Omega \setminus \mathrm{Sing}\, u, \mathbb{R}^N)$. Further $\mathrm{Sing}\, u \subset \Sigma_1 \cup \Sigma_2$, where*

$$\begin{aligned} \Sigma_1 &= \{x_0 \in \Omega \ : \ \liminf_{\rho \to 0^+} \fint_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \, dx > 0\}, \quad \textit{and} \\ \Sigma_2 &= \{x_0 \in \Omega \ : \ \sup_{\rho > 0} |(Du)_{x_0,\rho}| = \infty\}, \end{aligned}$$

and in particular, $\mathcal{L}^n(\mathrm{Sing}\, u) = 0$. In addition, for $\beta \in [\alpha, 1)$ and $x_0 \in \Omega \backslash \mathrm{Sing}\, u$ the derivative of u has modulus of continuity $r \mapsto r^\beta + \mathsf{M}(r)$ in a neighborhood of x_0.

2 Some preliminaries

For technical convenience we want to work with

$$\eta(t) := \mu(\sqrt{2t})^2\,.$$

Using η instead of μ, the continuity condition for $(x,\xi)\mapsto A(x,\xi,p)$, i.e (A3), becomes

$$|A(x,\xi,p)-A(\tilde{x},\tilde{\xi},p)| \le \kappa(|\xi|)\,\sqrt{\eta}(|x-\tilde{x}|^2+|\xi-\tilde{\xi}|^2)(1+|p|)\,. \tag{2-1}$$

Using (μ1)–(μ3) one readily verifies by elementary considerations:

(η1) η is continuous, nondecreasing, and $\eta(0)=0$;

(η2) η is concave, and $t\mapsto t^{-\alpha}\eta(t)$ is nonincreasing for some exponent $\alpha\in(0,1)$;

(η3) $\mathsf{H}(t) := 4\mathsf{M}(\sqrt{2t})^2 = \Big[\int_0^t \frac{\sqrt{\eta(\tau)}}{\tau}\,d\tau\Big]^2 < \infty$ for some $t>0$.

Changing κ by a constant, but keeping $\kappa\ge 1$, we can also assume that

(η4) $\eta(1)=1$, implying $\eta(t)\ge t$ for $t\in[0,1]$.

¿From (η2) we infer for $\theta\in(0,1)$, $t>0$, $i\in\mathbb{N}\cup\{0\}$,

$$\frac{2}{\alpha}(1-\theta^\alpha)\sqrt{\eta(\theta^{2i}t)} = \frac{\sqrt{\eta(\theta^{2i}t)}}{(\theta^{2i}t)^{\alpha/2}}\int_{\theta^{2(i+1)}t}^{\theta^{2i}t}\tau^{\alpha/2-1}\,d\tau \le \int_{\theta^{2(i+1)}t}^{\theta^{2i}t}\frac{\sqrt{\eta(\tau)}}{\tau}\,d\tau\,,$$

which implies

$$\sum_{i=0}^{\infty}\sqrt{\eta(\theta^{2i}t)} \le \frac{\alpha}{2(1-\theta^\alpha)}\sqrt{\mathsf{H}(t)}. \tag{2-2}$$

This yields in particular that $\eta(t)\le\frac{\alpha^2}{4}\mathsf{H}(t)$ for all $t\ge 0$. Moreover, $t\mapsto t^{-\alpha}\mathsf{H}(t)$ is nonincreasing.

¿From the fact that η is nondecreasing we deduce $s\eta(t)\le s\eta(s)$ for all $0\le t\le s$. We also note that $s\eta(t)\le t\eta(s)\le t$ for $0\le s\le t$, $0\le s\le 1$, by the nonincreasing property of $r\mapsto\eta(r)/r$ and $\eta(1)\le 1$. Combining both cases we obtain

$$s\eta(t)\le s\eta(s)+t \quad \text{for } s\in[0,1],\ t>0, \tag{2-3}$$

and similarly, replacing η by $\sqrt{\eta}$,

$$s\sqrt{\eta(t)}\le s\sqrt{\eta(s)}+t \quad \text{for } s\in[0,1],\ t\ge 0. \tag{2-4}$$

Using (A1) and (A2) we easily deduce

$$|A(x,\xi,p)-A(x,\xi,\tilde{p})|\le L|p-\tilde{p}|; \tag{2-5}$$

$$(A(x,\xi,p)-A(x,\xi,\tilde{p}))\cdot(p-\tilde{p})\ge\lambda|p-\tilde{p}|^2 \tag{2-6}$$

for $x\in\Omega$, $\xi\in\mathbb{R}^N$, and p, $\tilde{p}\in\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^N)$.

¿From (A3) we infer the existence of a modulus of continuity $\omega\colon[0,\infty)\times[0,\infty)\to[0,\infty)$ with $\omega(t,0)=0$ for all t such that $t\mapsto\omega(t,s)$ is nondecreasing for fixed s, $s\mapsto\omega(t,s)^2$ is concave and nondecreasing for fixed t, and such that for all (x,ξ,p), $(\tilde{x},\tilde{\xi},\tilde{p})$ in $\Omega\times\mathbb{R}^N\times\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^N)$ with $|\xi|+|p|\le M$ we have

$$\Big|\frac{\partial A}{\partial p}(x,\xi,p)-\frac{\partial A}{\partial p}(\tilde{x},\tilde{\xi},\tilde{p})\Big|\le\omega\Big(M,|x-\tilde{x}|^2+|\xi-\tilde{\xi}|^2+|p-\tilde{p}|^2\Big). \tag{2-7}$$

Concavity of $\omega(t,\,\cdot\,)^2$ can be shown analogously to $(\eta 2)$.
Finally, we note that in the weak formulation (1-2) of our system any $\varphi\in H_0^{1,2}(\Omega,\mathbb{R}^N)$ is admissible as test function. To show this we assume that $0\in\Omega$ and let $R:=\operatorname{diam}\Omega$. Then, for φ smooth we use (2-5), (A3), and the inequality $\mu(ts)\le t\mu(s)$ for $t\ge 1$, to deduce

$$\begin{aligned}
\Big|\int_\Omega A(x,u,Du)\cdot D\varphi\,dx\Big| &\le \int_\Omega |A(x,u,Du)-A(x,u,0)|\,|D\varphi|\,dx\\
&\quad+\int_\Omega |A(x,u,0)-A(0,0,0)|\,|D\varphi|\,dx+\int_\Omega |A(0,0,0)|\,|D\varphi|\,dx\\
&\le L\int_\Omega |Du|\,|D\varphi|\,dx+\kappa(0)\mu(R)\int_\Omega(1+R^{-1}|u|)|D\varphi|\,dx+|A(0,0,0)|\int_\Omega|D\varphi|\,dx\\
&< \infty.
\end{aligned}\tag{2-8}$$

The general case of $\varphi\in H_0^{1,2}(\Omega,\mathbb{R}^N)$ follows by approximation.

