Irregular Spectral Asymptotics

Jochen Brüning

Dedicated to Sergio Albeverio

1. Introduction

The resolvent trace expansion for elliptic operators is a well established tool with many important applications in Geometric Analysis, see e.g. [Gi] and its bibliography. It is known for a long time in the case of elliptic differential operators on compact manifolds [MP], [S1] and differential boundary value problems [S2], [Gre]. Recently, this expansion has been extended fully to pseudodifferential elliptic operators and boundary value problems [GruS], [Gru].

The method has been developed further to cover spaces with singularities, notably of conic type, including families of cones [Ch1], [BS2], [BS3]. Here, operators of Laplace type are considered which are most interesting in applications. Typically, near a conic singularity such an operator takes the form

$$\triangle^\circ = -\left(\frac{\partial}{\partial x}\right)^2 + x^{-2}A(x), \quad x \in (0, \varepsilon_0], \tag{1.1}$$

where A is a family of Laplace type operators on the base of the cone (a compact smooth manifold), and A varies smoothly with $x \in [0, \varepsilon_0]$. The next step towards more complicated (isolated) singularities will introduce operators of the form

$$\triangle^\circ = -\left(\frac{\partial}{\partial x}\right)^2 + x^{-2\alpha_0}A(x), \quad x \in (0, \varepsilon_0], \tag{1.2}$$

where $\alpha_0 \geq 1$ and A is no longer smooth but admits an asymptotic expansion in powers of x, as $x \to 0$. This comprises the case of "metric horns" which refers to Riemannian manifolds with metric singularities of the form $dx^2 + x^{2\alpha_0}g_N$

on $(0, \varepsilon_0] \times N$, with (N, g_N) a compact Riemannian manifold; in this case we have

$$A(x) = x^{-2\alpha_0} A_0 + x^{-\alpha_0 - 1} A_1 + x^{-2} A_2. \tag{1.3}$$

Various aspects of this class of singular spaces have been studied, like Hodge theory [Ch2] and index theorems [LPe]. In [B1], we have announced the full asymptotic expansion of the heat kernel for the Laplacian (Thm 1.2) and have used it to derive a Signature Theorem (Thm.1.3). The proof of the expansion result was based on an extension of the Singular Asymptotics Lemma (SAL) (Thm.2.2) which was introduced in [BS1] (cf. also the "push forward lemma" of Melrose [M]), a very convenient tool to produce asymptotic expansions in singular situations which admit scaling. Callias [Ca] has given a version of the SAL not requiring smoothness up to $x = 0$.

In this note we supply the proofs of [B1, Thms.1.2 and 2.2] and, at the same time, provide substantial generalizations. The main observation concerns the structure of the SAL. Recall that it deals with a function, $\sigma(x, \xi)$, of two variables $x, \xi > 0$, and derives an asymptotic expansion of the function

$$I(\sigma; z) := \int_0^\infty \sigma(x, xz)dx, \tag{1.4}$$

as $z \to \infty$. To do so one assumes a differentiable asymptotic expansion of σ as $\xi \to \infty$, with coefficients in $\mathcal{S}(\mathbb{R}_+)$ (as functions of x), and smoothness in $x \geq 0$, in order to use the Taylor expansion at $x = 0$; both expansions are matched by a special "integrability condition". The emphasis on smoothness at $x = 0$ links the SAL in this form closely to the special structure of the Taylor expansion, whereas the expansion at ∞ is unrestricted.

The main generalization we introduce here is that we drop the differentiability assumptions and put the expansions at 0 and at ∞ on an equal footing. This leads us to different remainder estimates which replace the "Integrability Condition" in [BS1]. It seems that they are more easily verified, but for the time being we do not see the precise relationship between both types of remainder estimates. The price we have to pay for now is the exclusion of "small eigenvalues" which can be easily treated, however, by ode methods. It should be noted that our expansion includes a new term which is not present if we assume smoothness at $x = 0$.

In Section 2, we prove Theorem 1, the generalization of the SAL, which was announced (in a more restrictive form) in [B1, Thm.1.2]. In Section 3, we use this result to deduce Theorem 2, as announced in [B1, Thm.2.2]. Thanks are due to the referee who pointed out a gap in the proof of Theorem 1.

2. The Generalized Singular Asymptotics Lemma

We begin with some terminology which is essentially taken from [L, Ch. II]. By Γ we denote a discrete closed subset of $\mathbb{C}$ with

$$\underline{\gamma} := \inf \mathrm{Re}\ \gamma > -\infty, \tag{2.1}$$

and

$$\Gamma_N := \{\gamma \in \Gamma; \mathrm{Re}\ \gamma \leq N\} \tag{2.2}$$

finite for all $N \in N$, and equipped with a "weight function" $\nu : \Gamma \to \mathbb{Z}_+$. In addition, we write $\Gamma_N^* := \Gamma_N \cap \{\mathrm{Re}\ z > 0\}$. Then we consider the following types of asymptotic expansions.

(1) For $f \in L^1_{\mathrm{loc}}(0,1]$, we want the expansion

$$f(x) = \sum_{\alpha \in \Gamma_N k \leq \nu(\alpha)} f^0_{\alpha k} x^\alpha \log^k x + R^0_N(f;x), \tag{2.3}$$

with certain numbers $f^0_{\alpha k} \in \mathbb{C}$ and remainder estimate

$$|R^0_N(f;x)| \leq C_{N\delta} x^{N-\delta}, \tag{2.4}$$

uniformly in $x \in (0,1]$, for every $\delta > 0$. This class of functions we denote by $\mathcal{F}^0_{\Gamma,\nu}$.

