ZEROS OF CONFORMAL VECTOR FIELDS AND TWISTOR SPINORS IN LORENTZIAN GEOMETRY. *

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Contents

Abstract

We study the zero set of conformal vector fields on Lorentzian manifolds that have properties like the associated conformal vector field of a twistor spinor. We prove that locally the zero set of such conformal vector fields lies on a lightlike smooth geodesic.

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0 Introduction

Let (M,g) be a semi-Riemannian manifold. A vector field $V \in \mathcal{X}(M)$ is called conformal if the Lie derivative of the metric g in direction of V satisfies

$$\mathcal{L}_V g = 2\alpha \cdot g$$

for some C^∞-function α on M. In particular, Killing vector fields on (M,g) are conformal. A conformal vector field V is called essential if V isn't a Killing vector field with respect to any

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metric $\tilde{g} \in [g]$ in the conformal class of g on M. Since a conformal vector field is a Killing vector field on $M \backslash \mathrm{zero}(\|V\|)$ with respect to the metric $\tilde{g} = \frac{1}{g(V,V)} \cdot g \in [g]$, the lenght $\|V\|$ of an essential conformal field V must vanish somewhere on (M,g). So it is nearby to investigate the behaviour of essential conformal fields and curvature properties of the underlying manifold in the near of the zero set.

On a semi-Riemannian spin manifold (M,g) there is a certain class of conformal vector fields, which are related to twistor spinors. A twistor spinor $\varphi \in \Gamma(S)$ on (M,g) is a solution of the conformally invariant spinor field equation

$$\nabla^S_X \varphi + \frac{1}{n} X \cdot D\varphi = 0 \qquad \forall X \in TM,$$

where ∇^S is the spinor derivative, D is the Dirac operator and the dot $\cdot$ denotes the Clifford multiplication. This equation was first studied by R. Penrose in General Relativity. The associated vector field V_φ to a twistor spinor φ on a space and time oriented semi-Riemannian manifold (M,g) given by

$$V_\varphi := (-i)^{k+1} \sum_{j=1}^{n} \varepsilon_j \langle \varphi, s_j \varphi \rangle_S s_j,$$

where $(s_1, \ldots, s_n)$, $\varepsilon_j := g(s_j, s_j)$, is a local orthonormal frame on (M,g), is conformal or vanishes. The zero set of φ is contained in that of V_φ and the conformal vector field V_φ satisfies

$$\nabla V_\varphi(p) = 0 \quad \forall p \in \mathrm{zero}(\varphi).$$

This property implies that a non-trivial, associated conformal field to a twistor spinor with zero is essential. More generally, a pair of twistor spinors (φ, ψ) on a semi-Riemannian manifold (M,g) generates the complex conformal field

$$V_{\varphi,\psi} = \sum \varepsilon_j \langle \varphi, s_j \psi \rangle_S s_j.$$

Example: Every conformal vector field on the (pseudo)-Euclidean space $\mathbb{R}^n_k := (\mathbb{R}^n, \langle\ ,\ \rangle^n_k)$ of index k is of the form (see [Sch97])

$$V(x) = 2\langle x, b\rangle^n_k x - \langle x, x\rangle^n_k b + \lambda x + \omega x + c,$$

where $b, c \in \mathbb{R}^n_k$, $\lambda \in \mathbb{R}$ and $\omega \in \mathfrak{o}(n,k)$. A conformal vector field of the form

$$W(x) = 2\langle x, b\rangle^n_k x - \langle x, x\rangle^n_k b, \ b \neq 0,$$

is essential, since $W(0) = 0$ and $\nabla W(0) = 0$. It is

$$\mathrm{zero}(W) = (b^\perp \cap L^n_k) \cup \{0\},$$

where $L^n_k := \{x \in \mathbb{R}^n_k | \langle x, x\rangle^n_k = 0, x \neq 0\}$ is the lightcone in $\mathbb{R}^n_k$. Let Δ^n_k denote the usual spinor modul. The twistor spinors on $\mathbb{R}^n_k$ are given by

$$\varphi(x) = x \cdot v + w, \quad v, w \in \Delta^n_k,$$

and the associated conformal field to a twistor spinor $\psi(x) = x \cdot v$ with a zero in the origin is

$$W_v = 2\langle x, b_v\rangle^n_k x - \langle x, x\rangle^n_k b_v, \quad b_v := -(-i)^{k+1} \sum \varepsilon_j \langle v, e_j v\rangle_{\Delta^n_k} e_j.$$

One can prove that every conformal vector field V on $\mathbb{R}^n_k$ can be generated by a pair of twistor spinors on $\mathbb{R}^n_k$.

Let us consider the Riemannian case. Essential conformal fields on a Riemannian manifold have been investigated by Obata, Lelong-Ferrand and Alekseevskii ([Ale72]). It is well known that a Riemannian manifold (M, g) is conformally flat in the neighborhood of a zero of an essential conformal field V. In particular, the zero set of an essential conformal field is discrete. Moreover, if the essential conformal field V is complete, i.e. there exists an one-parameter group Φ^V_t of essential conformal transformations on (M, g), then (M, g) is globally conformal to the Euclidean space $\mathbb{R}^n$ or to the standard sphere S^n (see [Yos75]).

Using the result of Alekseevskii and Yoshimata on essential conformal fields W. Kühnel and H.-B. Rademacher proved in [KR94] that a Riemannian spin manifold (M, g) admitting a twistor spinor with zero, whose associated conformal field doesn't vanish, is conformally flat. They also showed in [KR195] and [KR96] that there exist conformally non-flat Riemannian spin manifolds admitting twistor spinors with zeros.

Conformal maps and conformal vector fields were also intensively studied in pseudo-Riemannian geometry, especially in General Relativity. The situation in the pseudo-Riemannian case is more difficult and the most investigations make special assumptions on the conformal vector field or on curvature properties. We list here some papers and results concerning those cases. Kühnel and Rademacher investigated conformal gradient fields on pseudo-Riemannian manifolds in [KR295] and [KR197]. They proved that the zero set of a conformal gradient field is discrete and the manifold is conformally flat in a neighborhood of a zero. They also obtained global results for pseudo-Riemannian manifolds admitting conformal gradient fields with zeros. In case that (M, g) is Einstein and V is a non-homothetic conformal field, the gradient field

$$\mathrm{grad}(\mathrm{div}V) \in \mathcal{X}(M)$$

is also conformal on (M, g). Conformal fields on pseudo-Riemannian Einstein spaces and spaces with constant scalar curvature are discussed in [KR297]. In [CK78] it is proved that a locally symmetric Lorentzian manifold admitting a non-homothetic conformal field V is conformally flat.

In general, the zero set of a conformal vector field on a pseudo-Riemannian manifold (M, g) is neither discrete nor a submanifold. But in case that a conformal vector field V is linearizable, the connected components of $\mathrm{zero}(V)$ are submanifolds of (M, g). A homethetic field V is always linearizable. The connected components of $\mathrm{zero}(V)$ are then totally geodesic submanifolds and if V isn't a Killing vector field they are even totally isotropic submanifolds. Several results on the question when an algebra of conformal fields on a space-time reduces to an algebra of homothetic fields or when a single conformal field is linearizable can be found in [Hall90], [HS91] and [HCB97]. For these problems the algebraic properties of the Weyl tensor W and the conformal 2-form $F = d\omega_V$, $\omega_V := g(V, \cdot)$, in a zero of the conformal vector field V play an import role.

