A NOTE ON REAL KILLING SPINORS IN WEYL GEOMETRY [1]

VOLKER BUCHHOLZ[2]

Humboldt Universität zu Berlin, Institut für Reine Mathematik,
Ziegelstraße 13a , D-10099 Berlin.

Abstract: This text is dedicated to the real Killing equation on 3-dimensional Weyl manifolds. Any manifold admitting a real Killing spinor of weight 0 satisfies the conditions of a Gauduchon-Tod geometry. Conversely, any simply connected Gauduchon-Tod geometry has a 2-dimensional space of solutions of the real Killing equation on the spinor bundle of weight 0.

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1 Introduction

In [2] we introduced the spinor geometry on Weyl manifolds and investigated the Dirac-, Twistor- and Killing equation in this context. Concerning the real Killing equation we presented in [2] the following result:

Theorem 1.1 (see [2], Theorem 3.1) *Let $\psi \in \Gamma(S^w)$ be a real Killing spinor on a Weyl manifold (M^n, c, W), i.e. there exists a density $\beta \in \Gamma(\mathcal{L}^{-1})$ for which*

$$\nabla^{S,w}\psi = \beta \otimes \nu\psi, \quad \beta \in \Gamma(\mathbb{C} \otimes \mathcal{L}^{-1})$$

is satisfied. Then the following statements hold:

1. $R = 4n(n-1)\beta^2$.
2. $w \neq 0$*: W is exact and Einstein-Weyl.*
3. $w = 0$, $n \geq 4$*: W is exact and Einstein-Weyl.*

The following equations were obtained within the proof and will play an important role in the sequel:

$$\mu^2 Ric' \otimes \psi = 2\left(n-1-\frac{n-1}{n-2}\right)\nabla\beta\otimes\psi + \left(1-\frac{n-1}{n-2}\right)\mu^{12}\ Alt\nabla\beta\otimes c\otimes\psi + \frac{R}{n}\nu\psi - \mu^2 F\otimes\psi \quad (1)$$

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[2]e-mail: `bv@mathematik.hu-berlin.de`

$$F \cdot \psi = -\frac{4(n-1)}{n-2} \nabla\beta \cdot \psi. \tag{2}$$

Theorem 1.1 gives no statement for the case $n = 3$ and $w = 0$. In the next section we proof that in three dimensions the existence of a real Killing spinor of weight 0 is essentially equivalent to the fact that this manifold is a Gauduchon-Tod geometry:

Definition 1.2 (see [4], Proposition 5) *A 3-dimensional Weyl manifold (M^3, W, c) is called Gauduchon-Tod geometry, if there exists a density $\beta \in \Gamma(\mathcal{L}^{-1})$ such that the following conditions are satisfied:*

1. *W is Einstein-Weyl;*
2. *$R = 24\beta^2$;*
3. *$4\nabla\beta = *F$.*

Remark: The $\kappa \in C^\infty(M, \mathbb{R})$ in ([4], Proposition 5) is related to the $\beta \in \Gamma(\mathcal{L}^{-1})$ of Definition 1.2 in the following way: $\kappa l_{g_\Sigma}^{-1} = -4\beta$. For more information on Gauduchon-Tod geometries, e.g. their classification, see [4] and the references therein.

Hence, the main result of this text is as follows:

Theorem 1.3 *Let (M^3, c, W) be a CSpin-manifold.*

1. *If $\psi \in \Gamma(S^0)$ is a real Killing spinor then the space of solutions of the Killing equation is 2-dimensional and (M^3, c, W) is a Gauduchon-Tod geometry.*
2. *Conversely, any simply connected Gauduchon-Tod geometry has a two dimensional space of Killing spinors of weight 0.*

