Parametrization of Periodic Weighted Operators in Terms of Gap Lengths

Markus Klein
Universität Potsdam
Institut für Mathematik
e-mail: mklein@math.uni-potsdam.de

Evgeni Korotyaev *
Universität Potsdam
Institut für Mathematik
e-mail: evgeni@math.uni-potsdam.de

December 13, 1999

Abstract

We consider the periodic weighted operator $Ty = -\rho^{-2}(\rho^2 y')'$ in $L^2(\mathbb{R}, \rho(x)^2 dx)$, where ρ is a 1-periodic real positive function with $\rho(0) = 1$. We assume $q = \rho'/\rho \in L^2(0,1)$. The spectrum of T consists of intervals $\sigma_n = [\lambda_{n-1}^+, \lambda_n^-]$ separated by gaps $(\lambda_n^-, \lambda_n^+), n \geqslant 1$. Using essentially the square of the gap lengths, the centre of the gap and the Dirichlet eigenvalues on the unit interval we construct a gap length mapping $q \to \ell^2 \oplus \ell^2$ which provides a real analytic parametrization of the weight. For the proof we use nonlinear functional analysis in Hilbert space combined with sharp asymptotic estimates on the fundamental solution and the Lyapunov function in the high energy limit for complex q.

1 Introduction and main results

In this paper we shall consider the inverse problem for the periodic weighted operator

$$Tf = -\rho^{-2}(\rho^2 f')' = -f'' - 2qf', \qquad q \equiv \frac{\rho'}{\rho} \in L^2(\mathbb{R}/\mathbb{Z}), \tag{1.1}$$

*Partially supported by Russian Fund of Fundamental Research, INTAS and SFB 288. Permanent adress: Math. Dept. 2, ETU, 5 Prof. Popov Str., St.Petersburg, 197376 Russia

in the Hilbert space $L^2(R, \rho(x)^2 dx)$ where ρ is a 1-periodic real positive function, more precisely $\rho' \in L^2(\mathbb{R}/\mathbb{Z})$ and $\rho(x) \geqslant \rho_0$ for some $\rho_0 > 0$ and all $x \in [0,1]$. We shall normalize ρ by setting $\rho(0) = 1$. The weighted operator (1.1) is related to the Hill operator (for in general a very singular potential) as follows. Define the unitary transformation $U : L^2(\mathbb{R}, \rho^2 dx) \to L^2(\mathbb{R}, dx)$ as multiplication by ρ. Then T is unitarily equivalent to

$$T_1 = UTU^{-1} = -\Big(\frac{d}{dx} + q\Big)\Big(\frac{d}{dx} - q\Big) \tag{1.2}$$

which is formally given by

$$T_1 = -\frac{d^2}{dx^2} + V, \qquad V = q'(x) + q(x)^2 = \frac{\rho''(x)}{\rho(x)}. \tag{1.3}$$

Thus, if $q' \in L^2(\mathbb{R}/\mathbb{Z})$, then T corresponds to the Hill operator with L^2-potential. The situation considered in this paper ($\rho' \in L^2$, or $q \in L^2$) corresponds to a much more singular case which needs different asymptotic estimates. In particular, the gap length which behaves roughly as $2\pi n|q_n|$ in general is increasing. Here q_n denotes the Fourier coefficient of q.

The operator T was already considered by Lyapunov and Krein ([L]), [Kr]). In particular, they proved that the spectrum of T is absolutely continuous and consists of bands

$$\sigma(T) = \cup_{n \geqslant 1} \sigma_n, \quad \sigma_n = [\lambda_{n-1}^+, \lambda_n^-], \quad \lambda_n^- \leqslant \lambda_n^+ \tag{1.4}$$

Here λ_n^- and λ_n^+ are points of the 2-periodic spectrum of the following equation $-y'' - 2qy' = \lambda y$. The bands are separated by gaps $\gamma_n = (\lambda_n^-, \lambda_n^+)$ of length $|\gamma_n| \geqslant 0$. If $|\gamma_n| = 0$, i. e., $\lambda_n^- = \lambda_n^+$, the gap is called degenerate. We recall from $[K1]$ that

$$\lambda_n^\pm = (\pi n \pm |\int_0^1 e^{i2\pi nx} q(x) dx|)^2 + ..., \qquad (n \to \infty).$$

Thus the gap length $\lambda_n^+ - \lambda_n^- = 2\pi n |\int_0^1 e^{i2\pi nx} q(x) dx| + \ldots$ roughly, which does not, in general, decay as $n \to \infty$. These estimates need to be refined. Asymptotic estimates on the fundamental solution and the location of the band-edges $\lambda_n^\pm$ were obtained in [K1]. In [K2] the inverse problem was solved in terms of the data of vertical slits. In this paper we shall use different analytic mappings $\phi : H \to \ell^2 \oplus \ell^2$ and $\psi : H \to \ell^2_{-1} \oplus \ell^2_{-1}$ to parametrize the functions $q \in H$, where H denotes the real Hilbert space of functions $q \in L^2([0,1], \mathbb{R})$ with vanishing mean $\int_0^1 q(x) dx = 0$. Essentially, ϕ and ψ are constructed from the gap lengths in the $z = \sqrt{\lambda}$ -plane and the λ-plane; we shall call them *gap length mappings* and their components ϕ_n, ψ_n *coordinate functions* for q.

We remark that the band edges $\lambda_n^\pm$ (or $z_n^\pm = \sqrt{\lambda_n^\pm}$ - and correspondingly the gap lengths - are not analytic functions of $q \in H$. Symmetrization and antisymmetrization, however, lead to the analytic functions

$$b_n = \frac{1}{4}(z_n^+ - z_n^-)^2, \qquad \beta_n = \frac{1}{4}(\lambda_n^+ - \lambda_n^-)^2 \tag{1.5}$$

(essentially the square of the gap length) and the centre of the gap functions

$$z_n^0 = \frac{1}{2}(z_n^+ + z_n^-), \qquad \lambda_n^0 = \frac{1}{2}(\lambda_n^+ + \lambda_n^-), \tag{1.6}$$

which in some sense are useful analytic data to parametrize the potential. They belong, however, to no a priori useful sequence space. To correct this, it is useful to introduce beside (b_n, z_n^0) or (β_n, λ_n^0) additional spectral data: The Dirichlet eigenvalues $\mu_n(q), n \geqslant 1$, for the operator T^D formally given by T with Dirichlet boundary conditions on $[0,1]$, and the corresponding quantities in the z-plane

$$m_n(q) = \sqrt{\mu_n(q)} > 0, \tag{1.7}$$

which are both real analytic in $q \in H$. With this notation we are ready to introduce our coordinate functions $\phi = \{\phi_n\}_{n\geqslant 1}$ and $\psi = \{\psi_n\}_{n\geqslant 1}$ for $q \in H$ via

$$\phi_{sn} = z_n^0 - m_n, \qquad \phi_{cn} = |b_n - \phi_{sn}^2|^{1/2} \operatorname{sign} h_{cn} \tag{1.8}$$

and

$$\psi_{sn} = \lambda_n^0 - \mu_n, \qquad \psi_{cn} = |\beta_n - \psi_{sn}^2|^{1/2} \operatorname{sign} h_{cn}. \tag{1.9}$$

The additional signature $\operatorname{sign} h_{cn}$ is chosen for reasons of beauty; it serves to make ϕ_n asymptotic to the Fourier coefficients $q_n = (q_{sn}, q_{cn})$ of $q \in H$:

$$q_{cn} = \int_0^1 q(x) \cos 2\pi n x dx, \quad q_{sn} = \int_0^1 q(x) \sin 2\pi n x dx, \quad n \geqslant 1, \quad q \in H_C. \tag{1.10}$$

More precisely, h_{cn} is one component of the n-th vertical slit mapping introduced in [K2], i.e.,

$$h_{cn} = -\log[(-1)^n \varphi'(1, \mu_n(q), q)] \tag{1.11}$$

where $\varphi(x,z,q)$ (and the additional $\vartheta(x,z,q)$) are defined as the fundamental solutions for T satisfying $(y = \varphi, \vartheta)$

$$-y'' - 2qy' = z^2 y, \quad z \in \mathbb{C}, \quad q \in H_C, \tag{1.12}$$

with initial conditions

$$\varphi(0,z,q) = \vartheta'(0,z,q) = 0, \qquad \varphi'(0,z,q) = \vartheta(0,z,q) = 1. \tag{1.13}$$

We shall use the ℓ^2-norm on $\ell^2 \oplus \ell^2$ for the function ϕ

$$\phi_n^2 = \phi_{sn}^2 + \phi_{cn}^2, \qquad ||\phi||^2 = \sum_{n\geqslant 1} \phi_n^2. \tag{1.14}$$

We shall write H_C for the complexification of the real Hilbert space H with norm $||\cdot||_C$, and we denote a complex ball in H_C by

$$B_C(p,t) = \{q \in H_C : \|q - p\|_C < t\}, \quad p \in H_C, \tag{1.15}$$

and the analogous real ball by

$$B(p,t) = \{q \in H : \|q-p\| < t\}, \quad p \in H.$$

Introduce the weighted sequence spaces ℓ^r_p, $r \geqslant 1, p \in \mathbb{R}$ consisting of real sequences $f = \{f_n\}_{n\geqslant 1}$ with the norm

$$\|f\|^r_{p,r} = \sum_{n\geqslant 1} (2\pi n)^{2p} |f_n|^r < \infty,$$

(where in the Hilbert space case for $r=2$ we write $||f||_p = ||f||_{2,p}$). Furthermore, we write $a_n = b_n + \ell^d(n)$ iff the sequence $\{a_n - b_n\}_{n\geqslant 1} \in \ell^d$. With this notation we formulate our main result as

Theorem 1.1. *Each coordinate function $\phi_n(\cdot), n \geqslant 1$, is compact and real analytic on H, and they define a real analytic isomorphism*

$$\phi : H \to \ell^2 \oplus \ell^2,$$

which satisfies

$$||\phi(q)|| \leqslant ||q|| \leqslant 4||\phi(q)||(1 + 2||\phi(q)||), \quad q \in H. \tag{1.16}$$

Moreover, for any fixed $d > 1$ the following asymptotic estimates are fulfilled

$$\phi_n(q) = q_n + \ell^d(n), \qquad n \to \infty, \tag{1.17}$$

$$(d_q\phi_n(q))(x) = (\cos 2\pi nx, \sin 2\pi nx) + \ell^2(n), \qquad n \to \infty, \tag{1.18}$$

uniformly on $[0,1] \times B_C(p, \varepsilon_p), \varepsilon_p = 8^{-2} e^{-4||p||}$, for each fixed $p \in H$.