On an arbitrary time oriented Lorentzian spin manifold (M,g) it holds generally

$$\mathrm{zero}(V_\psi) = \mathrm{zero}(\psi)$$

for the zero set of the associated conformal field V_ψ to a twistor spinor ψ on (M,g) . This fact is a special feature of Lorentzian geometry. Hence, the associated conformal field V_ψ satisfies the condition

$$(I) \qquad \nabla V_\psi(p) = 0 \quad \forall p \in \mathrm{zero}(V_\psi).$$

Consider again the example above. There are three kinds of conformal vector fields of the form

$$W_b(x) = 2\langle x,b\rangle_1^n x - \langle x,x\rangle_1^n b, \quad b \neq 0,$$

on the Minkowski space $\mathbb{R}^n_1$ corresponding to the causal character of the vector b. In case that $b = b_s$ is a spacelike vector the zero set

$$\mathrm{zero}(W_{b_s}) = (b_s^\perp \cap L_1^n) \cup \{0\} \cong L_1^{n-1} \cup \{0\}$$

is not a submanifold of $\mathbb{R}^n_1$. It holds

$$W_{b_s}(0) = 0,\ \nabla W_{b_s}(0) = 0 \quad \text{and} \quad \nabla W_{b_s}(x) \neq 0 \quad \forall x \in \mathrm{zero}(W_{b_s})\backslash\{0\}.$$

If $b = b_t$ is a timelike vector we have

$$\mathrm{zero}(W_{b_t}) = \{0\} \quad \text{and} \quad \nabla W_{b_t}(0) = 0.$$

In the third case when $b = b_l$ is lightlike, the zero set is identical to the lightlike straight line $\mathbb{R} \cdot b_l$ in $\mathbb{R}^n_1$ and it holds

$$\nabla W_{b_l}(x) = 0 \quad \forall x \in \mathrm{zero}(W_{b_l}) = \mathbb{R} \cdot b_l.$$

Let

$$W_v(x) = 2\langle x,b_v\rangle_1^n x - \langle x,x\rangle_1^n b_v, \quad b_v := \sum \varepsilon_j \langle v, e_j v\rangle e_j,$$

be the associated conformal field to the twistor spinor $\psi_v(x) = x \cdot v,\ v \in \Delta_1^n$, on $\mathbb{R}^n_1$. Since $\mathrm{zero}(\psi_v) = \mathrm{zero}(W_v)$, it follows that

$$b_v \neq 0 \ \text{ and } \ \|b_v\| \leq 0.$$

In deed, the map

$$\begin{array}{rcl} i : \Delta_1^n & \to & K_1^n := \{x \in \mathbb{R}^n_1 | \langle x,x\rangle_1^n \leq 0,\ \langle x,e_1\rangle_1^n > 0\} \\ v & \mapsto & b_v \end{array}$$

is even surjective, i.e. up to a sign every conformal field on $\mathbb{R}^n_1$ satisfying property (I) is associated to a twistor spinor.

We investigate in this paper the zero set of conformal fields on arbitrary curved Lorentzian manifolds, which satisfy condition (I). Such conformal fields are neither gradient fields nor linearizable in the neighborhood of a zero. Our main result states:

Theorem *The zero set of a conformal vector field satisfying condition (I) on a Lorentzian manifold lies locally on a single lightlike smooth geodesic.*

However, there isn't a known example of a conformal field, which has a zero and satisfies condition (I), on a Lorentzian manifold that isn't conformally flat.

1 Some Lorentzian geometry

We will prove in this section three propositions that contain elementary properties of lightcones in a Lorentzian manifold. These propositions are the tool for proving our results on the shap of the zero set of a conformal vector field, what we will do in the next section. We use here some notations and facts concerning causality properties in Lorentzian geometry that can be found in a detailed manner in [BEE96].

Let $\mathbb{R}^n_1 := (\mathbb{R}^n, \langle\ ,\ \rangle^n_1)$ denote the n-dimensional Minkowski space, where

$$\langle x,y\rangle^n_1 := -x_1y_1 + \sum x_iy_i, \quad x,y \in \mathbb{R}^n_1,$$

and let

$$L^n_1 := \{x \in \mathbb{R}^n_1 \mid \langle x,x\rangle^n_1 = 0, x \neq 0\} \subset \mathbb{R}^n_1$$

be the lightcone of the Minkowski space $\mathbb{R}^n_1$. The lightcone L^n_1 is a submanifold in $\mathbb{R}^n_1$ of codimension 1. The tangent space $T_lL^n_1$ at every point $l \in L^n_1$ is lightlike, i.e. the restriction of the metric $\langle\ ,\ \rangle^n_1$ to $T_lL^n_1$ is degenerate. The line $\mathbb{R}\cdot l$ is the only totally lightlike subspace in $T_lL^n_1$.

Let $M^n_1 := (M^n, g)$, $n \geq 3$, be a n-dimensional Lorentzian manifold. Let L_p denote the lightcone in the tangent space $T_pM^n_1$ at $p \in M^n_1$ and

$$\exp_p : D_p \subset T_pM^n_1 \to M^n_1$$

the exponential map in the point $p \in M^n_1$, where D_p is the maximal domain of definition, which is an open starshaped neighborhood of the origin $0 \in T_pM$. The lightcone $\mathcal{L}_p$ in M^n_1 to $p \in M^n_1$ is then defined by

$$\mathcal{L}_p := \exp_p(D_p \cap L_p) \subset M^n_1,$$

i.e. $\mathcal{L}_p$ is exactly the set of points that can be connected with p by a smooth lightlike geodesic. In general, $\mathcal{L}_p$ isn't a submanifold of M^n_1.

Now, let U be a convex set in M^n_1. We remember that every point in a Lorentzian (semi-Riemannian) manifold admits a convex neighborhood. In a convex set U for any two points $p,q \in U$ an unique C^∞-geodesic γ_{pq} exists such that

$$\gamma_{pq}(0) = p,\ \gamma_{pq}(1) = q \quad \text{and} \quad \gamma_{pq}([0,1]) \subset U.$$

The quadratic distance function

$$\begin{array}{rcl} \Gamma^U : U \times U & \to & \mathbb{R} \\ (p,q) & \mapsto & \|\gamma'_{pq}\|^2 = g(\gamma'_{pq}(0), \gamma'_{pq}(0)) \end{array}$$

is a well defined and smooth function. In a time oriented open set U a causal vector $0 \neq v \in TU$, $g(v,v) \leq 0$, is either future directed ($\uparrow$-vector) or past directed ($\downarrow$-vector). We define the following subsets of a time oriented convex set $U \subset M_1^n$ to a point $p \in U$:

$$\begin{aligned}
I^{+(-)}(p,U) &:= \{q \in U : \|\gamma'_{pq}\| < 0, \gamma'_{pq}(0) \uparrow (\downarrow) - \text{vector}\} \\
J^{+(-)}(p,U) &:= \{q \in U : \|\gamma'_{pq}\| \leq 0, \gamma'_{pq}(0) \uparrow (\downarrow) - \text{vector}\} \cup \{p\} \\
\mathcal{L}_p^{U+(-)} &:= \{q \in U : \|\gamma'_{pq}\| = 0, \gamma'_{pq}(0) \uparrow (\downarrow) - \text{vector}\} \\
\mathcal{L}_p^U &:= \mathcal{L}_p^{U+} \cup \mathcal{L}_p^{U-}.
\end{aligned}$$

