Conjugate Harmonic Maps and Minimal Surfaces

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Abstract

We consider discrete harmonic maps that are conforming or non-conforming piecewise linear maps, and derive a bijection between the minimizers of the two corresponding Dirichlet problems. Pairs of harmonic maps with a conforming and a non-conforming component solve the discrete Cauchy-Riemann equations, and have vanishing discrete conformal energy.

As an application, the results of this work provide a thorough understanding of the conjugation algorithms of Pinkall/Polthier and Oberknapp/Polthier used in the computation of discrete minimal and constant mean curvature surfaces.

1 Introduction

Discrete harmonic maps have been well studied as a basic model problem in finite element theory, while the definition of the conjugate of a discrete harmonic map was not completely settled. In this paper we are interested in pairs of discrete harmonic maps on a Riemann surface M which are both minimizers of the Dirichlet energy

$$E(u) = \frac{1}{2}\int_M |\nabla u|^2 \, dx,$$

and are conjugate, i.e. solutions of the Cauchy Riemann equations

$$dv = *du.$$

We note that generically such pairs neither exist in the space of piecewise linear conforming Lagrange finite elements S_h, nor in the space of piecewise linear non-conforming Crouzeix-Raviart elements S_h^*. Each space alone is too rigid to contain a conjugate for each harmonic function.

In the present paper we compute the conjugate of discrete harmonic maps by simultaneously considering harmonic maps in S_h and S_h^* from M to $\mathbb{R}$. The main result derived in section 5 is

Theorem 1 *Let T_h be a triangulation of a domain on a Riemann surface M in R^n.*

1. *Let $u \in S_h$ be a minimizer of the Dirichlet energy in S_h. Then its conjugate map u^* is in S_h^* and is discrete harmonic.*
2. *Let $v \in S_h^*$ be a minimizer of the Dirichlet energy in S_h^*. Then its conjugate map v^* is in S_h and is discrete harmonic.*
3. *Let $u \in S_h$, respectively S_h^* be discrete harmonic in S_h, respectively S_h^*. Then $u^{**} = -u$.*

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Our interest in harmonic maps arose from the study of numerical algorithms to compute the conjugate of minimal and constant mean curvature surfaces in euclidean three-space, and thereby obtaining unstable solutions of the corresponding variational problems. In the algorithms [9] and [8], the conjugate of a minimal surface is obtained via the conjugate of a discrete harmonic map. Conjugate harmonic maps were defined on the dual graph of the edge graph of the original minimal surface. Although these methods were successful and allowed the numerical computation of a number of complicated minimal surfaces for the first time, they provided no further hints on the harmonicity properties of the conjugate harmonic maps. The results of the present paper provide a thorough understanding of the geometric conjugation algorithms used in Pinkall and Polthier [9] and in Oberknapp and Polthier [8] by relating the geometric discretization techniques to the context of finite element methods, and the convergence of the conjugation of minimal surfaces.

Convergence of conforming harmonic maps has been shown by Tsuchiya [10]. As a more general result for surfaces, Dziuk and Hutchinson [5] obtained optimal convergence results in the H^1 norm for the finite element procedure of the Dirichlet problem of surfaces with prescribed mean curvature have been obtained by . Compare Müller, Struwe, and Šverák [7] for harmonic maps on planar lattices using the five-point Laplacian.

In a sequel to our present paper we will apply the duality between discrete harmonic maps and their conjugates to define discrete conformal maps. We will extend a conformal energy proposed by Hutchinson [6] to the discrete spaces $S_h \times S_h^*$ and show that the discrete holomorphic maps have zero conformal energy, a property generically not available for conforming piecewise linear maps.

This paper starts with a review of the Dirichlet problem of harmonic maps in section 2, followed with its discretization using conforming Lagrange elements in section 3. In section 4 we discretize the same Dirichlet problem using the non-conforming Crouzeix-Raviart elements, and derive a pointwise expression of the discrete minimality condition. Section 5 contains the main results of the paper, namely, identifying solutions in both finite element spaces as pairs of discrete conjugate harmonic maps. An application of the results is given in section 6 to the conjugation of discrete minimal surface.

The author thanks Gerd Dziuk for a discussion on the relation of the conjugate minimal surface method [9] and non-conforming elements.

2 Review of the Dirichlet Problem

We start with a short review of the Dirichlet problem of harmonic maps. For simplicity, let $\Omega \subset \mathbb{R}^2$ be a simply connected, convex, polyhedrally bounded domain, and let $H^1(\Omega)$ be the Sobolev space of weakly differentiable functions. The weak derivative v of a function $u \in H^1(\Omega)$ is defined as the solution of

$$\int_\Omega v\phi dx = -\int_\Omega u\partial\phi dx \qquad \forall\phi \in C_0^\infty(\Omega),$$

and is denoted by $\partial u := v$. $L_2(\Omega)$ consists of all square integrable functions u equipped with the scalar product and norm

$$(u,v)_0 := \int_\Omega u(x)v(x)dx \text{ and } \|u\|_0 := \sqrt{(u,u)_0}.$$

On $H^1(\Omega)$ we have the scalar product and norm

$$(u,v)_m := \sum_{|\alpha|\le m} (\partial^\alpha u, \partial^\alpha v)_0 \text{ and } \|u\|_m := \sqrt{\|\partial^\alpha u\|_0}.$$

Let $V := \{u \in H^1(\Omega) \mid u_{|\partial\Omega} = g\}$ be the affine vector space of admissible functions and $V_0 := \{u \in H^1(\Omega) \mid u_{|\partial\Omega} = 0\}$ the space of possible variations with compact support. The Dirichlet problem of the Laplace equation in Ω for $u \in V$

$$\begin{aligned} \triangle u &= 0 \text{ in } \Omega \\ u_{|\partial\Omega} &= g \text{ on } \partial\Omega \end{aligned} \tag{1}$$

is the Euler-Lagrange equation of the corresponding variational problem of minimizing the Dirichlet energy

$$E_D(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \to \min \qquad \forall u \in V. \tag{2}$$