This result can also be transferred to the complex λ-plane using once more the real analyticity and the asymptotic behaviour of z^0_n established before and the estimates from [K1], [K9]. We have

Theorem 1.2. *Each coordinate function $\psi_n, n \geqslant 1$, is compact and real analytic on H and they define a real analytic isomorphism*

$$\psi : H \to \ell^2_{-1} \oplus \ell^2_{-1}, \tag{1.19}$$

which satisfies the two-sided estimate

$$\|q\| \leqslant 16\pi \|\psi(q)\|_{-1}(1 + 2\|\psi(q)\|_{-1}), \quad \|\psi(q)\|_{-1} \leqslant \|q\|(1 + \|q\|), \qquad q \in H. \tag{1.20}$$

Moreover, the following asymptotic estimates are fulfilled

$$\psi_n(q) = 2\pi n q_n + 2\pi n \ell^d(n), \tag{1.21}$$

$$(d_q\psi_n(q))(x) = 2\pi n(\cos 2\pi nx, \sin 2\pi nx) + 2\pi n \ell^2(n), \tag{1.22}$$

uniformly on $[0,1] \times B_C(p, \varepsilon_p), \varepsilon_p = 8^{-2} e^{-4||p||}$, for each fixed $p \in H$.

We shall briefly indicate how to prove these results. We need good control on the positions of the band edges $z_n^{\pm}, \lambda_n^{\pm}$ and the functions $(b_n, z_n^0), (\beta_n, \lambda_n^0)$ and the Dirichlet eigenvalues. A crucial role is, as usual, played by the Lyapunov function

$$\Delta(z,q) = \frac{1}{2}(\varphi'(1,z,q) + \vartheta(1,z,q)) \tag{1.23}$$

which yields the periodic spectrum $0 = \lambda_0^+ < \lambda_1^- \leqslant \lambda_1^+ < \dots$. $\Delta(z,q)$ and its derivatives with respect to q (denoted by $\partial = \partial/\partial q$) and z (denoted by $\dot{()} = \partial/\partial z$) enter the asymptotic formulae for z_n^0, b_n (and for their derivatives $d_q z_n^0, d_q b_n$). They must be estimated with care. Analyticity is proved by establishing analyticity (and estimates) for complex potentials in the ball $B_C(p, \varepsilon_p)$ for fixed $p \in H$. Together with control on the Dirichlet spectrum this gives asymptotic control on $\phi, d_q\phi$ (and $\phi, d_q\psi$, resp.) and proves most i.e.i) ii) iv)) of the sufficient conditions in the following "basic theorem" from [K2], which in an abstract situation states sufficient conditions for a map to be a real analytic isomorphism.

We recall definitions. Suppose that H, H_1 are real separable Hilbert spaces with norms $\|\cdot\|, \|\cdot\|_1$. The derivative of a map $f : H \to H_1$ at a point $y \in H$ is a bounded linear map from H into H_1, which we denote by $d_y f$. A map $f : H \to H_1$ is compact on H if it maps weakly convergent sequences in H into strongly convergent sequences in H_1. A map $f : H \to H_1$ is a real analytic isomorphism between H and H_1 if f is one-to-one and onto and both f and f^{-1} are real analytic maps of the Hilbert space. Then we have

Theorem A. *Let H, H_1 be real separable Hilbert spaces equipped with norms $\|\cdot\|, \|\cdot\|_1$. Suppose that a map $f : H \to H_1$ satisfies the following conditions:*
i) f is real analytic,
ii) the operator $d_q f$ has an inverse for all $q \in H$,
iii) there is a nondecreasing function $g : [0,\infty) \to [0,\infty), g(0) = 0$, such that $\|q\| \leqslant g(\|f(q)\|_1)$ for all $q \in H$,
iv) there exists a basis $\{e_n\}_1^\infty$ of H_1 such that each map $(f(\cdot), e_n)_1 : H \to \mathbb{R}, n \geqslant 1$, is compact,
v) for each $\zeta > 0$ the set $\{q : \sum n^2 (f(q), e_n)_1^2 < \zeta\} \subset H$ is compact.
Then f is a real analytic isomorphism between H, H_1.

To prove ii) (which implies that ϕ or ψ is a *local* isomorphism) we show that $d_q\phi$ (and $d_q\psi$) is Fredholm of index zero with zero-dimensional kernel. This relies on the fact that the functions

$$\{d_q h_{cn}, d_q \mu_n\}_{n\geqslant 1} \quad \text{are a basis of } H, \tag{1.24}$$

combined with a Theorem of Paley-Wiener about entire functions.

To prove the points iii) and v), we refer to results of [K1] and [K9], but we have to include few words of explanation. The proof of both results uses the definition of the quasimomentum for the weighted operator and its global properties as a map on a slit domain. Furthermore, we want to emphasize that the compactness result in v) is slightly subtle in the case of the weighted in marked distinction to the case of the Hill operator. There the asymptotic estimates on the gap length suffice to prove compactness of the difference of the gap length mapping and the Fourier transform. Thus the point v) is in some sense automatic. It could be shown from our asymptotic that a compactness result of this form is definitely false for the weighted operator. The condition v) in Theorem A is needed to remedy this situation. Its proof in the references [K1] and [K9] again crucially depends on the properties of the quasimomentum. In particular, we emphasize that it is essentially the need for this more general version of Theorem A which makes the proof of [K3] for the Hill operator inapplicable to the construction of the mapping ψ for the weighted operator considered in this paper. This applies equally well to the construction of the mapping ϕ treated in [KK1] for the Hill operator. We remind the reader, however, that this mapping was constructed in the paper [KK1] in the case of even potentials only. This is - in some sense - generalized by the results of this paper, even in the case of the Hill operator.

The plan of the paper is a follows. In Sect. 2 (Preliminaries) we collect from $[K1-2]$ the basic properties of the fundamental solution, the Lyapunov-function $\Delta(z,q)$, the Dirichlet eigenvalues $m_n(q)$ and the vertical-slit function $h_{cn}(q)$. This part is not new; we include this material in a minimalistic way for the sake of the reader. The main result of this section is Lemma 2.4 and Lemma 2.5 where we introduce a natural formal symplectic form and obtain a basis of the Hilbert space H via the gradients of the functions $m_n(q), h_{cn}(q)$ which play the role of the canonical coordinates.

Sect. 3 (Band edges in spectral planes) contains the main new technical results of this paper. We prove a counting Lemma (in the spirit of [PT]) for the zeroes of $\Delta^2(z,q)-1$ for complex potential $q \in B_C(p,\varepsilon_p)$ which gives a priori control on the band edges. Using the Lyapunov function we control the center of the gap z_n^0, the gap length square b_n (and their derivatives). Analogous results for (λ_n^0, β_n) then follow from essentially straightforward algebraic calculations.

In Sect. 4 (Coordinate functions) we combine the results of Sect. 3 with control of the Dirichlet eigenvalues to obtain real analyticity, compactness and asymptotic 0 of the coordinate functions.

In Section 5 we prove Theorem 1.1 and Theorem 1.2, using the results of Paley-Wiener and the estimates from [K1], [K9].

Finally we remark that these results (as those in [K3]) in some sense solve the inverse problem for the Hill operator with very singular potential

$$V = q' + q^2, \tag{1.25}$$

i.e. for the more regular case $V \in L^p([0,1]), 1 \leqslant p < 2$, where $T_0 = -\frac{d^2}{dx^2} + V$ can be defined as a quadratic form, and the very singular case $V = \rho''/\rho^2$ where $\rho' \in L^2(\mathbb{R}/\mathbb{Z})$.

Physically, this is motivated by the fact that the weighted operator T describes classical wave propagation (i.e. propagation of sound, see [6]) or a photonic crystal, since the operator

$$T_c f = -c^2(x) f'' \quad \text{in} \quad L^2(\mathbb{R}, c(x)^{-2}\, dx)$$

is related to the weighted operator $T = T_\rho$ if $c(x) = \rho(t(x))^{-2}$, where $t(x)$ is the inverse function of $x(t) = \int_0^t \rho(t)^{-2} dt$. Remark that the equation $-\ddot{y} = z^2 b(t)^2 y$ arises in the scattering of the few particles in an external homogeneous time-periodic magnetic field $B(t) = zb(t)(0,0,1) \in \mathbb{R}^3$, where z is the amplitude and $b(\cdot)$ is 1-periodic (see [K6]). Clearly, in all these problems it is natural to consider the case where the regularity of ρ is only $\rho' \in L^2(\mathbb{R}/\mathbb{Z})$. We remark, however, that the natural borderline case $\log \rho \in L^2$ is too singular to be treated by the methods of this paper. It would require basically new asymptotic estimates on the fundamental solution.

As a reference to results on the weighted operator and its history we refer to [A], [CML], [Kr], without any claim to completeness. In particular, many applications to physics are mentioned in [A].

2 Preliminaries

Below we need some properties of the fundamental solutions from [K1-2]. Denoting the Wronskian by $W(f,g) = fg' - f'g$, we have

$$W(\vartheta(x,z,q), \varphi(x,z,q)) = \rho(x)^{-2}, \quad x \in [0,1], \quad z \in \mathbb{C}, \quad q \in H_C. \tag{2.1}$$

For each $x \in [0,1]$ the functions $\varphi(x,z,q), \vartheta(x,z,q)$ are entire on $\mathbb{C} \times H_C$. Moreover, φ and ϑ are analytic as a map from $\mathbb{C} \times H_C$ into $W^2_{2C}(0,1)$ and the following estimates are fulfilled

$$\max\{|z\varphi(x,z,q)|, |\varphi'(x,z,q)|, |\vartheta(x,z,q)|, |z^{-1}\vartheta'(x,z,q)|\} \leqslant e^{|\operatorname{Im} z|x + 2\|q\|_C},$$

and their derivatives with respect to q have the forms

$$\partial\varphi'(1,z,q) = 2\varphi'(t,z,q)P_x(1,t,z,q), \quad x \in [0,1], \quad z \in \mathbb{C}, \quad q \in H_C, \tag{2.2}$$

and so on, where $P(x,t,z,q) = \rho(t)^2(\vartheta(x,z,q)\varphi(t,z,q) - \vartheta(t,z,q)\varphi(x,z,q))$. If the sequence q^ν converges weakly to q in H_C, as $\nu \to \infty$, then $\varphi'(x,z,q^\nu) \to \varphi'(x,z,q)$, uniformly on bounded subsets of $[0,1] \times \mathbb{C}$. The same kind of such convergence holds for the functions $\vartheta, \varphi, \vartheta'$.

We need some properties of the Lyapunov function from [K2]. The function $\Delta(z,q)$ is even with respect to $q \in H_C$ and with respect to $z \in \mathbb{C}$:

$$\Delta(z,q) = \Delta(z,-q) = \Delta(-z,q), \qquad q \in H_C, \quad z \in \mathbb{C}. \tag{2.3}$$

Define the function

$$\Delta_2(z,q) = \Delta(z,q) - \cos z - \Delta_1(z,q)$$

where

$$\Delta_1(z,q) = \int_0^1 q(t)dt \int_0^t q(s)\cos z(1-2t+2s)ds.$$

We need the properties of the function Δ from [K2].