(2) For $f \in L^1_{\mathrm{loc}}[1,\infty)$, we want the expansion

$$f(x) = \sum_{\beta \in \Gamma_N l \leq \nu(\beta)} f^\infty_{\beta l} x^{-\beta} \log^l x + R^\infty_N(f;x), \tag{2.5}$$

with certain numbers $f^\infty_{\beta l} \in \mathbb{C}$ and remainder estimate

$$|R^\infty_N(f;x)| \leq C_{N,\delta} x^{\delta - N}, \tag{2.6}$$

uniformly in $x \in [1,\infty)$, for every $\delta > 0$. This class of functions we denote by $\mathcal{F}^\infty_{\Gamma,\nu}$. Then we put

(2.7)
$$\mathcal{F}_{\Gamma,\nu} := \{f \in L^1_{\mathrm{loc}}(0,\infty); f_0 := f \mid (0,1] \in \mathcal{F}^0_{\Gamma,\nu}, f_\infty := f \mid [1,\infty) \in \mathcal{F}^\infty_{\Gamma,\nu}\};$$

this is the function space we are interested in. We can define the Mellin transform on $\mathcal{F}_{\Gamma,\nu}$ by

$$\mathcal{M}f(z) := \mathcal{M}f_0(z) + \mathcal{M}f_1(z). \tag{2.8}$$

$\mathcal{M}f$ is meromorphic in $\mathbb{C}$, hence we can define the "regularized integral" by

$$\text{(2.9)} \qquad \int\!\!\!\!-_0^\infty f(x)dx := \mathrm{Res}_0 \mathcal{M}f(1),$$

where "Res_0" denotes the constant term in the Laurent expansion. This is the appropriate notion for our purposes. We also employ the notion of "two-variable asymptotic expansion", associated with Γ and ν, as explained e.g. in [B2]. By this we mean an asymptotic expansion for functions, f, in $L^1_{\mathrm{loc}}((0,1]\times(0,1])$ of the form

$$\text{(2.10)} \qquad f(\varepsilon,\omega) = \sum_{\alpha,\beta\in\Gamma_N k\leq\nu(\alpha),l\leq\nu(\beta)} f^{kl}_{\alpha\beta}\varepsilon^\alpha \log^k\varepsilon \;\; \omega^\beta \log^l \omega + R_N(f;\varepsilon,\omega),$$

with certain $f^{kl}_{\alpha\beta}\in\mathbb{C}$ and remainder estimate

$$\text{(2.11)} \qquad |R_N(f;\varepsilon,\omega)| \leq C_{N\delta}(\varepsilon+\omega)^{N-\delta},$$

uniformly in $0<\varepsilon,\omega\leq 1$, for every $\delta>0$. We will mainly use the fact that (2.10) determines the coefficients $f^{kl}_{\alpha\beta}$ uniquely. We are interested in the asymptotic expansion of expressions of the form

$$\text{(2.12)} \qquad I(\sigma;z) := \int\!\!\!\!-_0^\infty \sigma(x,xz)dx, \quad 1\leq z\to\infty,$$

where σ is in $C((0,\infty)\times(0,\infty),\mathbb{C})$. To ensure the existence of the integral and its asymptotic expansion, we introduce two assumptions. (A_0) For any $C_0>0$ and $0<x\leq C_0$, $\xi\in(0,\infty)$, $N\in\mathbb{N}$, we have the expansion

$$\sigma(x,\xi) = \sum_{\alpha\in\Gamma_N k\leq\nu(\alpha)} \sigma^0_{\alpha k}(\xi)x^\alpha\log^k x + R^0_N(x,\xi),$$

where $\sigma^0_{\alpha k}\in\mathcal{F}_{\Gamma,\nu}$ and, for $0<\delta\leq 1$ and certain numbers $\alpha^0_0\leq 0, \alpha^0_\infty\geq 0$,

$$|R^0_N(x,\xi)| \leq C(N,\delta,C_0)x^{N-\delta}\xi^{\alpha^0_0}(1+\xi)^{\alpha^0_\infty}.$$

(A_∞) For any $C_0>0$ and $\xi\geq 1/C_0$, $x\in(0,\infty)$, $N\in\mathbb{N}$ we have the expansion

$$\sigma(x,\xi) = \sum_{\beta\in\Gamma_N l\leq\nu(\beta)} \sigma^\infty_{\beta l}(x)\xi^{-\beta}\log^l\xi + R^\infty_N(x,\xi),$$

where $\sigma^\infty_{\beta l}\in\mathcal{F}_{\Gamma,\nu}$ and, for $0<\delta\leq 1$ and certain numbers $\alpha^\infty_0\leq 0$, $\alpha^\infty_\infty\geq 0$,

$$|R^\infty_N(x,\xi)| \leq C(N,\delta,C_0)\xi^{\delta-N}x^{\alpha^\infty_0}(1+x)^{\alpha^\infty_\infty}.$$

We point out the following consequence of (A_0), (A_∞): as $x \to 0$, $\xi \to \infty$ we obtain the expansion, from (A_0),

$$\sigma(x,\xi) \sim \sum_{\alpha,\beta\in\Gamma k\leq\nu(\alpha),l\leq\nu(\beta)} [\sigma^0_{\alpha k}]^\infty_{\beta l} \; x^\alpha \log^k x \; \xi^{-\beta} \log^l \xi,$$

and from (A_∞),

$$\sigma(x,\xi) \sim \sum_{\alpha,\beta\in\Gamma k\leq\nu(\alpha),l\leq\nu(\beta)} [\sigma^\infty_{\beta l}]^0_{\alpha k} \; x^\alpha \log^k x \; \xi^{-\beta} \log^l \xi.$$

By uniqueness of the expansion coefficients, we conclude

$$[\sigma^0_{\alpha k}]^\infty_{\beta l} = [\sigma^\infty_{\beta l}]^0_{\alpha k} =: \sigma_{\alpha k,\beta l}. \tag{2.13}$$

Now we can formulate the first result of this paper.

THEOREM 2.1. *Let $\sigma \in C((0,\infty)\times(0,\infty),\mathbb{C})$. Under the assumptions (A_0) and (A_∞) we have the following asymptotic expansion, as $1 \leq z \to \infty$:*

$$I(\sigma;z) \sim \sum_{\alpha\in\Gamma k\leq\nu(\alpha)} ⨍_0^\infty (x/z)^\alpha \log^k(x/z)\sigma^0_{\alpha k}(x)dx/z \tag{2.14}$$

$$+ \sum_{\beta\in\Gamma l\leq\nu(\beta)} ⨍_0^\infty (xz)^{-\beta} \log^l(xz)\sigma^\infty_{\beta l}(x)dx \tag{2.15}$$

(2.16)
$$+ \sum_{\alpha\in\Gamma k\leq\nu(\alpha)0\leq k''\leq k,0\leq k'\leq\nu(\alpha+1)} z^{-\alpha-1}\log^{k+k'+1} zk()k''(-1)^{k-k'+1}(k+k'+1)^{-1}[\sigma^0_{\alpha k}]^0_{-\alpha-1,k'}$$

(2.17)
$$+ \sum_{\beta\in\Gamma k\leq\nu(\beta+1),l\leq\nu(\beta)0\leq k'\leq k} z^{-\beta}\log^{k+l+1} zk()k'(-1)^{k-k'}(k'+l+1)^{-1}\sigma_{\beta l;\beta-1,k}.$$

In particular, $I(\sigma;z) \in \mathcal{F}^\infty_{\Gamma,\tilde\nu}$ with $\tilde\nu(\alpha) = \nu(\alpha)+\nu(\alpha+1)+1$.