3 A Caccioppoli inequality

A key ingredient in the local regularity theory of weak solutions to nonlinear elliptic systems of type (1.2) is a Caccioppoli (or reverse Poincaré) inequality. We prove a version which is adapted to our situation.
For $s,t\ge 0$ let

$$\rho_1(s,t):=(1+t)^{-1}\kappa(s+t)^{-1}\,,\quad K_1(s,t):=(1+t)^4\kappa(s+t)^4.$$

Note that $\rho_1\le 1$ and that $s\mapsto\rho_1(s,t)$, $t\mapsto\rho_1(s,t)$ are nonincreasing functions.

Lemma 3.1 *Consider $\nu\in\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^N)$ and $\xi\in\mathbb{R}^N$ fixed. Let $u\in H^{1,2}(\Omega,\mathbb{R}^N)$ be a weak solution to (1-2) under the structure conditions (A1)–(A3), $(\eta 1)$, $(\eta 2)$, $(\eta 4)$. Then, for any $x_0\in\Omega$ and $\rho\le\rho_1(|\xi|,|\nu|)$ such that $B_\rho(x_0)\subset\subset\Omega$, the inequality*

$$\int_{B_{\rho/2}(x_0)}|Du-\nu|^2\,dx\le\frac{c_1}{\rho^2}\int_{B_\rho(x_0)}|u-\xi-\nu(x-x_0)|^2\,dx+c_2\alpha_n\rho^nK_1(|\xi|,|\nu|)\eta(\rho^2).$$

holds. Here $c_1=c_1(\lambda,L)$, $c_2=c_2(\lambda)$.

Proof. We let $v=u-\xi-\nu(x-x_0)$, and ψ be a cut-off function $\psi\in C_c^\infty(B_\rho(x_0))$ satisfying $0\le\psi\le 1$, $|\nabla\psi|<\frac{4}{\rho}$, $\psi\equiv 1$ on $B_{\rho/2}(x_0)$. Then $\varphi=\psi^2v$ is admissible as a test-function in (1-2), and we obtain

$$\int_{B_\rho(x_0)}A(x,u,Du)\cdot(Du-\nu)\psi^2\,dx=-2\int_{B_\rho(x_0)}A(x,u,Du)\cdot(v\otimes\psi\nabla\psi)\,dx.$$

Adding this to the equations

$$\begin{aligned}
&-\int_{B_\rho(x_0)}A(x,u,\nu)\cdot(Du-\nu)\psi^2\,dx\\
&= 2\int_{B_\rho(x_0)}A(x,u,\nu)\cdot(v\otimes\psi\nabla\psi)\,dx-\int_{B_\rho(x_0)}A(x,u,\nu)\cdot D\varphi\,dx
\end{aligned}$$

and

$$0 = \int_{B_\rho(x_0)} A(x_0, \xi, \nu) \cdot D\varphi \, dx,$$

we deduce

$$\begin{aligned}
&\int_{B_\rho(x_0)} (A(x,u,Du) - A(x,u,\nu)) \cdot (Du - \nu)\psi^2 dx \\
&= -2\int_{B_\rho(x_0)} (A(x,u,Du) - A(x,u,\nu)) \cdot (v \otimes \psi\nabla\psi)\, dx \\
&\quad - \int_{B_\rho(x_0)} (A(x,u,\nu) - A(x,\xi+\nu(x-x_0),\nu)) \cdot D\varphi \, dx \\
&\quad - \int_{B_\rho(x_0)} (A(x,\xi+\nu(x-x_0),\nu) - A(x_0,\xi,\nu)) \cdot D\varphi \, dx \\
&\le I + II + III + IV \,.
\end{aligned}$$

The terms I, II, III, IV are defined below and obtained by using the hypotheses (A1) and (A3), i.e. (2-5), and the modulus of continuity for $(x,\xi) \mapsto A(x,\xi,p)$ from (2-1). Using Young's inequality we obtain, for $\varepsilon > 0$ to be chosen later,

$$\begin{aligned}
I &= 2L\int_{B_\rho(x_0)} |Du-\nu|\,|v|\,|\nabla\psi|\,\psi\, dx \\
&\overset{\text{Young}}{\le} \varepsilon\int_{B_\rho(x_0)} |Du-\nu|^2\psi^2\, dx + \frac{16L^2}{\varepsilon\rho^2}\int_{B_\rho(x_0)} |v|^2\, dx \,.
\end{aligned}$$

In order to estimate II we use Jensen's inequality, (2-3), and the fact that $\eta(ts) \le t\eta(s)$ for $t \ge 1$, and deduce