Lemma 2.1. *i) If the sequence $q^\nu, \nu \geqslant 1$, converges weakly to q in H, then $\Delta(z,q^\nu) \to \Delta(z,q)$ uniformly on bounded subsets of $\mathbb{C}$. The function $\Delta(\cdot,\cdot)$ is entire on $\mathbb{C}\times H_C$, and the following estimates are fulfilled*

$$\max\{|\Delta(z,q)|, |\partial\Delta(t,z,q)|\} \leqslant e^{|\operatorname{Im} z|x + 2\|q\|_C}. \tag{2.4}$$

ii) If $|\gamma_n(q)|=0$ for some $q\in H$, then $\partial\Delta(z,q)=0$ at $z=m_n(q)$.
iii) Moreover, for any fixed $d>1$, the following asymptotic estimates are fulfilled:

$$\Delta_1(z,q)=\ell^d(n),\quad \Delta_2(z,q)=(\ell^d(n))^2,\quad \text{as}\quad |z|\to\infty,\ |z-\pi n|\leqslant\pi, \tag{2.5}$$

$$\Delta_1(\pi n,q)=\frac{(-1)^n}{2}q_n^2=\frac{(-1)^n}{2}(q_{cn}^2+q_{sn}^2), \tag{2.6}$$

and

$$(\partial\Delta(z,q))(t)=\int_0^1 q(x+t)\cos z(2x-1)dx+\ell^d(n),\quad |z-\pi n|\leqslant\pi,\quad n\in\mathbb{Z}, \tag{2.7}$$

$$(\partial\Delta(\pi n+\ell^2(n),q))(t)=d_q q_n^2+\ell^d(n), \tag{2.8}$$

$$(\partial\dot\Delta(\pi n,q))(t)=\ell^2(n),\quad (\partial\ddot\Delta(\pi n,q))(t)=\ell^2(n), \tag{2.9}$$

as $|n|\to\infty$. These estimates are infinitely differentiable with respect to z and are satisfied uniformly on bounded subsets of $[0,1]\times H_C$.

For given q and $\varepsilon>0$ we introduce the regularised smooth functions $\widetilde q,\widetilde\rho$ and the constant N_q by the following formulas

$$\widetilde\rho(x)=\exp\int_0^x\widetilde q(t)dt,\quad \widetilde q(x)=\sum_{-m}^{m}(q_{cn}-iq_{sn})e^{i2\pi nx},\qquad N_q=1+\frac{2^7}{\pi}[\|\widetilde q'\|+\|q\|^2]e^{2\|q\|}, \tag{2.10}$$

where m is defined by the following condition: $\|q-\widetilde q\|\leqslant\varepsilon$. The function $\widetilde q$ is smooth and the unitary transformation leads to the Hill operator with the smooth periodic potential

$$\widetilde V=\widetilde\rho''/\widetilde\rho=\widetilde q'+\widetilde q^2. \qquad . \tag{2.11}$$

Then

$$\int_0^1|\widetilde V(x)|dx\leqslant\int_0^1|\widetilde q(x)'|dx+\|\widetilde q\|^2\leqslant\|\widetilde q'\|+\|q\|^2. \tag{2.12}$$

For the operator $T_1f=-f''+\widetilde Vf$ in $L^2(\mathbb{R})$ we define the fundamental solutions y_1,y_2 of the following equation

$$-y''+\widetilde Vy=z^2y,$$

with the conditions

$$y_1(0,z,\widetilde V)=y_2'(0,z,\widetilde V)=1,\qquad y_1'(0,z,\widetilde V)=y_2(0,z,\widetilde V)=0.$$

Introduce the Lyapunov function by the formula $D(z,\widetilde V)=(1/2)(y_1(1,z,\widetilde V)+y_2'(1,z,\widetilde V))$ and it is well known that $D(z,\widetilde V)=\Delta(z,\widetilde q)$. Introduce the real Hilbert space $L^2((0,1),\rho^2dx)$ with the norm $\|f\|_{\rho^2}^2=\int_0^1 f^2(x)\rho^2(x)dx$. Below we shall use the following results on the Dirichlet problem for Eq. (1.12) on the interval $[0,1]$ from [K2].

Lemma 2.2. *i) Let $q\in H_C$ and $\varepsilon_q=8^{-2}\exp(-4\|q\|)$. Then for each integer $N>N_q$ and any $p\in B_C(q,\varepsilon_q)$ the function $\varphi(1,z,p)$ has exactly $2N$ roots, counted with multiplicity, in*

the disc $\{z : |z| < \pi(N+(1/2))\}$ and for each $|n| > N$, exactly one simple root in the domain $\{z : |z - \pi n| < 1\}$. There are no other roots.
ii) Each function $m_n(\cdot), n \geqslant 1$, is compact and real analytic on H. Its gradient is given by the formula

$$(d_q m_n(q))(x) = -\frac{\rho(x)^2}{2m_n(q)\|\varphi(\cdot, m_n(q), q)\|^2_{\rho^2}}(\varphi(x, m_n(q), q)^2)', \tag{2.13}$$

where

$$2m_n(q)\|\varphi(\cdot, m_n(q), q)\|^2_{\rho^2} = \varphi'(1, m_n(q), q)\dot{\varphi}(1, m_n(q), q) > 0. \tag{2.14}$$

Moreover, for any fixed $d > 1$ the following asymptotic estimates are fulfilled:

$$m_n(q) = \pi n - q_{sn} + \ell^d(n), \tag{2.15}$$

$$(d_q m_n(q))(x) = -\sin 2\pi n x + \ell^2(n), \tag{2.16}$$

as $n \to \infty$, uniformly on $[0,1] \times B_C(p, \varepsilon_p)$ for each fixed $p \in H$.

We need the following results concerning the mapping $h_{cn}(\cdot)$ from [K2]. Here we use the following functions

$$a_n = 2\rho^2(x)\varphi'(x, m_n(q), q)\vartheta(x, m_n(q), q)$$

which will be important below.

Lemma 2.3. *Each function $h_{cn}(\cdot), n \geqslant 1$, is compact and real analytic on H. Its gradient is given by the formulas*

$$d_q h_{cn} = -\frac{1}{\varphi'(1, m_n, q)}\{\dot{\varphi}'(1, m_n, q)d_q m_n + \partial\varphi'(1, m_n, q)\}, \tag{2.17}$$

$$d_q h_{cn} = \frac{(-1)^n}{\sinh h_{cn}}\{\dot{\Delta}(m_n, q)d_q m_n + \partial\Delta(\mu_n, q)\}, \quad h_{cn} \neq 0, \tag{2.18}$$

$$d_q h_{cn} = C_n(q)d_q m_n + a_n(q), \qquad C_n(q) = -\dot{\vartheta}(1, m_n, q)\varphi'(1, m_n, q), \tag{2.19}$$

where $m_n = m_n(q)$. Moreover, for any fixed $d > 1$ the following estimates are fulfilled:

$$h_{cn}(q) = q_{cn} + \ell^d(n), \tag{2.20}$$

$$(d_q h_{cn}(q))(x) = \cos 2\pi n x + \ell^2(n), \tag{2.21}$$

as $n \to \infty$, uniformly on $[0,1] \times B_C(p, \varepsilon_p)$ for each fixed $p \in H$.

Proof. We have to show only (2.19), since the others were proved in [K2]. Let $\varphi(t) = \varphi(t, m_n, q), \varphi'(t) = \varphi'(t, m_n, q), \vartheta(t) = \vartheta(t, m_n, q), \vartheta'(t) = \vartheta'(t, m_n, q)$ and so on. Using (2.17), and (2.2) we obtain

$$d_q h_{cn} = -\frac{\dot{\varphi}'(1)}{\varphi'(1)}d_q m_n - 2\rho^2(t)\varphi'(t)\frac{\vartheta'(1)\varphi(t) - \vartheta(t)\varphi'(1)}{\varphi'(1)}$$

$$= -\frac{\dot{\varphi}'(1)}{\varphi'(1)}d_q m_n - \rho^2(t)(\varphi^2(t))'\frac{\vartheta'(1)}{\varphi'(1)} + a_n.$$

Then the identities $\vartheta(1,z,q)\varphi'(1,z,q)-\vartheta'(1,z,q)\varphi(1,z,q)=1$ and (2.13-14) imply

$$d_q h_{cn}=-\frac{\dot\varphi'(1)}{\varphi'(1)}d_q m_n-\frac{\rho^2(t)(\varphi^2(t))'}{\varphi'(1)\dot\varphi(1)}+a_n$$

$$=-(\dot\varphi(1)\vartheta'(1)-\dot\varphi'(1)\vartheta(1))d_q m_n+a_n(q)=-\dot\vartheta(1)\varphi'(1)d_q m_n+a_n.\square$$

We need the following result concerning the vectors $a_n, d_q m_n$. Define the formal symplectic form

$$f\wedge g=\int_0^1 f(x)\frac{d}{dx}g(x)dx, \quad ,$$

where $f,g\in H$ and note that $f\wedge g=-g\wedge f$. Recall $a_n=2\rho^2(x)\varphi'(x,m_n,q)\vartheta(x,m_n,q)$.
Lemma 2.4. *For any $n,p\geqslant 1$, the following identities are fulfilled*

$$d_q m_n(q)\wedge d_q m_p(q)=0, \tag{2.22}$$

$$a_n(q)\wedge d_q m_p(q)=-m_n\delta_{np}, \tag{2.23}$$

$$a_n(q)\wedge a_p(q)=0. \tag{2.24}$$

Proof. We need the following identities. Let y_n, y_p be some solution of Eq. (1.12) with μ_n,μ_p respectively, then the derivative of the Wronskian $W=W(y_n,y_p)$ has the form

$$W'=W(y_n,y_p)'=y_n'y_p'+y_ny_p''-y_n'y_p'-y_n''y_p=2q(y_n'y_p-y_ny_p')+(\mu_n-\mu_p)y_ny_p,$$

and then we have

$$W'+2qW=(\mu_n-\mu_p)y_ny_p, \qquad W=W(y_n,y_p). \tag{2.25}$$

Define the following function

$$R=R(y_n,y_p)=\frac{\mu_ny_ny_p'-\mu_py_n'y_p}{\mu_n-\mu_p}.$$

Then using (1.12) we have

$$R'=(\mu_ny_ny_p'-\mu_py_n'y_p)'=\mu_ny_ny_p''+\mu_ny_n'y_p'-\mu_py_n''y_p-\mu_py_n'y_p'$$

$$=(-2qy_p'-\mu_py_p)y_n\mu_n+(2qy_n'+\mu_ny_n)y_p\mu_p+(\mu_n-\mu_p)y_n'y_p'$$

which yields

$$R'+2qR=y_n'y_p', \quad R=R(y_n,y_p). \tag{2.26}$$

Using again (1.12) we obtain

$$(\rho^2(y_n^2)')'=2((\rho^2y_n')y_n)'=2\rho^2((y_n')^2-\mu_ny_n^2). \tag{2.27}$$