REMARKS. Comparing with [BS1] we see that the terms (Ia) and (Ib) there - which equal (2.14) and (2.15) here - are now seen in a more symmetric way; indeed, our condition (A_0) replaces the Taylor expansion used before. Also, the fairly subtle integrability condition [BS1, (1.2b)] and the remainder estimate [BS1, (1.2a)] are now replaced by the rather straightforward estimates in (A_0) and (A_∞). They are enough for the application we have in mind here but it is not clear at the moment how they relate to the previous assumptions in general. In particular, it is not obvious how they imply the vanishing of (2.16). The systematic use of the regularized integral throughout also adds some transparency, as pointed out in [L]. In particular, we see that the "singular" contributions (2.17) - which corresponds to [BS1, (Ic)] - and the new contribution (2.16) arise simply from the formula for the change of variables [L, Lemma 2.1.4]; this was pointed out in [L,Ch.II].

Proof. We clearly need to split the integral into a part where x is small, $x \leq C_0$, and one where x is large, $xz \geq 1/C_0$, in order to use the different asymptotics (A_0), (A_∞). This will be the case if we split at $x = 1/C_0 z$ where the choice of C_0 is arbitrary. We take advantage of this by introducing two new expansion variables:

$$\varepsilon \in (0,1] \text{ and } w := \varepsilon z \geq 1. \tag{2.18}$$

Then we write

$$\begin{aligned} I(\sigma; z) &= \int\!\!\!\!\!-_0^\infty (\phi(xw) + (1-\phi(xw)))\sigma(x,xz)dx \\ &=: I_0(\sigma;\varepsilon,w) + I_\infty(\sigma;\varepsilon,w), \end{aligned} \tag{2.19}$$

with $0 \leq \phi \leq 1, \phi \in C_0^\infty(-2,2)$ and $\phi \mid [-1,1] = 1$. We will show that I_0 and I_∞ admit asymptotic expansions as $\varepsilon + w^{-1} \to 0$, which will imply the theorem. Beginning with I_0, we obtain from (A_0) and [L, Lemma 2.1.4.], since $x \leq 2$ on the support of the integrand,

$$\begin{aligned} I_0(\sigma;\varepsilon,w) &= \int\!\!\!\!\!-_0^\infty \phi(xw)\sigma(x,xz)dx \\ &= \int\!\!\!\!\!-_0^\infty \phi(xw)\left[\sum_{\alpha\in\Gamma_N k\leq\nu(\alpha)} x^\alpha \log^k x \ \ \sigma^0_{\alpha k}(xz) + R^0_N(x,xz)\right]dx \end{aligned}$$

$$=: \sum_{\alpha\in\Gamma_N k\leq\nu(\alpha)} \{\int\!\!\!\!\!-_0^\infty \phi(x\varepsilon)(x/z)^\alpha \log^k(x/z)\sigma^0_{\alpha k}(x)dx/z \tag{2.20}$$

$$+\sum_{\alpha\in\Gamma_N k\le\nu(\alpha) 0\le k''\le k, 0\le k'\le\nu(\alpha+1)} z^{-\alpha-1}\log^{k+k'+1} z k()k''(-1)^{k-k'+1}(k+k'+1)^{-1}[\sigma^0_{\alpha k}]^0_{-\alpha-1,k'} \tag{2.21}$$

$$+\tilde{R}^0_N(\sigma;\varepsilon,w). \tag{2.22}$$

Now, (2.21) already gives the first singular contribution, (2.16). (2.20) we rewrite as

$$\sum_{\alpha\in\Gamma_N k\le\nu(\alpha)} [\int\!\!\!\!-_0^\infty (x/z)^\alpha \log^k(x/z)\sigma^0_{\alpha k}(x)dx/z \tag{2.23}$$

$$-\int\!\!\!\!-_0^\infty (1-\phi(x\varepsilon))(x/z)^\alpha \log^k(x/z)\sigma^0_{\alpha k}(x)dx/z], \tag{2.24}$$

such that (2.23) gives (2.14), while (2.24) will be dealt with later. Using (A_0) again, we estimate the remainder (2.24), with $\delta = 1/2$:

$$\begin{aligned} |\tilde{R}^0_N(\sigma;\varepsilon,w)| &\le C_N \int_0^\infty \phi(xw)x^{N-1/2}(xz)^{\alpha^0_0}(1+xz)^{\alpha^0_\infty}dx \\ &\le C_N\ z^{\alpha^0_0+\alpha^0_\infty}\ w^{-N-1/2-\alpha^0_0}. \end{aligned} \tag{2.25}$$

I_∞ is now treated analogously. Since $xz \ge \varepsilon^{-1} \ge 1$ on the support of the integrand, we can use (A_∞) to get

$$I_\infty(\sigma;\varepsilon,w) = \int\!\!\!\!-_0^\infty (1-\phi(xw))\left[\sum_{\beta\in\Gamma_N l\le\nu(\beta)} \sigma^\infty_{\beta l}(x)(xz)^{-\beta}\log^l xz + R^\infty_N(x,xz)\right]dx$$

$$=: \sum_{\beta\in\Gamma_N l\le\nu(\beta)} \{\int\!\!\!\!-_0^\infty (xz)^{-\beta}\log^l xz\ \ \sigma^\infty_{\beta l}(x)dx \tag{2.26}$$

$$-\int\!\!\!\!-_0^\infty \phi(xw)(xz)^{-\beta}\log^l xz\ \ \sigma^\infty_{\beta l}(x)dx\} \tag{2.27}$$

$$+\ \tilde{R}^\infty_N(\sigma;\varepsilon,w). \tag{2.28}$$

Again, (2.26) gives the desired term (2.15); and the remainder we estimate using (A$_\infty$) and, again, $\delta = 1/2$:

$$
\begin{aligned}
|\tilde{R}_N^\infty(\sigma;\varepsilon,w)| &\leq C_N \int_{w^{-1}}^{\infty} (xz)^{1/2-N} x^{\alpha_0^\infty}(1+x)^{\alpha_\infty^\infty} dx \\
&= C_N \int_{\varepsilon^{-1}}^{\infty} z^{-1-\alpha_0^\infty} x^{1/2-N+\alpha_0^\infty}(1+x/z)^{\alpha_\infty^\infty} dx \\
&\leq C_N \; z^{-1-\alpha_0^\infty} \; \varepsilon^{N-3/2-\alpha_0^\infty-\alpha_\infty^\infty} \\
&\leq C_N \; z^{-\alpha_0^\infty} \; \varepsilon^{N-(3/2+\alpha_0^\infty+\alpha_\infty^\infty)}. \qquad (2.29)
\end{aligned}
$$