The functional E_D takes on its minimum in V at u if and only if the bilinear form $a : V \times V \to \mathbb{R}$ given by

$$a(u,v) := \int_\Omega \nabla u \nabla v dx$$

fulfills

$$a(u,v) = 0 \ \forall v \in V_0. \tag{3}$$

The minimizer is unique since

$$E_D(u+v) = E_D(u) + \frac{1}{2}a(v,v) > E_D(u) \ \forall v \in V_0 \setminus \{0\}.$$

Definition 1 *The solution $u \in H^1(\Omega)$ of the variational problem (2) or, equivalently, of the bilinear equation (3) is called a* weak solution *of the Laplace equation (1).*

For a discussion of the solutions of (1) and their numerics see e.g. [3][2]. By recalling two numerical methods based on conforming and non-conforming piecewise linear elements in the following two sections, we relate them with the geometric algorithms in [9] and [8]. The geometric point of view allows to interpret both solutions as pairs of conjugate discrete harmonic maps.

3 Conforming Finite Elements

We approximate V by a finite dimensional affine subspace $S_h \subset V$ where h denotes a discretization constant. For simplicity, we assume $\Omega \subset \mathbb{R}^2$ is a simply connected, convex, polygonally bounded domain divided into triangles. The results in section 5 are valid for more general domains with a regular Riemannian metric. Further, we restrict to Dirichlet boundary conditions for simplicity, see [9] for a discussion of Neumann boundary problems in this context.

Definition 2 *A subdivision $\mathfrak{T}_h = \{T_1, .., T_m\}$ of Ω into triangles T_i is a* conforming triangulation *if the following properties hold:*

1. $\overline{\Omega} = \bigcup_{i=1}^m T_i$
2. *For $i \neq j$, $T_i \cap T_j$ is empty, or it is a common vertex, or a common edge of both triangles.*

Furthermore, we assume each triangle in $\mathfrak{T}_h$ has diameter at most $2h$.

Definition 3 *For a triangulation $\mathfrak{T}_h$ of Ω, we define the space S_h of* conforming finite elements*:*

$$\begin{aligned} S_h \quad &: \quad = \left\{ v : \Omega \to \mathbb{R} \;\middle|\; v \in C^0(\overline{\Omega}) \text{ and } v \text{ is linear on each triangle of } \Omega \right\} \\ S_{h,0} \quad &: \quad = \left\{ v : \Omega \to \mathbb{R} \;\middle|\; v \in S_h \text{ and } v_{|\partial\Omega} = 0 \right\} \end{aligned}$$

$S_{h,0}$ is a finite dimensional subspace of $V = H^1(\Omega)$ spanned by the Lagrange basis functions $\{\varphi_1, .., \varphi_n\}$ corresponding to the set of interior vertices $\{x_1, ..., x_n\}$ of $\mathfrak{T}_h$, that is

$$\begin{gathered} \varphi_i : \Omega \to \mathbb{R},\, \varphi_i \in S_{h,0} \\ \varphi_i(x_j) = \delta_{ij} \ \forall i, j \in \{1, .., n\} \\ \varphi_i \text{ is linear on each triangle} \end{gathered} \tag{4}$$

where $\{x_1, ..., x_n\}$ is the set of interior vertices of $\mathfrak{T}_h$. Then each function $u_h \in S_h$ has a representation

$$u_h(x) = u_0(x) + \sum_{j=1}^{n} u_j \varphi_j(x)$$

where $u_j = u_h(x_j)$, and $u_0 \in S_h$ is an arbitrary function satisfying the Dirichlet boundary condition. We use $\vec{u}_h = (u_1, ..., u_n)$ to denote the unique vector representation of u_h.

Since S_h is a finite dimensional subspace of V, the minimization problem (2) for the Dirichlet energy has a unique solution u_h in S_h solving the system of equations

$$\frac{d}{du_j} E_D(u_h) = \frac{d}{du_j} \sum_{T_i \in \mathfrak{T}_h} \int_{T_i} |\nabla u_h|^2 = 0 \ \forall j \in \{1, .., n\}\,.$$

Definition 4 *The unique solution u_h in S_h of Dirichlet problem (2) in S_h is called a* discrete harmonic map*.*

3.1 Approximation

The following statements belong to the basic techniques in the approximation theory of finite elements and for details we refer the reader to [2][3]. Let u be the weak solution of the variational problem (2) in V, and u_h the solution in $S_h \subset V$. Since Ω is bounded it follows from the Poincaré-Friedrich inequality that the bilinear form a is V_0-elliptic, i.e. there exists constants $0 < \alpha \le C$ such that

$$a(u, u) = \alpha \|u\|_1^2 \text{ and } |a(u, v)| \le C \|u\|_1 \|v\|_1\,.$$

Therefore, the lemma of Céa gives the estimate

$$\|u - u_h\|_m \le \frac{C}{\alpha} \inf_{v_h \in S_h} \|u - v_h\|_m\,, \tag{5}$$

where the approximation quality of the discrete solution u_h of (2) is estimated by the minimizers of the distance of u to S_h. It is a standard technique to estimate the infimum of the distance u to S_h with the distance of the interpolating function $I_h u$ of u on $\mathfrak{T}_h$. Let $\mathfrak{T}_h$ be a quasi-uniform triangulation, i.e. there exists a positive constant uniformly bounding the quotient of the diameter of the cicumcribed circle over the diameter of the inscribed circle of all triangles in $\mathfrak{T}_h$.

Theorem 2 (Approximation Theorem) *Let $\mathfrak{T}_h$ be a quasi-uniform triangulation of Ω, and u the weak solution of (2). Then for the interpolation I_h with piecewise linear polynomials there exists a constant $c = c(\Omega, u)$ such that*

$$\|u - I_h u\|_{m,h} \le c h^{2-m} |u|_{2,\Omega} \ \forall u \in H^2(\Omega) \text{ and } 0 \le m \le 2 \tag{6}$$

Combining the estimates of equations (5) and (6), the piecewise linear solution $u_h \in S_h$ is estimated by

$$\|u - u_h\|_m \leq ch^{2-m} |u|_{2,\Omega}$$

on each discretization $\mathfrak{T}_h$ of Ω. Therefore, as $h \to 0$ one has quadratic convergence of u_h to u in the L_2 norm.