We introduce the normalized Dirichlet eigenfunctions $f_n\equiv r_n\varphi(t,m_n(q),q)$ where the constant $r_n>0$ is such that $\|f_n\|_{\rho^2}=1$. Let $n\neq p$, since the case $n=p$ is simple, then we get

$$d_q\mu_n(q)\wedge d_q\mu_p(q)=\frac{1}{2}\int_0^1 v_1(t)dt, \qquad v_1=d_q\mu_n(d_q\mu_p)'-(d_q\mu_n)'d_q\mu_p, \tag{2.28}$$

and by (2.13), (2.27),

$$v_1 = \rho^2 (f_n^2)'(\rho^2 (f_p^2)')' - \rho^2 (f_p^2)'(\rho^2 (f_n^2)')'$$
$$= 4\rho^4 [f_n f_n' ((f_p')^2 - \mu_p f_p^2) - f_p f_p' ((f_n')^2 - \mu_n f_n^2)]$$
$$= 4\rho^4 [f_n' f_p' W(f_n, f_p) + f_n f_p (\mu_n f_n f_p' - \mu_p f_p f_n')].$$

Let $W_1 = W(f_n, f_p), R_1 = R(f_n, f_p)$, then (2.25-26) imply

$$v_1(t) = 4\rho^4 [f_n' f_p' W_1 + (W_1' + 2qW_1)R_1] = 4\rho^4 [(R_1' + 2qR_1)W_1 + (W_1' + 2qW_1)R_1] = 4(\rho^4 W_1 R_1)'$$

The substitution of this identity into (2.28) gives

$$\int_0^1 v_1(t)dt = 2(\rho^4 W_1 R_1)|_0^1 = 0,$$

and (2.22) is proved. In order to show (2.24), we need the following identity

$$a_n \wedge a_p = \frac{1}{2} \int_0^1 v_3(t)dt, \qquad v_3(t) = a_n a_p' - a_n' a_p, \quad n \neq p,$$

since $n = p$ is simple. Let $W_3 = W(\vartheta_n, \vartheta_p), R_3 = R(\varphi_n, \vartheta_p)$. From (1.12) we obtain

$$v_3 = 2\rho^2 \varphi_n' \vartheta_n (2\rho^2 \varphi_p' \vartheta_p)' - (2\rho^2 \varphi_n' \vartheta_n)' 2\rho^2 \varphi_p' \vartheta_p$$

$$= 4\rho^4 \Big[\varphi_n' \vartheta_n (\varphi_p' \vartheta_p' - \mu_p \varphi_p \vartheta_p) - \varphi_p' \vartheta_p (\varphi_n' \vartheta_n' - \mu_n \varphi_n \vartheta_n)\Big]$$

$$= 4\rho^4 \Big[\vartheta_n \vartheta_p (\mu_n \varphi_n \varphi_p' - \mu_p \varphi_p \varphi_n') + \varphi_p' \varphi_p' W_3\Big] = 4\rho^4 \Big[\vartheta_n \vartheta_p R_3 + \varphi_p' \varphi_p' W_3\Big].$$

Then (2.25-26) imply

$$v_3 = 4\rho^4 \Big[(W_3' + 2qW_3)R + W_3(R_3' + 2qR_3)\Big] = (4\rho^4 W_3 R_3)',$$

which yields

$$a_n \wedge a_p = \frac{1}{2} \int_0^1 v_3(t)dt = \frac{1}{2} \int_0^1 (4\rho^4 W_3 R_3)' dt = 0.$$

Consider the case of (2.23). Using the simple identity

$$a_n \wedge d_q \mu_p = \frac{1}{2} \int_0^1 v_2(t)dt, \quad v_2(t) = a_n (d_q \mu_p)' - a_n' (d_q \mu_p),$$

we deduce that

$$v_2 = -2\rho^2 \varphi_n' \vartheta_n (\rho^2 (f_p^2)')' + (2\rho^2 \varphi_n' \vartheta_n)' \rho^2 (f_p^2)'$$

$$= 2\rho^4 \Big[(f_p^2)'(\varphi_n' \vartheta_n' - \mu_n \varphi_n \vartheta_n) - 2\varphi_n' \vartheta_n ((f_p')^2 - \mu_p f_p^2)\Big].$$

Then we have

$$v_2 = 4\rho^4 r_n^2 \Big[\varphi_n'\varphi_p'(\vartheta_n'\varphi_p - \vartheta_n\varphi_n') + \vartheta_n\varphi_p(\mu_p\varphi_n'\varphi_p - \mu_n\varphi_n\varphi_p')\Big]. \tag{2.29}$$

Now consider the case $n = p$. Then by (2.1), (2.29), $v_2 = -4\rho^2(f_n')^2$, and we have

$$a_n \wedge d_q\mu_n = -2\int_0^1 \rho^2(f_n')^2 dt = -2\mu_n \int_0^1 \rho^2(t) f_n^2(t) dt = -2\mu_n,$$

which yields (2.23), as $n = p$. Now consider the case $n \neq p$. Let $W_2 = W(\vartheta_n, \varphi_p), R_2 = R(\varphi_n, \vartheta_p)$. Then by (2.29), (2.25-26),

$$v_2 = -4\rho^4 r_n^2 \Big[\varphi_n'\varphi_p' W_2 + \vartheta_n\varphi_p R_2(\mu_n - \mu_p)\Big]$$

$$= -4\rho^4 p_n^2 \Big[(R_2' + 2qR_2)W_2 + (W_2' + 2qW_2)R_2\Big] = -4(\rho^4 W_2 R_2)'$$

and we get (2.23). □

Remark that Lemma 2.5 gives a new approach to the results in [CML]. The gradients of $m_n(q), h_{ns}$ satisfy the following relations which we will use later on.

Lemma 2.5. *For any $n, p \geqslant 1$, the following identities are fulfilled*

$$d_q m_n(q) \wedge d_q m_p(q) = 0, \tag{2.30}$$

$$d_q h_{cn}(q) \wedge d_q m_p(q) = -m_p\delta_{np}, \tag{2.31}$$

$$d_q h_{cn}(q) \wedge d_q h_{cp}(q) = 0. \tag{2.32}$$

Moreover, the sequence $\{d_q m_n(q), d_q h_{cn}(q),\ n \geqslant 1\}$, is a basis for H.

Proof. Having established the crucial Lemma 2.4, we follow the arguments in [PT]. For example, using Lemma 2.4 we obtain

$$d_q h_{n2} \wedge d_q m_p = \Big(a_n + C_n(q) d_q m_n)\Big) \wedge d_q m_p = a_n \wedge d_q m_p(q) = -m_p\delta_{np},$$

The proof of other identities is similar.

Using (2.21), (2.16) for the sequence $\{d_q m_n(q), d_q h_{cn}(q),\ n \geqslant 1\}$, we obtain

$$(d_q h_{cn}(q))(x) = \cos 2\pi n x + \ell^2(n), \qquad (d_q m_n(q))(x) = -\sin 2\pi n x + \ell^2(n).$$

The vectors on the right, without the error terms, are an orthonormal basis of H and the error terms are square summable. By (2.30-32), these vectors are linear independent, since for each vector of them there exist another one, which is perpendicular to all vectors except the given one. Then by well known result from the functional analysis (see [PT]) the sequence $\{d_q m_n(q), d_q h_{cn}(q),\ n \geqslant 1\}$, is a basis for H. □

3 Band edges in spectral planes

For $q \in H$ we introduce the domain $U_n^{\pm}(q) = \{z : |z - z_n^0(q)| < r_n^{\pm}\}$, and the contours

$$k_n(q) = \{z : |z - z_n^0(q)| = \frac{r_n^- + r_n^+}{2}\}, \quad r_n^{\pm} = \frac{1}{2}|\gamma_n(q)| + \frac{2 \pm 1}{6}\min\{|\sigma_n|, |\sigma_{n+1}|\}, \quad n \in \mathbb{Z}.$$

Let $\hat{q}_n = \sqrt{q_n^2}$. In the following Lemma we study general properties of $z_n^{\pm}(q)$.

Lemma 3.1. *i) Let $q \in H_C$ and $\varepsilon_q = 4^{-4}\exp(-4\|q\|)$. Then for each integer $N > N_q$ and any $p \in B_C(q, \varepsilon_q)$ the function $\Delta(z,p)^2 - 1$ has exactly $4N+2$ roots, counted with multiplicity, in the disc $\{z : |z| < \pi(N+(1/2))\}$ and for each $|n| > N$, exactly 2 roots $\widetilde{z}_n^{\pm}(q)$ in the domain $\{z : |z - \pi n| < 1\}$. There are no other roots and for any fixed $d > 1$, the following asymptotic estimates are fulfilled:*

$$\widetilde{z}_n^{\pm}(q) = \pi n \pm \sqrt{q_n^2} + \ell^d(n), \quad |n| \to \infty, \tag{3.1}$$

uniformly on $[0,1] \times B_C(p, \varepsilon_p)$ for each fixed $p \in H$.
ii) For each $p \in H$ and any $n \geqslant 1$ there exists $\varepsilon \in (0,1]$ such that for any $q \in B_C(p, \varepsilon)$ the function $\Delta(z,q)^2 - 1$ has exactly 2 roots, counted with multiplicity, in the disc $U_n^{\pm}(p)$.
iii) Each function $z_n^{\pm}(\cdot), n \geqslant 1$, is compact on H.