It remains to deal with the terms (2.27) and (2.24). In the integrand of (2.24) we have $x \geq \varepsilon^{-1}$, so we plug in the expansion of $\sigma_{\alpha k}^0$ as $x \to \infty$; the resulting remainder of order $L > N$ can be estimated by

$$
C_{LN} \int_{\varepsilon^{-1}}^{\infty} z^{1/2-\alpha} \; x^{\alpha-L-1/2} dx = C_{LN} \; z^{1/2-\alpha} \; \varepsilon^{L-\alpha-3/2} \leq C_{LN} \; z^{1/2-\underline{\gamma}} \; \varepsilon^{L-N-3/2}.
$$

Using [L, II (1.12)], this leads to the expansion

$$
\begin{aligned}
&- \sum_{\alpha\in\Gamma_N k\leq\nu(\alpha)} \int\!\!\!\!\!\!-_0^\infty (1-\phi(x\varepsilon))(x/z)^\alpha \log^k(x/z)\sigma_{\alpha k}^0(x)dx/z \\
&= \sum_{\alpha\in\Gamma_N,\beta\in\Gamma_L k\leq\nu(\alpha),l\leq\nu(\beta)} [\sigma_{\alpha k}^0]_{\beta l}^\infty \int\!\!\!\!\!\!-_0^\infty \phi(x\varepsilon)(x/z)^\alpha \log^k(x/z) x^{-\beta}\log^l x \;\; dx/z \\
&+ \; O_{NL}(z^{1/2-\underline{\gamma}}\varepsilon^{L-N-3/2}). \qquad (2.30)
\end{aligned}
$$

In the integrand of (2.27) we have $x \leq 2w^{-1} \leq 2$, so we can use the expansion of $\sigma_{\beta l}^\infty$ as $x \to 0$. Then we obtain quite similarly:

$$
\begin{aligned}
&- \sum_{\beta\in\Gamma_N l\leq\nu(\beta)} \int\!\!\!\!\!\!-_0^\infty \phi(xw)(xz)^{-\beta} \;\; \log^l(xz)\sigma_{\beta l}^\infty(x)dx \\
&= - \sum_{\beta\in\Gamma_N,\alpha\in\Gamma_L l\leq\nu(\beta),k\leq\nu(\alpha)} [\sigma_{\beta l}^\infty]_{\alpha k}^0 \int\!\!\!\!\!\!-_0^\infty \phi(x\varepsilon z)(xz)^{-\beta}\log^l(xz) x^\alpha \log^k x dx \\
&+ \; O_{NL}(z^{1/2-\underline{\gamma}} w^{1/2-(L-N)}). \qquad (2.31)
\end{aligned}
$$

But now we see from (2.13) and [L, Lemma 2.1.4 and II(2.12)] again that the sums in (2.30) and (2.31) with $\alpha, \beta \in \Gamma_N$ add up to the term

(2.32)
$$\sum_{\beta\in\Gamma_N k\in\nu(\beta+1),l\leq\nu(\beta)} \sigma_{\beta l;\beta-1,k} \; z^{-\beta} \; \log^{k+l+1} z \sum_{k'=0}^{k} (-1)^{k-k'} k() k' (k'+l+1)^{-1},$$

which coincides with the second singular contribution, (2.17). It now remains to estimate the terms in (2.30) with $N < \beta \leq L$, and in (2.31) with $N < \alpha \leq L$. For a term from (2.30), we have $\alpha \leq N < \beta$. We estimate with [L,Ch.2,(1.6)]

$$\begin{aligned} & |\int_0^\infty \phi(x\varepsilon)(x/z)^\alpha \log^k(x/z) x^{-\beta} \log^l x \; dx/z| \\ \leq \; & C_{NL} z^{-\alpha-1} \varepsilon^{\beta-\alpha-1}(1+|\log^k \varepsilon|) \log^k z \\ \leq \; & C_{NL} w^{-1} z^{-\alpha+1/2} \varepsilon^{(\beta-\alpha)/2}. \end{aligned}$$

If $\alpha \geq N/2$, we can estimate this by

$$C_{NL} z^{-(N-1)/2} \leq C_{NL} \; \varepsilon^{(N-1)/2},$$

and for $\alpha < N/2$, implying $\beta - \alpha > N/2$, by

$$C_{NL} z^{1/2-\underline{\gamma}} \; \varepsilon^{N/4},$$

hence altogether, with $D_1 := \max\{0, 1/2 - \underline{\gamma}\}$ and for $N \geq 2$:

$$\begin{aligned} & |\int_0^\infty \phi(x\varepsilon)(x/z)^\alpha \log^k(x/z) x^{-\beta} \log^l x \;\; dx/z| \\ \leq \; & C_{NL} \; z^{D_1} \varepsilon^{N/4}. \end{aligned} \tag{2.33}$$

A similar reasoning gives for a term in (2.31), with $N \geq 2$ and $\beta \leq N < \alpha$

$$\begin{aligned} & |\int_0^\infty \phi(xw)(xz)^{-\beta} \log^l(xz) x^\alpha \log^k x \; dx| \\ \leq \; & C_{NL}(z^{D_1} w^{-N/4} + \varepsilon^{(N-1)/2}). \end{aligned} \tag{2.34}$$

Putting now $L = 2N$ and collecting all the remainder terms, (2.25), (2.29), (2.30), (2.31), (2.33), and (2.34), we see that there is a problem only if one of the quantities

$$D_1, \;\; D_2 := \alpha_0^0 + \alpha_\infty^0, \;\; D_3 := -\alpha_0^\infty, \;\; D_4 := -\underline{\gamma},$$

is positive. Thus, we put

$$D := \sum_{i=1}^{4} |D_i|$$

and consider the expansion of $\tilde{I}(\sigma; z) := z^{-D} I(\sigma; z)$ resulting from our considerations. Then all remainder terms are small, so we obtain an asymptotic expansion of $\tilde{I}$ hence also of I. The proof is complete. □

3. Irregular Singular Equations

We want to apply Theorem 1 to obtain the resolvent expansion of certain elliptic equations with irregular singularities. The prototypical example is given by the differential operator

$$\triangle^\circ := -\left(\frac{\partial}{\partial x}\right)^2 + \frac{a}{x^{2\alpha_0}}, \quad x > 0, \tag{3.1}$$

with domain $C_0^2(0, \infty)$, here $a > 0$ and $\alpha_0 \geq 1$. $\triangle^\circ$ is symmetric in $L^2(\mathbb{R}_+)$ and bounded from below, hence we can consider its Friedrichs extension, $\triangle$, which is self-adjoint and bounded from below with the same lower bound. The expansion of the resolvent trace (or, almost equivalently, of the heat trace) for $\triangle$ has been announced by Callias [Ca]. In [B1], we have announced a different proof, in the spirit of 'regular singular analysis', and have applied it to derive the Signature Theorem for metric horns. Here we supply the missing details, in a more general and probably also more accessible form then studied before.