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2 The proof of Theorem 1.3

Let (M^3, c, W) be a CSpin-manifold. The curvature tensor of the Weyl structure W is given by

$$\mathcal{R} = Ric^N \triangle c + F \otimes c,$$

where $\triangle : T^{2,0} \times T^{2,0} \longrightarrow T^{4,0}$

$$\omega \triangle \eta := [(23) + (12)(24)(34) - (24) - (12)(23)]\omega \otimes \eta, \quad \omega, \eta \in T^{2,0}$$

is the so called Kulkarni-Nomizu product (see [1]) and

$$Ric^N := -\mathrm{sym}_0 Ric - \frac{1}{12} Rc + \frac{1}{2} F \tag{3}$$

is the normalized Ricci tensor of W (see [3]). sym_0 denotes the symmetric trace free part of a $(2,0)$-tensor. The following Lemma is a tool for calculations with Kulkarni-Nomizu products in spin geometry:

Lemma 2.1 *Let ω be a $(2,0)$-tensor. Then the following algebraic identity holds in any dimension:*

$$\mu^{34}\omega \triangle c = 2\ Alt\nu\mu^2\omega - 2\ Alt\omega.$$

PROOF:

$$\begin{aligned}
\mu^{34}\omega \triangle c &= \mu^{34}[(23)+(12)(24)(34)-(24)-(12)(23)]\omega\otimes c = [\mu^{24}+(12)\mu^{32}-\mu^{32}-(12)\mu^{24}]\omega\otimes c\\
&= [\mu^{23}+(12)\mu^{32}-\mu^{32}-(12)\mu^{23}]\omega\otimes c = [-2\mu^{32}+2tr^{23}+2(12)\mu^{32}-2(12)tr^{23}]\omega\otimes c\\
&= -2[\mu^{32}-(12)\mu^{32}]\omega\otimes c - 2\ Alt\omega = -2[1-(12)]\nu\mu^2\omega\otimes c - 2\ Alt\omega\\
&= 2\ Alt\nu\mu^2\omega - 2\ Alt\omega.
\end{aligned}$$

$\square$

Lemma 2.2 *Let $4\nabla\beta = *F$ be satisfied on (M^3, c, W). Then the following identities are true for any spinor $\psi \in \Gamma(S^w)$:*

$$\frac{1}{4}\ Alt\nu\mu^2 F\otimes\psi - \ Alt\nabla\beta\otimes\nu\psi - \frac{1}{2}F\otimes\psi = 0. \tag{4}$$

and

$$(\nu\nabla\beta - \nabla\beta\cdot\nu)\cdot\psi - \frac{1}{2}\mu^2 F\otimes\psi = 0. \tag{5}$$

PROOF: Denote by (e_1, e_2, e_3) a local weightless conformal frame on (M^3, c) as well as $(\sigma_1, \sigma_2, \sigma_3)$ its dual. In dimension 3 we have the important relation

$$e_i\cdot e_j\cdot\psi = -\sum_{k=1}^{3}\epsilon_{ijk}e_k\cdot\psi. \tag{6}$$

Here ϵ_{ijk} denotes the Levi-Civita symbol. Since the $*$-operator on 2-forms is defined by the formula

$$*F = \frac{1}{2}\sum_{i,j=1}^{3}F(e_i,e_j)*(\sigma_i\wedge\sigma_j) = \frac{1}{2}\sum_{i,j,k=1}^{3}F(e_i,e_j)\epsilon_{ijk}\sigma_k$$

we can rewrite the assumption as follows:

$$8\nabla_{e_k}\beta = \sum_{i,j=1}^{3}\epsilon_{ijk}F(e_i,e_j). \tag{7}$$

From the algebraic identity

$$F\cdot\psi = \sum_{i,j=1}^{3}F(e_i,e_j)e_i\cdot e_j\cdot\psi = -\sum_{i,j,k=1}^{3}\epsilon_{ijk}F(e_i,e_j)e_k\cdot\psi = -8\sum_{k=1}^{3}(\nabla_{e_k}\beta)e_k\cdot\psi$$

and (7) then follows:

$$F\cdot\psi = -8\nabla\beta\cdot\psi, \tag{8}$$

Using (7) and (6) we get:

$$\begin{aligned}
&\frac{1}{4}Alt\nu\mu^2 F\otimes\psi - Alt\nabla\beta\otimes\nu\psi - \frac{1}{2}F\otimes\psi\\
&= \frac{1}{4}\sum_{i,j,k=1}^{3}F(e_j,e_k)\sigma_i\wedge\sigma_j\otimes e_i\cdot e_k\cdot\psi - \sum_{i,j=1}^{3}(\nabla_{e_i}\beta)\sigma_i\wedge\sigma_j\otimes e_j\cdot\psi - \frac{1}{4}\sum_{i,j=1}^{3}F(e_i,e_j)\sigma_i\wedge\sigma_j\otimes\psi
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{4}\sum_{i,j=1}^{3} F(e_j,e_i)\sigma_i\wedge\sigma_j\otimes\psi + \frac{1}{4}\sum_{\substack{i,j,k=1\\ i\neq k}}^{3} F(e_j,e_k)\sigma_i\wedge\sigma_j\otimes e_i\cdot e_k\cdot\psi \\
&\quad - \sum_{i,j=1}^{3}(\nabla_{e_i}\beta)\sigma_i\wedge\sigma_j\otimes e_j\cdot\psi - \frac{1}{4}\sum_{i,j=1}^{3} F(e_i,e_j)\sigma_i\wedge\sigma_j\otimes\psi \\
&= \frac{1}{4}\sum_{i,j,k,l=1}^{3}\epsilon_{kil}F(e_j,e_k)\sigma_i\wedge\sigma_j\otimes e_l\cdot\psi - \frac{1}{8}\sum_{i,j,k,l=1}^{3}\epsilon_{kli}F(e_k,e_l)\sigma_i\wedge\sigma_j\otimes e_j\cdot\psi \\
&= \frac{1}{4}\sum_{i,j,k=1}^{3}\epsilon_{kij}F(e_j,e_k)\sigma_i\wedge\sigma_j\otimes e_j\cdot\psi - \frac{1}{8}\sum_{\substack{i,j,k,l=1\\ j=k}}^{3}\epsilon_{jli}F(e_j,e_l)\sigma_i\wedge\sigma_j\otimes e_j\cdot\psi \\
&- \frac{1}{8}\sum_{\substack{i,j,k,l=1\\ j=l}}^{3}\epsilon_{kji}F(e_k,e_j)\sigma_i\wedge\sigma_j\otimes e_j\cdot\psi \\
&= \frac{1}{4}\sum_{i,j,k=1}^{3}\epsilon_{kij}F(e_j,e_k)\sigma_i\wedge\sigma_j\otimes e_j\cdot\psi - \frac{1}{8}\sum_{i,j,l=1}^{3}\epsilon_{lij}F(e_j,e_l)\sigma_i\wedge\sigma_j\otimes e_j\cdot\psi \\
&- \frac{1}{8}\sum_{i,j,k=1}^{3}\epsilon_{kij}F(e_j,e_k)\sigma_i\wedge\sigma_j\otimes e_j\cdot\psi \\
&= 0,
\end{aligned}
$$

i.e. we have shown (4). Contracting (4) by Clifford multiplication, using (8) and $\mu^2\nu = -\nu\mu^1 - 2Id$ yields :

$$
\begin{aligned}
0 &= \frac{1}{4}\mu^2 Alt\nu\mu^2 F\otimes\psi - \mu^2 Alt\nabla\beta\otimes\nu\psi - \frac{1}{2}\mu^2 F\otimes\psi \\
&= \frac{1}{4}\mu^2\nu\mu^2 F\otimes\psi - \mu^2\nabla\beta\otimes\nu\psi - \frac{1}{2}\mu^2 F\otimes\psi - \frac{1}{4}\mu^1\nu\mu^2 F\otimes\psi + \mu^1\nabla\beta\otimes\nu\psi \\
&= -\frac{1}{4}\nu F\cdot\psi - \frac{1}{2}\mu^2 F\otimes\psi + 3\nabla\beta\otimes\psi - \frac{1}{2}\mu^2 F\otimes\psi + \frac{3}{4}\mu^2 F\otimes\psi + \nabla\beta\cdot\nu\psi \\
&= -\frac{1}{4}\nu F\cdot\psi - \frac{1}{2}\mu^2 F\otimes\psi + 3\nabla\beta\otimes\psi - \frac{1}{2}\mu^2 F\otimes\psi + \frac{3}{4}\mu^2 F\otimes\psi + \nabla\beta\cdot\nu\psi \\
&= \nu\nabla\beta\cdot\psi + \nabla\beta\otimes\psi - \frac{1}{4}\mu^2 F\otimes\psi \\
&= \frac{1}{2}(\nu\nabla\beta - \nabla\beta\cdot\nu)\cdot\psi - \frac{1}{4}\mu^2 F\otimes\psi.
\end{aligned}
$$

