L^2–RIEMANN–ROCH INEQUALITIES FOR COVERING MANIFOLDS

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ABSTRACT. We study the existence of L^2 holomorphic sections of invariant line bundles over Galois coverings of Zariski open sets in Moishezon manilolds. We show that the von Neuman dimension of the space of L^2 holomorphic sections is bounded below under reasonable curvature conditions. We also give criteria for a a compact complex space with isolated singularities and some related strongly pseudoconcave manifolds to be Moishezon. Their coverings are then studied with the same methods. As applications we give weak Lefschetz theorems using the Napier–Ramachandran proof of the Nori theorem.

CONTENTS

In this paper we wish to address the following problem. Let $\widetilde{M}$ be a complex manifold and assume there is a discrete group $\Gamma \subset \operatorname{Aut} \widetilde{M}$ acting freely and properly discontinuously on $\widetilde{M}$. Suppose that the quotient $M = \widetilde{M}/\Gamma$ is a Zariski open set in a Moishezon manifold X and let $E \longrightarrow X$ be a holomorphic line bundle on X. We denote by $p : \widetilde{M} \longrightarrow M$ the canonical projection.

Problem. Find non-trivial L^2 holomorphic sections in p^*E^k over $\widetilde{M}$ for large k provided E satisfies reasonable conditions in terms of curvature positivity.

Let us describe briefly the background of this problem. In two earth-breaking papers Siu [Si1, Si2] proved the Grauert–Riemenschneider conjecture [GR] by showing that if $E \longrightarrow X$ is a semipositive holomorphic line bundle on a compact manifold and it is positive at one point then E^k has a lot of holomorphic sections i.e. we have the Riemann–Roch inequality $\dim H^0(X, E^k) \geqslant C\, k^n$ for large k, where $n = \dim X$.

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Demailly [De1] developped a more powerful method based on Witten's work [Wi] to get asymptotic Morse inequalities. Takayama [Ta] generalized the Riemann–Roch inequality for the case when X is a compact complex space and E is positive in the neighbourhood of an analytic subset. In order to generalize these results to the case of coverings we shall use the framework of Atiyah [At] who computed the von Neumann index of a Γ–invariant elliptic operator. Our main technical device comes from a paper of Shubin [Sh] in which a proof in the spirit of Witten of the Novikov–Shubin inequalities is given. The present paper pertains also to the work of Gromov, Henkin and Shubin [GHS] in which the authors compute the von Neumann dimension of the space of L^2 holomorphic functions on coverings of strictly pseudoconvex domains. The von Neumann dimension turns out to be infinite generalizing thus Grauert's theorem.

Let us describe the content of our paper. In §1 we generalize the Weyl type formula of Demailly by describing the asymptotic behaviour of the spectrum of a Γ–invariant laplacian associated to high powers of a Γ–invariant line bundle. Using this tool we prove in §2 the main theorem which consists of studying manifolds with pointwise bounded torsion admitting an uniformly positive line bundle outside a compact set. In §3 we consider a special case of 1–concave manifolds and stongly pseudoconcave domains associated to compact complex spaces with isolated singularities. Our results are meaningful even for the trivial covering and they extend the Demailly–Siu criteria for algebraicity from the case of compact manifolds. For the type of manifolds under discussion we also prove some stability results for the perturbation of complex structure. In §4 we generalize the L^2 Riemann–Roch inequality of Takayama [Ta] by considering Galois coverings of smooth Zariski open sets in compact Moishezon spaces. We also remark that by using the $\bar\partial$–method as in Napier and Ramachandran [NR] we may extend the result for arbitrary coverings. Paragraph §5 gives applications to the quotients of bounded domains in $\mathbb{C}^n$. We remark that the von Neuman dimension of the space of L^2 holomorphic pluricanonical sections is infinite if the volume of the quotient in the Bergman metric is infinite. At the opposite side we give a positive partial answer to a question of Griffiths, by showing that the Bergman volume of the quotient is finite provided the quotient is the regular part of a compact complex space with only isolated singularities. Finally, §6 is devoted to proving weak Lefschetz theorems for Moishezon manifolds using the analytic proof (and generalization) of Nori's results due to [NR].