In order to apply the results to singular spaces like metric horns, we generalize (3.1) to an operator valued setting. Thus we consider a Hilbert space, H, a dense subspace, H_1, and a family of self-adjoint operators in H, $A(x)$, defined for $x > 0$, with common domain H_1. We assume in addition the following properties

$$(0, \infty) \ni x \mapsto A(x) \in \mathcal{L}(H_1, H) \text{ is smooth,} \tag{3.2}$$

$$A(x) \geq a > 0 \text{ for all } x > 0. \tag{3.3}$$

The smoothness of A in $\mathbb{R}_+$ - which holds in the conic case - is replaced by assuming an asymptotic expansion

$$A(x) = A_0 + \sum_{\gamma \in \Gamma^*_{N-1}} A_\gamma x^\gamma + R_N(x), \tag{3.4}$$

where all A_γ are defined and symmetric on H_1, and

$$|R_N(x)| \leq C_N x^{\mu_n} A_0^{1-\delta_N}, \tag{3.5}$$

for some $\delta_N \in (0,1]$ and some sequence $\mu_N \nearrow \infty$ (note that A_0 is self-adjoint with domain H, and $A_0 \geq a$). We also assume, with $\mathcal{C}_p$ the Schatten-von Neumann class of order p, that

$$A_0^{-1} \in \mathcal{C}_p(H) \tag{3.6}$$

for some $p > 0$. Then we can consider the analogue of (3.1),

$$\triangle^\circ := -\left(\frac{\partial}{\partial x}\right)^2 + \frac{A(x)}{x^{2\alpha_0}}, \tag{3.7}$$

with domain $C_0^2((0,\infty), H_1)$, and also its Friedrichs extension, $\triangle$, in the Hilbert space $\mathcal{H} := L^2(\mathbb{R}_+, H)$. Because $\mathbb{R}_+$ is not compact, the resolvent may not be trace class. To remedy this, we choose $\phi \in C_0^\infty(-2,2)$ with $\phi = 1$ in a neighbourhood of 0 and consider for $z \geq 1$ and $l \in \mathbb{N}$, l sufficiently large, the function

$$r(z) := \operatorname{tr}_{\mathcal{H}}[\phi(\triangle + z^2)^{-l}]. \tag{3.8}$$

We will show that r is well defined and admits an asymptotic expansion in powers of z and $\log z$, as $z \to \infty$; this will be a consequence of Theorem 1. To put r into the framework of the SAL we use the natural scaling, as in [BS 2,3], and the Trace Lemma from [BS 2]. Starting with the latter, we have the following result. LEMMA 3.1 *For $l \geq p + 1/2$, $(\triangle + z^2)^{-l}$ has an operator kernel, $(\triangle + z^2)^{-l}(x,y)$, on $\mathbb{R}_+$ such that*

$$(\triangle + z^2)^{-l}(x,x) \in \mathcal{C}_1(H), \quad x > 0. \tag{3.9}$$

Moreover,

$$r(z) = \int_0^\infty \phi(x) \operatorname{tr}_H[(\triangle + z^2)^{-l}(x,x)]dx. \tag{3.10}$$

PROOF. We write $\mathcal{S} := C_0^\infty(\mathbb{R}_+)$ and note that, by construction,

$$\psi \mathcal{D}(\triangle) \subset \mathcal{D}(\triangle) \ \text{ for } \ \psi \in \mathcal{S}. \tag{3.11}$$

We also write $\psi_2 > \psi_1$, for $\psi_1, \psi_2 \in \mathcal{S}$, if $\psi_2 = 1$ in a neighbourhood of supp ψ_1. Then, by the Trace Lemma, it is enough to prove that

$$\psi_2(\triangle + z^2)^{-l}\psi_1 \in \mathcal{C}_1(\mathcal{H}), \tag{3.12}$$

for all such ψ_1 and ψ_2 in $\mathcal{S}$ if l is large enough. Thus, we fix $\psi_i \in \mathcal{S}$, $1 \le i \le 3$, with $\psi_3 > \psi_2 > \psi_1$ and supp $\psi_3 \subset [0,T)$, for some $T > 0$. We want to prove that

$$\psi_2(\triangle + z^2)^{-l} \ \psi_1 \in \mathcal{C}_{(p+1/2)/l}(\mathcal{H}), \tag{3.13}$$

with p from (3.6). To show this for $l = 1$, we denote by $\triangle_T$ the Friedrichs extension of $\triangle^0$ in $L^2([0,T],H)$ with domain $C_0^2((0,T),H_1)$. Then we obtain from the max-min principle and (3.4), (3.5) the estimate

$$\begin{aligned} \| (\triangle_T + C_1)^{-1} \|^{p+1/2}_{\mathcal{C}_{p+1/2}(\mathcal{H}_T)} &\le C_2 \sum_{i,j\ge 1} \left(\frac{4\pi^2}{T^2} i^2 + \lambda_j \right)^{-(p+1/2)} \\ &\le C_2 \sum_{j\ge 1} \int_0^\infty \left(\frac{4\pi^2}{T^2} x^2 + \lambda_j \right)^{-(p+1/2)} dx \\ &\le C_3 \| A_0^{-1} \|_{\mathcal{C}_p(H)}, \end{aligned} \tag{3.14}$$

where we have written $\mathcal{H}_T = L^2((0,T),H)$, $(\lambda_j)_{j\ge 1} = \operatorname{spec} \mathrm{A}_0$, and the constants C_i depend only on A_0 and p. Now we find by standard arguments

$$\begin{aligned} \psi_2(\triangle + z^2)^{-1}\psi_1 &= \psi_2(\triangle_T + z^2)^{-1}\psi_1 \\ &+ \psi_2(\triangle_T + z^2)^{-1}[\triangle_T, \psi_3](1-\psi_2)(\triangle + z^2)^{-1}\psi_1. \end{aligned} \tag{3.15}$$

