# Dual group actions on C*–algebras and their description by Hilbert extensions

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**Abstract**

Given a $C^*$-algebra $\mathcal{A}$, a discrete abelian group $\mathcal{X}$ and a homomorphism $\Theta\colon\ \mathcal{X} \to \mathrm{Out}\,\mathcal{A}$, defining the dual action group $\Gamma \subset \mathrm{aut}\,\mathcal{A}$, the paper contains results on existence and characterization of Hilbert extensions of $\{\mathcal{A}, \Gamma\}$, where the action is given by $\hat{\mathcal{X}}$. They are stated at the (abstract) C*–level and can therefore be considered as a refinement of the extension results given for von Neumann algebras for example by Jones [16] or Sutherland [20, 21]. A Hilbert extension exists iff there is a generalized 2–cocycle. These results generalize those in [10], which are formulated in the context of superselection theory, where it is assumed that the algebra $\mathcal{A}$ has a trivial center, i.e. $\mathcal{Z} = \mathbb{C}\mathbb{1}$. In particular the well–known "outer characterization" of the second cohomology $H^2(\mathcal{X}, \mathcal{U}(\mathcal{Z}), \alpha_{\mathcal{X}})$ can be reformulated: there is a bijection to the set of all $\mathcal{A}$–module isomorphy classes of Hilbert extensions. Finally, a Hilbert space representation (due to Sutherland [20, 21] in the von Neumann case) is mentioned. The C*–norm of the Hilbert extension is expressed in terms of the norm of this representation and it is linked to the so–called regular representation appearing in superselection theory.

## 1 Introduction

In the Doplicher/Roberts theory (e.g. [12, 14]) it is a central assumption that the center of the C*–algebra $\mathcal{A}$ with which one starts the analysis is trivial, i.e. $\mathcal{Z} = \mathcal{Z}(\mathcal{A}) = \mathbb{C}\mathbb{1}$ (in a more categorial notation the assumption reads $(\iota, \iota) = \mathbb{C}\mathbb{1}$, where $\iota$ denotes the unit object of the strict monoidal C*–category, cf. [13]). From a systematical point of view it is interesting to study the properties and structural modifications of this theory if one assumes the presence of a nontrivial center $\mathcal{Z} \supset \mathbb{C}\mathbb{1}$. For example, if $(\mathcal{F}, \alpha_{\mathcal{G}})$ is a Hilbert C*–system for a compact group $\mathcal{G}$ and if the corresponding fixed point algebra $\mathcal{A}$ has a nontrivial center that satisfies the relation $\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$, then the Galois correspondence does not hold anymore, i.e. we have the proper inclusion $\alpha_{\mathcal{G}} \subset \mathrm{stab}\,\mathcal{A}$ in $\mathrm{aut}\,\mathcal{F}$ (cf. [6, Section 7]). Recall, that in the trivial center situation it is a fundamental result of the theory that $\alpha_{\mathcal{G}} = \mathrm{stab}\,\mathcal{A}$. As a further justification we can also mention that in other generalizations of the Doplicher/Roberts theory as well as in some applications in mathematical physics a nontrivial center plays, to a certain extent, a distinguished role [17, 24, 15].

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In the present paper we continue the analysis of the presence of a nontrivial center in the construction of an extension algebra $\mathcal{F}$ (cf. [4, 5]). In particular, we study what we call dual group actions in the simple case where the group $\mathcal{X}$ is discrete and abelian (cf. with [10] in the special case where $\mathcal{Z} = \mathbb{C}\mathbb{1}$). This investigations will be done at the abstract C*–level which is the context of the Doplicher/Roberts theory mentioned above (cf. also [3]). On the other hand the results can be considered as a refinement of the study of twisted group algebras (twisted crossed products) on the concrete von Neumann algebra level (see e.g. [9, 16, 20, 21]). For example, the decisive C*–norm for the extension is defined intrisically and the natural representation (discussed e.g. by Sutherland) is related to the so–called regular representation that appears in the superselection theory [2]. We hope that the present analysis will be useful to obtain a more general 'inversion' theorem, where endomorphisms of $\mathcal{A}$ are involved. Indeed, the main theorems in Section 3 suggest that for a more general inversion theory in the nontrivial center situation the cohomological aspects may be essential.

The paper is structured in 5 sections: in the following section we will introduce the notion of a Hilbert C*–system and study some properties of the group homomorphism $\Theta: \mathcal{X} \to \mathrm{Out}\,\mathcal{A}$. Hilbert C*–systems are the result of the extension procedure mentioned above. In Section 3 we begin the study of the inverse (extension) problem: in particular it contains the result that a Hilbert extension exists iff there is a generalized 2–cocycle (to be defined there), and that in this case the set of all Hilbert extensions can be described in terms of the set of center–valued 2–cocycles of $H^2(\mathcal{X}, \mathcal{U}(\mathcal{Z}), \alpha_{\mathcal{X}})$ (cf. Theorems 3.4 and 3.8). In the next section we relate the previously obtained results to the special case of the Doplicher/Roberts frame, where $\mathcal{Z} = \mathbb{C}\mathbb{1}$. Finally, in Section 5 we give a representation of the Hilbert extension, which was already studied by Sutherland [20, 21] in the von Neumann case. In particular, we show that if there is a faithful state of $\mathcal{A}$, this representation coincides with the so–called regular representation that appears in superselection theory (cf. e.g. [2]) and the intrinsic C*–norm turns out to be the operator norm of this representation.

## 2 Hilbert C*–systems

A C*-algebra $\mathcal{F}$ together with a pointwise norm-continuous group homomorphism $\mathcal{G} \ni g \to \alpha_g \in \mathrm{aut}\,\mathcal{F}$ of a locally compact group $\mathcal{G}$ is called a C*-system $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$. Let $\mathcal{A} \subseteq \mathcal{F}$ be its fixed point algebra, i.e. $\mathcal{A} := \{A \in \mathcal{F} \mid \alpha_g A = A, g \in \mathcal{G}\}$. We denote by $\mathcal{A}^c := \mathcal{F} \cap \mathcal{A}' \subseteq \mathcal{F}$ the relative commutant of $\mathcal{A}$ w.r.t. $\mathcal{F}$. As is well-known, $\alpha_g \restriction \mathcal{A}^c$ is an automorphism of $\mathcal{A}^c$, so $\{\mathcal{A}^c, \alpha_{\mathcal{G}}\}$ is also a C*-system. We call it the *assigned* C*-system. The center $\mathcal{Z}(\mathcal{A})$ is denoted by $\mathcal{Z}$.

In the following let $\mathcal{G}$ be compact and abelian so that $\hat{\mathcal{G}} =: \mathcal{X}$ is abelian and discrete. The corresponding spectral projections w.r.t. $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ are denoted by $\Pi_\chi$, $\chi \in \mathcal{X}$. Note that $\Pi_\iota \mathcal{F} = \mathcal{A}$, where $\iota$ is the unit element of $\mathcal{X}$.

**2.1 Definition** *A C*-system $\{\mathcal{F}, \alpha_{\mathcal{G}}\}, \mathcal{G}$ compact abelian, is called a* Hilbert C*-system *if $\mathrm{spec}\,\alpha_{\mathcal{G}} = \mathcal{X}$ and if each spectral subspace $\Pi_\chi \mathcal{F}$ contains a unitary $U_\chi$, i.e. $\mathcal{U}(\Pi_\chi \mathcal{F}) \neq \emptyset$.*

If $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ is Hilbert, then $\beta_\chi := \mathrm{ad}\,U_\chi \restriction \mathcal{A}$ is an automorphism of $\mathcal{A}$, i.e. $\beta_\chi \in \mathrm{aut}\,\mathcal{A}$. We denote by $\pi$ the canonical homomorphism of $\mathrm{aut}\,\mathcal{A}$ onto $\mathrm{Out}\,\mathcal{A} := \mathrm{aut}\,\mathcal{A}/\mathrm{int}\,\mathcal{A}$, where $\mathrm{int}\,\mathcal{A}$ denotes the normal subgroup of all inner automorphisms of $\mathcal{A}$. Then

$$\mathcal{X} \ni \chi \to \Theta(\chi) := \pi(\beta_\chi) \in \mathrm{Out}\,\mathcal{A} \tag{1}$$

is a group homomorphism of $\mathcal{X}$ into $\mathrm{Out}\,\mathcal{A}$, i.e. we have

**2.2 Lemma** *To each Hilbert C*–system $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$, where $\mathcal{G}$ is compact abelian, there is canonically assigned a group homomorphism $\Theta$: $\mathcal{X} \to \mathrm{Out}\,\mathcal{A}$ given by (1).*