3.2 Discrete Harmonic Maps

There exists a geometric description of the minimality condition of the Dirichlet energy (2) as a balancing condition of weighted edges which was an essential ingredient in the algorithm [9]. Now we use the explicit representation of the basis functions to derive the same formulas via the finite element approach.

Let $T = \{V_1, V_2, V_3\}$ be a triangle with oriented edges $\{c_1, c_2, c_3\}$, and $\varphi_{V_i} : T \to \mathbb{R}$ be the Lagrange basis function at vertex V_i with $\varphi_{V_i}(V_j) = \delta_{ij}$. Then its gradient is

$$\nabla\varphi_{V_i}(x) = \frac{1}{2\,\mathrm{area}\,T} Jc_i, \tag{7}$$

where J denotes rotation by $\frac{\pi}{2}$ oriented such that Jc_i points into the triangle. The basis functions have mutual scalar products

$$\begin{aligned} \left\langle \nabla\varphi_{V_{i-1}}, \nabla\varphi_{V_{i+1}} \right\rangle &= -\frac{\cot\alpha_i}{2\,\mathrm{area}\,T} \\ \left\langle J\nabla\varphi_{V_i}, \nabla\varphi_{V_{i+1}} \right\rangle &= \frac{1}{2\,\mathrm{area}\,T} \\ |\nabla\varphi_{V_i}|^2 &= \frac{\cot\alpha_{i-1} + \cot\alpha_{i+1}}{2\,\mathrm{area}\,T}. \end{aligned} \tag{8}$$

Note, that equation 7 implies $\nabla\varphi_i = -\nabla\varphi_{i-1} - \nabla\varphi_{i+1}$. Let $u_h \in S_h$, then on a single triangle T the gradient of $u_{h|T} : T \longrightarrow R$ is obtained from $u_{h|T}(x) = \sum u_i\varphi_i(x)$

$$\nabla u_{h|T} = \frac{1}{2\,\mathrm{area}\,T} \sum_{j=1}^{3} u_j J c_j. \tag{9}$$

Theorem 3 *Let Ω be a domain with triangulation $\mathfrak{T}_h$, and S_h the set of continuous and piecewise linear functions on $\mathfrak{T}_h$. Then the discrete Dirichlet energy of any function $u_h \in S_h$ is given by*

$$E_D(u_h) = \frac{1}{4} \sum_{edges\ (x_i,x_j)} (\cot\alpha_{ij} + \cot\beta_{ij}) |u(x_i) - u(x_j)|^2. \tag{10}$$

Further, the unique minimizer of the Dirichlet functional (2) solves

$$\frac{d}{du_i} E_D(u_h) = \frac{1}{2} \sum_{edges\ (x_i,x_{i_j})\ at\ x_i} (\cot\alpha_{ii_j} + \cot\beta_{ii_j})(u(x_i) - u(x_{i_j})) = 0 \tag{11}$$

at each interior vertex x_i of $\mathfrak{T}_h$. The first summation runs over all edges of the triangulation, and the second summation over all edges emanating from x_i. The angles α_{ii_j} and β_{ii_j} are vertex angles lying opposite to the edge (x_i, x_{i_j}) in the two triangles adjacent to (x_i, x_{i_j}).

Proof. Using the explicit representation (7) of the basis functions and using $\nabla\varphi_i = -\nabla\varphi_{i-1} - \nabla\varphi_{i+1}$, we obtain the Dirichlet energy of $u_{h|T}$:

$$E_D(u_{h|T}) = \frac{1}{2}\int_T -\sum_{j=1}^{3} |u_{j+1} - u_{j-1}|^2 \langle\nabla\varphi_{j-1}, \nabla\varphi_{j+1}\rangle = \frac{1}{4}\sum_{j=1}^{3} \cot\alpha_j\, |u_{j+1} - u_{j-1}|^2 .$$

Summation over all triangles of $\mathfrak{T}$ and combining the two terms corresponding to the same edge leads to equation (10).

At each interior vertex x_i of $\mathfrak{T}_h$, the gradient of E_D with respect to variations of $u_i = u_h(x_i)$ in the image of u_h is obtained by partial differentiation and easily derived from

$$\frac{d}{du_i} E_D(u_h) = \int_\Omega \langle\nabla u_h, \nabla\varphi_i\rangle . \tag{12}$$

The explicit representations are essential for the results in section (5). ■

Example 1 *On a rectangular regular grid in $\mathbb{R}^2$ which is triangulated by subdividing along either diagonal of each rectangle, the interpolating functions of*

$$\mathrm{Re}\, z, \mathrm{Re}\, z^2, \mathrm{Re}\, z^3, \ \textit{and}\ \mathrm{Im}\, z^4$$

are discrete harmonic maps, and so are the interpolants of some other polynomials. On this regular grid, the weight of each diagonal is zero and, therefore, only the discrete values of the five-star contribute to the Dirichlet gradient.

4 Non-Conforming Finite Elements

We now review the discretization of the Dirichlet problem (2) in the space of non-conforming finite elements, and refer to [3][2] for a detailed discussion.