Remark. *Note that the function $\widetilde{z}_n^{\pm}(q)$ is complicated and is not real analytic. Below we need only their sum $\widetilde{z}_n^-(q) + \widetilde{z}_n^+(q) = z_n^-(q) + z_n^+(q)$ which is analytic, and we study this function in Lemma 3.3.*

Proof. i) Fix $N > N_q$, and let $N_1 > N$ be another integer. Consider the contours

$$|z| = \pi(N + \frac{1}{2}), \quad |z| = \pi(N_1 + \frac{1}{2}), \quad |z - \pi n| = 1, \quad |n| > N.$$

We have $\Delta(z, \widetilde{q}) = D(z, V)$ where $V = \widetilde{q}' + \widetilde{q}^2, \varepsilon = \varepsilon_q$ (see (2.11)). In the book [PT] there exists the following estimate

$$|D(z, \widetilde{q}) - \cos z| \leqslant Ae^{A + |\operatorname{Im} z|}, \quad A = \frac{1}{|z|}\int_0^1 |V(x)|dx,$$

and note that

$$A \leqslant \frac{1}{\pi 4^3} \equiv a.$$

Then using these estimates we have

$$|\Delta^2(z, \widetilde{q})) - \cos^2 z| \leqslant e^{2|\operatorname{Im} z|}Ae^A(2 + Ae^A) \leqslant 4^{-2}e^{2|\operatorname{Im} z|}C,$$

where

$$C = ae^a(2 + ae^a) \leqslant \frac{1}{3}.$$

Hence using the simple estimate $\exp(|\operatorname{Im} z|) \leqslant 4|\sin z|$ as $|z - \pi m| \geqslant \pi/4, m \in \mathbb{Z}$, (see [PT]) we obtain

$$|\Delta^2(z, \widetilde{q})) - \cos^2 z| \leqslant \frac{1}{3}|\sin^2 z|, \tag{3.2}$$

on all contours. We have the equality

$$\Delta(z,p)-\Delta(z,\widetilde{q})=\int_0^1(\partial\Delta(z,\widetilde{q}+tv),v)dt,\qquad v=p-\widetilde{q},\tag{3.3}$$

where $(\cdot,\cdot)$ is the scalar product in H_C. Estimates (2.4) yields

$$|\partial\Delta(z,\widetilde{q}+tv)|\leqslant e^{|\operatorname{Im}z|+2\|\widetilde{q}+tv\|}\tag{3.4}$$

and the simple inequalities $\|v\|\leqslant\|p-q\|+\|q-\widetilde{q}\|\leqslant 2\varepsilon$ and $\|\widetilde{q}+tv\|\leqslant\|q\|+2\varepsilon$ give

$$|\Delta(z,p)-\Delta(z,\widetilde{q})|\leqslant 2\varepsilon e^{|\operatorname{Im}z|+2\|q\|+4\varepsilon}.\tag{3.5}$$

Then using (3.5), (2.4) we get

$$|\Delta^2(z,p)-\Delta^2(z,\widetilde{q})|\leqslant[|\Delta(z,p)|+|\Delta(z,\widetilde{q})|]2\varepsilon e^{|\operatorname{Im}z|+2\|q\|+4\varepsilon}$$

$$\leqslant 4\varepsilon e^{2|\operatorname{Im}z|+4\|q\|+6\varepsilon}\leqslant\frac{1}{8}e^{2|\operatorname{Im}z|},$$

and again due to the simple estimate $\exp(|\operatorname{Im}z|)\leqslant 4|\sin z|$ as $|z-\pi m|\geqslant\pi/4, m\in\mathbb{Z}$, (see [PT]) we obtain

$$|\Delta^2(z,p)-\Delta^2(z,\widetilde{q})|\leqslant\frac{1}{2}|\sin^2 z|.\tag{3.6}$$

Using estimates (3.2), (3.6) we obtain

$$|(1-\Delta^2(z,p))-\sin^2 z|\leqslant|\Delta^2(z,p)-\Delta^2(z,\widetilde{q})|+|\Delta^2(z,\widetilde{q}))-\cos^2 z|<\frac{5}{6}|\sin^2 z|$$

on all contours. Hence, by Rouche's theorem, $\Delta(z,q)^2-1$ has as many roots, counted with multiplicity, as $\sin^2 z$ in each of the bounded regions and the remaining unbounded domain. Since $\sin^2 z$ has only the roots $\pi n, n\in\mathbb{Z}$, and since $N_1>N$ can be chosen arbitrarily large, i) of the lemma follows, without (3.1).
Now we improve the last result. Using (2.5) we obtain

$$\Delta(z,q)=\cos z+o(1),\quad|z-\pi n|\leqslant 1,\quad|z|\to\infty,\tag{3.7}$$

which implies

$$\widetilde{z}_n^{\pm}=\pi n+o(1),\quad|n|\to\infty.\tag{3.8}$$

We improve the last result again. Let $r_n=\widetilde{z}_n^{\pm}-\pi n$. Then using (2.5) we deduce that

$$(-1)^n=(-1)^n\cos r_n+\ell^d(n),$$

for any fixed $d>1$, and $2\sin^2(r_n/2)=\ell^d(n)$, which yields

$$r_n^2=\ell^d(n),\quad|n|\to\infty.\tag{3.9}$$

Using again (2.5-6), (3.9) we get the following asymptotic estimates

$$2\sin^2(r_n/2) = (-1)^n \Delta_1(\pi n + r_n, q) + (\ell^d(n))^2,$$

$$(-1)^n \Delta_1(\pi n + r_n, q) = (1/2)q_n^2 + x_n r_n + (\ell^d(n))^2, \qquad x_n = (-1)^n \dot{\Delta}_1(\pi n, q) = \ell^d(n) \quad (3.10)$$

and then last relations and (3.9-10) yield

$$r_n^2 = q_n^2 + 2x_n r_n + (\ell^d(n))^2, \quad n \to \infty,$$

which implies

$$r_n = x_n \pm \sqrt{q_n^2 + (\ell^d(n))^2} = \pm\sqrt{q_n^2} + \ell^d(n), \quad n \to \infty.$$

Using last estimates we have (3.1).

ii) The analyticity (see Lemma 2.1) of Δ yields

$$\sup_{z \in U_n^{\pm}(p), \|q-p\|_{H_C} \leqslant \varepsilon} |\Delta(z,q) - \Delta(z,p)| < C_0 \varepsilon, \tag{3.11}$$

for some small $\varepsilon > 0$ and constant $C_0 = C_0(p,n)$. Introduce the constants

$$C_1 = \max_{z \in \partial U_n^{\pm}(p)} |\Delta(z,p)|, \quad C_2 = \min_{z \in \partial U_n^{\pm}(p)} |\Delta(z,p)^2 - 1| > 0.$$

Let $S = \varepsilon(C_0/C_2)(C_0\varepsilon + 2C_1)$, then estimate (3.11) implies

$$|(\Delta(z,q)^2 - 1) - (\Delta(z,p)^2 - 1)| < C_0\varepsilon(C_0\varepsilon + 2C_1) \leqslant S|\Delta(\lambda,p)^2 - 1|,$$

where $\|q-p\|_{H_C} \leqslant \varepsilon, z \in \partial U_n^{\pm}(p)$, and if we take the number ε such that $S < 1$ hence by Rouche's theorem, $\Delta(z,q)^2 - 1$ has as many roots, counted with multiplicity, as $\Delta(z,p)^2 - 1$ in the disc $U_n^{\pm}(p)$. Since $\Delta(z,p)^2 - 1$ has two roots, counted with multiplicity, we have proved ii).

iii) In order to verify compactness of $z_n^{\pm}$ suppose a sequence $q^{\nu}, \nu \geqslant 1$, converges weakly to q in H. For some q and $N > 1, \varepsilon > 0$, we introduce the set $I = (\cup I_n^0) \cup (\cup I_n^{\pm})$ and the intervals:

$$I_n^{\pm} = [z_n^{\pm}(q) - \varepsilon, z_n^{\pm}(q) + \varepsilon], \quad \text{if} \quad z_n^-(q) \neq z_n^+(q)$$

$$I_n^0 = [z_n^+(q) - \varepsilon, z_n^+(q) + \varepsilon], \quad \text{if} \quad z_n^-(q) = z_n^+(q), \quad -N \leqslant n \leqslant N.$$

If ε is sufficiently small, then these intervals are all disjoint and contained in the disk $D = \{z : |z| < r\}$ for some $r > 0$. The function $f(z,q) \equiv \Delta^2(z,q) - 1$ changes sign on each $I_n^{\pm}$, that is $f(z,q)$ has a simple root on each $I_n^{\pm}$ and $|\dot{f}(z,q)| > B > 0$ on each $I_n^{\pm}$, for some $B > 0$. Moreover, the function $f(z,q)$ has a root of multiplicity 2 and $|\ddot{f}(z,q)| > B > 0$ on each I_n^0.

The entire functions $f(z,q^{\nu})$ converge to $f(z,q)$, as $\nu \to \infty$, uniformly on D by Lemma 2.1. Then, for sufficiently large ν, any function $f(z,q^{\nu})$ has a simple root in each interval $I_n^{\pm}$ and $f(z,q^{\nu})$ has exactly two roots in each interval I_n^0, which must be the eigenvalues $z_n^{\pm}(q^{\nu})$. This yields $|z_n^{\pm}(q^{\nu}) - z_n^{\pm}(q)| < \varepsilon, |n| \leqslant N$, for all sufficiently large ν. It follows that

$z_n^\pm(q^\nu) \to z_n^\pm(q)$, $\nu \to \infty$, since N and $\varepsilon > 0$ were arbitrary. Thus, $z_n^\pm(\cdot)$ are the compact functions of q. $\square$

Introduce the functions

$$F(z,q) \equiv \frac{\Delta(z,q)\Delta_z(z,q)}{\Delta^2(z,q)-1}, \qquad G(z,q) \equiv \frac{\Delta(z,q)\partial\Delta(z,q)}{\Delta^2(z,q)-1},$$

and the contour $c_n(r) = \{z : |z-\pi n| = r\}, n \geqslant 0, r > 0$. Define $z_n^0 = (z_n^- + z_n^+)/2$. In order to study ϕ_n, ψ_n we have to consider the function $z_n^0(q), q \in H$. Introduce the functions

$$v_n^\pm(q) = \frac{\partial\Delta(z_n^\pm, q)}{\Delta_z(z_n^\pm, q)}, \qquad z_n^- \neq z_n^+, \quad \text{and} \quad v_n^0(q) = \frac{\partial\dot\Delta(z_n^0, q)}{\ddot\Delta(z_n^0, q)}, \qquad z_n^0 = z_n^- = z_n^+.$$

We prove the needed properties of the center of the gap.

Lemma 3.2. *Each function $z_n^0(\cdot), n \geqslant 1$, is compact and real analytic on H. Its gradient is given by the formulas*

$$d_q z_n^0 = -\frac{1}{2}(v_n^-(q) + v_n^+(q)), \qquad z_n^- \neq z_n^+, \tag{3.12}$$

$$d_q z_n^0 = -v_n^0(q), \qquad z_n^0 = z_n^- = z_n^+. \tag{3.13}$$

Moreover, for any fixed $d > 1$, the following asymptotic estimates are fulfilled:

$$z_n^0 = \pi n + \ell^d(n), \tag{3.14}$$

$$d_q z_n^0 = \partial\dot\Delta(\pi n, q) + \ell^d(n) = \ell^2(n), \tag{3.15}$$

$n \to \infty$ uniformly on $[0,1] \times B_C(p, \varepsilon_p)$ for any fixed $p \in H$.

Proof. By Lemma 3.1, the mapping $z_n^0(\cdot)$ is compact.

To prove real analyticity, fix $p \in H$ and consider $z_n^0(q)$ for some $n \geqslant 1$. By Lemma 3.1, there exists $\varepsilon \in (0,1)$ such that for all $q \in B_C(p,\varepsilon)$ the function $\Delta(z,q)^2 - 1$ has exactly two roots $\widetilde{z}_n^\pm(q)$ in each disc $U_n^-(p) \subset U_n^+(p)$. We have the identity

$$z_n^0(q) = \frac{1}{2\pi i}\int_{k_n(p)} zF(z,q)dz, \tag{3.16}$$

for all real $q \in B(p, \varepsilon_p)$. The function $F(z,q), z \in k_n(p),\ q \in B(p,\varepsilon)$ has analytic extension on the ball $B_C(p,\varepsilon_p)$, then by (3.16), $z_n^0(\cdot)$ also has analytic extension on the ball $B_C(p,\varepsilon)$. Hence $z_n^0(\cdot)$ is a real analytic function on H.