*Proof:* Note that for $\chi_1, \chi_2 \in \mathcal{X}$ we have that $U_{\chi_1\chi_2} U^*_{\chi_2} U^*_{\chi_1} \in \mathcal{A}$ and this implies that $\beta_{\chi_1\chi_2} \circ \beta^{-1}_{\chi_2} \circ \beta^{-1}_{\chi_1} \in \operatorname{int} \mathcal{A}$. ∎

We mention next the characterization of those Hilbert C*-systems where $\Theta$ is an isomorphism and of those where the classes $\Theta(\chi)$ are pairwise disjoint. Recall that $\alpha, \beta \in \operatorname{aut} \mathcal{A}$ are called disjoint if

$$(\alpha, \beta) := \{X \in \mathcal{A} \mid X\alpha(A) = \beta(A)X \quad \text{for all} \quad A \in \mathcal{A}\} = 0\,.$$

**2.3 Proposition** *(i) $\Theta$ is a monomorphism iff no spectral subspace $\Pi_\chi \mathcal{A}^c, \chi \neq \iota$, of the assigned C*–system contains a unitary.*

*(ii) The classes $\Theta(\chi)$ are pairwise disjoint iff $\mathcal{A}^c = \mathcal{Z}$, i.e. the relative commutant coincides with the center of $\mathcal{A}$.*

*Proof:* For one of the directions of part (i) take a unitary $U_\chi \in \Pi_\chi(\mathcal{A}^c)$ with $\iota \neq \chi \in \mathcal{X}$, so that the corresponding $\beta_\chi = \mathrm{id}$ and $\pi(\beta_\chi) = \operatorname{int} \mathcal{A}$. Thus $\Theta$ is not injective. For the other implication take $\mathcal{X} \ni \chi_0 \neq \iota$ with $\chi_0 \in \ker \Theta$, i.e. $\Theta(\chi_0) = \operatorname{int} \mathcal{A}$. Thus there exists a unitary $V \in \mathcal{U}(\mathcal{A})$ with $\operatorname{ad} V = \operatorname{ad} U_{\chi_0}$. From this we get $V^* U_{\chi_0} \in \mathcal{U}(\mathcal{A}^c) \cap \Pi_{\chi_0}(\mathcal{F})$, i.e. $\Pi_{\chi_0} \mathcal{A}^c \neq \emptyset$.

Finally, part (ii) follows from [7, Lemma 10.1.8]. ∎

We mention several useful concepts for Hilbert C*-systems $\{\mathcal{F}, \alpha_\mathcal{G}\}$ with a compact abelian group.

**2.4 Definition** *$\beta \in \operatorname{aut} \mathcal{A}$ is called a* canonical automorphism *if $\beta := \operatorname{ad} V \restriction \mathcal{A}\,, V \in \bigcup_{\chi \in \mathcal{X}} \mathcal{U}(\Pi_\chi \mathcal{F})$. The set of all canonical automorphisms is denoted by $\Gamma$.*

**2.5 Remark** Note that for the set of canonical automorphisms we have $\operatorname{int} \mathcal{A} \subseteq \Gamma \subseteq \operatorname{aut} \mathcal{A}$ and that for $\alpha, \beta \in \Gamma$ the automorphisms $\alpha \circ \beta$ and $\beta \circ \alpha$ are unitarily equivalent. Furthermore, $\mathcal{X} \cong \Gamma/\mathrm{int}\mathcal{A}$ and the set $\Gamma$ is sometimes called *dual action* on $\mathcal{A}$.

For any $\gamma_1, \gamma_2 \in \Gamma$ we write

$$\gamma_1 \circ \gamma_2 \circ \gamma_1^{-1} \circ \gamma_2^{-1} = \operatorname{ad} \epsilon(\gamma_1, \gamma_2),$$

where $\epsilon(\gamma_1, \gamma_2) \in \mathcal{U}(\mathcal{A})$ and the class $\widehat{\epsilon}(\gamma_1, \gamma_2) := \epsilon(\gamma_1, \gamma_2) \bmod \mathcal{U}(\mathcal{Z})$ is uniquely defined.

**2.6 Lemma** *The permutators $\epsilon(\cdot, \cdot)$ satisfy the following relations:*

$$\begin{aligned}
\epsilon(\gamma_1, \gamma_2)\epsilon(\gamma_2, \gamma_1) &\equiv \mathbb{1} \bmod \mathcal{U}(\mathcal{Z})\,, \qquad \gamma_1, \gamma_2 \in \Gamma\,, \\
\epsilon(\iota, \gamma) \equiv \epsilon(\gamma, \iota) &\equiv \mathbb{1} \bmod \mathcal{U}(\mathcal{Z})\,, \qquad \gamma \in \Gamma\,, \\
\gamma_1(\epsilon(\gamma_2, \gamma_3))\epsilon(\gamma_1, \gamma_3) &\equiv \epsilon(\gamma_1\gamma_2, \gamma_3) \bmod \mathcal{U}(\mathcal{Z})\,, \qquad \gamma_1, \gamma_2, \gamma_3 \in \Gamma\,, \\
A\gamma_1(B)\epsilon(\gamma_1, \gamma_2) &\equiv \epsilon(\gamma_1', \gamma_2')B\gamma_2(A) \bmod \mathcal{U}(\mathcal{Z})\,, \qquad \gamma_1, \gamma_2, \gamma_1', \gamma_2' \in \Gamma \ \text{ and} \\
& \qquad A \in (\gamma_1, \gamma_1') \cap \mathcal{U}(\mathcal{A})\,,\ B \in (\gamma_2, \gamma_2') \cap \mathcal{U}(\mathcal{A}).
\end{aligned}$$

*Proof:* The first and second equations above are obvious. To prove the third one consider the the inner automorphism characterized by the l.h.s. of the equation:

$$\begin{aligned}
\operatorname{ad}\Big(\gamma_1(\epsilon(\gamma_2, \gamma_3))\epsilon(\gamma_1, \gamma_3)\Big) &= \operatorname{ad}\Big(\gamma_1(\epsilon(\gamma_2, \gamma_3))\Big) \circ \operatorname{ad}\Big(\epsilon(\gamma_1, \gamma_3)\Big) \\
&= \gamma_1 \operatorname{ad}(\epsilon(\gamma_2, \gamma_3))\, \gamma_1^{-1} \circ \operatorname{ad}(\epsilon(\gamma_1, \gamma_3)) \\
&= \gamma_1\,(\gamma_2\gamma_3\gamma_2^{-1}\gamma_3^{-1})\,\gamma_1^{-1}\,(\gamma_1\gamma_3\gamma_1^{-1}\gamma_3^{-1}) = (\gamma_1\gamma_2)\,\gamma_3\,(\gamma_1\gamma_2)^{-1}\,\gamma_3^{-1} \\
&= \operatorname{ad}\Big(\epsilon(\gamma_1\gamma_2, \gamma_3)\Big)\,,
\end{aligned}$$

and this shows the desired relation. Finally, to prove the last equation recall that from the assumptions we have $\gamma_1' = \mathrm{ad}(A)\circ\gamma_1$ and $\gamma_2' = \mathrm{ad}(B)\circ\gamma_2$. From this we compute

$$\begin{aligned}\mathrm{ad}(\epsilon(\gamma_1',\gamma_2')) &= (\mathrm{ad}(A)\circ\gamma_1)\circ(\mathrm{ad}(B)\circ\gamma_2)\circ(\mathrm{ad}(A)\circ\gamma_1)^{-1}\circ(\mathrm{ad}(B)\circ\gamma_2)^{-1}\\ &= \mathrm{ad}(A)\circ\mathrm{ad}(\gamma_1(B))\circ\underbrace{\gamma_1\circ\gamma_2\circ\gamma_1^{-1}\circ\gamma_2^{-1}}_{\mathrm{ad}(\epsilon(\gamma_1,\gamma_2))}\circ\,\mathrm{ad}(\gamma_2(A))^{-1}\circ\mathrm{ad}(B)^{-1}\,.\end{aligned}$$

Therefore we get

$$\mathrm{ad}\Big(\epsilon(\gamma_1',\gamma_2')B\gamma_2(A)\Big) = \mathrm{ad}\Big(A\gamma_1(B)\epsilon(\gamma_1,\gamma_2)\Big)$$

which implies the last equation of the statement. ∎

**2.7 Definition** *Let $\beta_\chi \in \Theta(\chi)$, $\chi \in \mathcal{X}$, with $\beta_\iota = id_{\mathcal{A}}$, be a system of representatives, i.e. $\pi(\beta_\chi) = \Theta(\chi)$. Then $\beta_{\mathcal{X}}$ is called a* lifting *of $\Theta$ if $\mathcal{X} \ni \chi \to \beta_\chi \in aut\mathcal{A}$ is a homomorphism.*

**2.8 Remark** For the notion of lifting see for example Jones [16]. Sutherland [20, 21] says that $\Theta$ *splits* if there is a lifting of $\Theta$. If $\Theta$ is an isomorphism then a lifting is also called *monomorphic section* (this latter name is used by Doplicher/Haag/Roberts [10]).