The sets $I^+(p,U)$, $I^-(p,U)$ are open and it holds

$$\begin{aligned}
J^{\pm}(p,U) &= \overline{I^{\pm}(p,U)}^U \\
\mathcal{L}_p^{U\pm} &= \partial_U I^{\pm}(p,U) \setminus \{p\} \\
\mathcal{L}_p^U &\subset \mathcal{L}_p \cap U.
\end{aligned}$$

Notice that if $q \in I^+(p,U)$ then $J^+(q,U) \subset I^+(p,U)$ and if $q \in J^+(p,U)$ then $I^+(q,U) \subset I^+(p,U)$ (see [Pen72]). Furthermore, there exists an open set $V_p \subset T_pM$ such that $\exp_p : V_p \to U$ is a diffeomorphism. Then it holds

$$\mathcal{L}_p^U = \exp_p(V_p \cap L_p)$$

and $\mathcal{L}_p^U$ is a submanifold in M_1^n of codimension 1. From the Gauss lemma it follows that the induced symmetric bilinear form of g on $T\mathcal{L}_p^U$ is degenerate in every point $l \in \mathcal{L}_p^U$.

This is the first of the announced propositions:

Proposition 1 *Let $U \subset M_1^n$ be convex and $p, q \in U$, $p \neq q$. For the intersection $\mathcal{L}_{pq}^U := \mathcal{L}_p^U \cap \mathcal{L}_q^U$ of the lightcones to p and q one of the following assertions is true:*

i) $\mathcal{L}_{pq}^U = \emptyset$,

ii) $\mathcal{L}_{pq}^U \neq \emptyset$, $\|\gamma'_{pq}\| \neq 0$ *and* $\mathcal{L}_{pq}^U$ *is a $(n-2)$-dimensional spacelike submanifold of M,*

iii) $\mathcal{L}_{pq}^U \neq \emptyset$, $\|\gamma'_{pq}\| = 0$ *and* $\mathcal{L}_{pq}^U = \mathrm{Im}\gamma_{pq} \cap U$ *is a 1-dimensional totally isotropic submanifold of M.*

PROOF: Suppose that $\mathcal{L}_{pq}^U \neq \emptyset$ and $\|\gamma'_{pq}\| \neq 0$. Then we have

$$\gamma'_{pl}(1) \nparallel \gamma'_{ql}(1) \qquad \forall l \in \mathcal{L}_{pq}^U,$$

which implies

$$\gamma'_{pl}(1) \notin T_l\mathcal{L}_q^U \qquad \forall l \in \mathcal{L}_{pq}^U.$$

Hence, the submanifolds $\mathcal{L}_p^U$ and $\mathcal{L}_q^U$ in M_1^n are transversal and $\mathcal{L}_{pq}^U$ is a $(n-2)$-dimensional submanifold in M_1^n. The tangent space

$$T_l(\mathcal{L}_{pq}^U) = T_l(\mathcal{L}_p^U) \cap T_l(\mathcal{L}_q^U)$$

is spacelike for every $l \in \mathcal{L}_{pq}^U$.

Suppose now that $\mathcal{L}^U_{pq} \neq \emptyset$ and $\|\gamma'_{pq}\| = 0$. Obviously, it holds $\mathrm{Im}\gamma_{pq} \cap U \subset \mathcal{L}^U_{pq}$. In a convex set of a Lorentzian manifold there are never lightlike triangles, i.e. if $p, q, r \in U$ and $\|\gamma'_{pq}\| = \|\gamma'_{pr}\| = \|\gamma'_{qr}\| = 0$, then $r \in \mathrm{Im}\gamma_{pq} \cap U$. Therefore equality holds: $\mathrm{Im}\gamma_{pq} \cap U = \mathcal{L}^U_{pq}$. □

The intersection $\mathcal{L}^U_{pq}$, $p, q \in U$, of cones in a convex set U isn't empty, if p and q are sufficiently close together:

Proposition 2 *Let $p \in M^n_1$. There is a neighborhood $U(p)$ of p contained in a convex set U with the property*

$$\mathcal{L}^U_{qr} \neq \emptyset \qquad \forall q, r \in U(p).$$

PROOF: Let U be a time oriented convex neighborhood of $p \in M^n_1$. In an arbitrary neighborhood $V(p) \subset U$ exist points $u, v \in V(p)$ such that the open set $\langle u, v\rangle_U := I^+(u, U) \cap I^-(v, U)$ is a neighborhood of p in $V(p)$:

$$p \in \langle u, v\rangle_U \subset V(p) \subset U$$

(see [Gun88] p. 15 or [Fri75]). So let $\tilde{V}(p) \subset \mathrm{int}U$ be a relative compact neighborhood of p and $a, b \in \tilde{V}(p)$ such that $p \in \langle a, b\rangle_U \subset \tilde{V}(p)$. We show that the neighborhood $U(p) := \langle a, b\rangle_U \subset U$ has the desired property.

1) Suppose that $q, r \in \langle a, b\rangle_U$ and $\|\gamma'_{qr}\| > 0$. Consider the geodesic γ_{qb}. It holds $\|\gamma'_{qr}\| > 0$ and $\|\gamma'_{br}\| < 0$. Therefore, it exists a $t \in [0, 1]$, $\tilde{b} := \gamma_{qb}(t)$, with $\|\gamma'_{\tilde{b}r}\| = 0$. Similar, one can find a $\hat{t} \in [0, 1]$ such that $\|\gamma'_{\gamma_{r\tilde{b}}(\hat{t})q}\| = 0$ and then $\gamma_{r\tilde{b}}(\hat{t}) \in \mathcal{L}^U_{qr}$.

2) Suppose that $q, r \in \langle a, b\rangle_U$, $\|\gamma'_{qr}\| < 0$ and $\gamma'_{qr}(0)$ a $\uparrow$-vector. The set $\overline{\langle a, r\rangle_U} \subset U$ is compact. Let $\gamma_q : I^U_q \to U$ be an arbitrary maximal lightlike $\uparrow$-geodesic in U with $\gamma_q(0) = q$. Since $q \in I^+(a, U)$ and $\gamma_q(t) \in J^+(q, U)$ for every $t \in I^U_q \cap \mathbb{R}_+$, it holds

$$\gamma_q(I^U_q \cap \mathbb{R}_+) \subset I^+(a, U).$$

The set $\gamma_q(I^U_q \cap \mathbb{R}_+)$ isn't contained in a compact subset of U and therefore it exists a $t \in I^U_q \cap \mathbb{R}_+$ with

$$\gamma_q(t) \notin J^-(r, U) \text{ and } \|\gamma'_{r\gamma_q(t)}\| > 0.$$

But then $\hat{t} \in I^U_q \cap \mathbb{R}_+$ exists such that $\|\gamma'_{r\gamma_q(\hat{t})}\| = 0$, which implies $\mathcal{L}^U_{qr} \neq 0$.