Next we observe the easy a priori estimate

$$\| [\triangle_{(T)}, \psi]u \|_{\mathcal{H}_{(T)}} \le C(\psi)((\triangle_{(T)}u, u) + \| u \|^2_{\mathcal{H}_{(T)}}), \tag{3.16}$$

which holds for any $\psi \in \mathcal{S}_{(T)}$ and $u \in C_0^2((0,\infty),H_1)$, $(C_0^2((0,T),H_1))$, where $\mathcal{S}_T = C_0^\infty[0,T]$. But then it is readily seen that $(\triangle_{(T)} + I, \mathcal{S}_{(T)})$ satisfy the axioms (Co1) through (Co5) in [B3, Sec.4]; in particular, from (Co2) we see that

$$(\triangle_T + I)^{-1/2}[\triangle_T, \psi_3] \text{ is bounded in } \mathcal{H}_T. \tag{3.17}$$

Thus, we derive from (3.14), (3.15), and (3.17) that

$$\psi_1(\triangle + z^2)^{-1}\psi_1 \in \mathcal{C}_{2p+1}(\mathcal{H}),$$

hence

$$\psi_1(\triangle + z^2)^{-1/2} \in \mathcal{C}_{4p+2}(\mathcal{H}).$$

But then it follows from [B3, Lemma 4.1] that

$$(1-\psi_2)(\triangle + z^2)^{-1}\psi_1 \in \mathcal{C}_q(\mathcal{H}), \text{ for all } q > 0,$$

which gives, with (3.15), (3.13) for $l = 1$. If we have proved this for some $l \geq 1$, then we write

$$\begin{aligned}\psi_2(\triangle + z^2)^{-l-1}\psi_1 &= \psi_2(\triangle + z^2)^{-l}\psi_3(\triangle + z^2)^{-1}\psi_1 \\ &+ \psi_2(\triangle + z^2)^{-l}(1-\psi_3)(\triangle + z^2)^{-1}\psi_1.\end{aligned}$$

Using [B3, Lemma 4.1] again and the Hölder inequality for $\mathcal{C}_p$-norms, we complete the induction; hence the assertion. $\square$ The scaling mentioned above is effected through the unitary family

$$U_\varepsilon u(X) := \varepsilon^{1/2}u(\varepsilon x),\ \ \varepsilon > 0,\ \ u \in \mathcal{H}. \tag{3.18}$$

Clearly, with $\alpha_0 =: 1 + \beta_0$,

$$\begin{aligned}U_\varepsilon \triangle^\circ U_\varepsilon^\star &=: \varepsilon^{-2}\left(-\partial_x^2 + \varepsilon^{-2\beta_0}\frac{A(\varepsilon x)}{x^{2\alpha_0}}\right) \\ &= \varepsilon^{-2}\triangle_\varepsilon^\circ.\end{aligned} \tag{3.19}$$

It follows that for the Friedrichs extension, $\triangle_\varepsilon$, we have

$$\varepsilon^2 U_\varepsilon \triangle U_\varepsilon^* = \triangle_\varepsilon. \tag{3.20}$$

It follows also that

$$\varepsilon^{-2l}U_\varepsilon(\triangle + z^2)^{-l}U_\varepsilon^\star = (\triangle_\varepsilon + (\varepsilon z)^2)^{-l}, \tag{3.21}$$

and for the kernels

$$\varepsilon^{1-2l}(\triangle + z^2)^{-l}(\varepsilon x, \varepsilon y) = (\triangle_\varepsilon + (\varepsilon z)^2)^{-l}(x, y). \tag{3.22}$$

Thus, we obtain with Lemma 3.1 (and $x = y = 1, \varepsilon = y$)

$$r(z) = \int_0^\infty y^{2l-1}\phi(y)\mathrm{tr}_H[(\triangle_y + (yz)^2)^{-l}(1,1)]dy. \tag{3.23}$$

In the conic case (when $\alpha_0 = 1$) we have applied the SAL directly to (3.9), cf. [BS2, Sec.7]. To be able to do this also in this case, we rewrite our operator as

$$\triangle_y + (yz)^2 = -\left(\frac{\partial}{\partial x}\right)^2 + y^{-2\beta_0}\left(\frac{A(yx)}{x^{2\alpha_0}} + (yz^{1/2\alpha_0})^{2\alpha_0}\right). \tag{3.24}$$

This suggests that we introduce $\tilde{z} := z^{1/2\alpha_0}$ and write

$$\tilde{r}(\tilde{z}) = r(z) =: \int_0^\infty \sigma(y, y\tilde{z})dy, \tag{3.25}$$

where

$$\sigma(y,\zeta) = y^{2l-1}\phi(y)\mathrm{tr}_H[(-(\frac{\partial}{\partial x})^2 + y^{-2\beta_0}(\frac{A(yx)}{x^{2\alpha_0}} + \zeta^{2\alpha_0}))^{-l}(1,1)]. \tag{3.26}$$

In what follows, it is convenient to write

$$\tilde{A}(x) := \frac{A(x)}{x^{2\alpha_0}}, \tag{3.27}$$

such that

$$y^{-2\beta_0}\frac{A(yx)}{x^{2\alpha_0}} = y^2\tilde{A}(yx) =: \tilde{A}_y(x). \tag{3.28}$$

We also write

$$P(y,\xi;\zeta) := (\xi^2 + y^2\tilde{A}(y) + \zeta^{2\alpha_0}/y^{2\beta_0})^{-1}. \tag{3.29}$$

Now, for any fixed $y > 0$, we have to determine the resolvent kernel expansion of the elliptic operator with operator coefficients, (3.24), at a *regular* point, in order to verify the conditions of Theorem 1. To do so, we have to add some assumptions on resolvent expansions for the operator family $\tilde{A}$. First of all, we assume now that the expansion (3.4) can be differentiated any number of times, i.e. that for $j \in \mathbb{Z}_+$ we have

$$A^{(j)}(x) = \sum_{\gamma\in\Gamma_N^*} A_\gamma j!\gamma()jx^{\gamma-j} + R_{Nj}(x),$$

where, with μ_N and δ_N as above,

$$|R_{Nj}(x)| \leq C_N x^{\mu_N - j} A_0^{1-\delta_N}. \tag{3.30}$$

Moreover, we will have to consider expressions of the form

$$I_\alpha(A, B_1, \ldots, B_{j-1}; x, \eta)$$

$$:= \mathrm{tr}_H[(A(x)+\eta^2)^{-\alpha_1}B_1(A(x)+\eta^2)^{-\alpha_2}B_2\cdots B_{j-1}(A(x)+\eta^2)^{-\alpha_j}]. \tag{3.31}$$