Results on the existence of liftings when $\mathcal{A}$ is a von Neumann algebra and in a more general context w.r.t. the group $\mathcal{X}$ (theory of Q-kernels) are due to Sutherland [20, 21]. Further, recall also the result of Doplicher/Haag/Roberts [10] in the "automorphism case" of the superselection theory, where $\mathcal{Z} = \mathbb{C}\mathbb{1}$ and $\mathcal{A}$ is a so-called quasilocal algebra w.r.t. a net of local von Neumann algebras (see also [2]).

# 3 Hilbert extensions

The question concerning the description of $\{\mathcal{F},\alpha_{\mathcal{G}}\}$ by $\mathcal{A}$ and 'something else' is called the *reconstruction problem.* It is posed, for example, by Takesaki [23, p. 202] and by Bratteli/Robinson [8, p. 137]. Also the superselection structures in algebraic quantum field theory are connected with the reconstruction problem (for the automorphism case see Doplicher/Haag/Roberts [10]).

From Lemma 2.2 it seems natural to consider the corresponding *inverse problem*, which is an extension problem. This is just the emphasis in the mentioned papers by Sutherland and Jones (see also Nakamura/Takeda [19, 22]) as well as an essential aspect of the superselection theory (cf. [10, 2]).

**3.1 Definition** *Let a system $\{\mathcal{A},\Theta(\mathcal{X})\}$ be given where $\mathcal{X}$ is a discrete abelian group and where $\Theta\colon \mathcal{X} \to Out\mathcal{A}$ is a homomorphism and put $\mathcal{G} := \hat{\mathcal{X}}$. A Hilbert $C^*$-system $\{\mathcal{F},\alpha_{\mathcal{G}}\}$ is called a* Hilbert extension *of $\{\mathcal{A},\Theta(\mathcal{X})\}$ if $\mathcal{A} = \Pi_\iota\mathcal{F}$ and $\Theta(\mathcal{X})$ coincides with the homomorphism given by Lemma 2.2.*

Now let $\{\mathcal{A},\Theta(\mathcal{X})\}$ and $\mathcal{G}$ be given as in the previous definition. As it is pointed out, for example in [16], a crucial object for the extension problem is the so-called *obstruction* $\mathrm{Ob}\,\Theta$. We recall the relevant relations: Choose a system $\beta_\chi \in \Theta(\chi)$, $\chi \in \mathcal{X}, \beta_\iota := \mathrm{id}_{\mathcal{A}}$ of representatives. Then

$$\beta_{\chi_1}\circ\beta_{\chi_2} = \mathrm{ad}\,(\omega(\chi_1,\chi_2))\circ\beta_{\chi_1\chi_2}, \tag{2}$$

where

$$\mathcal{X}\times\mathcal{X} \ni (\chi_1,\chi_2) \to \omega(\chi_1,\chi_2) \in \mathcal{U}(\mathcal{A}) \tag{3}$$

and we have the intertwining property

$$\omega(\chi_1,\chi_2) \in (\beta_{\chi_1\chi_2},\beta_{\chi_1}\circ\beta_{\chi_2}), \tag{4}$$

which is implied by (2). Moreover we have

$$\omega(\iota,\chi)=\omega(\chi,\iota)=\mathbb{1}. \tag{5}$$

Now associativity yields

$$\mathrm{ad}\,(\omega(\chi_1,\chi_2)\omega(\chi_1\chi_2,\chi_3))=\mathrm{ad}\,(\beta_{\chi_1}(\omega(\chi_2,\chi_3))\omega(\chi_1,\chi_2\chi_3))$$

so that there is $\gamma(\chi_1,\chi_2,\chi_3)\in\mathcal{U}(\mathcal{Z})$ with

$$\omega(\chi_1,\chi_2)\omega(\chi_1\chi_2,\chi_3)=\gamma(\chi_1,\chi_2,\chi_3)\beta_{\chi_1}(\omega(\chi_2,\chi_3))\omega(\chi_1,\chi_2\chi_3).$$

If $\gamma(\chi_1,\chi_2,\chi_3)=\mathbb{1}$ for all $\chi_1,\chi_2,\chi_3\in\mathcal{X}$ we obtain the equation

$$\omega(\chi_1,\chi_2)\omega(\chi_1\chi_2,\chi_3)=\beta_{\chi_1}(\omega(\chi_2,\chi_3))\omega(\chi_1,\chi_2\chi_3). \tag{6}$$

Obviously, the existence of a system of representatives $\beta_{\mathcal{X}}$ such that equation (6) has a solution $\omega$ equipped with the properties (3)–(5) is necessary for the existence of a Hilbert extension. Even more, the existence of such a solution is also sufficient for the existence of a Hilbert extension.

**3.2 Definition** *A function $\omega$, assigned to a given system $\beta_{\mathcal{X}}$ of representatives of $\Theta(\mathcal{X})$, equipped with the properties (3)–(6) is called a generalized 2-cocycle.*

One calculates easily that the existence of a generalized 2-cocycle is independent of the choice of the system $\beta_{\mathcal{X}}$ of representatives. Further, a generalized cocycle $\omega$ for $\beta_{\mathcal{X}}$ satisfies the relation

$$\mathrm{ad}\,(\omega(\chi_1,\chi_2)\omega(\chi_2,\chi_1)^{-1})=\beta_{\chi_1}\circ\beta_{\chi_2}\circ\beta_{\chi_1}^{-1}\circ\beta_{\chi_2}^{-1}.$$

The existence of a lifting of $\Theta$ can be expressed in terms of generalized 2-cocycles as follows.

**3.3 Lemma** *There exists a lifting $\beta_{\mathcal{X}}$ of $\Theta$ iff to each system $\gamma_{\mathcal{X}}$ of representatives there corresponds a generalized 2–cocycle $\omega$ of the form*

$$\omega(\chi_1,\chi_2)\equiv\gamma_{\chi_1}(V_{\chi_2}^{-1})V_{\chi_1}^{-1}V_{\chi_1\chi_2}\quad\mathrm{mod}\,\mathcal{U}(\mathcal{Z})\,,$$

*where $V_\chi\in\mathcal{U}(\mathcal{A})$, $V_\iota=\mathbb{1}$. In this case, i.e. if there is a lifting $\beta_{\mathcal{X}}$, then a corresponding generalized 2–cocycle $\omega$ is given by $\omega(\chi_1,\chi_2)=\mathbb{1}$ for all $\chi_1,\chi_2\in\mathcal{X}$.*

*Proof:* Let $\beta_\chi=\mathrm{ad}(V_\chi)\circ\gamma_\chi$, $V_\chi\in\mathcal{U}(\mathcal{A})$, $\chi\in\mathcal{X}$. Now if $\omega(\chi_1,\chi_2)=\gamma_{\chi_1}(V_{\chi_2}^{-1})V_{\chi_1}^{-1}V_{\chi_1\chi_2}Z$ for some $Z\in\mathcal{U}(\mathcal{Z})$, then we have on the one hand $\beta_{\chi_1\chi_2}=\mathrm{ad}(V_{\chi_1\chi_2})\circ\gamma_{\chi_1\chi_2}$ and on the other

$$\beta_{\chi_1}\circ\beta_{\chi_2}=(\mathrm{ad}(V_{\chi_1})\circ\gamma_{\chi_1})\circ(\mathrm{ad}(V_{\chi_2})\circ\gamma_{\chi_2})=\mathrm{ad}\Big(V_{\chi_1}\gamma_{\chi_1}(V_{\chi_2})\,\omega(\chi_1,\chi_2)\Big)\circ\gamma_{\chi_1\chi_2}\,,$$

which using the assumption on $\omega$ and the fact that $\mathrm{ad}(V_{\chi_1\chi_2}Z)=\mathrm{ad}(V_{\chi_1\chi_2})$, implies that $\beta_{\chi_1\chi_2}=\beta_{\chi_1}\circ\beta_{\chi_2}$, i.e. there is a lift of $\Theta$. To prove the converse let $\beta_{\chi_1\chi_2}=\beta_{\chi_1}\circ\beta_{\chi_2}$, so that from the above relations we have

$$\mathrm{ad}(V_{\chi_1\chi_2})=\mathrm{ad}\Big(V_{\chi_1}\gamma_{\chi_1}(V_{\chi_2})\,\omega(\chi_1,\chi_2)\Big)\,,$$

which implies $\omega(\chi_1,\chi_2)=\gamma_{\chi_1}(V_{\chi_2}^{-1})V_{\chi_1}^{-1}V_{\chi_1\chi_2}\,\mathrm{mod}\,\mathcal{U}(\mathcal{Z})$. ■