Obviously, it is $\mathcal{L}^U_{qr} \neq \emptyset$ for $q, r \in \langle a, b\rangle_U$ with $\|\gamma'_{qr}\| = 0$. □

Proposition 3 *Let $N^1 \subset M^n_1$ be a 1-dimensional, spacelike submanifold. Then an open set $U_N \subset M^n_1$ exists with the property that for every point $r \in U_N$ there are lightlike vectors*

$$v_r \nparallel w_r \in T_rM \quad \textit{with} \quad \exp_r v_r, \exp_r w_r \in N.$$

The proposition isn't true in general for a lightlike, 1-dimensional submanifold $N^1 \subset M^n_1$. To prove it we need some preparation.

Lemma 1 *Let $U \subset M_1^n$ be a time oriented convex set, $r \in U$ and $a, b, c \in \mathcal{L}_r^{U^-}$ points in the lightcone of the past with*

$$\|\gamma'_{ab}\|, \|\gamma'_{ac}\|, \|\gamma'_{bc}\| > 0.$$

Then there exists in every neighborhood $U(r)$ of r a point $\tilde{r} \in U(r)$ such that

$$a, b \in I^-(\tilde{r}, U) \text{ and } c \notin J^-(\tilde{r}, U) \cup J^+(\tilde{r}, U).$$

Proof: It is

$$r \in \mathcal{L}_{abc}^U := \mathcal{L}_a^U \cap \mathcal{L}_b^U \cap \mathcal{L}_c^U \neq \emptyset.$$

For every $\hat{r} \in \mathcal{L}_{abc}^U$ the lightlike vectors

$$\gamma'_{a\hat{r}}(1), \gamma'_{b\hat{r}}(1), \gamma'_{c\hat{r}}(1) \in T_{\hat{r}} M_1^n$$

aren't pairwise parallel by assumption and thus they are linearly independent. It follows

$$T_{\hat{r}} \mathcal{L}_{ab}^U = \operatorname{span}\{\gamma'_{a\hat{r}}(1), \gamma'_{b\hat{r}}(1)\}^\perp \not\subset (\gamma'_{c\hat{r}}(1))^\perp = T_{\hat{r}} \mathcal{L}_c^U$$

and the submanifolds $\mathcal{L}_c^U$ and $\mathcal{L}_{ab}^U$ of M_1^n are transversal. Hence, $\mathcal{L}_{abc}^U$ is a submanifold in $\mathcal{L}_{ab}^U$ of codimension 1. This implies the existence of a C^∞-curve $\alpha : (-\delta, \delta) \to \mathcal{L}_{ab}^U \subset M_1^n$ with $\alpha(0) = r$ and $\alpha'(0) \notin T_r \mathcal{L}_c^U$, i.e. the curve α intersects the lightcone $\mathcal{L}_c^U$ at the point r. Then there must be a $\hat{t} \neq 0$ such that $c \notin J^-(\alpha(\hat{t}), U) \cup J^+(\alpha(\hat{t}), U)$. Since $\alpha(\hat{t}) \in \mathcal{L}_{ab}^U$, we can find a point $\tilde{r} \in I^+(\alpha(\hat{t}), U)$ in the near of $\alpha(\hat{t})$ such that $c \notin J^-(\tilde{r}, U) \cup J^+(\tilde{r}, U)$ and $a, b \in I^-(\tilde{r}, U)$. The point $\tilde{r}$ can be chosen arbitrary close to r. $\square$

We use the following notation. Let $a, b, r \in U$ points in a convex set such that

$$\#(\ \{a, b\} \ \cap \ I^-(r, U)\ ) = 1$$
$$\#(\ \{a, b\} \ \cap \ (J^-(r, U) \cup J^+(r, U))\ ) = 1.$$

Then we call $r \in U$ an (a, b)-separating point in U.

Lemma 2 *Let $U \in M_1^n$ be a time oriented convex set, $r \in U$ and $a_1, a_2, b_1, b_2 \in I^-(r, U)$ such that*

$$\|\gamma'_{xy}\| > 0 \qquad \forall x, y \in \{a_1, a_2, b_1, b_2\}, \quad x \neq y.$$

Then there exists a point $s \in U$, which separates the pairs (a_1, a_2) and (b_1, b_2) in U.

Proof: Consider the geodesic $\gamma_{a_1 r}$. There are real numbers $t_{a_2}, t_{b_1}, t_{b_2} \in (0, 1)$ such that

$$x \in \mathcal{L}_{\gamma_{a_1 r}(t_x)}^{U^-} \qquad \forall x \in \{a_2, b_1, b_2\}.$$

1. case: One of the numbers $t_{a_2}, t_{b_1}, t_{b_2}$ is greater then the others. Then clearly $\hat{t} < \max\{t_{a_2}, t_{b_1}, t_{b_2}\}$ exists such that for $\tilde{r} := \gamma_{a_1 r}(\hat{t})$

$$\#(\ \{a_1, a_2, b_1, b_2\} \cap I^-(\tilde{r}, u)\ ) = \#(\ \{a_1, a_2, b_1, b_2\} \cap (J^-(\tilde{r}, u) \cup J^+(\tilde{r}, u)\ ) = 3. \qquad (*)$$

2. case: It is $t_{a_2} = t_{b_1} = t_{b_2}$. Then by Lemma 1 there exists a $\tilde{r} \in U$ such that (*) is satisfied.

In case that two of the numbers $t_{a_2}, t_{b_1}, t_{b_2}$ are equal and greater then the third, Lemma 1 is all the more applicable. Altogether, we have in any case a point $\tilde{r} \in U$ such that after eventually changing the notation

$$a_1, b_1, b_2 \in I^-(\tilde{r}, U), \quad a_2 \notin J^-(\tilde{r}, U) \cup J^+(\tilde{r}, U).$$

With the same procedure as before applied to the points $\{a_1, b_1, b_2\}$ it follows the existence of a point $s \in U$ such that

$$\begin{array}{ll} & a_1, b_1 \in I^-(s, U),\ b_2 \notin J^-(s, U) \cup J^+(s, U) \\ \text{or} & a_1, b_2 \in I^-(s, U),\ b_1 \notin J^-(s, U) \cup J^+(s, U). \end{array}$$

The point s can be chosen such that still $a_2 \notin J^-(s, U) \cup J^+(s, U)$. $\square$

Proof of Proposition 3: Let $U \subset M_1^n$ be a convex set and $s \in U \subset M_1^n$ such that

$$N^1 \cap I^-(s, U) \neq \emptyset.$$

Since N^1 is spacelike, there is a spacelike C^∞-curve $\alpha : (-1, 1) \to N^1 \cap I^-(s, U)$. An easy consideration shows that there are real numbers $t_1, t_2, t_3, t_4 \in (-1, 1),\ t_1 < t_2 < t_3 < t_4$, such that

$$\begin{array}{ll} \|\gamma'_{\alpha(t_1)\alpha(t_2)}\|,\ \|\gamma'_{\alpha(t_3)\alpha(t_4)}\| > 0 & \text{and} \\ \|\gamma'_{xy}\| > 0 & \forall x \in \alpha([t_1, t_2]),\ y \in \alpha([t_3, t_4]). \end{array}$$