$$\begin{aligned}
II &= (1+|\nu|)\,\kappa(|\xi|+|\nu|)\int_{B_\rho(x_0)} \sqrt{\eta}(|v|^2)|Du-\nu|\psi^2\, dx \\
&\le \varepsilon\int_{B_\rho(x_0)} |Du-\nu|^2\psi^2\, dx \\
&\quad + \frac{1}{\varepsilon}(1+|\nu|)^2\kappa(|\xi|+|\nu|)^2\alpha_n\rho^n \fint_{B_\rho(x_0)} \eta(|v|^2)\, dx \\
&\le \varepsilon\int_{B_\rho(x_0)} |Du-\nu|^2\psi^2\, dx \\
&\quad + \frac{\alpha_n\rho^{n-2}}{\varepsilon}(1+|\nu|)^2\kappa(|\xi|+|\nu|)^2\rho^2\eta\Big(\fint_{B_\rho(x_0)} |v|^2\, dx\Big) \\
&\le \varepsilon\int_{B_\rho(x_0)} |Du-\nu|^2\psi^2\, dx + \frac{1}{\varepsilon\rho^2}\int_{B_\rho(x_0)} |v|^2\, dx \\
&\quad + \frac{\alpha_n\rho^n}{\varepsilon}(1+|\nu|)^2\kappa(|\xi|+|\nu|)^2\eta\Big((1+|\nu|)^2\kappa(|\xi|+|\nu|)^2\rho^2\Big) \\
&\le \varepsilon\int_{B_\rho(x_0)} |Du-\nu|^2\psi^2\, dx + \frac{1}{\varepsilon}\rho^{-2}\int_{B_\rho(x_0)} |v|^2\, dx \\
&\quad + \frac{\alpha_n\rho^n}{\varepsilon}(1+|\nu|)^4\kappa(|\xi|+|\nu|)^4\eta(\rho^2)\,.
\end{aligned}$$

Note that the application of (2-3) in the second last inequality is possible by our choice $\rho \le \rho(|\xi|,|\nu|)$. In the same way we derive

$$III = 2(1+|\nu|)\,\kappa(|\xi|+|\nu|)\int_{B_\rho(x_0)} \sqrt{\eta}(|v|^2)|v|\,|\nabla\psi|\,\psi\, dx$$

$$
\begin{aligned}
&\leq 8(1+|\nu|)\,\kappa(|\xi|+|\nu|)\int_{B_\rho(x_0)}\sqrt{\eta}(|v|^2)\frac{|v|}{\rho}\,dx\\
&\leq \frac{16}{\rho^2}\int_{B_\rho(x_0)}|v|^2\,dx+(1+|\nu|)^2\kappa(|\xi|+|\nu|)^2\alpha_n\rho^n ⨍_{B_\rho(x_0)}\eta(|v|^2)\,dx\\
&\leq \frac{17}{\rho^2}\int_{B_\rho(x_0)}|v|^2\,dx+\alpha_n\rho^n(1+|\nu|)^4\kappa(|\xi|+|\nu|)^4\eta(\rho^2)
\end{aligned}
$$

and

$$
\begin{aligned}
IV &= (1+|\nu|)\,\kappa(|\xi|+|\nu|)\int_{B_\rho(x_0)}\sqrt{\eta}\Big(|x-x_0|^2+|\nu(x-x_0)|^2\Big)\\
&\qquad\cdot\Big(\psi^2|Du-\nu|+2\psi\,|v|\,|\nabla\psi|\Big)\,dx\\
&\leq (1+|\nu|)\,\kappa(|\xi|+|\nu|)\int_{B_\rho(x_0)}\sqrt{\eta}\Big(\rho^2(1+|\nu|)^2\Big)\Big(\psi|Du-\nu|+\frac{8}{\rho}|v|\Big)\,dx\\
&\leq \varepsilon\int_{B_\rho(x_0)}|Du-\nu|^2\psi^2\,dx+\frac{32\varepsilon}{\rho^2}\int_{B_\rho(x_0)}|v|^2\,dx\\
&\qquad+\frac{\alpha_n\rho^n}{\varepsilon}(1+|\nu|)^2\kappa(|\xi|+|\nu|)^2\eta\Big(\rho^2(1+|\nu|)^2\Big)\\
&\leq \varepsilon\int_{B_\rho(x_0)}|Du-\nu|^2\psi^2\,dx+\frac{32\varepsilon}{\rho^2}\int_{B_\rho(x_0)}|v|^2\,dx\\
&\qquad+\frac{\alpha_n\rho^n}{\varepsilon}(1+|\nu|)^4\kappa(|\xi|+|\nu|)^4\eta(\rho^2).
\end{aligned}
$$

Here we have used $\kappa\geq 1$ in the last inequality.

Combining these estimates and applying (A2) to the left hand side we finally arrive at the following estimate

$$
\begin{aligned}
&(\lambda-3\varepsilon)\int_{B_\rho(x_0)}|Du-\nu|^2\psi^2\,dx\\
&\leq \Big(\frac{32L^2+1}{\varepsilon}+17+16\varepsilon\Big)\frac{1}{\rho^2}\int_{B_\rho(x_0)}|v|^2\,dx+\Big(\frac{2}{\varepsilon}+1\Big)\alpha_n\rho^nK_1(|\xi|,|\nu|)\eta(\rho^2)\,.
\end{aligned}
$$

Choosing $\varepsilon=\lambda/6$ we obtain the desired inequality. ■

4 Approximatively Γ–harmonic functions

A function $h\in H^{1,2}(\Omega,\mathbb{R}^n)$ is called Γ–harmonic for $\Gamma\in\mathrm{Bil}(\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^N))$ if

$$
\int_\Omega\Gamma(Dh,D\varphi)\,dx=0\text{ for all }\varphi\in C_c^1(B_\rho(x_0),\mathbb{R}^N).
$$

The core of our prove is the Γ–**harmonic approximation technique** adapted from [Si], [DS], [DG]. We cite the corresponding lemma from [DS; Lemma 3.3]:

Lemma 4.1 *Consider fixed positive λ and L, and $n,N\in\mathbb{N}$ with $n\geq 2$. Then for any given $\varepsilon>0$ there exists $\delta=\delta(n,N,\lambda,L,\varepsilon)\in(0,1]$ with the following property: for any $\Gamma\in\mathrm{Bil}(\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^N))$ satisfying $\Gamma(\xi,\xi)\geq\lambda|\xi|^2$ for all $\xi\in\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^N)$ and $\Gamma(\xi,\tilde\xi)\leq L|\xi|\,|\tilde\xi|$ for all $\xi,\tilde\xi\in\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^N)$, for any $g\in H^{1,2}(B_\rho(x_0),\mathbb{R}^N)$ (for some $\rho>0$, $x_0\in\mathbb{R}^n$) satisfying*

$$
\begin{aligned}
⨍_{B_\rho(x_0)}|Dg|^2\,dx &\leq 1\ \textit{and}\\
\Big|⨍_{B_\rho(x_0)}\Gamma(Dg,D\varphi)\,dx\Big| &\leq \delta\sup_{B_\rho(x_0)}|D\varphi|\ \textit{for all}\ \varphi\in C_c^1(B_\rho(x_0),\mathbb{R}^N)
\end{aligned}
$$

there exists an Γ*–harmonic function* $h \in H^{1,2}(B_\rho(x_0), \mathbb{R}^N)$ *such that*

$$\fint_{B_\rho(x_0)} |Dh|^2\,dx \le 1 \quad \textit{and} \quad \rho^{-2} \fint_{B_\rho(x_0)} |g-h|^2\,dx \le \varepsilon\,.$$