**3.4 Theorem** *Let $\omega$ be a generalized 2–cocycle for the system $\beta_{\mathcal{X}}$ of representatives. Then there is a Hilbert extension $\{\mathcal{F},\alpha_{\mathcal{G}}\}$ of $\{\mathcal{A},\Theta(\mathcal{X})\}$.*

*Proof:* The proof consists of several steps that correspond to gradually imposing a richer structure on an initially considered $\mathcal{A}$–left module:

1. Indeed, choose first system of 1-dimensional linear spaces, generated by abstract elements $U_\chi$, $\chi \in \mathcal{X}$, $U_\iota := \mathbb{1} \in \mathcal{A}$. Form the $\mathcal{A}$–left modules $\mathcal{A} \otimes \mathbb{C}U_\chi$ and $\mathcal{F}_0 := \bigoplus_\chi (\mathcal{A} \otimes \mathbb{C}U_\chi)$. By identification $A \otimes \mathbb{1} \leftrightarrow A$, $\mathbb{1} \otimes U_\chi \leftrightarrow U_\chi$ one has

$$\mathcal{F}_0 = \left\{ \sum_{\chi\,,\ \text{finite sum}} A_\chi U_\chi \mid \quad A_\chi \in \mathcal{A} \right\},$$

where $\{U_\chi \mid \chi \in \mathcal{X}\}$ forms an abstract $\mathcal{A}$–module basis.

2. Next we want to equip $\mathcal{F}_0$ with a multiplication structure. First $\mathcal{F}_0$ becomes an $\mathcal{A}$–bimodule extending linearly the following definition

$$U_\chi\, A := \beta_\chi(A)\, U_\chi\,, \quad A \in \mathcal{A}\,, \chi \in \mathcal{X}\,,$$

where $\beta_{\mathcal{X}}$ is the system of representatives to which we associate the generalized cocycle $\omega$. Now the product structure is finally specified by putting

$$U_{\chi_1} \cdot U_{\chi_2} := \omega(\chi_1, \chi_2)\, U_{\chi_1\chi_2}\,, \quad \chi_1, \chi_2 \in \mathcal{X}\,,$$

where the cocycle equation (6) guarantees that the product is associative and the boundary conditions (5) lead to $U_\chi \cdot \mathbb{1} = \mathbb{1} \cdot U_\chi = U_\chi$. Note that the preceding product structure already implies that the $U_\chi$ are invertible. Indeed, it can be checked easily that the inverse is given explicitly by

$$U_\chi^{-1} := \beta_{\chi^{-1}}\Big(\omega(\chi, \chi^{-1})^{-1}\Big)U_{\chi^{-1}}$$

(use for example the relation $\beta_\chi(\omega(\chi^{-1}, \chi)) = \omega(\chi, \chi^{-1})$, which follows from the cocycle equation (6) by putting $\chi_1 := \chi$, $\chi_2 := \chi^{-1}$ and $\chi_3 = \chi$).

3. The following step consists in defining a *–structure on $\mathcal{F}_0$. This is done by putting

$$U_\chi^* := \omega(\chi^{-1}, \chi)^* U_{\chi^{-1}} \quad \text{and} \quad (AU_\chi)^* := U_\chi^* A^*.$$

We still have to check that this definition is consistent, in particular with the product structure in $\mathcal{F}_0$, i.e. we have to verify:

$$(U_\chi^*)^* = U_\chi\,, \quad (U_\chi A)^* = A^* U_\chi^* \quad \text{and} \quad (U_{\chi_1} \cdot U_{\chi_2})^* = U_{\chi_2}^* \cdot U_{\chi_1}^*\,. \tag{7}$$

For the first equation we have

$$\begin{aligned}
(U_\chi^*)^* &= \Big(\omega(\chi^{-1}, \chi)^*\, U_{\chi^{-1}}\Big)^* = U_{\chi^{-1}}^*\, \omega(\chi^{-1}, \chi) = \omega(\chi, \chi^{-1})^*\, U_\chi\, \omega(\chi^{-1}, \chi) \\
&= \omega(\chi, \chi^{-1})^*\, \beta_\chi\Big(\omega(\chi^{-1}, \chi)^*\Big) U_\chi = \omega(\chi, \chi^{-1})^*\, \omega(\chi, \chi^{-1})\, U_\chi \\
&= U_\chi
\end{aligned}$$

The second equation in (7) can also be checked immediately from the definitions considered above. For the last equation we will consider the two sides separately: for the r.h.s. we have

$$\begin{aligned}
U_{\chi_2}^* \cdot U_{\chi_1}^* &= \omega(\chi_2^{-1}, \chi_2)^*\, U_{\chi_2^{-1}} \cdot \omega(\chi_1^{-1}, \chi_1)^*\, U_{\chi_1^{-1}} \\
&= \omega(\chi_2^{-1}, \chi_2)^*\, \beta_{\chi_2^{-1}}\Big(\omega(\chi_1^{-1}, \chi_1)^*\Big) U_{\chi_2^{-1}}\, U_{\chi_1^{-1}} \\
&= \omega(\chi_2^{-1}, \chi_2)^*\, \beta_{\chi_2^{-1}}\Big(\omega(\chi_1^{-1}, \chi_1)^*\Big) \omega(\chi_2^{-1}, \chi_1^{-1})\, U_{(\chi_1\chi_2)^{-1}} \\
&= \omega(\chi_2^{-1}, \chi_2)^*\, \omega((\chi_1\chi_2)^{-1}, \chi_1)^*\, \underbrace{\omega(\chi_2^{-1}, \chi_1^{-1})^* \omega(\chi_2^{-1}, \chi_1^{-1})}_{\mathbb{1}}\, U_{(\chi_1\chi_2)^{-1}}\,,
\end{aligned}$$

where we have used the relation

$$\beta_{\chi_2^{-1}}(\omega(\chi_1^{-1},\chi_1)) = \omega(\chi_2^{-1},\chi_1^{-1})\,\omega(\chi_2^{-1}\chi_1^{-1},\chi_1)\,,$$

which again follows from the cocycle equation (6) taking now $\chi_1 := \chi_2^{-1}$, $\chi_2 := \chi_1^{-1}$ and $\chi_3 = \chi_1$. Now the l.h.s. reads

$$\begin{aligned}(U_{\chi_1}\cdot U_{\chi_2})^* &= U^*_{\chi_1\chi_2}\,\omega(\chi_1,\chi_2)^* = \omega((\chi_1\chi_2)^{-1},\chi_1\chi_2)^*\,U_{(\chi_1\chi_2)^{-1}}\,\omega(\chi_1,\chi_2)^* \\ &= \omega((\chi_1\chi_2)^{-1},\chi_1\chi_2)^*\beta_{(\chi_1\chi_2)^{-1}}\Big(\omega(\chi_1,\chi_2)^*\Big)\,U_{(\chi_1\chi_2)^{-1}}\,.\end{aligned}$$

Thus to show the last equation in (7) we need to prove that

$$\omega((\chi_1\chi_2)^{-1},\chi_1\chi_2)^*\,\beta_{(\chi_1\chi_2)^{-1}}\Big(\omega(\chi_1,\chi_2)^*\Big) = \omega(\chi_2^{-1},\chi_2)^*\,\omega\Big((\chi_1\chi_2)^{-1},\chi_1\Big)^*$$

or taking adjoints

$$\beta_{(\chi_1\chi_2)^{-1}}\Big(\omega(\chi_1,\chi_2)\Big)\,\omega((\chi_1\chi_2)^{-1},\chi_1\chi_2) = \omega\Big((\chi_1\chi_2)^{-1},\chi_1\Big)\,\omega(\chi_2^{-1},\chi_2)\,.$$

But the preceding equation is nothing else than the cocycle equation (7) with $\chi_1 := (\chi_1\chi_2)^{-1}$, $\chi_2 := \chi_1$ and $\chi_3 := \chi_2$. Finally, note that since $\beta_{\chi^{-1}}\Big(\omega(\chi,\chi^{-1})^{-1}\Big) = \omega(\chi^{-1},\chi)^*$ we also have that the $U_\chi$, are unitary, i.e. $U_\chi^* = U_\chi^{-1}$, $\chi\in\mathcal{X}$.