From Lemma 2 it follows the existence of a point $\tilde{s} \in U$, which separates the pairs $(\alpha(t_1), \alpha(t_2))$ and $(\alpha(t_3), \alpha(t_4))$ in U. Moreover, there is a neighborhood U_N of $\tilde{s}$ such that every point $\hat{s} \in U_N$ separates these pairs in U. This shows for every $\hat{s} \in U_N$ the existence of points

$$x_{\hat{s}} \in \mathcal{L}_{\hat{s}}^{U^-} \cap \alpha([t_1, t_2]) \ \text{ and } \ y_{\hat{s}} \in \mathcal{L}_{\hat{s}}^{U^-} \cap \alpha([t_3, t_4]).$$

Since $\|\gamma'_{x_{\hat{s}} y_{\hat{s}}}\| > 0$, it follows

$$\gamma'_{\hat{s} x_{\hat{s}}}(0) \nparallel \gamma'_{\hat{s} y_{\hat{s}}}(0). \qquad \square$$

2 The zero set of a conformal vector field

Let $M_1^n := (M^n, g)$, $n \geq 3$, be a n-dimensional Lorentzian manifold. A vector field $V \in \mathcal{X}(M_1^n)$ is called conformal, if

$$\mathcal{L}_V g = 2\alpha \cdot g$$

for some function $\alpha \in C^\infty(M_1^n)$. A conformal vector field V on a connected Lorentzian manifold M_1^n is uniquely determined by the values of

$$V(x_o),\ \nabla V(x_o),\ \alpha(x_o),\ d\alpha(x_o)$$

at an arbitrary point $x_o \in M^n_1$. Let $\Phi^V : A^V \subset \mathbb{R} \times M^n_1 \to M^n_1$ denote the maximal local flow of the conformal vector field V, i.e. for every point $p \in M^n_1$ the map

$$\rho_p(t) := \Phi^V_t(p) = \Phi^V(t,p), \quad t \in I_p := A^V \cap (\mathbb{R} \times \{p\}),$$

is the maximal integral curve of the field V through the point $p \in M^n_1$. In case that the flow Φ^V is defined on an open subset $W \subset M^n_1$ for all $t \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, the mapping

$$\Phi^V_t : W \to \Phi^V_t(W) \subset M^n_1$$

is a conformal diffeomorphism for every $t \in (-\varepsilon, \varepsilon)$. The zero set of the conformal vector field V is denoted by $\mathrm{zero}(V)$. The property $\nabla V(p) = 0$ for a conformal vector field V in $p \in \mathrm{zero}(V)$ implies for the maximal local flow Φ^V in p that $d\Phi^V_t(p) = \mathrm{id}|_{T_pM}$ for all $t \in I_p$ holds.

Lemma 3 *Let V be a conformal vector field on a Lorentzian manifold M^n_1 and $p \in M^n_1$ a zero of V with $\nabla V(p) = 0$. For every point $q \in \mathcal{L}_p$ and every lightlike smooth geodesic $\gamma_q : [0,1] \to M^n_1$, $\gamma_q(0) = p$, $\gamma_q(1) = q$, it holds*

$$V(q) \parallel \gamma_q'(1) \in T_qM^n_1 \quad \textit{or} \quad V(q) = 0.$$

Proof: Let W_{γ_q} be an open neighborhood of the compact set $\gamma_q([0,1]) \subset M^n_1$ and $\varepsilon > 0$ such that the flow Φ^V_t on W_{γ_q} is defined for every $t \in (-\varepsilon, \varepsilon)$. Because $\Phi^V_t : W_{\gamma_q} \to M^n_1$ is a conformal transformation for every $t \in (-\varepsilon, \varepsilon)$, every C^∞-curve $\gamma_{q_t} := \Phi^V_t \circ \gamma_q : [0,1] \to M^n_1$ is a lightlike pregeodesic in M^n_1 with $\gamma_{q_t}(0) = p$. Since

$$d\Phi^V_t(p) = \mathrm{id}|_{T_pM} \qquad \forall t \in (-\varepsilon, \varepsilon),$$

it holds

$$\gamma_{q_t}'(0) = \gamma_q'(0) \qquad \forall t \in (-\varepsilon, \varepsilon),$$

which implies the existence of a smooth function $\lambda : (-\varepsilon, \varepsilon) \to \mathbb{R}$ with

$$\Phi^V_t(q) = \gamma_{q_t}(1) = \exp_p(\lambda(t)\gamma_q'(0)).$$

It follows $V(q) = \frac{d}{dt}|_{t=0}\Phi^V_t(q) = \lambda'(0)\gamma_q'(1)$. $\square$

We call a point $q \in M^n_1$ lightlike conjugated to $p \in M^n_1$, if there exist lightlike C^∞-geodesics

$$\gamma_i : [0,1] \to M^n_1,\ \gamma_i(0) = p,\ \gamma_i(1) = q,\ i = 1,2,$$

with $\gamma_1'(1) \nparallel \gamma_2'(1)$ (comp. [BEE96], Chap. 9 and 10) and let $\mathrm{lc}(p)$ denote the set of lightlike conjugated points to p in M^n_1. Lemma 3 implies the

Conclusion: If $p \in M^n_1$ is a zero of a conformal vector field V on M^n_1 with $\nabla V(p) = 0$ then

$$\mathrm{lc}(p) \subset \mathrm{zero}(V).$$

With the results of the first section we prove now:

Theorem 1 *Let $0 \not\equiv V$ be a conformal vector field on a connected Lorentzian manifold M_1^n with the property*

$$\nabla V(p) = 0 \qquad \forall p \in \mathrm{zero}(V).$$

Then there exists for every $p \in \mathrm{zero}(V)$ a neighborhood $U(p) \subset M_1^n$ and a lightlike C^∞-geodesic γ_p such that

$$\mathrm{zero}(V) \cap U(p) \subset \mathrm{Im}\gamma_p \cap U(p).$$

Proof: From Proposition 2 it follows the existence of a neighborhood $U(p)$ of p, which is contained in a convex set U, such that

$$\mathcal{L}_{qr}^U \neq \emptyset \qquad \forall q, r \in U(p).$$

Suppose that there are points $q, r \in \mathrm{zero}(V) \cap U(p)$ with $\|\gamma'_{qr}\| \neq 0$. Then we have

$$\gamma'_{ql}(1) \nparallel \gamma'_{rl}(1) \qquad \forall l \in \mathcal{L}_{qr}^U$$

and by Lemma 3 it follows $V(l) = 0$ for $l \in \mathcal{L}_{qr}^U$. Proposition 1 says that $\mathcal{L}_{qr}^U \subset \mathrm{zero}(V)$ is a spacelike submanifold of M_1^n. But then Proposition 3 implies the existence of an open set $U_{qr} \subset \mathrm{zero}(V)$. This isn't possible, since M_1^n is connected and $V \not\equiv 0$. We can conclude that

$$\|\gamma'_{qr}\| = 0 \qquad \forall q, r \in \mathrm{zero}(V) \cap U(p).$$

We mentioned already that lightlike triangles don't exist in a convex subset of a Lorentzian manifold. Hence, the set $\mathrm{zero}(V) \cap U(p)$ must be contained in the image of a single lightlike C^∞-geodesic γ_p. □

On the Minkowski space $\mathbb{R}_1^n$ we know from the explicit form of the conformal vector fields given in the introduction that the zero set of a conformal vector field V with

$$\nabla V(p) = 0 \qquad \forall p \in \mathrm{zero}(V)$$

is a lightlike straight line or a single point. The assertion of Theorem 1 for arbitrary curved Lorentzian manifolds is only a bit weeker. We can also prove a more global version of Theorem 1:

Theorem 2 *Let M_1^n be a connected Lorentzian manifold and $V \in \mathcal{X}(M_1^n)$ a conformal vector field. If $p, q \in \mathrm{zero}(V)$ and lightlike vectors $v_p \in T_pM$, $v_q \in T_q(M)$ exist with*

i) $\mathrm{rg}(d\exp_p(v_p)) = \mathrm{rg}(d\exp_q(v_q)) = n$

ii) $r := \exp_p(v_p) = \exp_q(v_q) \in M_1^n$

iii) $\frac{d}{dt}|_{t=1} \exp_p tv_p \nparallel \frac{d}{dt}|_{t=1} \exp_q tv_q$,

then the conformal vector field V vanishes identically.