In Chapter 5 we want to apply this Lemma to suitably rescaled solutions of (1-2). In the next lemma we abbreviate for $u \in H^{1,2}(\Omega, \mathbb{R}^N)$, $B_\rho(x_0) \subset\subset \Omega$, and $\nu \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$:

$$\Phi(x_0, \rho, \nu) \;:=\; \fint_{B_\rho(x_0)} |Du-\nu|^2\,dx,$$

Then we have

Lemma 4.2 *Suppose* $u \in H^{1,2}(\Omega, \mathbb{R}^N)$ *is a weak solution to (1-2). Moreover assume that the structure conditions (A3), (η1), (η2), and (η4) are satisfied. Then, with a constant* $c_3 = c_3(n) \ge 1$, *for any* $B_\rho(x_0) \subset\subset \Omega$ *with* $\rho \le \rho_1(|\xi|, |\nu|)$, $\nu \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$, *and* $\xi = u_{x_0,\rho}$ *the inequality*

$$\begin{aligned}\Big|\fint_{B_\rho(x_0)} \frac{\partial A}{\partial p}(x_0, \xi, \nu)(Du-\nu)D\varphi\,dx\Big| &\le c_3\Big\{\Phi(x_0,\rho,\nu) \\ &+ \omega\Big(|\xi|+|\nu|, \Phi(x_0,\rho,\nu)\Big)\Phi(x_0,\rho,\nu)^{1/2} + K_1(|\xi|,|\nu|)\sqrt{\eta}(\rho^2)\Big\} \sup_{B_\rho(x_0)} |D\varphi|\end{aligned}$$

holds for all functions $\varphi \in C^1_c(B_\rho(x_0), \mathbb{R}^N)$.

Proof. We may assume $\sup_{B_\rho(x_0)} |D\varphi| \le 1$. Using (1-2) and the fact that $\int_{B_\rho(x_0)} A(x_0,\xi,\nu)\cdot D\varphi\,dx = 0$ we deduce

$$\fint_{B_\rho(x_0)} \frac{\partial A}{\partial p}(x_0,\xi,\nu)(Du-\nu)D\varphi\,dx = I + II + III$$

where

$$\begin{aligned} I &= \fint_{B_\rho(x_0)} \Big[\int_0^1 \Big(\frac{\partial A}{\partial p}(x_0,\xi,\nu) - \frac{\partial A}{\partial p}(x_0,\xi,\nu+t(Du-\nu))\Big)(Du-\nu)\cdot D\varphi\Big]\,dt\,dx\,, \\ II &= \fint_{B_\rho(x_0)} \Big(A(x_0,\xi,Du) - A(x,\xi+\nu(x-x_0),Du)\Big)\cdot D\varphi\,dx\,, \\ III &= \fint_{B_\rho(x_0)} \Big(A(x,\xi+\nu(x-x_0),Du) - A(x,u,Du)\Big)\cdot D\varphi\,dx\,. \end{aligned}$$

We estimate I using the modulus of continuity $\omega(\cdot,\cdot)$ for $(x,\xi) \mapsto \frac{\partial A}{\partial p}(x,\xi,p)$ from (2-7), the fact that $t \mapsto \omega^2(s,t)$ is concave, and Jensen's inequality

$$\begin{aligned} I &\le \fint_{B_\rho(x_0)} \omega(|\xi|+|\nu|, |Du-\nu|^2)|Du-\nu|\,dx \\ &\le \Big(\fint_{B_\rho(x_0)} \omega^2(|\xi|+|\nu|, |Du-\nu|^2)\,dx\Big)^{1/2}\Big(\fint_{B_\rho(x_0)} |Du-\nu|^2\,dx\Big)^{1/2} \\ &\le \omega\Big(|\xi|+|\nu|, \Phi(x_0,\rho,\nu)\Big)\Phi(x_0,\rho,\nu)^{1/2}\,. \end{aligned}$$

In the same way, using the modulus of continuity for $(x,\xi) \mapsto A(x,\xi,p)$ from (2-1) and the fact that $\eta(ts) \le t\eta(s)$ for $t \ge 1$, we deduce

$$II \;\le\; \kappa(|\xi|+|\nu|)\sqrt{\eta}(\rho^2(1+|\nu|)^2)\fint_{B_\rho(x_0)} (1+|\nu|+|Du-\nu|)\,dx$$

$$\begin{aligned}
&\leq \fint_{B_\rho(x_0)} |Du-\nu|^2\,dx + (1+|\nu|)^2\kappa(|\xi|+|\nu|)\sqrt{\eta}(\rho^2) \\
&\qquad + (1+|\nu|)^2\kappa(|\xi|+|\nu|)^2\eta(\rho^2) \\
&\leq \Phi(x_0,\rho,\nu) + 2(1+|\nu|)^2\kappa(|\xi|+|\nu|)^2\sqrt{\eta}(\rho^2)\,.
\end{aligned}$$