4. Here we will define a representation of the compact abelian group $\mathcal{G} = \widehat{\mathcal{X}}$ in terms of automorphisms of the $*$–algebra $\mathcal{F}_0$. The automorphisms are fixed by putting

$$\alpha_g(U_\chi) := \chi(g)\,U_\chi \quad\text{and}\quad \alpha_g(AU_\chi) := A\,\alpha_g(U_\chi) = \chi(g)\,A\,U_\chi\,,\quad g\in\mathcal{G},\,A\in\mathcal{A},\,\chi\in\mathcal{X}\,.$$

First we check that with the definition above the $\alpha_g$ is indeed an automorphism compatible with the structure in $\mathcal{F}_0$:

$$\begin{aligned}\alpha_g\Big(U_{\chi_1}U_{\chi_2}\Big) &= \alpha_g\Big(\omega(\chi_1,\chi_2)\,U_{\chi_1\chi_2}\Big) = (\chi_1\chi_2)(g)\,\omega(\chi_1,\chi_2)\,U_{\chi_1\chi_2} \\ &= \chi_1(g)\chi_2(g)\,U_{\chi_1}\,U_{\chi_2} = \alpha_g\Big(U_{\chi_1}\Big)\alpha_g\Big(U_{\chi_2}\Big)\end{aligned}$$

and

$$\begin{aligned}\alpha_g\Big(U_\chi^*\Big) &= \alpha_g\Big(\omega(\chi^{-1},\chi)^*\,U_{\chi^{-1}}\Big) = (\chi^{-1})(g)\,\omega(\chi^{-1},\chi)^*\,U_{\chi^{-1}} \\ &= \overline{\chi}(g)\,U_\chi^* = \alpha_g\Big(U_\chi\Big)^*\,.\end{aligned}$$

It can be also easily seen that the assignment $\mathcal{G}\ni g\to\alpha_g\in\operatorname{aut}\mathcal{F}_0$ is an injective group homomorphism. Finally, note that the fixed point algebra of the previous action coincides with $\mathcal{A}$, i.e. for $F\in\mathcal{F}_0$, $\alpha_g(F) = F$ for all $g\in\mathcal{G}$ iff $F\in\mathcal{A}$. Indeed, for an arbitrary element $\sum_\chi A_\chi U_\chi\in\mathcal{F}_0$ the equation $\sum_\chi \chi(g)A_\chi U_\chi = \sum_\chi A_\chi U_\chi$, $g\in\mathcal{G}$, implies by the base property of the $U_\chi$ that $\chi(g)A_\chi = A_\chi$ , $g\in\mathcal{G}$, $\chi\in\mathcal{X}$. Therefore if $\chi_0\neq\iota$, then there is a $g_0\in\mathcal{G}$ with $\chi_0(g_0)\neq 1$ and this shows that $A_{\chi_0} = 0$. The converse implication is obvious.

5. Finally, to specify a C*–norm on $\mathcal{F}_0$ we introduce the following $\mathcal{A}$–valued scalar product (note the variation w.r.t. the definition in [2, p. 101]):

$$\langle F_1,F_2\rangle := \sum_\chi \beta_\chi^{-1}(A_\chi^* B_\chi)\,,\quad\text{where}\quad F_1 = \sum_\chi A_\chi U_\chi\,,\ F_2 = \sum_\chi B_\chi U_\chi\in\mathcal{F}_0.$$

This scalar product satisfies the properties

$$\langle F_1,F_2\rangle^* = \langle F_2,F_1\rangle\,,\quad \langle F_1,F_1\rangle\geq 0\quad\text{and}\quad\langle F_1,F_1\rangle = 0 \text{ iff } F_1 = 0\,.$$

Next we show that

$$\langle F_1, F_2\rangle = \Pi_\iota(F_1^* F_2)\,,$$

Indeed, using the definitions above we have

$$F_1^* F_2 = \sum_{\chi_1,\chi_2} U^*_{\chi_1} A^*_{\chi_1} B_{\chi_2} U_{\chi_2} = \sum_{\chi_1,\chi_2} \omega(\chi_1^{-1},\chi_1)^*\, \beta_{\chi_1^{-1}}(A^*_{\chi_1} B_{\chi_2})\, \omega(\chi_1^{-1},\chi_2) U_{\chi_1^{-1}\chi_2}\,.$$

Putting, $\chi_1 = \chi_2 = \chi$ in the preceding expression we get

$$\begin{aligned}
\Pi_\iota(F_1^* F_2) &= \sum_\chi \omega(\chi^{-1},\chi)^*\, \beta_{\chi^{-1}}(A^*_\chi B_\chi)\, \omega(\chi^{-1},\chi) \\
&= \sum_\chi \omega(\chi^{-1},\chi)^* \omega(\chi^{-1},\chi)\, \beta_\chi^{-1}(A^*_\chi B_\chi)\, \omega(\chi^{-1},\chi)^* \omega(\chi^{-1},\chi) \\
&= \langle F_1, F_2\rangle\,,
\end{aligned}$$

where for the second equation before we have used eq. (2) in the form $\beta_{\chi^{-1}} = \mathrm{ad}\,(\omega(\chi^{-1},\chi))\circ\beta_\chi^{-1}$. In particular the relation above implies the following invariance property: $\langle\alpha_g(F_1), \alpha_g(F_2)\rangle = \langle F_1, F_2\rangle$, $g \in \mathcal{G}$.

Define next the following norm on $\mathcal{F}_0$ by

$$|F| := \|\langle F, F\rangle\|^{\frac{1}{2}}\,, \quad F \in \mathcal{F}_0\,,$$

and the representation of $\mathcal{F}_0$ on $(\mathcal{F}_0, |\cdot|)$ in terms of multiplication operators

$$\rho(F)X := FX\,, \quad F, X \in \mathcal{F}_0\,.$$

Note that by the definition of the $\mathcal{A}$–valued scalar product the property $\rho(F^*) = \rho(F)^*$, $F \in \mathcal{F}_0$, holds. Now using the corresponding operator norm we introduce

$$\|F\|_* := |\rho(F)|_{op}\,, \quad F \in \mathcal{F}_0\,,$$

which by similar arguments as in [2, p. 102-103] satisfies the C*–property $\|F^*F\|_* = \|F\|_*^2$. Further, it satisfies also (cf. again the previous reference)

$$\|A\|_* = \|A\|\,, \quad A \in \mathcal{A} \quad \text{and} \quad \|\alpha_g(F)\|_* = \|F\|_*\,, \quad g \in \mathcal{G}\,,\, F \in \mathcal{F}_0\,.$$

Therefore, we can finally extend $\alpha_g$ isometrically from $\mathcal{F}_0$ to

$$\mathcal{F} := \mathrm{clo}_{\|\cdot\|_*}(\mathcal{F}_0)\,.$$

Further, $\alpha_{\mathcal{G}} \subset \mathrm{aut}\,\mathcal{F}$ is norm continuous w.r.t. the pointwise norm convergence, because for any $F_0 = \sum_\chi A_\chi U_\chi \in \mathcal{F}_0$ we have

$$\|\alpha_{g_1}(F_0) - \alpha_{g_2}(F_0)\|_* = \|\sum_\chi \Big(\chi(g_1) - \chi(g_2)\Big) A_\chi U_\chi\|_* \leq \sum_\chi |\chi(g_1) - \chi(g_2)|\, \|A_\chi\|\,.$$

By construction we also have that $U_\chi \in \Pi_\chi(\mathcal{F})$, $\chi \in \mathcal{X}$. Therefore from the definitions of Sections 2 and 3 we have constructed a Hilbert C*–extension $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ of $\{\mathcal{A}, \Gamma\}$ and the proof is concluded. ■

Using now Lemma 3.3 one has

**3.5 Corollary** *If there is a lifting of $\Theta$, then there is a Hilbert extension of $\{\mathcal{A}, \Theta(\mathcal{X})\}$, corresponding to $\omega = \mathbb{1}$.*

**3.6 Remark** The construction in the proof of the previous theorem generalizes to the nontrivial center situation the procedure already presented (with small modifications) in [2, Section 3.6].