Definition 5 *For a triangulation $\mathfrak{T}_h$ of Ω, we define the space of* non-conforming finite elements *by*

$$\begin{aligned} S_h^* \ &: = \left\{ v : \Omega \to \mathbb{R} \;\middle|\; \begin{array}{l} v_{|T} \textit{ is linear for each } T \in \mathfrak{T}_h, \textit{ and} \\ v \textit{ is continuous at the midpoints of all triangle edges} \end{array} \right\} \\ S_{h,0}^* \ &: = \{ v \in S_h^* \,|\, v = 0 \textit{ at the midpoints of all boundary edges} \} \end{aligned}$$

The space S_h^* is no longer a finite dimensional subspace of $V = H^1(\Omega)$ as in the case of conforming elements, but S_h^* is a superset of S_h. Let $\{y_i\}$ denote the set of edge midpoints of $\mathfrak{T}_h$, then the basis functions $\psi_i \in S_{h,0}^*$ of non-conforming elements are linear on each triangle,

$$\begin{array}{c} \psi_i : \Omega \to \mathbb{R},\ \psi_i \in S_{h,0}^* \\ \psi_i(y_j) = \delta_{ij}\ \forall i, j \in \{1, .., n\} \\ \psi_i \text{ is linear on each triangle.} \end{array} \tag{13}$$

The support of a function ψ_i consists of the (at most two) triangles adjacent to the edge e_i, and ψ_i is usually not continuous on Ω. Each function $v \in S_h^*$ has a representation

$$v_h(x) = v_0(x) + \sum_{\text{edges } e_i} v_i \psi_i(x)$$

where $v_i = v_h(y_i)$ is the value of v_h at the edge midpoint y_i of e_i, and $v_0 \in S_h^*$ an arbitrary function satisfying the boundary conditions.

4.1 Approximation

The space S_h^* is not a subset of $H^1(\Omega)$, so the quadratic form a is not directly defined. One extends it to a grid dependent bilinear form a_h to be used in the definition of the Dirichlet energy in S_h^*:

$$a_h(u,v) := \sum_{T\in\mathfrak{T}_h} \int_T \nabla u \nabla v \text{ for } u,v \in H^1(\Omega)\oplus S_h^*$$

and norm

$$\|v\|_{m,h}^2 := \sum_{T\in\mathfrak{T}_h} \|v\|_{m,T}^2 = a_h(v,v) \text{ for } v \in H^1(\Omega)\oplus S_h^*.$$

Note that for functions $u \in H^m(\Omega)$ both norms agree: $\|u\|_{m,h} = \|u\|_m$.

In the following Ansatz for the Dirichlet problem, one obtains an additional boundary term that did not appear in the conforming situation:

$$\int_{\mathfrak{T}_h} \triangle u\psi = -\sum_{T\in\mathfrak{T}_h} \int_T \nabla u \nabla\psi ds + \sum_{T\in\mathfrak{T}_h} \int_{\partial T} \partial_\nu u\psi ds \text{ for } \psi \in S_{h,0}^*. \tag{14}$$

In the definition of the Dirichlet energy, we exclude the boundary term since by equation (18), it vanishes for discrete harmonic maps.

Definition 6 *For $u_h \in S_h^*$ we define its* Dirichlet energy *by*

$$E_D(u_h) = \frac{1}{2}\sum_{T\in\mathfrak{T}_h} \int_T |\nabla u_h|^2 .$$

Compared to the approximation of a weak solution $u \in V$ in S_h, there occurs in S_h^* a further error, the consistency error, in addition to the approximation error.

The following discussion uses the approximation results given in [1]. Let α and C be the ellipticity constants of a with respect to S_h^*, then the second lemma of Strang gives the estimate of Strang:

Lemma 4 (Strang 2) *Let u be the weak solution of equation (2) in $V = H^1(\Omega)$ and u_h the discrete solution in S_h^*. Then there exists a number c independent of the discretization h such that*

$$\|u-u_h\|_{m,h} \le c\left\{\inf_{u_h\in S_h^*} \|u-u_h\|_{m,h} + \sup_{w_h\in S_h} \frac{|a_h(u,w_h)|}{\|w_h\|_{m,h}}\right\}.$$

The first term is the approximation error that appears in Céa's lemma, and the second term is the consistency error of S_h^*. One estimates the terms on the right-hand side of the lemma and obtains [3]:

Theorem 5 *Let Ω be a convex domain or have smooth boundary. Then the non-conforming discrete harmonic maps $v_h \in S_h^*$ approximate a weak solution $u \in H^1(\Omega)$ with the following convergence*

$$\|u-v_h\|_{0,h} + h\|u-v_h\|_{1,h} \le ch^2 |u|_2 .$$

4.2 Discrete Non-Conforming Harmonic Maps

Using the identities in an Euclidean triangle T with vertices $\{V_1, V_2, V_3\}$ and oriented edges $\{c_1, c_2, c_3\}$, $c_1 + c_2 + c_3 = 0$, we obtain on T the following representation of the basis functions $\psi_i \in S_h^*$ corresponding to edge c_i :

$$\nabla\psi_i = -2\nabla\varphi_{V_i} = \frac{-1}{\text{area}\,\triangle} Jc_i \ , \tag{15}$$

where $\varphi_{V_i} \in S_h$ is the conforming basis function corresponding to the triangle vertex V_i opposite to the edge c_i, and J is the rotation of an edge by $\frac{\pi}{2}$ such that Jc points from edge c in the direction of V.

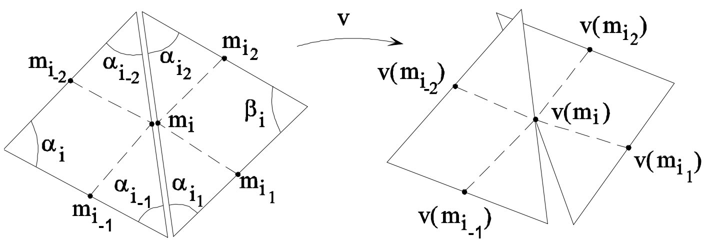

Figure 1: A non-conforming map is given by its values on edge midpoints.

Theorem 6 *Let $v \in S_h^*$ be a non-conforming function on a triangulation $\mathfrak{T}_h$ of Ω. Then the Dirichlet energy of v_h has the explicit representation*

$$E_D(v) = \sum_{all\ edges\ c_i} \cot\alpha_i \left|v_{i_{-2}} - v_{i_{-1}}\right|^2 + \cot\beta_i \left|v_{i_1} - v_{i_2}\right|^2. \tag{16}$$

where $\{i_{-2}, i_{-1}, i_1, i_2,\}$ denote subindices of adjacent edge midpoints as shown in figure 1, and v_i denote the value $v(m_i)$. The angles are measured in Ω.