Here, $\alpha \in \mathbb{N}^j$ is a multiindex satisfying

$$|\alpha| \geq j + p, \tag{3.32}$$

and the operators B_i are closed with domain containing H_1; in addition, we assume that

$$A_0^{-1}B_i \in \mathcal{L}(H), \tag{3.33}$$

and that there is a locally convex space, $\mathcal{B}$, such that

$$B_i \in \mathcal{B}, \;\; 1 \leq i \leq j. \tag{3.34}$$

We observe that (3.30), (3.32), and (3.33) imply that (3.31) is, indeed, well defined. If we now add the assumption that

$$A_\gamma \in \mathcal{B} \text{ and } R_N(x) \in \mathcal{B}, \tag{3.35}$$

for all $\gamma \in \Gamma^*$, $x > 0$, $N \in \mathbb{N}$, then we see from (3.4,5) that, for x sufficiently small, the Neumann series

$$\begin{aligned} (A(x)+\eta^2)^{-1} &= (A_0+\eta^2)^{-1}(I + R_1(x)(A_0+\eta^2)^{-1})^{-1} \\ &= \sum_{j\geq 0}(-1)^j (A_0+\eta^2)^{-1}(R_1(x)(A_0+\eta^2)^{-1})^j \end{aligned} \tag{3.36}$$

is asymptotic in $\mathcal{C}_1(H)$ as $\eta \to \infty$, in the sense that the trace norm of a suitable remainder decays faster than any given power of $(1+\eta)^{-1}$. Hence the η-expansion of I_α is reduced to the expansion of

$$I^0_\alpha(B_1, \cdots, B_{j-1}; \eta) :=$$

$$\operatorname{tr}_H[(A_0+\eta^2)^{-\alpha_1} B_1 (A_0+\eta^2)^{-\alpha_2} B_2 \dots B_{j-1}(A_0+\eta^2)^{-\alpha_j}], \tag{3.37}$$

for $\alpha \in \mathbb{N}^j$, $|\alpha| \geq j + p$, and $B_i \in \mathcal{B}$ (possibly also depending on x). Thus, we are lead to the following assumption:

There is an asymptotic expansion of the form

$$\begin{aligned} I^0_\alpha(B_1, \dots, B_{j-1}; \eta) &= \sum_{\gamma\in\Gamma_N m\leq\nu(\gamma)} I^0_{\alpha,\gamma m}(B_1, \dots, B_{j-1})\eta^\gamma \log^m \eta \\ &+ R_N(B_1, \dots, B_{j-1}; \eta), \end{aligned} \tag{3.38}$$

where the multilinear forms $I^0_{\alpha,\gamma m}$ and $\eta^{\mu_N} R_N(B_1, \dots, B_{j-1}; \eta)$ are bounded on bounded sets in $\mathcal{B}$, uniformly in $\eta \geq 0$.

Note that in view of (3.3) we can in fact allow $\eta \geq 0$ for the resolvent estimates. With these preparations we can turn to the expansion theorem we are after.

We begin with the following lemma which follows e.g. from the material in [BS2, Sec.2]; a detailed treatment will be given in a forthcoming publication.

LEMMA 3.2 *For $l \geq p$, $j \geq 1$, and $\alpha \in \mathbb{N}_+^j$, $\beta \in \mathbb{Z}_+^{j-1}$ there are homogeneous polynomials $p_{\alpha\beta}$ in $\mathbb{R}$, of degree $d_{\alpha\beta}$, such that for $y, \zeta > 0$*

$$\begin{aligned} \sigma(y,\zeta) &= \sum_{1\leq j\leq N \alpha,\beta\in\mathcal{M}_j} y^{2l-1}\phi(y)\int_{\mathbb{R}} p_{\alpha\beta}(\xi)\mathrm{tr}_H[P(y,\xi;\zeta)^{\alpha_1}y^{2+\beta_1}\tilde{A}^{(\beta_1)}(y)..\\ &\quad ..\ y^{2+\beta_{j-1}}\tilde{A}^{(\beta_{j-1})}(y)P(y,\xi;\zeta)^{\alpha_j}]d\xi \\ &+ R_N(y,\zeta). \end{aligned} \tag{3.39}$$

Here, the index set is

$$\mathcal{M}_j := \{(\alpha,\beta) \in \mathbb{N}_+^j \times \mathbb{Z}_+^{j-1}; |\alpha| \geq l+j-1, d_{\alpha\beta} - 2|\alpha| = -2(l+j-1) - |\beta|\}, \tag{3.40}$$

and we have the uniform estimate

$$|R_N(y,\zeta)| \leq C_N(\| \tilde{A}(y)^{-1}A_{2\alpha} \|_H)(a/2 + \zeta/y^{2\beta})^{-N}, \tag{3.41}$$

with some continuous function $C_N : \mathbb{R}_+ \to \mathbb{R}_+$. Now we are ready for the proof of our main result. THEOREM 2. *Assume that the operator family, $A(x)$, satisfies the assumptions* (3.2) *through* (3.6) *and* (3.30), (3.35), *and* (3.38). *Then the function σ introduced in* (3.26) *satisfies the assumptions of Theorem* 1. *This implies a complete asymptotic expansion for*

$$r(z) = \mathrm{tr}_{\mathcal{H}}[\phi(\triangle + z^2)^{-l}] \text{ as } z \to \infty,$$

whenever $l \geq p + 1/2$. In particular, $r \in \mathcal{F}^{\infty}_{\Gamma,\tilde{\nu}}$, with $\tilde{\nu}$ as in Theorem 1.