In order to calculate the gradient we use the residue theorem. Using (3.16) and integrating by parts, we deduce that

$$d_q z_n^0 = \frac{1}{2\pi i}\int_{k_n} z\partial F(z,q)dz = -\frac{1}{2\pi i}\int_{k_n} G(z,q)dz, \tag{3.17}$$

which implies (3.12-13). Note that if $z_n^-(q) = z_n^+(q)$, then Lemma 2.1 yields $\partial\Delta(z_n^-, q) = 0$.

We show (3.14-15). First, (3.1) yields (3.14). Let $q \in B_C(p, \varepsilon_p)$. Then by i) of Lemma 3.1, $|\widetilde{z}_n^{\pm}(q) - \pi n| < 1$ for all $|n| > N_p$ that is the distance between $\widetilde{z}_n^{\pm}(q)$ and the contour $c_n = c_n(r), r = 3/2$, is greater than $1/2$. Rewriting integral (3.17) in the form

$$d_q z_n^0 = \frac{1}{2\pi i} \int_{c_n} z \partial F(z, q) dz = -\frac{1}{2\pi i} \int_{c_n} G(z, q) dz$$

and substituting estimates (2.10), (2.12), we obtain

$$d_q z_n^0 = -\frac{1}{2\pi i} \int_{c_n(r)} \frac{(\cos z + \ell^d(n)) \partial \Delta(z, q) dz}{-\sin^2 z + \ell^d(n)} = \partial \dot{\Delta}(\pi n, q) + \ell^d(n) = \ell^2(n),$$

as $n \to \infty$, uniformly on $[0, 1] \times B_C(p, \varepsilon_p)$. $\square$

In the following Lemma we will study the function $b_n = (z_n^+ - z_n^-)^2/4$.

Lemma 3.3. *Each function $b_n = |g_n(\cdot)|^2/4, n \geqslant 1$, is compact and real analytic on H. Its gradient is given by the formulas*

$$d_q b_n = -\frac{(z_n^+(q) - z_n^-(q))}{2} [v_n^+(q) - v_n^-(q)], \qquad z_n^-(q) \neq z_z^+(q), \tag{3.18}$$

$$d_q b_n(q) = 0, \qquad z_n^-(q) = z_n^+(q). \tag{3.19}$$

Moreover, for any fixed $d > 1$, the following asymptotic estimates are fulfilled:

$$b_n = q_n^2 + \ell^d(n) \ell^2(n), \tag{3.20}$$

$$d_q b_n = d_q \hat{q}_n^2 + \ell^d(n), \tag{3.21}$$

$n \to \infty$, uniformly on $[0, 1] \times B_C(p, \varepsilon_p)$ for any fixed $p \in H$.

Proof. The proof for b_n repeats the one for z_n^0. By Lemma 3.1, the mapping $b_n(\cdot)$ is compact.

To prove real analyticity, fix $p \in H$ and consider $b_n(q)$ for some $n \geqslant 1$. By Lemma 3,1, there exists $\varepsilon \in (0, 1)$ such that for all $q \in B_C(p, \varepsilon)$ the function $\Delta(z, q)^2 - 1$ has exactly two roots in each disc $U_n^-(p) \subset U_n^+(p)$. We have the identity

$$b_n = \frac{1}{4\pi i} \int_{k_n(p)} (z - z_n^0(q))^2 F(z, q) dz, \tag{3.22}$$

for all real $q \in B(p, \varepsilon)$. The function $F(z, q), z \in k_n(p), q \in B(p, \varepsilon)$ has analytic extension on the ball $B_C(p, \varepsilon)$. The function $b_n(\cdot)$ also has analytic extension on the ball $B_C(p, \varepsilon)$. Hence by (3.22), $b_n(\cdot)$ is a real analytic function on H.

Differentiating (3.22) we obtain

$$d_q b_n = \frac{1}{4\pi i} \int_{k_n} 2(z_n^0(q) - z) F(z, q) dz (d_q z_n^0) + \frac{1}{4\pi i} \int_{k_n} (z_n^0(q) - z)^2 \partial F(z, q) dz, \tag{3.23}$$

and integrating by parts we deduce that

$$d_q b_n = \frac{1}{2\pi i} \int_{k_n} (z_n^0(q) - z)(F(z, q) d_q z_n^0 + G(z, q)) dz, \tag{3.24}$$

hence using the residue theorem we have (3.18-19). Note that if $z_n^- = z_n^+$ then we get $\partial\Delta(z_n^-, q) = 0$, (see Lemma 2.1).

Using (3.1) we obtain (3.20). To prove (3.21) fix $q \in B_C(p, \varepsilon_p$, for some $p \in H$. Then by i) of Lemma 3.1, $|\widetilde{z}_n^{\pm}(q) - (\pi n)| < 1$ for all $|n| > N_p$, that is the distance between $\widetilde{z}_n^{\pm}(q)$ and the contour $c_n(r), r = 3/2$, is greater than $1/2$. We rewrite integral (3.24) in the form

$$d_q b_n = \frac{1}{2\pi i} \int_{c_n(r)} (z_n^0(q) - z)\Delta(z, q) \frac{(\Delta_z(z, q) d_q z_n^0 + \partial\Delta(z, q))dz}{\Delta^2(z, q) - 1}. \tag{3.25}$$

Using (3.25), (3.14-15), (2.5), we have

$$d_q b_n = \frac{1}{\pi i} \int_{c_n(r)} \frac{(z - \pi n)\cos z}{\sin^2 z} (\sin z d_q z_n^0 - \partial\Delta(z, q))(1 + \ell^d(n))dz = (-1)^n \partial\Delta(\pi n, q) + \ell^d(n),$$

and (2.13) implies (3.21). □

4 Coordinate functions

Introduce the functions

$$f_n(z, q) = \frac{\Delta^2(z, q) - 1)}{w_n(z, q)}, \quad w_n(z) = (z_n^+ - z)(z - z_n^-) \quad z \in \mathbb{C}, \quad q \in H_C.$$

In order to get the analyticity of ϕ_n, ψ_n we need some properties of f_n.

Lemma 4.1. *Each function $f_n(\mu_n(\cdot), \cdot), n \geqslant 1$, is compact, real analytic and positive on H. Moreover, for any fixed $d > 1$, the following asymptotic estimates are fulfilled:*

$$f_n(m_n(q), q) = 1 + \ell^d(n), \tag{4.1}$$

$$d_q f_n(m_n(q), q) = \ell^2(n), \tag{4.2}$$

$n \to \infty$, uniformly on $[0, 1] \times B_C(p, \varepsilon_p)$ for any fixed $p \in H$.

Proof. In order to prove real analyticity, fix $p \in H$ and consider $m_n(q)$ for some $n \geqslant 1$. By Lemma 3.1, there exists $\varepsilon \in (0, 1)$ such that for all $q \in B_C(p, \varepsilon)$ the function $\Delta(z, q)^2 - 1$ has exactly two roots in each disc $U_n^-(p) \subset U_n^+(p)$. Moreover, by Lemma 2.2, there exists $\varepsilon \in (0, 1)$ such that for all $q \in B_C(p, \varepsilon)$ we get $m_n(q) \in U_n^-(p) \subset U_n^+(p)$. We have the identity

$$f_n(m_n(q), q) = \frac{1}{2\pi i} \int_{k_n} \frac{f_n(z, q)dz}{z - m_n(q)}, \tag{4.3}$$

for all real $q \in B(p, \varepsilon)$ and rewrite the function w_n in the form

$$w_n(z, q) = b_n(q) - (z_n^0(q) - z)^2. \tag{4.4}$$

Then by Lemmas 2.1-2,2 and Lemma 3.2-3, the function $f_n(z, q), z \in k_n(p), q \in B(p, \varepsilon)$ has analytic extension on the ball $B_C(p, \varepsilon)$. Therefore, by Lemma 2.3, $f_n(m_n(\cdot), \cdot)$ also has analytic extension on the ball $B_C(p,\varepsilon)$. Hence $f_n(m_n(\cdot),\cdot)$ is a real analytic function on H and by (4.3), Lemmas 2.1, 3.1, 3.3, 3.5, $f_n(m_n(\cdot),\cdot)$, is compact and positive on H.

To prove (4.1) fix $q\in B_C(p,\varepsilon_p)$, for some $p\in H$. Then by i) of Lemma 3.1, $|\widetilde{z}_n^{\pm}(q)-\pi n|<1$ for all $|n|>N_p$, that is the distance between $\widetilde{z}_n^{\pm}(q)$ and the contour $c_n(r), r=3/2$, is greater than $1/2$. Using (2.5), (3.14), (3.20), we obtain

$$f_n(z,q)=\frac{\sin^2 z+\ell^d(n)}{(z-\pi n)^2(z-m_n(q))},\quad z\in c_n(r),\ q\in B(p,\varepsilon_p). \tag{4.5}$$

Substituting (4.5) into (4.3), with $c_n=k_n$, we get

$$f_n(m_n(q),q)=\frac{1}{2\pi i}\int_{c_n}\frac{(\sin^2 z+\ell^d(n))dz}{(z-\pi n)^2(z-m_n)}=\frac{1}{2\pi i}\int_{c_n}\frac{(\sin^2 z+\ell^d(n))dz}{(z-\pi n)^3}=1+\ell^d(n),$$

Differentiating (4.3), with $c_n=k_n$, we have

$$d_qf_n(m_n(q),q)=\frac{1}{2\pi i}\int_{c_n}\frac{f_n(z,q)dz}{(z-m_n(q))^2}d_qm_n(q)+\frac{1}{2\pi i}\int_{c_n}\frac{\partial f_n(z,q)dz}{z-m_n(q)}, \tag{4.6}$$

Asymptotic estimates (4.5), (2.15) give

$$\frac{1}{2\pi i}\int_{c_n}\frac{f_n(z,q)dz}{(z-m_n(q))^2}=\frac{1}{2\pi i}\int_{c_n}\frac{(\sin^2 z+\ell^d(n))dz}{(z-\pi n)^2(z-m_n(q))^2}=\ell^2(n).$$

In order to estimate the second term in (4.6) we write ∂f_n in the form

$$\partial f_n(z,q)=\frac{1}{w_n(z,q)}\{-f_n(z,q)\partial w_n(z,q)+\partial\Delta(z,q)^2\},\quad z\in c_n(r). \tag{4.7}$$

Identity (4,4) and (3.20-21), (3.14-15) yield

$$\partial w_n(z,q)=d_qb_n+2(z-z_n^0)d_qz_n^0=d_qq_n^2+\ell^2(n)=\ell^2(n),\quad z\in c_n(r).$$

Then (4.5), (4.7), (3.20), (3.14) imply

$$\partial f_n(z,q)=-\frac{\partial\Delta(z,q)^2+\ell^2(n)}{(z-\pi n)^2},\quad z\in c_n(r).$$

Using these estimates and (2.5), (2.8-9), (2.15) we obtain

$$\int_{c_n}\frac{\partial f_n(z,q)dz}{z-m_n(q)}=-\int_{c_n}\frac{\partial\Delta(z,q)^2+\ell^2(n)}{(z-\pi n)^3}=-2\int_{c_n}\frac{\cos z\partial\Delta(z,q)+\ell^2(n)}{(z-\pi n)^3}=\ell^2(n).\ \square$$

After Lemma 4.1 it is easy to get the needed properties of functions ϕ_n.