Here we have used $\eta(\rho) \leq \sqrt{\eta}(\rho)$ which follows from the nondecreasing property of $t \mapsto \eta(t)$, ($\eta 4$), and our assumption $\rho \leq \rho_1 \leq 1$.
Similarly, we derive, using again the modulus of continuity for $(x,\xi) \mapsto A(x,\xi,p)$ from (2-1), Jensen's inequality, (2-3), (2-4), Poincaré's inequality (with a constant $c_p = c_p(n)$; note that $c_p = 2^{2n}$ is a possible choice), $\kappa \geq 1$, and again the fact that $\eta(ts) \leq t\eta(s)$ for $t \geq 1$,

$$\begin{aligned}
III &\leq \kappa(|\xi|+|\nu|)\fint_{B_\rho(x_0)} \sqrt{\eta}(|v|^2)(1+|\nu|+|Du-\nu|)\,dx \\
&\leq \fint_{B_\rho(x_0)} |Du-\nu|^2\,dx + (1+|\nu|)\kappa(|\xi|+|\nu|)\fint_{B_\rho(x_0)} \sqrt{\eta}(|v|^2)\,dx \\
&\qquad + \kappa(|\xi|+|\nu|)^2\fint_{B_\rho(x_0)} \eta(|v|^2)\,dx \\
&\leq \Phi(x_0,\rho,\nu) + 2\rho^{-2}\fint_{B_\rho(x_0)} |v|^2\,dx \\
&\qquad + (1+|\nu|)\kappa(|\xi|+|\nu|)\sqrt{\eta}\Big((1+|\nu|)\kappa(|\xi|+|\nu|)\rho^2\Big) \\
&\qquad + \kappa(|\xi|+|\nu|)^2\eta\Big(\kappa(|\xi|+|\nu|)^2\rho^2\Big) \\
&\leq \Phi(x_0,\rho,\nu) + 2\rho^{-2}\fint_{B_\rho(x_0)} |v|^2\,dx \\
&\qquad + (1+|\nu|)^2\kappa(|\xi|+|\nu|)^2\sqrt{\eta}(\rho^2) + \kappa(|\xi|+|\nu|)^4\eta(\rho^2) \\
&\leq (1+2c_p)\Phi(x_0,\rho,\nu) + 2(1+|\nu|)^2\kappa(|\xi|+|\nu|)^4\sqrt{\eta}(\rho^2)\,.
\end{aligned}$$

Combining these estimates we obtain the conclusion estimate with $c_3 = 1+2c_p$. ∎

5 Proof of the regularity theorem

For given $\varepsilon > 0$ (to be determined later) let $\delta = \delta(n,N,\lambda,L,\varepsilon) \in (0,1]$ be the constant from Lemma 4.1. We consider a weak solution $u \in H^{1,2}(\Omega,\mathbb{R}^N)$ of (1-2), $\xi \in \mathbb{R}^N$, and $\nu \in \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^N)$. We define

$$\gamma = 4c_3\sqrt{\Phi(x_0,\rho,\nu) + \delta^{-2}K_1(|\xi|,|\nu|)^2\eta(\rho^2)}\,,$$

and

$$w = \gamma^{-1}\Big(u - \xi - \nu(x-x_0)\Big). \tag{5-1}$$

We consider $B_\rho(x_0) \subset\subset \Omega$ such that

$$\rho \leq \rho_1(|\xi|,|\nu|). \tag{5-2}$$

Applying Lemma 4.2 on $B_\rho(x_0)$ to u, we obtain

$$\begin{aligned}
&\Big|\fint_{B_\rho(x_0)} \frac{\partial A}{\partial p}(x_0,\xi,\nu)Dw\,D\varphi\,dx\Big| \\
&\leq \ [\omega(|\xi|+|\nu|,\Phi(x_0,\rho,\nu))^2 + \Phi(x_0,\rho,\nu) + \tfrac{1}{2}\delta^2]^{1/2}\sup_{B_\rho(x_0)}|D\varphi|
\end{aligned} \tag{5-3}$$

for all $\varphi \in C_c^\infty(B_\rho(x_0), \mathbb{R}^n)$. Moreover, we have

$$\fint_{B_\rho(x_0)} |Dw|^2 \, dx \leq \frac{1}{16c_3^2} \leq 1. \tag{5-4}$$

We now fix $\Gamma(\xi, \eta) := \frac{\partial A}{\partial p}(x_0, \xi, \nu)\xi \cdot \eta$ and apply Lemma 4.1 to obtain a Γ–harmonic $h \in H^{1,2}(B_\rho(x_0), \mathbb{R}^N)$ such that

$$\fint_{B_\rho(x_0)} |Dh|^2 \, dx \leq 1 \tag{5-5}$$

and

$$\rho^{-2} \fint_{B_\rho(x_0)} |w - h|^2 \, dx \leq \varepsilon \,, \tag{5-6}$$

provided

$$\omega\Big(|\xi| + |\nu|, \Phi(x_0, \rho, \nu)\Big)^2 + \Phi(x_0, \rho, \nu) \leq \tfrac{1}{2}\delta^2 \,. \tag{5-7}$$

Note that (A1) and (A2) imply that Γ fulfills the hypotheses of Lemma 4.1. From the a-priori-estimate for solutions to homogeneous constant coefficient elliptic systems [Ca, Teorema 9.2]

$$\rho^{-2} \sup_{B_{\rho/2}(x_0)} |Dh|^2 + \sup_{B_{\rho/2}(x_0)} |D^2 h|^2 \leq \frac{c_4(n, \lambda, L)}{\rho^2} \fint_{B_\rho(x_0)} |Dh|^2 dx$$

we deduce by Taylor expansion and (5-5) for $\theta \in (0, 1/4]$

$$\sup_{x \in B_{2\theta\rho}(x_0)} |h(x) - h(x_0) - Dh(x_0)(x - x_0)|^2 \leq \frac{c_4}{\rho^2}(2\theta\rho)^4 = 16c_4\theta^4\rho^2. \tag{5-8}$$

Combining (5-8) with (5-6) we get

$$\begin{aligned}
&(2\theta\rho)^{-2} \fint_{B_{2\theta\rho}(x_0)} |w - h(x_0) - Dh(x_0)(x - x_0)|^2 dx \\
&\leq\; 2(2\theta\rho)^{-2}\Big(\fint_{B_{2\theta\rho}(x_0)} |w - h|^2 dx + \fint_{B_{2\theta\rho}(x_0)} |h - h(x_0) - Dh(x_0)(x - x_0)|^2 dx \Big) \\
&\leq\; 2(2\theta\rho)^{-2} \Big((2\theta)^{-n}\rho^2\varepsilon + 16c_4\theta^4\rho^2 \Big) \\
&=\; 2^{-n-1}\theta^{-n-2}\varepsilon + 8c_4\theta^2 \,.
\end{aligned} \tag{5-9}$$