PROOF. We have to analyze the expansion of σ given in Lemma 3.2. Note first that, since supp $\phi \subset [-2,2]$, we derive from (3.41) the remainder estimate

$$|R_N(y,\zeta)| \leq C_N(C_0)y^{2N\beta_0}(a/2+\zeta)^{-N}, \tag{3.42}$$

uniformly in $\zeta > 0$. This estimate is good for (A_0) and (A_∞) since $\sigma(y,\zeta) = 0$ for $y > 2$. Hence it only remains to verify that the individual terms in the sum (3.39) satisfy the assumptions (A_0) and (A_∞). We begin with (A_0). Using (3.3), and (3.40), we obtain for a typical term in (3.20) the form

$$y^{2l-1+2\beta_0|\alpha|}\phi(y)\int_{-\infty}^{+\infty} p_{\alpha\beta}(\xi)\tilde{I}_\alpha(y, y^{\beta_0}\xi,\zeta)d\xi$$

$$= y^{2l-1+\beta_0(2|\alpha|-d_{\alpha\beta}-1)}\phi(y) \int_{-\infty}^{+\infty} p_{\alpha\beta}(\xi)\tilde{I}_\alpha(y,\xi,\zeta)d\xi, \tag{3.43}$$

where

$$\tilde{I}_\alpha(y,\tilde{\xi},\zeta)$$

$$:= I_\alpha(A(y)-a/2, y^{2+\beta_1}\tilde{A}^{(\beta_1)}, \dots, y^{2+\beta_{j-1}}\tilde{A}^{(\beta_{j-1})}(y); y, (\tilde{\xi}^2+\zeta^{2\alpha_0}+a/2)^{1/2}).$$

Now, we observe that, from (3.27) and (3.30),

$$y^{2+i}\tilde{A}^{(i)}(y) = O_i(y^{-2\beta_0}), \tag{3.44}$$

and admits a full asymptotic expansion with coefficients in $\mathcal{B}$. Plugging this into (3.43) and using (3.36), we obtain an asymptotic expansion as $y \to 0+$, with typical term

$$y^\delta\phi(y)\int_{\mathbb{R}} p(\xi)I_\alpha^0(B_1,\dots,B_{k-1};(\xi^2+\zeta^{2\alpha_0}+a/2)^{1/2})d\xi, \tag{3.45}$$

for some δ and some polynomial p, and with A_0 replaced by $A_0 - a/2$. Using (3.38) we see that each term of the form (3.45) is in $\mathcal{F}_{\Gamma,\nu}$, as required. For (A_∞), we use (3.36) and (3.38) first, as $\zeta \to \infty$, and plug in the asymptotic expansions, as $y \to 0+$, for the operators (3.28) in each term. This completes the proof. □ We remark in conclusion that that condition (3.5) is unnecessarily

restrictive, even though it suffices to deal with Laplace type operators in metric horns, as explained in the Introduction. It is fairly obvious how to modify the assumptions on the functions I_α in (3.38) in such a way that they are satisfied if $A(y)$ is a family of elliptic differential operators on a compact manifold, and still imply Theorem 2. It would be desirable, however, to find a more general and less clumsy set of conditions also implying this result. Let us finally spell out the result for the "metric horns". Corollary 3. *Denote by* $\triangle$ *the*

Friedrichs extension of the Laplacian on forms on a manifold with a metric horn, as in (1.3). Then we have for $l > m/2$

$$\mathrm{tr}(\triangle + z^2)^{-l} \in \mathcal{F}^\infty_{\Gamma,\nu},$$

where, with $\alpha_0 =: 1 + \beta_0$*,*

$$\Gamma := \{k + l\beta_0; k, l \in \mathbb{Z}, k \geq 2l - m, l \geq 0\},$$

and $\nu(\alpha) \equiv 1$. Proof: Since the conic case has been dealt with in [Ch1] and [BS2], we assume $\beta_0 > 0$. In view of (1.3), Theorem 2 applies if we can prove that $A(x) \geq a > 0$ for x near 0. The explicit form of the coefficients in (1.3), given in [B1, (2.4)] shows that this holds if ker $A_2 = 0$. Otherwise, the full operator function $A(x)$ is reduced by $\ker A_2 \cap \ker A_0$ which splits the problem as a sum of a "small eigenvalue part" and a "large eigenvalue part". To the latter, Theorem 2 applies, and the former is dealt with by standard ode methods. □

References

[B1] J. Brüning: The signature theorem for manifolds with metric horns. Proceedings "Equations aux derivées partielles", St.Jean de Monts, 1996

[B2] J. Brüning: On the asymptotic expansion of some integrals. Arch. Math. **42** (1984), 253-259

[BS1] J. Brüning and R.T. Seeley: Regular singular asymptotics. Advances in Math. **58** (1985), 133-148

[BS2] J. Brüning and R.T. Seeley: The resolvent expansion for regular singular operators. J. Funct. Anal. **73** (1987), 369-429

[BS3] J. Brüning and R.T. Seeley: The expansion of the resolvent near a singular stratum of conic type. J. Funct. Anal. **95** (1991), 255-290

[Ca] C.J. Callias: On the existence of small-time heat expansions for operators with irregular singularities in the coefficients. Math. Res. Lett. **2** (1995), 129-146

[Ch1] J. Cheeger: Spectral geometry of singular Riemannian spaces. J. Diff. Geom. **18** (1983), 575-657

[Ch2] J. Cheeger: On the Hodge theory of Riemannian pseudomanifolds. In: Proc. Sympos. Pure Math. **36** (1980), 21-45

[Gi] P. Gilkey: Invariance theory, the heat equation, and the Atiyah-Singer index theorem. Boca Raton 1995

[Gre] P. Greiner: An asymptotic expansion for the heat equation. Arch. Rat. Mech. Anal. **41** (1971), 163-218

[Gru] G. Grubb: A weakly polyhomogeneous calculus for pseudodifferential boundary problems. University of Copenhagen, Institute for Mathematical Sciences, Preprint **13** (1999)

[GruS] G. Grubb and R.T. Seeley: Weakly parametric pseudodifferential operators and Atiyah-Patodi-Singer boundary problems. Inventiones Math. **121** (1995), 481-529

[L] M. Lesch: Operators of Fuchs type, conical singularities, and asymptotic methods. Teubner-Texte zur Mathematik **136**, Stuttgart, Leipzig 1997

[LPe] M. Lesch and N. Peyerimhoff: On index formulas for manifolds with metric horns. Commun. Partial Differ. Equations **23** (1998), 649-684

[M] R.B. Melrose: Calculus of conormal distributions on manifolds with corners. Internat. Math. Res. Notices **3** (1992), 51-61

[MPl] S. Minakshisundaram and Å. Pleijel: Some properties of the eigenfunctions of the Laplace operator on Riemannian manifolds. Can. J. Math. **1** (1949), 242-256

[S1] R.T. Seeley: Complex powers of an elliptic operator. In: Proceedings Sympos. Pure Math. **10** (1968), 308-315

[S2] R.T. Seeley: The resolvent of an elliptic boundary value problem. Amer. J. Math. **91** (1969), 963-983