Proof: The three assumptions imply the existence of neighborhoods $V_p \subset T_pM$ of v_p and $V_q \subset T_qM$ of v_q such that

$$\mathcal{L}_1 := \exp_p(V_p \cap L_p) \quad \text{and} \quad \mathcal{L}_2 := \exp_p(V_q \cap L_q)$$

are transversal submanifolds in M_1^n and $W_{pq} := \mathcal{L}_1 \cap \mathcal{L}_2$ is a spacelike submanifold of M_1^n. With the same arguments as in the proof of Theorem 1 we can conclude that $V \equiv 0$ on M_1^n. □

3 The zero set of a twistor spinor

Every twistor spinor on a time oriented Lorentzian spin manifold induces a conformal vector field. Since the zero sets of the twistor spinor and the associated conformal field are identical, the results of the previous section can be applied to twistor spinors. That will be done here. We start with recalling briefly the definition of a twistor spinor and its main properties (comp. [Baum198] and [BFGK91]).

Let M^n_1, $n \geq 3$, be a Lorentzian spin manifold with spin structure (Q, f) and let S be the associated spinor bundle over M^n_1, $r := \dim_{\mathbb{C}} S = 2^{[\frac{n}{2}]}$. We denote by

$$\nabla^S : \Gamma(S) \to \Gamma(TM \otimes S)$$

the spinor derivative and by

$$\mu : TM \otimes S \to S$$

the Clifford multiplication. In case that M^n_1 is time oriented, there exists a non-degenerate, indefinite Hermitian product

$$\langle \cdot, \cdot \rangle_S : S \times S \to \mathbb{C}$$

on the spinor bundle S. It holds

$$\begin{aligned} \langle X \cdot \varphi, \psi \rangle_S &= \langle \varphi, X \cdot \psi \rangle_S, \\ X \langle \varphi, \psi \rangle_S &= \langle \nabla^S_X \varphi, \psi \rangle_S + \langle \varphi, \nabla^S_X \psi \rangle_S \end{aligned}$$

for all $X \in \Gamma(TM)$ and $\varphi, \psi \in \Gamma(S)$. The Dirac operator is defined by

$$D : \mu \circ \nabla^S : \Gamma(S) \to \Gamma(S)$$

and the twistor operator is defined by

$$\mathcal{D} := \mathrm{proj}_{\ker \mu} \circ \nabla^S : \Gamma(S) \to \Gamma(\ker \mu),$$

where $\mathrm{proj}_{\ker \mu} : TM \otimes S \to \ker \mu$ denotes the projection onto the kernel of the Clifford multiplication. A spinor field $\varphi \in \Gamma(S)$, which satisfies $\mathcal{D}\varphi = 0$, is called twistor spinor. For a twistor spinor $\varphi \in \Gamma(S)$ holds

1) $\nabla^S_X \varphi + \frac{1}{n} X \cdot D\varphi = 0$
2) $D^2 \varphi = \frac{1}{4} R \frac{n}{n-1} \varphi$
3) $\nabla^S_X D\varphi = \frac{n}{2} L(X) \varphi$,

where R is the scalar curvature and L is the Schouten tensor. A twistor spinor φ on a connected spin manifold M^n_1 is uniquely determined by the values of $\varphi(x_o)$ and $D\varphi(x_o)$ in an arbitrary point $x_o \in M^n_1$.

To every spinor field $\psi \in \Gamma(S)$ on a time oriented spin manifold M^n_1 there is a vector field V_ψ associated by

$$V_\psi := -\Sigma \varepsilon_i \langle \psi, s_i \psi \rangle_S s_i,$$

where $(s_1, \dots, s_n)$ is a local orthonormal frame on M^n_1. If $\varphi \in \Gamma(S)$ is a twistor spinor, then the associated field V_φ is conformal on M^n_1 and it has the properties

1) $\mathrm{zero}(V_\varphi) = \mathrm{zero}(\varphi)$

2) $\nabla V_\varphi(p) = 0 \qquad \forall p \in \mathrm{zero}(V_\varphi) \qquad$ (see [Baum198]).

Consider now a smooth geodesic $\gamma(t)$ on a Lorentzian spin manifold M^n_1 admitting a twistor spinor $\varphi \in \Gamma(S)$. Let $\mathrm{Im}\gamma$ denote the image of the geodesic γ in M^n_1. Let $p \in \mathrm{Im}\gamma$ be a point. The set $U_\gamma := \exp_p(D_p) \subset M^n_1$ is a time orientable neighborhood of $\mathrm{Im}\gamma$ in M^n_1. Furthermore, let $\{f_i(t) : i \in \{1, \dots, r\}\}$ be a parallel basis field of S along the geodesic $\gamma(t)$:

$$\nabla^S_{\dot\gamma} f_i = 0 \qquad \forall i \in \{1, \dots, r\}.$$

We choose on U_γ a time orientation and define the functions

$$u_i(t) := \langle \varphi(\gamma(t)), f_i(\gamma(t)) \rangle_S, \quad i \in \{1, \dots, r\}.$$

It holds

$$\begin{aligned} \frac{du_i}{dt} &= \dot\gamma(u_i) = \langle \nabla^S_{\dot\gamma}\varphi, f_i\rangle_S = -\frac{1}{n}\langle \dot\gamma D\varphi, f_i\rangle_S, \\ \frac{d^2u_i}{dt^2} &= \dot\gamma\dot\gamma(u_i) = -\frac{1}{n}\langle \nabla^S_{\dot\gamma}(\dot\gamma D\varphi), f_i\rangle_S = -\frac{1}{2}\langle \dot\gamma L(\dot\gamma)\varphi, f_i\rangle_S \\ &= -\frac{1}{2}\langle \varphi, L(\dot\gamma)\dot\gamma f_i\rangle_S \qquad \forall i \in \{1, \dots, r\}. \end{aligned}$$

For the vector $U(t) := \begin{pmatrix} u_1(t) \\ \vdots \\ u_r(t) \end{pmatrix}$ we obtain the linear differential equation system

$$U'' = -\frac{1}{2}\overline{C} \cdot U,$$

where $C(t) \in M(r, \mathbb{C})$ is the complex matrix of the endomorphism $s \in S_{\gamma(t)} \mapsto L(\dot\gamma)\dot\gamma(t)s \in S_{\gamma(t)}$ with respect to the basis $\{f_i(t)\}$.