Next, we estimate, using the fact that $u - (\nu + \gamma Dh(x_0))(x - x_0)$ has mean value $u_{x_0,2\theta\rho}$ on $B_{2\theta\rho}(x_0)$, the definition (5-1) of w, and (5-9),

$$\begin{aligned}
&(2\theta\rho)^{-2} \fint_{B_{2\theta\rho}(x_0)} |u - u_{x_0,2\theta\rho} - (\nu + \gamma Dh(x_0))(x - x_0)|^2 dx \\
&\leq\; (2\theta\rho)^{-2} \fint_{B_{2\theta\rho}(x_0)} |u - \xi - \nu(x - x_0) - \gamma(h(x_0) + Dh(x_0)(x - x_0))|^2 dx \\
&=\; \gamma^2(2\theta\rho)^{-2} \fint_{B_{2\theta\rho}(x_0)} |w - h(x_0) - Dh(x_0)(x - x_0)|^2 dx \\
&\leq\; 16c_3^2 \left(2^{-n-1}\theta^{-n-2}\varepsilon + 8c_4\theta^2\right) \left(\Phi(x_0, \rho, \nu) + \delta^{-2}K_1(|\xi|, |\nu|)^2\eta(\rho^2)\right) \\
&\leq\; c_5 \left(\theta^{-n-2}\varepsilon + \theta^2\right) \left(\Phi(x_0, \rho, \nu) + \delta^{-2}K_1(|\xi|, |\nu|)^2\eta(\rho^2)\right),
\end{aligned} \tag{5-10}$$

where $c_5 := c_5(n, \lambda, L) \geq 1$. We now fix $\xi = u_{x_0,\rho}$, $\nu = (Du)_{x_0,\rho}$, and abbreviate $\Phi(x_0, r) = \Phi(x_0, r, (Du)_{x_0,r})$. Furthermore, we let $P = (Du)_{x_0,\rho} + \gamma Dh(x_0)$. If we impose the additional smallness condition

$$\rho \leq \rho_1(|u_{x_0,2\theta\rho}|, |P|) \tag{5-11}$$

on ρ, the Caccioppoli inequality (Lemma 3.1) with ξ, ν, $B_\rho(x_0)$ replaced by $u_{x_0,2\theta\rho}$, P, $B_{2\theta\rho}(x_0)$ implies

$$\begin{aligned}
\Phi(x_0,\theta\rho) &\leq \fint_{B_{\theta\rho}(x_0)} |Du-P|^2\,dx \\
&\leq 2^n c_1(2\theta\rho)^{-2}\fint_{B_{2\theta\rho}(x_0)} |u-u_{x_0,2\theta\rho}-P(x-x_0)|^2\,dx \\
&\qquad + 2^n c_2 K_1(|u_{x_0,2\theta\rho}|,|P|)\eta((2\theta\rho)^2) \\
&\leq 2^n c_1 c_5\Big(\theta^{-n-2}\varepsilon+\theta^2\Big)\Big[\Phi(x_0,\rho)+\delta^{-2}K_1(|u_{x_0,\rho}|,|(Du)_{x_0,\rho}|)^2\eta(\rho^2)\Big] \\
&\qquad + 2^n c_2 K_1(|u_{x_0,2\theta\rho}|,|P|)\eta(\rho^2)\,,
\end{aligned} \tag{5-12}$$

the last inequality following from (5-11) and the nondecreasing property of η.
We now specify $\varepsilon := \theta^{n+4}$ and choose $\theta \in (0,\frac{1}{4}]$ such that $2^{n+1}c_1c_5\theta^2 \leq \theta^{2\beta}$. Note that δ is fixed now and that ε and θ are constants depending on n, λ, L, and β, while δ depends also on N. If we require

$$c_6\gamma^2 \leq 1 \tag{5-13}$$

with $c_6 = \max\{c_4,(2\theta)^{-n}\}$ we obtain using (5-5) and (5-8),

$$\begin{aligned}
|P| &\leq |(Du)_{x_0,\rho}| + \gamma|Dh(x_0)| \\
&\leq |(Du)_{x_0,\rho}| + \sqrt{c_4}\gamma\Big(\fint_{B_\rho(x_0)}|Dh|^2\,dx\Big)^{1/2} \\
&\leq |(Du)_{x_0,\rho}| + \sqrt{c_4}\gamma \\
&\leq |(Du)_{x_0,\rho}| + 1\,,
\end{aligned}$$

the second last inequality following from the apriori estimate for h. Moreover, taking into account Poincaré's inequality and recalling the definition of γ and c_3 we deduce

$$\begin{aligned}
|u_{x_0,2\theta\rho}| &\leq |u_{x_0,\rho}| + (2\theta)^{-n/2}\Big(\fint_{B_\rho(x_0)}|u-u_{x_0,\rho}-(Du)_{x_0,\rho}(x-x_0)|^2\,dx\Big)^{1/2} \\
&\leq |u_{x_0,\rho}| + \frac{\sqrt{c_p}}{(2\theta)^{n/2}}\rho\Phi(x_0,\rho)^{1/2} \\
&\leq |u_{x_0,\rho}| + \frac{\sqrt{c_p}\gamma}{c_3(2\theta)^{n/2}} \\
&\leq |u_{x_0,\rho}| + 1\,.
\end{aligned}$$

Combining these estimates with (5-12) and (5-10), we arrive at

$$\Phi(x_0,\theta\rho) \leq \theta^{2\beta}\Phi(x_0,\rho) + K(|u_{x_0,\rho}|,|(Du)_{x_0,\rho}|)\eta(\rho^2)\,, \tag{5-14}$$

where $K(s,t) = (2^nc_2+\delta^{-2})K_1(1+s,1+t)$ (here we have used the nondecreasing property of $s \mapsto K_1(s,t)$, $t \mapsto K_1(s,t)$ and $K_1 \leq 1$), provided we assume the **radius condition**

$$\rho \leq \rho_1\Big(|u_{x_0,\rho}|+1,|(Du)_{x_0,\rho}|+1\Big) \leq 1 \tag{5-15}$$

and the **smallness conditions** (cf. (5-7), (5-13))