The second problem consists in the description of all Hilbert extensions. For this purpose let $\Omega(\mathcal{X},\mathcal{U}(\mathcal{Z}),\beta_{\mathcal{X}})$ be the set of all $\mathcal{U}(\mathcal{Z})$–valued 2–cocycles $\lambda$, i.e. $\lambda$ satisfies equation (6) and condition (5), but (3),(4) are replaced by $\lambda(\chi_1,\chi_2)\in\mathcal{U}(\mathcal{Z})$. For example, $\lambda(\chi_1,\chi_2):=\mathbb{1}$ for all $\chi_1,\chi_2\in\mathcal{X}$ is such a cocycle. Further let $\Omega_0(\mathcal{X},\mathcal{U}(\mathcal{Z}),\beta_{\mathcal{X}})$ be the set of all $\mathcal{U}(\mathcal{Z})$–valued coboundaries $\partial Z$, i.e.

$$\partial Z(\chi_1,\chi_2):=\frac{Z(\chi_1)\beta_{\chi_1}(Z(\chi_2))}{Z(\chi_1\chi_2)},$$

where $Z(\cdot)$ is a $\mathcal{U}(\mathcal{Z})$-valued 1-cycle, $Z(\iota)=\mathbb{1}$. Then $\partial Z$ is a $\mathcal{U}(\mathcal{Z})$-valued 2-cocycle, $\Omega\supseteq\Omega_0$. As usual, $\Omega$ and $\Omega_0$ are abelian groups w.r.t. pointwise multiplication and the second cohomology is given by $H^2(\mathcal{X},\mathcal{U}(\mathcal{Z}),\beta_{\mathcal{X}}):=\Omega/\Omega_0$.

Next we need the concept of *$\mathcal{A}$–module isomorphism* of Hilbert extensions.

**3.7 Definition** *Let $\{\mathcal{F}^1,\alpha^1_{\mathcal{G}}\}$, $\{\mathcal{F}^2,\alpha^2_{\mathcal{G}}\}$ be Hilbert extensions of $\{\mathcal{A},\Theta(\mathcal{X})\}$. They are called $\mathcal{A}$–module isomorphic if there is an algebraic isomorphism $\Phi$: $\mathcal{F}^1\to\mathcal{F}^2$, with $\Phi(A)=A$ for all $A\in\mathcal{A}$ and $\Phi\circ\alpha^1_g=\alpha^2_g\circ\Phi$ for all $g\in\mathcal{G}$.*

**3.8 Theorem** *Let $\omega_0$ be a generalized 2–cocycle. Then:*

*(i) Each $\mathcal{U}(\mathcal{Z})$–valued 2–cocycle $\lambda$ yields a Hilbert extension generated by the generalized 2–cocycle $\omega:=\lambda\cdot\omega_0$ and each Hilbert extension is generated by some $\mathcal{U}(\mathcal{Z})$–valued 2–cocycle $\lambda$ via $\omega:=\lambda\cdot\omega_0$.*

*(ii) Two Hilbert extensions are $\mathcal{A}$–module isomorphic iff the generating generalized 2–cocycles $\omega_1,\omega_2$ differ only by a $\mathcal{U}(\mathcal{Z})$–valued coboundary $\partial Z$, i.e. $\omega_1=\partial Z\cdot\omega_2$.*

*Proof:* (i) If two generalized 2–cocycles $\omega_1,\omega_2$ are given, then note first that $\lambda(\chi_1,\chi_2):=\omega_1(\chi_1,\chi_2)\omega_2(\chi_1,\chi_2)^{-1}\in\mathcal{U}(\mathcal{Z})$ for all $\chi_1,\chi_2$, because of condition (4). Further, eq. (5) follows from the corresponding properties of $\omega_1$ and $\omega_2$. Finally, the cocycle equation for $\lambda(\chi_1,\chi_2)$ is a consequence of the following computation:

$$\begin{aligned}
\lambda(\chi_1,\chi_2)\lambda(\chi_1\chi_2,\chi_3) &= \omega_1(\chi_1,\chi_2)\omega_2(\chi_1,\chi_2)^{-1}\cdot\omega_1(\chi_1\chi_2,\chi_3)\omega_2(\chi_1\chi_2,\chi_3)^{-1}\\
&= \omega_1(\chi_1,\chi_2)\omega_1(\chi_1\chi_2,\chi_3)\omega_2(\chi_1\chi_2,\chi_3)^{-1}\omega_2(\chi_1,\chi_2)^{-1}\\
&= (\omega_1(\chi_1,\chi_2)\omega_1(\chi_1\chi_2,\chi_3))\cdot(\omega_2(\chi_1,\chi_2)\omega_2(\chi_1\chi_2,\chi_3))^{-1}\\
&= \beta_{\chi_1}(\omega_1(\chi_2,\chi_3))\omega_1(\chi_1,\chi_2\chi_3)\cdot(\beta_{\chi_1}(\omega_2(\chi_2,\chi_3))\omega_2(\chi_1,\chi_2\chi_3))^{-1}\\
&= \beta_{\chi_1}(\omega_1(\chi_2,\chi_3))\omega_1(\chi_1,\chi_2\chi_3)\omega_2(\chi_1,\chi_2\chi_3)^{-1}\beta_{\chi_1}(\omega_2(\chi_2,\chi_3))^{-1}\\
&= \beta_{\chi_1}(\omega_1(\chi_2,\chi_3)\omega_2(\chi_2,\chi_3)^{-1})\omega_1(\chi_1,\chi_2\chi_3)\omega_2(\chi_1,\chi_2\chi_3)^{-1}\\
&= \beta_{\chi_1}(\lambda(\chi_2,\chi_3))\cdot\lambda(\chi_1,\chi_2\chi_3),
\end{aligned}$$

i.e. if one fixes a generalized 2-cocycle $\omega_0$, then $\omega:=\lambda\cdot\omega_0$ runs through all generalized 2–cocycles $\omega$ if $\lambda$ runs through all $\mathcal{U}(\mathcal{Z})$–valued 2–cocycles in $\Omega(\mathcal{X},\mathcal{U}(\mathcal{Z}),\beta_{\mathcal{X}})$.

(ii) Let $\{\mathcal{F}^1,\alpha^1_{\mathcal{G}}\}$ and $\{\mathcal{F}^2,\alpha^2_{\mathcal{G}}\}$ be two Hilbert extensions of $\{\mathcal{A},\Theta(\mathcal{X})\}$ and denote the corresponding set of abstract unitaries by $\{U_\chi\mid\chi\in\mathcal{X}\}$ resp. $\{V_\chi\mid\chi\in\mathcal{X}\}$.

Suppose first that there exists coboundary $\partial Z\in\Omega_0(\mathcal{X},\mathcal{U}(\mathcal{Z}),\beta_{\mathcal{X}})$, where $\beta_{\mathcal{X}}$ is system of representatives in $\Theta$, such that the corresponding generalized cocycles $\omega_1$ and $\omega_2$ satisfy $\omega_1=\partial Z\cdot\omega_2$. In this case we will show that the extensions are isomorphic. Indeed, define the isomorphism by

$$\Phi(AU_\chi):=A\,Z(\chi)\,V_\chi\,,\quad A\in\mathcal{A}\,,\ \chi\in\mathcal{X}\,,$$

and extend it by linearity to the corresponding left $\mathcal{A}$–module. Now $\Phi$ is even a *–homomorphism between the *–algebras $\mathcal{F}_0^1$ and $\mathcal{F}_0^2$ that are defined in step 3 of the proof of Theorem 3.4. This follows from the following computations:

$$\begin{aligned}
\Phi(U_\chi A) &= \Phi\Big(\beta_\chi(A)U_\chi\Big) = Z(\chi)\,V_\chi A = \Phi(U_\chi)\Phi(A)\,,\\
\Phi(U_\chi U_{\chi'}) &= \Phi\Big(\omega_1(\chi,\chi')U_{\chi\chi'}\Big) = \partial Z(\chi,\chi')\cdot\omega_2(\chi,\chi')\,Z(\chi\chi')\,V_{\chi\chi'}\\
&= \frac{Z(\chi)\beta_\chi(Z(\chi'))}{Z(\chi\chi')}\cdot Z(\chi\chi')\,V_\chi V_{\chi'} = Z(\chi)V_\chi\,Z(\chi')V_{\chi'} = \Phi(U_\chi)\Phi(U_{\chi'})\,,\\
\Phi(U_\chi^*) &= \Phi\Big(\omega_1(\chi^{-1},\chi)^* U_{\chi^{-1}}\Big) = \partial Z(\chi^{-1},\chi)^*\cdot\omega_2(\chi^{-1},\chi)^* Z(\chi^{-1})V_{\chi^{-1}}\\
&= Z(\chi^{-1})^*\beta_{\chi^{-1}}(Z(\chi))^* Z(\chi^{-1})\,\omega_2(\chi^{-1},\chi)^* V_{\chi^{-1}} = (Z(\chi)V_\chi)^* = \Phi(U_\chi)^*\,,
\end{aligned}$$

where $\chi,\chi'\in\mathcal{X}$, $A\in\mathcal{A}$. Note further that on $\mathcal{F}_0^1$ we already have $\Phi\circ\alpha_g^1 = \alpha_g^2\circ\Phi$, $g\in\mathcal{G}$, since for any $\chi\in\mathcal{X}$ we have

$$\Phi\circ\alpha_g^1(AU_\chi) = \chi(g)\,A\,Z(\chi)V_\chi = \alpha_g^2(A\,Z(\chi)V_\chi) = \alpha_g^2\circ\Phi(AU_\chi)\,.$$