The unique minimizer of the Dirichlet functional on $\mathfrak{T}_h$ solves the system of equations

$$\frac{d}{dv_i} E_D(v) = 2 \left(\begin{array}{c} \cot\alpha_{i_{-2}}(v_i - v_{i_{-1}}) + \cot\alpha_{i_{-1}}(v_i - v_{i_{-2}}) \\ + \cot\alpha_{i_1}(v_i - v_{i_2}) + \cot\alpha_{i_2}(v_i - v_{i_1}) \end{array} \right) = 0. \tag{17}$$

at each edge midpoint m_i.

Proof. Since $\nabla\psi_i = -2\nabla\varphi_i$, the representation of the Dirichlet energy is a direct consequence of the explicit representation for conforming elements (10). On a single triangle T,

$$E_D(v_{|T}) = \frac{1}{2}\int_T -\sum_{j=1}^{3} |v_{j+1} - v_{j-1}|^2 \langle\nabla\psi_{j-1}, \nabla\psi_{j+1}\rangle = \sum_{j=1}^{3} \cot\alpha_j \left|v_{j+1} - v_{j-1}\right|^2.$$

The support of a component of the gradient of the Dirichlet energy consists of those two triangles adjacent to the edge corresponding to this variable. Equation (17) follows directly from the representation on a single triangle T with edges $\{c_1, c_2, c_3\}$ and $c_1 + c_2 + c_3 = 0$

$$\begin{aligned} \frac{d}{dv_i} E_D(v_{|T}) &= \int_T \langle\nabla v_{|T}, \nabla\psi_i\rangle = \frac{1}{\text{area}(T)} \sum_{j=1}^{3} v_j \langle c_j, c_i\rangle \\ &= 2\cot\alpha_{i-1}(v_i - v_{i+1}) + 2\cot\alpha_{i+1}(v_i - v_{i-1}). \end{aligned}$$

by combining the expression for the two triangles in the support of ψ_i. ■

For $v \in S_h^*$ and $\psi \in S_{h,0}^*$ the boundary integral in equation (14) gives

$$\int_{\partial T} \partial_\nu v \cdot \psi ds = -\sum_{i=1}^{3} \int_{c_i} < \nabla v, \frac{Jc_i}{|c_i|} > \psi ds,$$

since the outer normal along an edge c_i is given by $\nu_i = -\frac{Jc_i}{|c_i|}$. Using $\langle \nabla v, c_i \rangle = 2(v_{i+1} - v_{i-1})$ and $Jc_i = \cot \alpha_{i+1} c_{i+1} - \cot \alpha_{i-1} c_{i-1}$ we get

$$\int_{\partial T} \partial_\nu v \cdot \psi ds = \sum_{i=1}^{3} 2(\cot \alpha_{i+1}(v_{i-1} - v_i) + \cot \alpha_{i-1}(v_{i+1} - v_i))\psi(m_i), \tag{18}$$

where m_i is midpoint of c_i for $i = 1, 2, 3$. Since ψ is continuous at inner edge midpoints, the value $\psi(m_i)$ is equal for adjacent triangles and, therefore, the boundary integral vanishes at inner edges if and only if $v \in S_h^*$ is discrete harmonic. Summing up, for each discrete harmonic map $v \in S_h^*$ and function $\psi \in S_{h,0}^*$ the boundary integral vanishes.

5 Conjugate Harmonic Maps

In this section we define the conjugate harmonic maps of discrete harmonic maps in S_h and in S_h^*. A smooth harmonic map $u : M \to \mathbb{R}$ on an oriented Riemannian surface M and its conjugate harmonic map $u^* : M \to \mathbb{R}$ solve the Cauchy-Riemann equations

$$du^* = *du$$

where $*$ is the Hodge star operator with respect to the metric in M. In the discrete version, we denote by J the rotation through $\frac{\pi}{2}$ in the oriented tangent space of M, and start with a locally equivalent definition as Ansatz:

Definition 7 *Let $u \in S_h$, respectively S_h^*, be a discrete harmonic map on a triangulation $\mathfrak{T}$ with respect to the Dirichlet energies in S_h, respectively S_h^*. Then its conjugate harmonic map u^* is defined by the requirement that it locally fulfills*

$$\nabla u^*_{|T} = J \nabla u_{|T} \text{ for each triangle } T \in \mathfrak{T}. \tag{19}$$

The rest of the section is devoted to showing that the discrete conjugate map is well-defined (showing the closedness of the differential $*du$) and to proving the harmonicity properties of the integral u^*.

To avoid case distinctions we represent each function with respect to the basis functions ψ_i of S_h^* such that on each triangle

$$u_{|T} = \sum_{i=1}^{3} u_i \psi_i,$$

where u_i is the function value of u at the midpoint of edge c_i. We use the same notation for $u^*_{|T}$, and obtain by definition 19

$$\sum_{i=1}^{3} u_i^* \nabla \psi_i = \sum_{i=1}^{3} u_i J \nabla \psi_i. \tag{20}$$

Lemma 7 *Let T be a triangle with oriented edges $\{c_1, c_2, c_3\}$, $c_1 + c_2 + c_3 = 0$. A pair of linear functions u and u^* related by equation (20), has values at edge midpoints related by*

$$\begin{pmatrix} u_3^* - u_1^* \\ u_3^* - u_2^* \end{pmatrix} = \begin{pmatrix} \cot\alpha_3(u_2 - u_1) + \cot\alpha_1(u_2 - u_3) \\ \cot\alpha_3(u_2 - u_1) + \cot\alpha_2(u_3 - u_1) \end{pmatrix} \tag{21}$$

Proof. The representation (15) of $\nabla\psi_i$ converts equation (20) to

$$\sum_{i=1}^{3} u_i^* J c_i = \sum_{i=1}^{3} u_i c_i.$$

Using $-c_3 = c_1 + c_2$, we express the left side of the above equation as a vector in the span of $\{Jc_1, Jc_2\}$

$$(u_3^* - u_1^*)\, Jc_1 + (u_3^* - u_2^*)\, Jc_2 = \sum_{i=1}^{3} u_i c_i.$$

If the triangle T is nondegenerate, then the matrix (Jc_1, Jc_2) has rank 2, and scalar multiplication with c_1 and c_2 yields

$$\begin{pmatrix} u_3^* - u_1^* \\ u_3^* - u_2^* \end{pmatrix} = \frac{2}{\text{area}(T)} \sum_{i=1}^{3} u_i \begin{pmatrix} \langle c_2, c_i\rangle \\ -\langle c_1, c_i\rangle \end{pmatrix},$$

which easily transforms to equation (21). ■

Now we consider a discrete harmonic map $u \in S_h$ and prove local exactness of its discrete conjugate differential.