$$\omega\Big(|u_{x_0,\rho}|+|(Du)_{x_0,\rho}|,\Phi(x_0,\rho)\Big)^2 + \Phi(x_0,\rho) \leq \tfrac{1}{2}\delta^2\,, \tag{5-16}$$

$$c_7K_1(|u_{x_0,\rho}|,|(Du)_{x_0,\rho}|)^2\eta(\rho^2) \leq \delta^2 \tag{5-17}$$

where $c_7 = 32c_3^2c_6$. Here we have assumed that $16c_3^2c_6\delta^2 \le 1$ which is no restriction.
For $T > 0$ we find $\Phi_0(T) > 0$, (depending on n, N, λ, L, β, and $\omega(\,\cdot\,)$) such that

$$\omega(2T, 2\Phi_0(T))^2 + 2\Phi_0(T) \le \tfrac{1}{2}\delta^2\,, \tag{5-18}$$

$$4c_3\Phi_0(T) \le \theta^n(1-\theta^\beta)^2T^2. \tag{5-19}$$

With $\Phi_0(T)$ from (5-18) and (5-19) we choose $\rho_0(T) \in (0,1]$ (depending also on n, N, λ, L, α, β, $\omega(\,\cdot\,)$, $\eta(\,\cdot\,)$, and $\kappa(\,\cdot\,)$) such that

$$\rho_0(T) \le \rho_1(2T+1, 2T+1)\,, \tag{5-20}$$

$$c_7K_1(2T,2T)^2\eta(\rho_0(T)^2) \le \delta^2\,, \tag{5-21}$$

$$K_0(T)\eta(\rho_0(T)^2) \le (\theta^{2\alpha} - \theta^{2\beta})\Phi_0(T)\,, \tag{5-22}$$

$$2(1+c_p)K_0(T)\mathsf{H}(\rho_0(T)^2) \le \theta^n(1-\theta^\alpha)^2(\theta^{2\alpha}-\theta^{2\beta})T^2, \tag{5-23}$$

where $K_0(T) := K(2T, 2T)$.

Lemma 5.1 *Assume that for some $T_0 > 0$ and $B_\rho(x_0) \subset\subset \Omega$ we have*

$$\begin{array}{rl} (i) & |u_{x_0,\rho}| + |(Du)_{x_0,\rho}| \le T_0, \\ (ii) & \rho \le \rho_0(T_0), \\ (iii) & \Phi(x_0,\rho) \le \Phi_0(T_0). \end{array}$$

Then the smallness conditions (5-15)–(5-18) are satisfied on $B_{\theta^j\rho}(x_0)$ for $j \in \mathbb{N}\cup\{0\}$. Moreover, the limit

$$\Lambda_{x_0} := \lim_{j\to\infty} (Du)_{x_0,\theta^j\rho}$$

exists, and the inequality

$$⨍_{B_\rho(x_0)} |Du - \Lambda_{x_0}|^2\,dx \le c_8\Big(\Big(\frac{r}{\rho}\Big)^{2\beta}\Phi(x_0,\rho) + \mathsf{H}(r^2)\Big) \tag{5-24}$$

is valid for $0 < r \le \rho$ with a constant $c_8 = c_8(n, N, \lambda, L, \beta, T_0)$.

Proof. Inductively we shall derive for $j \in \mathbb{N}\cup\{0\}$ the following four assertions:

$$\begin{array}{rl} (I_j) & \Phi(x_0,\theta^j\rho) \le 2\Phi_0(T_0), \\ (I'_j) & \Phi(x_0,\theta^j\rho) \le \theta^{2\beta j}\Phi(x_0,\rho) + \dfrac{K_0(T_0)}{\theta^{2\alpha}-\theta^{2\beta}}\eta((\theta^j\rho)^2), \\ (\mathit{II}_j) & |u_{x_0,\theta^j\rho}| + |(Du)_{x_0,\theta^j\rho}| \le 2T_0, \\ (\mathit{III}_j) & \theta^j\rho \le \rho_1\Big(|u_{x_0,\theta^j\rho}| + 1, |(Du)_{x_0,\theta^j\rho}| + 1\Big). \end{array}$$

We first note that (III_j), (5-20), (II_j), and the nonincreasing property of $s \mapsto \rho_1(s,t)$, $t \mapsto \rho_1(s,t)$ imply $(5\text{-}15)_j$, i.e. (5-15) with $\theta^j\rho$ instead of ρ. Next, we observe that (I_j), (II_j), and (5-18) yield $(5\text{-}16)_j$. Finally, from (II_j), (ii), and (5-21) we infer $(5\text{-}17)_j$.
For $j = 0$ we note that (III_0) is a consequence of (ii), (5-20), and the monotonicity property of ρ_1. Now suppose that for some $j \in \mathbb{N}$ we have (I_ℓ), (II_ℓ), and (III_ℓ) for $\ell = 0\ldots j-1$. Then

we can use $(5\text{-}15)_\ell$ and (II_ℓ) for $\ell = 0 \dots j-1$, $(\eta 2)$, (ii), and (5-22) to deduce

$$\begin{aligned}
\Phi(x_0,\theta^j\rho) &\leq \theta^{2\beta j}\Phi(x_0,\rho) + \sum_{\ell=0}^{j-1}\theta^{2\beta\ell}\eta(\theta^{2(j-\ell-1)}\rho^2)K\Big(|u_{x_0,\theta^{j-\ell-1}\rho}|,|(Du)_{x_0,\theta^{j-\ell-1}\rho}|\Big)\\
&\leq \theta^{2\beta j}\Phi(x_0,\rho) + K_0(T_0)\sum_{\ell=0}^{j-1}\theta^{2\beta\ell}\eta(\theta^{2(j-\ell-1)}\rho^2)\\
&\leq \theta^{2\beta j}\Phi(x_0,\rho) + K_0(T_0)\eta((\theta^j\rho)^2)\theta^{-2\alpha}\sum_{\ell=0}^{j-1}\theta^{2(\beta-\alpha)\ell}\\
&\leq \theta^{2\beta j}\Phi(x_0,\rho) + \frac{K_0(T_0)}{\theta^{2\alpha}-\theta^{2\beta}}\eta((\theta^j\rho)^2)\\
&\leq 2\Phi_0(T_0)\,.
\end{aligned}$$