Recall that $\Phi$ is a bijection between $\mathcal{F}_0^1$ and $\mathcal{F}_0^2$ and we will finish this part of the proof if we can also show that $\Phi$ is even an isometry w.r.t the corresponding C*–norms, because in this case we can isometrically extend $\Phi$ to the desired Hilbert extension isomorphism $\Phi\colon\ \mathcal{F}^1\to\mathcal{F}^2$. Now denote by $\langle\cdot,\cdot\rangle_k$ the $\mathcal{A}$–valued scalar products on $\mathcal{F}_0^k$, $k=1,2$, given in step 5 of the proof of Theorem 3.4. For any $F=\sum_\chi A_\chi U_\chi\in\mathcal{F}_0^1$, so that $\Phi(F)=\sum_\chi A_\chi\,Z(\chi)\,V_\chi\in\mathcal{F}_0^2$, we have the following invariance

$$\langle\Phi(F),\Phi(F)\rangle_2 = \sum_\chi \beta_\chi^{-1}\Big(Z(\chi)^* A_\chi^* A_\chi Z(\chi)\Big) = \sum_\chi \beta_\chi^{-1}\Big(A_\chi^* A_\chi\Big) = \langle F,F\rangle_1\,.$$

From this and recalling the definition of the C*–norm again in step 5 of the proof of Theorem 3.4 we immediately get the desired isometry property:

$$\|\Phi(F)\|_* = \sup_{\substack{X_2\,\in\,\mathcal{F}_0^2\\ |X_2|\,\le\,1}} |\Phi(F)X_2| = \sup_{\substack{X_1\,\in\,\mathcal{F}_0^1\\ |X_1|\,\le\,1}} |\Phi(F)X_1| = \|F\|_*\,.$$

To prove the converse implication assume that $\Phi\colon\ \mathcal{F}_1\to\mathcal{F}_2$ specifies the isomorphy of the Hilbert extensions. Use the unitaries $\{U_\chi\mid\chi\in\mathcal{X}\}$ and $\{V_\chi\mid\chi\in\mathcal{X}\}$ in $\mathcal{F}_1$ resp. $\mathcal{F}_2$ to define the unitary

$$Z(\chi) := \Phi(U_\chi)\,V_\chi^*\,,\quad \chi\in\mathcal{X}\,,$$

that satisfies $Z(\iota)=\mathbb{1}$. Even more $Z(\chi)\in\mathcal{U}(\mathcal{Z})$, since for any $A\in\mathcal{A}$ we have

$$A\,Z(\chi) = \Phi(AU_\chi)\,V_\chi^* = \Phi\Big(U_\chi\beta_\chi^{-1}(A)\Big)V_\chi^* = \Phi(U_\chi)\,(A^*V_\chi)^* = Z(\chi)\,A\,.$$

Finally, for $\chi,\chi'\in\mathcal{X}$ we have

$$\begin{aligned}
Z(\chi\chi') &= \Phi\Big(\omega_1(\chi,\chi')^{-1}U_\chi U_{\chi'}\Big)\cdot V_{\chi'}^* V_\chi^*\,(\omega_2(\chi,\chi')^{-1})^*\\
&= \omega_1(\chi,\chi')^{-1}\,\Phi(U_\chi)\,Z(\chi')\,V_\chi^*\,\omega_2(\chi,\chi')\\
&= \omega_1(\chi,\chi')^{-1}\,\Phi(U_\chi)\Big(\beta_\chi(Z(\chi')^*)V_\chi\Big)^*\omega_2(\chi,\chi')\\
&= \omega_1(\chi,\chi')^{-1}\,Z(\chi)\,\beta_\chi(Z(\chi'))\,\omega_2(\chi,\chi')\,.
\end{aligned}$$

Now recalling the definition of the coboundary $\partial Z$, the preceding equations imply that $\omega_1(\chi,\chi') = \partial Z(\chi,\chi')\cdot\omega_2(\chi,\chi')$, $\chi,\chi'\in\mathcal{X}$, and the prove is concluded. ■

**3.9 Remark** (i) Note that the results are independent of the choice of the system $\beta_{\mathcal{X}}$ of representatives of $\Theta(\mathcal{X})$. Theorem 3.8 means that there is a bijection between $H^2(\mathcal{X},\mathcal{U}(\mathcal{Z}),\beta_{\mathcal{X}})$ and the set of all $\mathcal{A}$–module isomorphy classes of Hilbert extensions of $\{\mathcal{A},\Theta(\mathcal{X})\}$ if there is one extension. In other words, the theorem gives an *outer* characterization of $H^2(\mathcal{X},\mathcal{U}(\mathcal{Z}),\beta_{\mathcal{X}})$ by the set of all $\mathcal{A}$–module isomorphy classes of Hilbert extensions.

(ii) For a closer analysis of the second cohomology in the special cases were $\Gamma \cong \mathbb{Z}_N$ and $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ see [1]. Consider also the abstract results in [18, Chapter 4].

# 4 The case of a trivial center

In this case we have $\mathcal{Z} = \mathbb{C}\mathbb{1}$, thus $\mathcal{U}(\mathcal{Z}) = \mathbb{T}\mathbf{1}$ and this implies that two automorphisms $\alpha, \beta \in \Gamma$ are either unitarily equivalent or otherwise disjoint. The following result is a special case of the famous Doplicher/Roberts theorem (see [13, 3]) in the present automorphism context.

**4.1 Proposition** *If there is a system of representatives $\epsilon(\alpha,\beta)$ of the permutator classes $\widehat{\epsilon}(\alpha,\beta)$ which satisfy the equations*

$$\begin{aligned} \epsilon(\gamma_1,\gamma_2)\epsilon(\gamma_2,\gamma_1) &= \mathbb{1}\,, \\ \epsilon(\iota,\gamma) = \epsilon(\gamma,\iota) &= \mathbb{1}\,, \\ \gamma_1(\epsilon(\gamma_2,\gamma_3))\epsilon(\gamma_1,\gamma_3) &= \epsilon(\gamma_1\gamma_2,\gamma_3)\,, \\ A\beta_{\chi_1}(B)\epsilon(\chi_1,\chi_2) &= \epsilon'(\chi_1,\chi_2)B\beta_{\chi_2}(A)\,, \end{aligned}$$

*for all $A \in (\beta_{\chi_1},\beta'_{\chi_1})$, $B \in (\beta_{\chi_2},\beta'_{\chi_2})$, where $\epsilon'$ belongs to $\beta'_{\mathcal{X}}$, then there is a generalized 2–cocycle $\omega_0$ w.r.t. some system $\beta_\chi$ of representatives of the classes $\chi \in \Gamma/\mathrm{int}\mathcal{A}$, with*

$$\omega_0(\chi_1,\chi_2)\omega_0(\chi_2,\chi_1)^{-1} = \epsilon(\beta_{\chi_1},\beta_{\chi_2}).$$

*In this case there is a Hilbert extension $\mathcal{F}$ of $\{\mathcal{A},\Gamma\}$.*

*Conversely, if there is a Hilbert extension $\mathcal{F}$ of $\{\mathcal{A},\Gamma\}$, then to each $\alpha \in \Gamma$ there corresponds a unitary $V_\alpha \in \bigcup_{\chi\in\mathcal{X}}\mathcal{U}(\Pi_\chi\mathcal{F})$, such that $\alpha = ad V_\alpha\restriction\mathcal{A}$ and*

$$\epsilon(\alpha,\beta) := V_\alpha\, V_\beta\, V_\alpha^{-1}\, V_\beta^{-1}\,,$$