Proposition 8 *Let $\mathfrak{T}_h$ be a triangulation and $u \in S_h$ with $u : \mathfrak{T}_h \longrightarrow \mathbb{R}$ an edge continuous discrete harmonic function. Then the discrete Cauchy-Riemann equations (19) have a globally defined solution $u^* : \mathfrak{T}_h \longrightarrow \mathbb{R}$ with $u^* \in S_h^*$. Two solutions u_1^* and u_2^* differ by an additive integration constant.*

Proof. We define the discrete differential du^* of u^* such that on each triangle T

$$du^*_{|T} := *du_{|T}.$$

Since $u_{|T}$ is a linear map, the conjugate differential $du^*_{|T}$ is well defined and there exists a unique smooth solution $u^*_{|T}$ of the smooth Cauchy-Riemann equations on T, up to an additive constant. By Lemma 7, $u^*_{|T}$ is explicitly given in terms of $u_{|T}$ and T.

If $u \in S_h$ is a discrete harmonic map then it turns out that du^* is closed along closed paths on Ω that cross edges only at their midpoints. Since du^* is closed inside each triangle, it is sufficient to prove closedness for a path γ in the vertex star of a vertex $p \in \Omega$ such that $\gamma_{|T}$ linearly connects the midpoints of the two edges of T having p in common, see Figure 2. Let $\{e_0, .., e_{m-1}\}$ be the sequence of edge midpoints determining γ. The edges $d_j := e_{j+1} - e_j$ of γ are parallel to c_j with $c_j = 2d_j$. We use equation (21) in each triangle to derive

$$\begin{aligned} \int_\gamma du^* &= \sum_{j=1}^{m} \int_{\gamma_{|T_j}} *du_{|T_j} = \sum_{j=1}^{m} < J\nabla u_{|T_j}, d_j > \\ &= -\frac{1}{2}\sum_{j=1}^{m} < \nabla u_{|T_j}, Jc_j >= 0, \end{aligned}$$

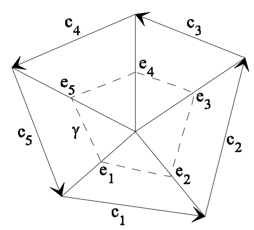

Figure 2: Dual edge graph γ around a vertex.

since u is harmonic in S_h, see equation (12). Therefore, du^* is closed along the dual edge graph through the edge midpoints of $\mathfrak{T}$, and $u^* \in S_h^*$ is globally defined on simply connected regions of Ω. ■

For a harmonic map $u \in S_h$, the following proposition proves harmonicity of the conjugate map $u^* \in S_h^*$.

Proposition 9 *Let $u \in S_h$ be a discrete harmonic map on a triangulation $\mathfrak{T}_h$ and let $u^* \in S_h^*$ be a solution of the discrete Cauchy-Riemann equations (19) given by Proposition 8. Then u^* has the same Dirichlet energy as u, and u^* is discrete harmonic in S_h^*.*

Proof. Let u^* be the solution of the discrete Cauchy-Riemann equations (19) for a discrete harmonic map $u \in S_h$. Then we show that u^* is a critical point of the non-conforming Dirichlet energy in S_h^* by rewriting the Dirichlet gradient (17) of u^* in terms of values of u.

On a single triangle $T = \{c_1, c_2, c_3\}$ with edge midpoints m_i on c_i,we note that

$$\left\langle J\nabla u_{|T}, \nabla\psi_i \right\rangle = \frac{2}{\text{area}(T)}(u(m_{i-1}) - u(m_{i+1})) \; \forall i \in \{1,2,3\}\,, \tag{22}$$

which follows directly from $\nabla u = \sum_{j=1}^{3} u(m_j)\nabla\psi_j$ and

$$\langle J\nabla\psi_j, \nabla\psi_i \rangle = \begin{cases} 0 & j = i \\ \frac{2}{\text{area}(T)} & j = i-1 \\ \frac{-2}{\text{area}(T)} & j = i+1 \end{cases} .$$

Let $T_1 \cup T_2$ denote the two triangles forming the support of ψ_i as shown in figure 3. Using equation (22) we obtain

$$\begin{aligned} \frac{d}{du_i^*} E_D(u^*) &= \int_{T_1 \cup T_2} \langle \nabla u^*, \nabla\psi_i \rangle \\ &= 2(u(m_{i_{-2}}) - u(m_{i_{-1}})) + 2(u(m_{i_1}) - u(m_{i_2})). \end{aligned}$$

Since u is linear we can rewrite the differences at edge midpoints as differences of u at vertices on the common edge of T_1 and T_2, and obtain

$$\frac{d}{du_i^*} E_D(u^*) = u(V_{j_{-1}}) - u(V_{j_{-2}}) + u(V_{j_2}) - u(V_{j_1}). \tag{23}$$

This equation relates the energy gradient of u^* to the function values at vertices of u. We emphasize the fact that the derivation of the equation does not use edge continuity of u, which will allow us to use 23 in the proof of Theorem 10. The right hand side of (23)

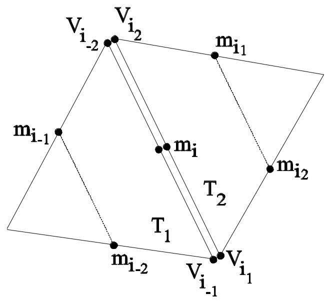

Figure 3: Notation of edge midpoints in pair of triangles.

vanishes if and only if

$$u_{|e_i \text{ in } T_1} = u_{|e_i \text{ in } T_2} + constant.$$

Therefore, the harmonicity of u^* follows from, and is equal to, the edge continuity of $u \in S_h$. ■

The following main theorem states the complete relationship between harmonic maps in S_h and S_h^*, and includes the previous propositions as special cases.