This proves (I_j) and (I'_j). We next want to show (II_j). Using Poincaré's inequality, (I'_j) for $\ell = 0 \dots j-1$, the hypotheses (i)–(iii), the estimate (2.2) from section 2, and (5-19), (5-23) we obtain

$$\begin{aligned}
&|u_{x_0,\theta^j\rho}| + |(Du)_{x_0,\theta^j\rho}|\\
&\leq |u_{x_0,\rho}| + |(Du)_{x_0,\rho}| + \frac{1+\sqrt{c_p}}{\theta^{n/2}}\sum_{\ell=0}^{j-1}\sqrt{\Phi(x_0,\theta^\ell\rho)}\\
&\leq T_0 + \frac{1+\sqrt{c_p}}{\theta^{n/2}}\sum_{\ell=0}^{j-1}\left(\theta^{\ell\beta}\sqrt{\Phi(x_0,\rho)} + \sqrt{\frac{K_0(T_0)}{\theta^{2\alpha}-\theta^{2\beta}}\,\eta(\theta^{2\ell}\rho^2)}\right)\\
&\leq T_0 + \frac{1+\sqrt{c_p}}{\theta^{n/2}}\left(\frac{\sqrt{\Phi(x_0,\rho)}}{1-\theta^\beta} + \sqrt{\frac{K_0(T_0)}{\theta^{2\alpha}-\theta^{2\beta}}}\,\frac{\sqrt{\mathsf{H}(\rho^2)}}{2(1-\theta^\alpha)}\right)\\
&\leq T_0 + \frac{1+\sqrt{c_p}}{\theta^{n/2}}\left(\frac{\sqrt{\Phi_0(T_0)}}{1-\theta^\beta} + \sqrt{\frac{K_0(T_0)}{\theta^{2\alpha}-\theta^{2\beta}}}\,\frac{\sqrt{\mathsf{H}(\rho_0(T_0)^2)}}{2(1-\theta^\alpha)}\right)\\
&\leq 2T_0\,.
\end{aligned}$$

¿From (II_j), hypothesis (ii), (5-20), and the nonincreasing property of $s \mapsto \rho_1(s,t)$, $t \mapsto \rho_1(s,t)$ we easily derive (III_j).

We next want to prove that $(Du)_{x_0,\theta^j\rho}$ converges to some limit Λ_{x_0} in $\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^N)$. Arguing as in the proof of (II_j) we deduce for $k > j$

$$\begin{aligned}
|(Du)_{x_0,\theta^j\rho} - (Du)_{x_0,\theta^k\rho}| &\leq \sum_{\ell=j+1}^{k}|(Du)_{x_0,\theta^\ell\rho} - (Du)_{x_0,\theta^{\ell-1}\rho}|\\
&\leq \theta^{-n/2}\left(\frac{\sqrt{\Phi(x_0,\rho)}}{1-\theta^\beta}\theta^{\beta j} + \sqrt{\frac{K_0(T_0)}{\theta^{2\alpha}-\theta^{2\beta}}}\,\frac{\sqrt{\mathsf{H}((\theta^j\rho)^2)}}{2(1-\theta^\alpha)}\right). \qquad (5\text{-}25)
\end{aligned}$$

Taking into account our assumption $(\eta 3)$ we see that $\{(Du)_{x_0,\theta^j\rho}\}$ is a Cauchy sequence in $\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^N)$. Therefore the limit $\Lambda_{x_0} = \lim_{j\to\infty}(Du)_{x_0,\theta^j\rho}$ exists and from (5-25) we infer for $j \in \mathbb{N}\cup\{0\}$

$$|(Du)_{x_0,\theta^j\rho} - \Lambda_{x_0}| \leq c_9\sqrt{\theta^{2\beta j}\Phi(x_0,\rho) + \mathsf{H}((\theta^j\rho)^2)} \qquad (5\text{-}26)$$

where $c_9 = \frac{\sqrt{2}}{\theta^{-n/2}(1-\theta^\beta)}\Big(1+\frac{1}{2}\sqrt{\frac{K_0(T_0)}{\theta^{2\alpha}-\theta^{2\beta}}}\Big)$. Combining this with (I'_j), and recalling the estimate $\eta(t) \leq \frac{\alpha^2}{4}\mathsf{H}(t)$ from Section 2 we arrive at

$$\fint_{B_{\theta^j\rho}(x_0)}|Du-\Lambda_{x_0}|^2\,dx \quad\leq\quad 2\Phi(x_0,\theta^j\rho) + 2|(Du)_{x_0,\theta^j\rho} - \Lambda_{x_0}|^2$$

$$\leq \quad c_{10}\Big(\theta^{2\beta j}\Phi(x_0,\rho) + \mathsf{H}((\theta^j\rho)^2)\Big). \tag{5-27}$$

with $c_{10} = 2(1 + c_9^2 + \frac{K_0(T_0)}{\theta^{2\alpha}-\theta^{2\beta}})$.

For $0 < r \leq \rho$ we find $j \in \mathbb{N} \cup \{0\}$ such that $\theta^{j+1}\rho < r \leq \theta^j\rho$. Then (5-27) and the nonincreasing property of $t \mapsto t^{-\alpha}\mathsf{H}(t)$ imply

$$\begin{aligned}
⨍_{B_r(x_0)} |Du - \Lambda_{x_0}|^2\, dx \quad &\leq \quad c_{10}\theta^{-n}\Big(\theta^{2j\beta}\Phi(x_0,\rho) + \mathsf{H}((\theta^j\rho)^2)\Big) \\
&\leq \quad c_{10}\theta^{-n}\Big[\theta^{-2\alpha}\Big(\frac{r}{\rho}\Big)^{2\beta}\Phi(x_0,\rho) + \theta^{-2\alpha}\mathsf{H}((\theta^{j+1}\rho)^2)\Big] \\
&\leq \quad c_{10}\theta^{-n-2\beta}\Big[\Big(\frac{r}{\rho}\Big)^{2\beta}\Phi(x_0,\rho) + \mathsf{H}(r^2)\Big].
\end{aligned}$$

This proves (5-24) with $c_8 = c_{10}\theta^{-n-2\beta}$. ∎

It is now standard to deduce Theorem 1.1 from this lemma.

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