Lemma 4 *Let M^n_1 be a Lorentzian spin manifold, $\varphi \in \Gamma(S)$ a twistor spinor on M^n_1, $p \in \mathrm{zero}(\varphi)$ and $\gamma_p(t)$ a C^∞-geodesic on M^n_1 with $\gamma_p(0) = p$.*

i) If $\dot\gamma_p(0) \cdot D\varphi(p) = 0$, then $\mathrm{Im}\gamma_p \subset \mathrm{zero}(\varphi)$.

ii) If $\dot\gamma_p(0) \cdot D\varphi(p) \neq 0$, then there exists a neighborhood $U(p)$ of p with

$$\mathrm{zero}(\varphi) \cap \mathrm{Im}\gamma_p \cap U(p) = \{p\}.$$

PROOF: With the notations as above we have the differential equation system

$$U'' = -\frac{1}{2}\overline{C} \cdot U, \qquad U(0) = 0,$$

for the functions $u_i(t) := \langle \varphi(\gamma(t)), f_i(\gamma(t)) \rangle_S$ with respect to a parallel frame $\{f_i(t)\}$. It holds

$$U'(0) = 0 \text{ iff } \dot\gamma_p(0) D\varphi(p) = 0. \qquad \square$$

Definition 1 *Let M_1^n be a Lorentzian spin manifold, $\varphi \in \Gamma(S)$ a twistor spinor and*

$$\gamma_p : \{t \in \mathbb{R} : tv_p \in D_p\} \to M_1^n, \quad v_p \in T_pM,$$

a maximal geodesic such that $\mathrm{Im}\gamma_p \subset \mathrm{zero}(\varphi)$. Then the set $Z_{\gamma_p} := \mathrm{Im}\gamma_p$ is called a zero set geodesic of φ.

Obviously, the image $\mathrm{Im}\gamma$ of a maximal C^∞-geodesic γ in M_1^n is a zero set geodesic to a twistor spinor $\varphi \not\equiv 0$ iff there is a point $p \in \mathrm{Im}\gamma \cap \mathrm{zero}(\varphi)$ with $\dot{\gamma} \cdot D\varphi(p) = 0$.

Theorem 3 *Let M_1^n be a connected Lorentzian spin manifold and $0 \not\equiv \varphi \in \Gamma(S)$ a twistor spinor on M_1^n.*

i) Every zero set geodesic to φ in M_1^n is a totally lightlike, 1-dimensional submanifold of M_1^n.

ii) Every zero set geodesic to φ is isolated, i.e. for every zero set geodesic $Z_\gamma = \mathrm{Im}\gamma$ there is an open set $U(\gamma)$ with

$$\mathrm{zero}(\varphi) \cap \overline{U(\gamma)} = \mathrm{Im}\gamma.$$

iii) The zero set $\mathrm{zero}(\varphi)$ is the countable union of isolated points and isolated zero set geodesics:

$$\mathrm{zero}(\varphi) = \bigcup_{i\in\mathbb{N}} \mathrm{Im}\gamma_i \cup \bigcup_{i\in\mathbb{N}} \{p_i\}.$$

Proof: By Theorem 1, it exists a time oriented neighborhood $U(p)$ of $p \in \mathrm{zero}(\varphi)$ and a C^∞-geodesic γ_p with

$$\mathrm{zero}(\varphi) \cap U(p) = \mathrm{zero}(V_\varphi) \cap U(p) \subset \mathrm{Im}\gamma_p,$$

where V_φ is the associated conformal field to φ on $U(p)$. Moreover, from Lemma 4 it follows the existence of an open neighborhood $\tilde{U}(p) \subset U(p)$ of p with

$$\begin{aligned} \mathrm{zero}(\varphi) \cap \tilde{U}(p) &= \mathrm{Im}\gamma_p \qquad \text{or} \\ \mathrm{zero}(\varphi) \cap \tilde{U}(p) &= \{p\}. \end{aligned}$$

This proves that every zero set geodesic $Z_\gamma = \mathrm{Im}\gamma$ of φ is a submanifold in M_1^n and $Z_\gamma = \mathrm{Im}\gamma$ is isolated. In particular, the third assertion is then clear. It remains to prove that a zero set geodesic is lightlike. So let $\mathrm{Im}\gamma$ be a zero set geodesic, then for $p \in \mathrm{Im}\gamma$ we have

$$\dot{\gamma} D\varphi(p) = 0 \text{ and } D\varphi(p) \neq 0.$$

It follows $\|\dot{\gamma}\| = 0$ $\square$

One should notice that we used Theorem 1 in the proof. Theorem 3 isn't a direct consequence of Lemma 4, which is proved only using spinor calculus.

For arbitrary vectors $X, Y \in T_pM_1^n$ the mapping

$$s \in S_p \mapsto X \cdot Y \cdot s \in S_p$$

is a complex linear endomorphism. Let $w \in S_p$ be an eigenspinor of $X \cdot Y$ to the eigenvalue $c \neq 0$. We have

$$\begin{aligned} XYw = cw, \quad YXw = \frac{g(X,X)g(Y,Y)}{c} w = (-c - 2g(X,Y))w \qquad \text{and} \\ c = -g(X,Y) \pm \sqrt{g(X,Y)^2 - g(X,X)g(Y,Y)} \in \mathbb{C}. \end{aligned}$$

If there is in addition a spinor $v \in S_p$ with $XYv = 0$, then $g(X,X)g(Y,Y) = 0$. Hence, the endomorphism $XY \in \mathrm{End}(S_p)$, $X, Y \in T_pM$, has at most the eigenvalues

$$\begin{aligned} C^+_{XY} &:= -g(X,Y) + \sqrt{g(X,Y)^2 - g(X,X)g(Y,Y)}, \\ C^-_{XY} &:= -g(X,Y) - \sqrt{g(X,Y)^2 - g(X,X)g(Y,Y)}. \end{aligned}$$

The endomorphism $X \cdot Y \in End(S_p)$ has no positive eigenvalues if and only if

$$\begin{aligned} & g(X,Y)^2 - g(X,X) \cdot g(Y,Y) < 0 \quad \text{or} \\ & \|X\| \cdot \|Y\| \geq 0, \quad g(X,Y) \geq 0. \end{aligned}$$

Theorem 4 *Let M^n_1 be a Lorentzian spin manifold with $\nabla \mathrm{Ric} = 0$, $0 \not\equiv \varphi \in \Gamma(S)$ a twistor spinor on M^n_1, $p \in \mathrm{zero}(\varphi)$ and $\gamma_p(t)$ a C^∞-geodesic with $\gamma_p(0) = p$ and $\dot\gamma_p D\varphi(p) \neq 0$. If*

$$\begin{aligned} & g(\dot\gamma_p, L(\dot\gamma_p))^2 - g(\dot\gamma_p, \dot\gamma_p) \cdot g(L(\dot\gamma_p), L(\dot\gamma_p)) < 0 \quad \textit{or} \\ & \|\dot\gamma_p\| \cdot \|L(\dot\gamma_p)\| \geq 0, \quad g(\dot\gamma_p, L(\dot\gamma_p)) \geq 0, \end{aligned}$$

then $\mathrm{Im}\gamma_p \cap \mathrm{zero}(\varphi) = \{p\}$.