Lemma 4.2. *Each function $\phi_n(\cdot), n\geqslant 1$, is compact, real analytic on H and the following identity is fulfilled:*

$$\phi_{cn}(q)=\frac{1}{\sqrt{f_n(m_n(q),q)}}\sinh h_{cn}(q). \tag{4.8}$$

Moreover, for any fixed $d > 1$, we have

$$\phi_n(q) = q_n + \ell^d(n), \tag{4.9}$$

$$d_q\phi_n = (\sin 2\pi nx, \cos 2\pi nx) + \ell^2(n), \tag{4.10}$$

$n \to \infty$, uniformly on $[0,1] \times B_C(p, \varepsilon_p)$ for any fixed $p \in H$.
Proof. For any $q \in H$ the following identity is fulfilled:

$$\sinh^2 h_{cn} = \Delta^2(m_n(q), q) - 1 = f_n(m_n(q), q)\phi_{cn}^2(q). \tag{4.11}$$

Recall that since $f_n > 0$, we deduce that the function $1/\sqrt{f_n}$ has an analytic extension in some ball $B_C(q, \varepsilon)$ for some $\varepsilon > 0$ and since $h_{cn}(\cdot)$ is real analytic we get (4.8). Using Lemmas 4.1 2.3 and (4.11) we derive that $\phi_{cn}(\cdot)$ is compact and real analytic on H and estimates (4.9-10) are fulfilled. $\square$

After Lemma 4.1-2 we will obtain the needed properties of functions ψ_n.
Lemma 4.3. *Each function $\psi_n(\cdot), n \geqslant 1$, is compact, real analytic on H and the following identities are fulfilled:*

$$\psi_{sn} = \phi_{cn}^2 + 2z_n^0\phi_{sn} \tag{4.12}$$

$$\psi_{cn}(q) = \sqrt{B}_n(z_n^+ + m_m)(z_n^- + m_n)\phi_{cn}, \qquad B_n = (z_n^+ + m_m)(z_n^- + m_n) = (2z_n^0)^2 - \psi_{sn}, \tag{4.13}$$

where the functions $B_n, \sqrt{B}_n$ are compact, real analytic and positive on H. Moreover, for any fixed $d > 1$, we have

$$\psi_n(q) = 2\pi n(q_n + \ell^d(n)), \tag{4.14}$$

$$d_q\psi_n = 2\pi n[(\sin 2\pi nx, \cos 2\pi nx) + \ell^2(n)], \tag{4.15}$$

$n \to \infty$, uniformly on $[0,1] \times B_C(p, \varepsilon_p)$ for any fixed $p \in H$.
Proof. We have

$$\psi_{sn} = \frac{\lambda_n^+ + \lambda_n^-}{2} - \mu_n = \frac{(z_n^+)^2 + (z_n^-)^2}{2} - (z_n^0)^2 + [(z_n^0)^2 - m_n^2] =$$

$$= \frac{(z_n^+ - z_n^-)^2}{4} + (z_n^0 - m_m)(z_n^0 + m_n) = \phi_n^2 + \phi_{sn}(2z_n^0 - \phi_{sn})$$

which yields (4.12). Due to the identity

$$\psi_{cn}^2 = (\lambda_n^+ - \mu_n)(\mu_n - \lambda_n^-) = (z_n^+ + m_m)(z_n^- + m_n)\phi_{cn}^2 = B_n\phi_{cn}^2,$$

we get (4.13). Using the following identities

$$B_n = (z_n^+ + m_m)(z_n^- + m_n) = (z_n^0 + m_n)^2 - b_n,$$

and Lemmas 3.2-3, 2.2, we deduce that the function B_n is compact, real analytic and positive on H and $\sqrt{B}_n$ is so.

Lemmas 4.3, 3.2 and identities (4.12-13) show that ψ_n is compact, real analytic on H.

Due to (4.12), (3.14), (4.9) we obtain

$$\psi_{sn} = \ell^1(n) + [2\pi n + \ell^d(n)][q_{sn} + \ell^d(n)] = 2\pi n[q_{sn} + \ell^d(n)]$$

and

$$\psi_{cn} = 2\pi n(1 + \frac{1}{n}\ell^d(n))[q_{cn} + \ell^d(n)] = 2\pi n[q_{cn} + ell^d(n)]$$

Using (4.12), (4.9-10), (3.14-15) we have

$$d_q\psi_{sn} = d_q\phi_{cn}^2 + 2(d_q z_n^0)\phi_{sn} + 2qz_n^0 d_q\phi_{sn}$$

$$= \big(2\pi n + \ell^d(n)\big)\big(\sin 2\pi nx + \ell^2(n)\big) = 2\pi n\big(\sin 2\pi nx + \ell^2(n)\big)$$

Analogously we get the needed result for $d_q\psi_{cn}$ which yields (4.15).□

5 Analytic isomorphism

Now we recall the well known Paley-Wiener result (see [Le]).
Lemma 5.1. *Suppose that $f(z)$ is an entire function with*

$$\sup_{z\in\mathbb{C}} |f(z)e^{-|z|}| < \infty, \qquad \int_{\mathbb{R}} |f(x)|^2 dx < \infty,$$

and $f(z_n) = 0$ for some sequence $\{z_n\}_{-\infty}^{\infty}$ of distinct real numbers such that $z_n = \pi n + o(1)$ as $|n| \to \infty$. Then $f \equiv 0$. □

Define the Fourier transformation $\Phi : H_C \to \ell_C^2 \oplus \ell_C^2$ such that $(\Phi q)_n = \sqrt{2}q_n, n \geqslant 1$, where $q_n = (q_{cn}, q_{sn})$, and the coefficients q_{cn}, q_{sn} have the form (1.10). We are now ready to prove Theorem. 1.1.

Proof of Theorem 1.1. We check all conditions of Theorem A for the mapping $\phi : H \to \ell^2 \oplus \ell^2$. i) By Lemma 4.2, each function $\phi_n(\cdot), n \geqslant 1$, is compact and real analytic on H, and estimates (1.17-18) are fulfilled. Relations (1.7) show that the map $\phi(q)$ is locally bounded; by the uniform boundedness principle, $\phi(q)$ is real analytic.
ii) Using asymptotic estimate (1.18), we see that $d_q\phi$ is the sum of the Fourier transform Φ and a compact operator for all $q \in H$. That is $d_q\phi - (1/\sqrt{2})\Phi$ is a compact operator; consequently, $d_q\phi$ is a Fredholm operator. We prove that the operator $d_q\phi$ is invertible by contradiction. Let $g \in H$ be a solution of the equation

$$(d_q\phi(q))g = 0, \qquad \text{or} \qquad \{(d_q\phi_n(q), g) = 0, n \geqslant 1\}, \tag{5.1}$$

for some fixed $q \in H$. In order to prove $g = 0$ we use the approach from [K2] and [K3]. We introduce the function

$$f(z) = \int_0^1 (\partial\Delta)(t, z, q)g(t)dt, \quad z \in \mathbb{C}.$$

First, assume $\gamma_n = 0$, then Lemma 2.1 yields $f(z_n) = 0$, for $z_n^0 = z_n^\pm$. Second, let $\gamma_n \neq 0$, then using (5.1) we get $(d_q b_n, g) = 0$, and identity (3.18) implies

$$-\frac{f(z_n^-)}{\dot\Delta(z_n^-)} + \frac{f(z_n^+)}{\dot\Delta(z_n^+)} = 0.$$

Also, $\dot\Delta(z_n^+)\dot\Delta(z_n^-) < 0$. Hence $f(z_n^-)f(z_n^+) \leqslant 0$ and there exists a point $z_n \in [z_n^-, z_n^+]$ such that $f(z_n) = 0$. Moreover, using the identity $\Delta(0, q) = 1$ for all $q \in H$ we have $\partial\Delta(0, q) = 0$, then $f(0) = 0$ and $z_0 = 0$. Hence $f(z_n) = 0$ for any $n \in \mathbb{Z}$, and (3.1) implies $z_n = \pi n + o(1)$ as $|n| \to \infty$. Now we have to show that $f \in L^2(\mathbb{R})$. The asymptotic estimate (2.7) yields

$$f(z) = \int_0^1 (\partial\Delta(z, q))(t)g(t)dt = \int_0^1 \int_0^1 \cos z(2y-1)q(y+t)g(t)dtdy + \ell^2(n)$$

$$= \int_0^1 \cos z(2y-1)G(y)dy + \ell^2(n), \qquad |z - \pi n| \leqslant \pi, \quad |n| \to \infty,$$

where $G(y) = \int_0^1 q(y+t)g(t)dt$. Hence $f \in L^2(\mathbb{R})$. So, using (2.4) we see that all conditions of Lemma 5.1 are fulfilled, and we get $f \equiv 0$.
Using (3.12-13) we deduce that $(d_q z_n^0, g) = 0$ and then $(d_q\phi_{sn}, g) = -(d_q m_n, g) = 0$ for all $n \geqslant 1$.Suppose that $(d_q h_{cn}, g) = 0$ for any $n \geqslant 1$. Then $g = 0$ since the vectors $\{(d_q h_{cn}), (d_q m_n), n \geqslant 1\}$ form a basis of H (see Lemma 2.5). Then the operator $d_q\phi$ is invertible.
In order to show $(d_q h_{cn}, g) = 0$ we consider two cases. First, let $h_{cn} \neq 0$. Then by (2.18), $(d_q h_{cn}, g) = 0$.
Second, if $h_{cn} = 0$, then $\phi_{cn} = 0$, and by (4.8),

$$\sqrt{f_n}(d_q\phi_{cn}, g) = (d_q h_{cn}, g) = 0,$$

which yields $(d_q h_{cn}, g) = 0$.
iii) Estimates (1.16) were proved in [K1].
iv) By Lemma 4.2, each mapping $\phi_n : H \to \mathbb{R}^2, n \geqslant 1$, is compact.
v) We need some results about the Hill operator $T_1 = -\frac{d^2}{dx^2} + V(x)$ in $L^2(\mathbb{R})$ with a real 1-periodic potential $V \in L^2(0, 1)$. In the paper [K9] (see also [K8], [K5]) the following estimate has been proved

$$\|V\| \leqslant 2(1 + \|\phi\|_1)\|\phi\|_1, \qquad \|\phi\|_1^2 = \sum_{n\geqslant 1}(2\pi n)^2|\phi_n|^2.$$

Then identity (1.3) yields $\|q'\| \leqslant 2(1+\|\phi\|_1\|\phi\|_1$. Hence, for each $\xi > 0$ the set $\{q : \|\phi\|_1 < \xi\}$ is compact.
Therefore, all conditions of Theorem A are fulfilled and then ϕ is the real analytic isomorphism between H and $\ell^2 \oplus \ell^2$. □

Remark. In order to show $f \equiv 0$ we use that this function has a zero in each gap. Hence we need additional zero z_0 and we proved that there exists such zero $z_0 = 0$. Then we can use the well known Paley-Wiener result (see Lemma 5.1). Here we simplify the corresponding proof in [K2].