*is a system of representatives of the permutators $\widehat{\epsilon}(\alpha,\beta)$ satisfying the equations above.*

**4.2 Remark** (i) In the present case the 2-cocycles $\lambda$ of the preceding section are $\mathbb{T}\mathbf{1}$-valued and the relation (6) becomes the usual cocycle equation

$$\lambda(\chi_1,\chi_2)\lambda(\chi_1\chi_2,\chi_3) = \lambda(\chi_2,\chi_3)\lambda(\chi_1,\chi_2\chi_3).$$

(ii) In the particular case where $\mathcal{A}$ is the inductive limit of a net of von Neumann algebras (which is a standard situation in algebraic quantum field theory, $\mathcal{A}$ being the so–called quasilocal algebra) it can be shown that there is a lift $\gamma_{\mathcal{X}}$ of a given system of representatives $\beta_{\mathcal{X}}$, $\beta_\chi \in \chi$ (cf. Definition 2.7), and by Corollary 3.5 we have that $\omega(\chi_1,\chi_2) = 1$ is an admissible 2–cocycle of the system $\gamma_{\mathcal{X}}$. For a detailed construction of the lift see [10], [2, Section 3.2].

# 5 A Hilbert space representation of $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$

Following Sutherland [20, 21] one can introduce a faithful Hilbert space representation of a Hilbert extension $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ of $\{\mathcal{A}, \Theta(\mathcal{X})\}$.

First let $\mathcal{H}$ be a Hilbert space and let $\pi$ be a faithful representation of $\mathcal{A}$ on $\mathcal{H}$. Form the Hilbert space $\mathcal{K} := l^2(\mathcal{X}, \mathcal{H})$ by completion of $C_0(\mathcal{X} \to \mathcal{H})$ w.r.t. the norm $\|f\|^2 := \sum_\chi \|f(\chi)\|^2_{\mathcal{H}}$. Choose a system $\beta(\mathcal{X})$ of representatives of $\Theta(\mathcal{X})$ and let $\omega$ be a corresponding generalized 2-cocycle such that $U_{\chi_1} \cdot U_{\chi_2} = \omega(\chi_1, \omega_2) U_{\chi_1\chi_2}$. Now define a representation $\Phi$ of $\mathcal{F}_0 \subset \mathcal{F}$ on $\mathcal{K}$ by

$$\begin{aligned}
(\Phi(A)f)(\chi) &:= \pi(\beta_{\chi^{-1}}(A))f(\chi), \quad A \in \mathcal{A}, \\
\Phi(U_{\chi_0})f)(\chi) &:= \pi(\omega(\chi^{-1}, \chi_0))f(\chi_0^{-1}\chi), \quad \chi_0 \in \mathcal{X}, \\
\Phi(AU_\chi) &:= \Phi(A)\Phi(U_\chi), \quad A \in \mathcal{A}, \chi \in \mathcal{X}.
\end{aligned}$$

Note that $\Phi(\mathbb{1}) = \mathbb{1}_{\mathcal{K}}$ and $\|\Phi(A)\|_{\mathcal{K}} = \|A\|$. One calculates easily

$$\begin{aligned}
\Phi(U_{\chi_1})\Phi(U_{\chi_2}) &= \Phi(\omega(\chi_1, \chi_2))\Phi(U_{\chi_1\chi_2}), \\
\Phi(U_\chi)\Phi(A) &= \Phi(\beta_\chi(A))\Phi(U_\chi), \\
\Phi(A^*) = \Phi(A)^*, \quad \Phi(U_\chi^*) &= \Phi(U_\chi)^*.
\end{aligned}$$

Further $\Phi(\sum_\chi A_\chi U_\chi) = 0$ implies $\sum_\chi A_\chi U_\chi = 0$, i.e. $\Phi$ is a *-isomorphism from $\mathcal{F}_0$ onto $\Phi(\mathcal{F}_0) \subset \mathcal{L}(\mathcal{K})$. Recall that

$$\|\Phi(F)\|_{\mathcal{K}} = \sup_{\|f\| \leq 1} \|\Phi(F)f\|_{\mathcal{K}}.$$

We have

**5.1 Lemma** *The relation*

$$\sup_{g \in \mathcal{G}} \|\Phi(\alpha_g F)\|_{\mathcal{K}} < \infty, \quad F \in \mathcal{F}_0, \tag{8}$$

*holds.*

*Proof:* With $F = \sum_\chi A_\chi U_\chi$ we have

$$\begin{aligned}
\|\Phi(F)f\|^2 &= \sum_{y \in \mathcal{X}} \| \sum_\chi \pi(\alpha_{y^{-1}}(A_\chi)\omega(y^{-1}, \chi))f(y^{-1}\chi)\|^2_{\mathcal{H}} \\
&\leq \sum_{y \in \mathcal{X}} (\sum_\chi \|\pi(\alpha_{y^{-1}}(A_\chi)\omega(y^{-1}, \chi))f(y^{-1}\chi)\|)^2 \\
&\leq \sum_{y \in \mathcal{X}} (\sum_\chi \|A_\chi\| \cdot \|f(y^{-1}\chi)\|)^2 \leq \sum_{y \in \mathcal{X}} (\sum_\chi \|A_\chi\|^2)(\sum_\chi \|f(y^{-1}\chi)\|^2) \\
&= (\sum_\chi \|A_\chi\|^2) \sum_\chi \sum_{y \in \mathcal{X}} \|f(y^{-1}\chi)\|^2 = N(F)\|f\|^2 \sum_\chi \|A_\chi\|^2,
\end{aligned}$$

where $N(F)$ denotes the number of terms of $F$. Hence we obtain

$$\|\Phi(F)\|_{\mathcal{K}} \leq N(F)^{1/2} (\sum_\chi \|A_\chi\|^2)^{1/2} =: C_F.$$

and this implies

$$\|\Phi(\alpha_g F)\|_{\mathcal{K}} \leq C_F, \quad g \in \mathcal{G},$$

because the number of terms of $\alpha_g F$ equals that of $F$ and $\|\chi(g)A_\chi\| = \|A_\chi\|$. This implies the inequality (8). ■

This result means that

$$\|\Phi(F)\|_{sup} := \sup_{g \in \mathcal{G}} \|\Phi(\alpha_g F)\|_{\mathcal{K}}$$

is a C*-norm on $\mathcal{F}_0$.

**5.2 Theorem** *The relation*

$$\|\Phi(F)\|_{sup} = \|F\|_*, \quad F \in \mathcal{F}_0\,,$$

*holds, and in particular* $\|\Phi(F)\|_{\mathcal{K}} \leq \|F\|_*$, $F \in \mathcal{F}_0$.

*Proof:* The norm $\mathcal{F}_0 \ni F \to \|\Phi(F)\|_{sup}$ has the properties $\|\Phi(A)\|_{sup} = \|A\|$ for all $A \in \mathcal{A}$ and $\|\Phi(\alpha_g F)\|_{sup} = \|\Phi(F)\|_{sup}$ for all $g \in \mathcal{G}$. However, according to Doplicher/Roberts [11, p. 105] there is at most one C*-norm on $\mathcal{F}_0$ with the mentioned properties. ■

**5.3 Remark** If there is a faithful state $\phi_0$ of $\mathcal{A}$, then Theorem 5.2 can be improved. In this case

$$\|\Phi(F)\|_{\mathcal{K}} = \|F\|_*, \quad F \in \mathcal{F}_0,$$

holds. This is implied by the fact that in this case Sutherland's representation $\Phi$ of $\mathcal{F}_0$ on $\mathcal{K}$ is unitarily equivalent to the so–called regular representation of $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ (restricted to $\mathcal{F}_0$) given by the (faithful) GNS-representation $\pi$ of $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ on the GNS-Hilbert space $\mathcal{H}_\pi$ w.r.t. the $\mathcal{G}$-invariant state $\phi(F) := \phi_0(\Pi_\iota F)$, $F \in \mathcal{F}$, such that $\|\Phi(F)\|_{\mathcal{K}} = \|\pi(F)\|_{\mathcal{H}_\pi}$ for all $F \in \mathcal{F}_0$, but $\|\pi(F)\|_{\mathcal{H}_\pi} = \|F\|_*$ for all $F \in \mathcal{F}$ (see, for example, [2, p. 108 ff.]).

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