Proof: Let $\{f_i(t) : i \in \{1, \ldots, r\}\}$ be a parallel basis field along $\gamma_p(t)$. From $\nabla \mathrm{Ric} = 0$ it follows that the scalar curvature R is constant and therefore $\nabla L = 0$. Then

$$\dot\gamma_p \langle L(\dot\gamma_p)\dot\gamma_p f_i, f_j\rangle_S = 0,$$

i.e. the matrix function $C(t) \equiv C$ is constant and it exists a parallel eigenspinor field $s(t)$ on $\mathrm{Im}\gamma_p$ such that the function $u_s(t) := \langle \varphi(\gamma_p(t), s(t)\rangle_S \not\equiv 0$ satisfies

$$\frac{d^2 u_s(t)}{dt^2} = -\frac{1}{2}\overline{c} \cdot u_s,$$

where $\mathbb{C} \ni c = \text{const.}$ is an eigenvalue of C. Hence, the function u_s is of the form

$$u_s(t) = A\left(e^{\sqrt{-\frac{1}{2}\overline{c}} \cdot t} - e^{-\sqrt{-\frac{1}{2}\overline{c}} \cdot t} \right), \quad A \neq 0.$$

If $u_s(t) = 0$ for $t \neq 0$, then $\sqrt{-\frac{1}{2}\overline{c}} \in i\mathbb{R}$ and thus $c > 0$. But $L(\dot\gamma_p)\dot\gamma_p$ has no positive eigenvalues and such a $t \neq 0$ doesn't exist. $\square$

Theorem 5 *Let M^n_1 be a Lorentzian Einstein spin manifold, $0 \not\equiv \varphi \in \Gamma(S)$ a twistor spinor on M^n_1, $p \in \mathrm{zero}(\varphi)$ and $\gamma_p(t)$ a C^∞-geodesic with $\gamma_p(0) = p$ and $\dot\gamma_p D\varphi(p) \neq 0$.*

i) If $g(\dot\gamma_p, \dot\gamma_p) \cdot R \leq 0$, then

$$\mathrm{zero}(\varphi) \cap \mathrm{Im}\gamma_p = \{p\}.$$

ii) If $g(\dot\gamma_p,\dot\gamma_p)\cdot R>0$, then

$$\mathrm{zero}(\varphi)\cap \mathrm{Im}\gamma_p=\{\gamma_p(\frac{n\cdot\pi}{d})|n\in\mathbb{N}\},\qquad d=\sqrt{\frac{R\cdot g(\dot\gamma_p,\dot\gamma_p)}{4n(n-1)}},$$

i.e. the zero set is periodic on $\mathrm{Im}\gamma_p$.

PROOF: On an Einstein space the Schouten tensor L equals $\frac{-R}{2n(n-1)}\mathrm{id}$ and therefore

$$L(\dot\gamma_p)\dot\gamma_p=\frac{R\cdot g(\dot\gamma_p,\dot\gamma_p)}{2n(n-1)}\mathrm{id}_S.$$

The first assertion is a special case of Theorem 4. If $g(\dot\gamma_p,\dot\gamma_p)\cdot R>0$, then the solution of

$$U''=-\frac{R\cdot g(\dot\gamma_p,\dot\gamma_p)}{4n(n-1)}U,\quad U=\begin{pmatrix}u_1(t)\\ \vdots\\ u_r(t)\end{pmatrix}\in\mathbb{C}^r,$$

is given by

$$U=\sin(dt)\cdot U'(0),\quad d=\sqrt{\frac{R\cdot g(\dot\gamma_p,\dot\gamma_p)}{4n(n-1)}}.$$

This proves the second assertion. □

There is a direct consequence of Theorem 5. In case that M^n_1 is an Einstein spin manifold admitting a twistor spinor $\varphi\not\equiv 0$ and the points $p,q\in\mathrm{zero}(\varphi)$ don't lie on a common zero set geodesic of φ, then the intersection of the lightcones to p and q is empty:

$$\mathcal{L}_p\cap\mathcal{L}_q=\emptyset.$$

Example: The pseudosphere

$$\iota: S^{2n}_1:=\{x\in\mathbb{R}^{2n+1}_1|\langle x,x\rangle^{2n+1}_1=1\}\hookrightarrow\mathbb{R}^{2n+1}_1$$

is a totally umbilic hypersurface in $\mathbb{R}^{2n+1}_1$. The spin structure on $\mathbb{R}^{2n+1}_1$ induces via the embedding ι a spin structure on S^{2n}_1. The zero set of a twistor spinor φ on $\mathbb{R}^{2n+1}_1$ is empty, a single point or a lightlike straight line. The restriction $\varphi|_{S^{2n+1}_1}$ to the totally umbilic hypersurface S^{2n}_1 of a twistor spinor φ on $\mathbb{R}^{2n+1}_1$ is again a twistor spinor (comp. [Baum298]) and the zero set $\mathrm{zero}(\varphi|_{S^{2n}_1})=\mathrm{zero}(\varphi)\cap S^{2n}_1$ is also empty, a single point or a lightlike geodesic (that is a straight line in $S^{2n}_1\subset\mathbb{R}^{2n+1}_1$). Consider now a twistor spinor $\varphi|_{S^{2n}_1}$ on S^{2n}_1 with $e_2\in\mathrm{zero}(\varphi|_{S^2_1})$ and the spacelike geodesic $\gamma(t)=\cos(t)e_2+\sin(t)e_3$. It is $d=\sqrt{\frac{R\cdot g(\dot\gamma,\dot\gamma)}{4n(n-1)}}=\frac{1}{2}$ and in fact

$$\gamma(t)\in\mathrm{zero}(\varphi|_{S^{2n}_1})\quad\text{iff}\quad t=2\pi\cdot n=\frac{\pi\cdot n}{d}.$$

This is in accordance with Theorem 5, since S^{2n}_1 is Einstein.

Consider the universal covering

$$\pi : \tilde{S}^2_1 \to S^2_1$$

with induced metric $g_{\tilde{S}^2_1}$ and induced spin structure. The space $\tilde{S}^2_1$ is geodesically complete and conformally flat. Every twistor spinor $\tilde{\varphi}$ on $\tilde{S}^2_1$ is induced by a twistor spinor $\varphi|_{S^2_1}$ on S^2_1:

$$\pi_*(\tilde{\varphi}) = \varphi|_{S^2_1}.$$

If $\varphi|_{S^2_1}$ on S^2_1 admits a zero or a zero set geodesic, then $\tilde{\varphi}$ on $\tilde{S}^2_1$ admits infintely many zeros or zero set geodesics. The product $\mathbb{R} \times \tilde{S}^2_1$ with metric $dt \oplus g_{\tilde{S}^2_1}$ is a geodesically complete and conformally flat Lorentzian spin manifold of dimension 3. On $\mathbb{R} \times \tilde{S}^2_1$ a set of twistor spinors with maximal dimension 4 exists. Every twistor spinor $\tilde{\varphi}$ on $\tilde{S}^2_1$ can be extended to a twistor spinor ψ on $\mathbb{R} \times \tilde{S}^2_1$ such that $\psi|_{\{0\}\times\tilde{S}^2_1} = \tilde{\varphi}$ (comp. [BFGK91]). The space $\mathbb{R} \times \tilde{S}^2_1$ is an example of a geodesically complete Lorentzian spin manifold that admits twistor spinors with infinetly many zero set geodesics.

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