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. We check all conditions of Theorem A for the mapping $\psi : H \to \ell^2_{-1} \oplus \ell^2_{-1}$. i) By Lemma 4.3, each function $\psi_n(\cdot), n \geqslant 1$, is compact and real analytic on H, and the estimates (1.21-22) are fulfilled. The relations (1.21) show that the map $\psi(q)$ is locally bounded; by the uniform boundedness principle, $\psi(q)$ is real analytic.
ii) Using estimate (1.22), we see that $d_q\psi$ is the sum of the operator Φ and a compact operator for all $q \in H$. That is $(\sqrt{2}\pi n)^{-1}(d_q\psi f)_n - (\Phi f)_n$ is a compact operator; consequently, $d_q\psi$ is a Fredholm operator. We prove that the operator $d_q\psi$ is invertible by contradiction. Let $g \in H$ be a solution of the equation

$$(d_q\psi(q))g = 0, \qquad \text{or} \qquad \{(d_q\psi_n(q), g) = 0, n \geqslant 1\}, \tag{5.2}$$

for some fixed $q \in H$. We introduce the function

$$f(z) = \int_0^1 (\partial\Delta)(t, z, q)g(t)dt, \quad z \in \mathbb{C}.$$

First, assume $\gamma_n = 0$, then Lemma 2.1 yields $f(z_n) = 0$, for $z_n^0 = z_n^{\pm}$. Second, assume $\gamma_n \neq 0$. By Lemma 4.3, each function $\beta_n = (\lambda_n^+ - \lambda_n^-)^2/4 = \psi_{cn}^2 + \psi_{sn}^2, n \geqslant 1$, is compact and real analytic on H. Using the following identity $\beta_n = (2z_n^0)^2 b_n$ and (3.12), (3.18) we obtain

$$d_q\beta_n = 8z_n^0(d_q z_n^0)b_n + (2z_n^0)^2 d_q b_n = (z_n^0)(2b_n d_q z_n^0 + z_n^0 d_q b_n)$$

$$= -z_n^0(4b_n(v_n^+ + v_n^-) + 2z_n^0(z_n^+ - z_n^-)(v_n^+ - v_n^-)) =$$

$$-z_n^0(z_n^+ - z_n^-)[(z_n^+ - z_n^-)(v_n^+ + v_n^-) + (z_n^+ + z_n^-)(v_n^+ - v_n^-)] = -2z_n^0(z_n^+ - z_n^-)[z_n^+ v_n^+ - z_n^- v_n^-]$$

which yields

$$d_q\beta_n = (\lambda_n^+ - \lambda_n^-)[z_n^- v_n^-(q) - z_n^+ v_n^+(q)], \qquad z_n^- \neq z_z^+, \quad q \in H. \tag{5.3}$$

Then using (5.2) we get $(d_q\beta_n, g) = 0$, and identity (5.3) implies

$$-\frac{z_n^- f(z_n^-)}{\dot\Delta(z_n^-)} + \frac{z_n^+ f(z_n^+)}{\dot\Delta(z_n^+)} = 0.$$

Also, $\dot\Delta(z_n^+)\dot\Delta(z_n^-) < 0$. Hence $f(z_n^-)f(z_n^+) \leqslant 0$ and there exists a point $z_n \in [z_n^-, z_n^+]$ such that $f(z_n) = 0$. Moreover, using the identity $\Delta(0, q) = 1$ for all $q \in H$ we have $\partial\Delta(0, q) = 0$, then $f(0) = 0$ and $z_0 = 0$ Thus, there exists $z_n = \pi n + o(1)$ as $n \to \pm\infty$ (see (3.1)).

Now we have to show that $f \in L^2(\mathbb{R})$. The asymptotic estimate (2.7) yields

$$f(z) = \int_0^1 (\partial\Delta(z, q))(t)g(t)dt = \int_0^1 \int_0^1 \cos z(2y - 1)q(y + t)g(t)dtdy + \ell^2(n)$$

$$= \int_0^1 \cos z(2y - 1)G(y)dy + \ell^2(n)), \qquad |z - \pi n| \leqslant \pi, \quad |n| \to \infty,$$

where $G(y) = \int_0^1 q(y+t)g(t)dt$. Hence $f \in L^2(\mathbb{R})$. So, using (2.4) we see that all conditions of Lemma 5.1 are fulfilled, and we get $f \equiv 0$.
Using identity $\lambda_n^0 = (z_n^0)^2 + b_n$ and (3.18), (3.13) we deduce that $(d_q\lambda_n^0, g) = 0$ and then $(d_q\psi_{sn}, g) = -2m_n(d_q m_n, g) = 0$ for all $n \geqslant 1$.Suppose that $(d_q h_{cn}, g) = 0$ for any $n \geqslant 1$. Then $g = 0$ since the vectors $\{(d_q h_{cn}), (d_q m_n), n \geqslant 1\}$ form a basis of H (see Lemma 2.5). Then the operator $d_q\phi$ is invertible.
In order to show $(d_q h_{cn}, g) = 0$, we consider two cases. First, let $h_{cn} \neq 0$. Then by (2.18), $(d_q h_{cn}, g) = 0$.
Second, if $h_{cn} = 0$, then by (4.8), (4.13), $\phi_{cn} = 0$, and we have

$$(d_q\psi_{cn}, g) = A(d_q h_{cn}, g) = 0, \quad A = \sqrt{\frac{(z_n^+(q) + m_n(q))(z_n^-(q) + m_n(q))}{f_n(m_n(q), q)}} \neq 0,$$

which yields $(d_q h_{cn}, g) = 0$,
iii) Estimates (1.20) were proved in [K1].
iv) By Lemma 4.3, each mapping $\psi_n : H \to \mathbb{R}^2, n \geqslant 1$, is compact.
v) We need some results about the Hill operator $T_1 = -\frac{d^2}{dx^2} + V(x)$ in $L^2(\mathbb{R})$ with a real 1-periodic potential $V \in L^2(0,1)$. In the paper [K9], [K7], the following estimate has been proved

$$\|V\| \leqslant 2(1 + \|\psi\|)\|\psi\|.$$

Then the identity (1.3) yields $\|q'\| \leqslant 2(1 + \|\psi\|)\|\psi\|$. Hence, for each $\xi > 0$ the set $\{q : \|\psi\| < \xi\}$ is compact.

Therefore, all conditions of Theorem A are fulfilled and then ψ is the real analytic isomorphism between H and $\ell^2_{-1} \oplus \ell^2_{-1}$. $\square$

References

[A] Andersson L., : Inverse eigenvalue problems for a Sturm-Liouville equation in impedance form.Inverse Problems, 4, 929-271, 1988

[CML] Coleman C., McLaughlin J.: Solution of the inverse problem for an impedance with integrable derivative, Commun. Pure Appl. Math. 1993, 46, 145-184; 185-212

[CMK] Constantin A., McKean H.: A shallow water equation on the circle, to appear in Commun. Pure Appl. Math.

[GT] Garnett J., Trubowitz E.: Gaps and bands of one dimensional periodic Schrödinger operator, Comment. Math. Helv. 59, 258-312 (1984)

[KK1] Kargaev P., Korotyaev E.: The inverse problem for the Hill operator, the direct approach. Invent. Math. 129, no 3, 567-593 (1997)

[KK2] Kargaev P., Korotyaev E.: Erratum, The inverse problem for the Hill operator, the direct approach. Invent. Math.138, 1999, 227

[KK3] Kargaev P., Korotyaev E.: Effective masses and conformal mappings. Commun. Math. Phys. 169, 597-625 (1995).

[K1] Korotyaev E. : Periodic weighted operator. preprint SFB-288, 1999

[K2] E. Korotyaev: Inverse problem for periodic weighted operator, J. Func. Anal.168, 1999, 1-31

[K3] E. Korotyaev: The inverse problem and trace formula for the Hill operator, II. Math. Zeit. 231, 1999, 345-368

[K4] E. Korotyaev: The inverse problem for the Hill operator I. Internat. Math. Res. Notices 1997, 3, 113-125.

[K5] Korotyaev E. : Metric properties of conformal mappings on the complex plane with parallel slits. Internat. Math. Res. Notices. 1996, 10, 493-503.

[K6] Korotyaev E. : The propagation of the waves in the one-dimensional periodic media. Asymptotic Analysis, 15, 1-24, (1997)

[K7] E. Korotyaev: The estimate of periodic potentials in terms of effective masses. Commun. Math. Phys. 1997, 183, 383-400

[K8] Korotyaev E. Estimate for the Hill operator.I, preprint FIM ETH, Zurich, Nov. 1997, will be published in J. Diff. Eq.

[K9] Korotyaev E.: Estimate of periodic potentials in terms of gap lengths. Commun. Math. Phys. 1998, 197, no. 3, 521-526

[K10] Korotyaev E.: On the scattering in an external, homogeneous time-periodic magnetic field. Mat. Sb. 180(1989), no. 4, 491-512, English transl. in Math. USSR Sb. 66(1990).

[Kr] Krein M. On the characteristic function $A(\lambda)$ of a linear canonical system differential equation of the second order with periodic coefficients (Russian) , Prikl. Mat. Meh. 21, 320-329 (1957)

[Le] Levinson N. Gap and density theorem. Amer. Math. Soc. Coll. Publ., New York, 1940

[L] Lyapunov,A: Sur une equation transcendante et les equations differentielles lineares du second ordre a coefficients periodiques, C. R., t. CXXVIII, 18, 1085-1088 (1899).

[MO1] Marchenko V., Ostrovski I.: A characterization of the spectrum of the Hill operator. Math. USSR Sb. 26, 493-554 (1975).

[MO2] Marchenko V., Ostrovski I.: Approximation of periodic by finite-zone potentials. Selecta Math. Sovietica. 1987, 6, No 2, 101-136.

[N] Nojima S. Exitonic polaritons in one-dimensional photonic crystal, Phys. Rev. B, 1998, 54, no. 4, R2057-R2060

[PT] Pöschel, P., Trubowitz E.: Inverse Spectral Theory. Boston: Academic Press, 1987.

[T] Titchmarsh E.: The theory of functions. Sec. ed., Univ. Press, London, 1975