Theorem 10 *Let $\mathfrak{T}_h$ be a triangulation of a Riemann surface M.*

1. *Let $u \in S_h$ be a minimizer of the Dirichlet energy in S_h. Then its conjugate map u^* is in S_h^* and is discrete harmonic.*
2. *Let $v \in S_h^*$ be a minimizer of the Dirichlet energy in S_h^*. Then its conjugate map u is in S_h and discrete harmonic.*
3. *Let $u \in S_h$, respectively S_h^* , be discrete harmonic in S_h, respectively. S_h^*. Then $u^{**} = -u$.*

Proof. 1. The first statement was proved in propositions 8 and 9.

2. Let $v \in S_h^*$ given by $v = \sum v_i \psi_i$ be discrete harmonic. Along the lines of the proof for the corresponding Proposition 8 concerning S_h, we define $v^*_{|T}$ (up to an additive integration constant) as the well-defined integral of

$$dv^*_{|T} := *dv_{|T} \ \forall T \in \mathfrak{T}_h,$$

which uniquely exists since $v_{|T}$ is linear. Using the same arguments as in the proof of Proposition 9 and $\nabla v^* = J\nabla v$, we derive an equation for v that is identical to equation (23) for u :

$$\frac{d}{dv_i} E_D(v) = v^*(V_{j_{-1}}) - v^*(V_{j_{-2}}) + v^*(V_{j_2}) - v^*(V_{j_1}),$$

where V_{j_k} are vertices as denoted in Figure 3. Since v is harmonic, we can choose the integration constants of v^* such that v^* becomes edge continuous and lies in S_h.

The harmonicity property of v^* follows from the closedness of v. Let $v^* = \sum v_i^* \varphi_i \in S_h$, and then splitting $\nabla \psi_i = -\nabla \psi_{i_j} - \nabla \psi_{i_{j+1}}$ in each triangle, we obtain

$$\frac{d}{dv_i^*} E_D(v^*) \quad = \quad \int_{\mathfrak{T}_h} \left\langle \nabla v^*, \frac{d}{dv_i^*} \nabla v^* \right\rangle = \int_{star(p_i)} \langle J\nabla v, \nabla \varphi_i \rangle$$

$$
\begin{aligned}
&= \sum_j \int_{T_{i_j}} \left\langle J\nabla v, -\frac{1}{2}(\nabla\psi_{i_j} + \nabla\psi_{i_{j+1}} \right\rangle \\
&= \sum_j \int_{T_{i_j}} \frac{1}{\operatorname{area} T_{i_j}}((v_{i_{j+1}} - v_{i_{j-1}}) + (v_{i_{j-1}} - v_{i_j})) \\
&= \sum_j v_{i_{j+1}} - v_{i_j} = 0
\end{aligned}
$$

since $v \in S_h^*$ is closed on the path around each vertex p_i. Therefore v^* is critical for the Dirichlet energy in S_h.

3. The third statement is a direct consequence of twice applying the $*$ operator twice, which rotates the gradient in each triangle by π in the plane of the gradient. ■

Corollary 11 *The conjugation is a bijection between discrete harmonic maps in S_h and S_h^*, where each pair (u, v) fulfills the discrete Cauchy Riemann equations. Further, both maps have the same Dirichlet energy.*

Proof. The proof of Theorem 10 and the previous propositions show that, for a pair (u, v) of harmonic conjugate functions $u \in S_h$ and $v \in S_h^*$, the harmonicity condition of u is equal to the closedness condition of v, and the closedness condition of u is equal to the harmonicity condition of v.

The equality of the Dirichlet energies follows directly from the Cauchy-Riemann equations. ■

6 Application to Minimal Surfaces

We apply the results of the previous sections to the computation of the conjugate of a minimal surface. Since these results are corollaries of the previous sections applied to the conjugation method in [9] we refer to the original work for details on the minimization algorithm.

Let $F : M \to \mathbb{R}^3$ be a minimal immersion of a Riemann surface M into $\mathbb{R}^3$ with normal vector field $N : M \to \mathbb{S}^2$ and differential dF. Then the conjugate minimal surface $F^* : M \to \mathbb{R}^3$ with normal field $N^* : M \to \mathbb{S}^2$ is defined as solution of the system

$$
\begin{aligned}
dF^* &= *dF \qquad (24)\\
N^* &= N
\end{aligned}
$$

where $*$ is the Hodge star operation with respect to the induced metric on M. From the system of differential equations it is obvious that a minimal surface and its conjugate are isometric and have parallel normal vectors at corresponding points in the smooth case.

Currently, only the method of Pinkall and Polthier [9] seems to allow the conjugation of a numerically computed discrete minimal surface with reasonable results. The main difficulties are to provide accurate C^1 data from numerically obtained minimal surfaces required for the system 24. In [9] the discrete conjugation algorithm is based the concept of discrete harmonic maps, but the method did not unveil the variational properties of the conjugate surface. In the following we first show the area minimality of the conjugate discrete minimal surface, and second, describe a practical algorithm by reformulating the conjugation method of [9] in terms of the conjugation of harmonic maps using conforming and non-conforming functions derived in the present paper.

For simplicity, consider a Plateau problem with Dirichlet conditions: for a given closed boundary curve $\Gamma \subset \mathbb{R}^3$ find a simply connected minimal surface $M \subset \mathbb{R}^3$ with $\partial M = \Gamma$.

Corollary 12 *Let M be a conforming (respectively non-conforming) triangulation with vanishing gradient of the discrete area functional at all interior vertices. Its conjugate surface M^* is constructed by rotating all triangles by $\pi/2$ in the oriented triangle planes, and by assembling pairs of adjacent triangles at the midpoint of their common edge.*

M^* is a critical point of the area functional in the space of non-conforming (respectively conforming) triangulations with fixed Dirichlet boundary, and M^* is unique up to translation.