Hence (5) is true. □

After these preliminary calculations we get to the proof of Theorem 1.3: Let $\psi \in \Gamma(S^0)$ be a real Killing spinor. The first statement follows immediately from the existence of an equivariant quaternionic structure **j** on the spinor module, which commutes with the Clifford multiplication. By Theorem 1.1 and its proof we have

$$R = 24\beta^2;\quad F\cdot\psi = -8\nabla\beta\cdot\psi.$$

Since ψ vanishes nowhere the second equation is equivalent to

$$4\nabla\beta = *F$$

by (6). Therefore we have already verified the conditions *2.* and *3.* of the definition of a Gauduchon-Tod geometry. It remains to proof that the manifold is Einstein-Weyl. To this end we have to simplify (1) by means of $R = 24\beta^2$ and $Ric' = sym_0 Ric + \frac{1}{3}Rc - \frac{1}{2}F$ to

$$\mu^2 sym_0 Ric' \otimes \psi = -(\nabla\beta \cdot \nu - \nu\nabla\beta) \cdot \psi - \frac{1}{2}\mu^2 F \otimes \psi$$

But the righthandside vanishes according to (5). Hence W is Einstein-Weyl.

Conversely, let W be a simply connected Gauduchon-Tod geometry. Is is sufficient to show that S^0 is flat with respect to $\nabla^\beta = \nabla^{S,0} - \beta \otimes \nu$. To this end, we have to proof that the curvature of ∇^β vanishes. We use the properties of Gauduchon-Tod geometries given in Definition 1.2, the result of Lemma 2.1 and the equations (4) and $\mathcal{R}^{S,0} = \frac{1}{4}\mu^{34} Ric^N \triangle c = -\frac{1}{4}\mu^{34}\left(\frac{1}{12}Rc - \frac{1}{2}F\right) \triangle c$.

$$\begin{aligned}
\mathcal{R}^\beta &= Alt\nabla^\beta \circ \nabla^\beta = Alt\nabla^\beta \circ (\nabla^{S,0} - \beta \otimes \nu) \\
&= Alt\left(\nabla^{S,0} \circ \nabla^{S,0} - \nabla\beta \otimes \nu - (12)\beta \otimes \nu\nabla^{S,0} - \beta\nu\nabla^{S,0} + \beta^2\nu\nu\right) \\
&= \mathcal{R}^{S,0} - Alt(\nabla\beta)\nu + \beta^2\ Alt\nu^{12} = -\frac{1}{4}\mu^{34}\left(\frac{1}{12}Rc - \frac{1}{2}F\right) \triangle c - Alt(\nabla\beta)\nu + \beta^2\ Alt\nu^{12} \\
&= -\frac{1}{4}\left(\frac{1}{6}R\ Alt\nu\mu^2 c - Alt\nu\mu^2 F + 2F\right) - Alt(\nabla\beta)\nu + \beta^2\ Alt\nu^{12} \\
&= -\frac{1}{24}R\ Alt\nu^{12} + \frac{1}{4}\ Alt\nu\mu^2 F - \frac{1}{2}F - Alt(\nabla\beta)\nu + \beta^2\ Alt\nu^{12} \\
&= \frac{1}{4}\ Alt\nu\mu^2 F - \frac{1}{2}F - Alt(\nabla\beta)\nu \\
&= 0.
\end{aligned}$$

$\square$

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