Proof. Since M is area minimal, the identity map

$$id : M \to M$$

is a conformal, discrete harmonic map with

$$\text{area}(id(M)) = E_D(id : M \to M).$$

Applying the conjugation algorithm of the previous sections to the harmonic coordinate functions of id gives a conjugate map $id^* : M \to M^*$ which is discrete harmonic. Since id^* is rotation by $\pi/2$ in each triangle, it is isometric on each triangle and conformal. Therefore, M^* is a critical point of the discrete area functional in the class of non-conforming (respectively conforming) triangulations. ■

In practical applications the assumption of the previous corollary, i.e. having a discrete minimal surface to start with, is hardly satisfied. Often, minimal surfaces are computed by solving a variational problem where the numerical method stops before reaching the absolute zero of the gradient. The method of [9] allows to circumvent this difficulty by applying the conjugation to harmonic maps instead of minimal surfaces. Harmonic maps are quadratic problems which can be computed exactly compared to the non-linear process of area minimization.

We approximate a minimal surface M via a sequence of harmonic maps on surfaces. We recall the iteration process introduced in Dziuk [4] for minimal surfaces in $\mathbb{R}^3$, and extended by Oberknapp and Polthier [8] to $\mathbb{S}^3$:

Algorithm 13 *To compute the conjugate M_h^* of the Plateau problem M_h with Dirichlet boundary condition Γ:*

1. Let M_0 be a triangulated initial surface with boundary $\partial M_0 = \Gamma$.

2. Let M_i be a surface with boundary Γ, then compute the surface M_{i+1} as minimizer of the Dirichlet energy

$$\frac{1}{2}\int_{M_i} |\nabla(F_i : M_i \to M_{i+1})|^2 = \min_{M, \partial M = \Gamma} \frac{1}{2}\int_{M_i} |\nabla(F : M_i \to M)|^2 .$$

   This uniquely defines a Laplace-Beltrami harmonic function F_i whose image $F_i(M_i) = M_{i+1}$ is the next surface.

3. Set $i \to i+1$, and continue with step 2 until the area gradient of M_i is reasonable small.

4. Compute the harmonic conjugate F_i^* of $F_i : M_i \to M_{i+1}$.

5. Set $M_h := M_{i+1}$ as numerical approximation of the Plateau solution, and set $M_h^* := F_i^*(M_i)$ as approximation of the conjugate minimal surface.

This algorithm generates a sequence of discrete surfaces $\{M_i\}$ and vector-valued harmonic maps $\{F_i : M_i \to M_{i+1}\}$ which converges to a minimal surface if no degeneration occurs. In order to extend the conjugation technique of the previous sections to the computation of the conjugate of a minimal surface we allow the surfaces M_i to be either all conforming or all non-conforming triangulations. In this case the coordinate functions of each F_i are discrete harmonic functions either in S_h or S_h^*, and the image $F_i^*(M_i)$ of the conjugate harmonic of F_i is a good approximation of the conjugate minimal surface. The two approximations M_h and M_h^* are either a conforming and a non-conforming triangulation, or vice-versa.

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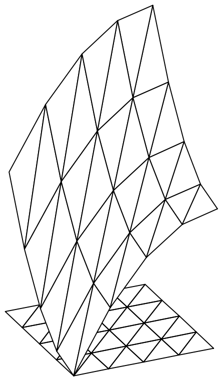
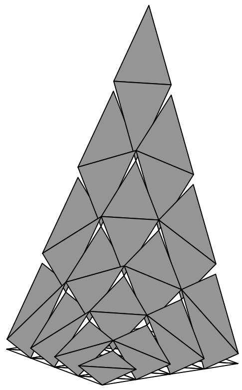
7 Figure Appendix

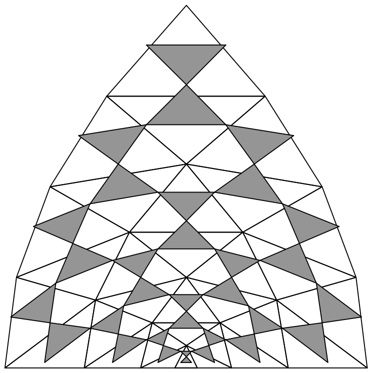
7.1 Harmonic and Holomorphic Maps on a Square Grid



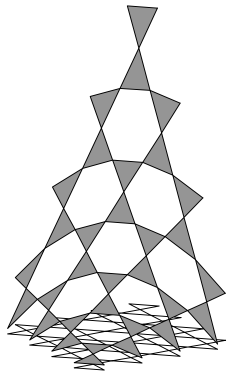
Conforming harmonic map $u \in S_h$
interpolating $\operatorname{Re} z^2$



Conjugate harmonic $u^* \in S_h^*$ is
a non-conforming map

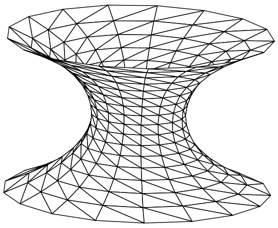


Holomorphic pair (u, u^*) (1/4 drawing)
compared with exact solution (grid)

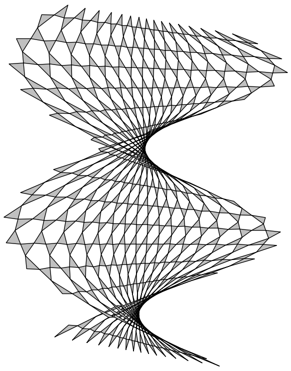


u^* displayed middle 1/4 of each triangle.

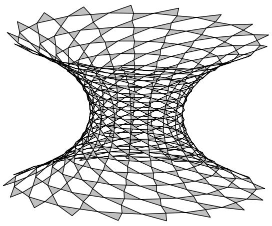
7.2 Application to Minimal Surfaces



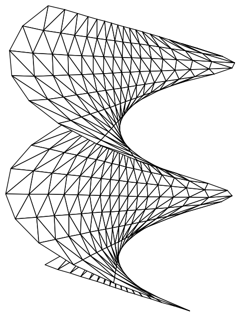
Conforming catenoid



Non-conforming Conjugate



Non-conforming dual of Helicoid



Conforming Helicoid