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§**1. Estimates of the spectrum distribution function.** Let $\widetilde{M}$ be a complex analytic manifold of complex dimension n on which a discrete group Γ acts freely and properly discontinuously. Let $X = \widetilde{M}/\Gamma$ let $\pi : \widetilde{M} \longrightarrow X$ be the canonical projection. We assume $\widetilde{M}$ paracompact so that Γ will be countable. Suppose we are given a holomorphic vector bundle F on X and take its pull-back $\widetilde{F} = \pi^*F$, which is a Γ invariant bundle on $\widetilde{M}$. We also fix a Γ invariant hermitian metric on $\widetilde{M}$ and on

$\widetilde{F}$. We consider a relatively compact open set $\Omega \Subset X$ and its preimage $\widetilde{\Omega} = \pi^{-1}\Omega$; Γ acts on $\widetilde{\Omega}$ and $\widetilde{\Omega}/\Gamma = \Omega$. In general we will decorate by tildes the preimages of objects living on the quotient. Let U be a fundamental domain of the action of Γ on $\widetilde{\Omega}$. This means that (see e.g. [At]): a) $\widetilde{\Omega}$ is covered by the translations of $\overline{U}$, b) different translations of U have empty intersection and c) $\overline{U} \smallsetminus U$ has zero measure (since $\partial\Omega$ is smooth). Since Ω is relatively compact U has the same property. Let us define the space of square integrable sections $L^2(\widetilde{\Omega}, \widetilde{F})$ with respect to a Γ invariant metric on $\widetilde{M}$ (and its volume form) and a Γ invariant metric on $\widetilde{F}$. Then $L^2(U, \widetilde{F})$ is constructed with respect to the same. There is a unitary action of Γ on $L^2(\widetilde{\Omega}, \widetilde{F})$. In fact it is easy to see that $L^2(\widetilde{\Omega}, \widetilde{F}) \cong L^2\Gamma \otimes L^2(U, \widetilde{F}) \cong L^2\Gamma \otimes L^2(\Omega, F)$.
We have a unitary action of Γ on $L^2\Gamma$ by left translations: $\gamma \longmapsto l_\gamma$ where $l_\gamma f(x) = f(\gamma^{-1}x)$ for $x \in \Gamma$, $f \in L^2\Gamma$. It induces an action on $L^2(\widetilde{\Omega}, \widetilde{F})$ by $\gamma \longmapsto L_\gamma = l_\gamma \otimes \mathrm{Id}$. Finally we denote by $\mathcal{D}(.,.)$ the various spaces of smooth compactly supported sections.

Let us consider a formally self-adjoint, strongly elliptic, positive differential operator P on M acting on sections of F. Denote by $\widetilde{P}$ the Γ–invariant differential operator which is its pull-back to $\widetilde{M}$. From $\widetilde{P}$ we construct the following operators: the Friedrichs extension in $L^2(\widetilde{\Omega}, \widetilde{F})$ of $\widetilde{P}$ with domain $\mathcal{D}(\widetilde{\Omega}, \widetilde{F})$ and the Friedrichs extension in $L^2(U, \widetilde{F})$ of $\widetilde{P}$ with domain $\mathcal{D}(U, \widetilde{F})$. From now on we denote these extensions by $\widetilde{P}$ and P_0. They are closed self-adjoint positive operators. It is known that $\widetilde{P}$ is also Γ invariant i.e. it commutes with all L_γ. This amounts of saying that E_λ commutes with L_γ, $\gamma \in \Gamma$, where $(E_\lambda)_\lambda$ is the spectral family of $\widetilde{P}$. On the other hand the Rellich lemma tells that P_0 has compact resolvent and hence discrete spectrum. We will take the task of comparing the distribution of the two spectra. Namely since E_λ is Γ invariant its image $R(E_\lambda)$ is a Γ invariant closed subspace of the free Hilbert Γ–module $L^2\Gamma \otimes L^2(U, \widetilde{F}) \cong L^2(\widetilde{\Omega}, \widetilde{F})$. In general for any Hilbert space $\mathcal{H}$ we call the Hilbert space $L^2\Gamma \otimes \mathcal{H}$ a free Hilbert Γ–module. The action of Γ is defined as above by $\gamma \longmapsto L_\gamma = l_\gamma \otimes \mathrm{Id}$. For Γ invariant closed spaces (called Γ modules) one can associate a positive, possibly infinite real number, called von Neumann or Γ–dimension, denoted $\dim_\Gamma$. For notions involving the Γ–dimension and linear algebra for Γ–modules we refer the reader to [At], [Sh] and [Ko] (in the latter proofs from scratch are given). We give here the barest discussion of this score. Let us denote by $\mathcal{A}_\Gamma$ the von Neumann algebra which consists of all bounded linear operators in $L^2\Gamma \otimes \mathcal{H}$ which commute to the action of Γ. To describe $\mathcal{A}_\Gamma$ let us consider the von Neumann $\mathcal{R}_\Gamma$ algebra of all bounded operators on $L^2\Gamma$ which commute with all L_γ. It is generated by all right translations. If we consider the orthonormal basis $(\delta_\gamma)_\gamma$ in $L^2\Gamma$ where δ_γ is the Dirac delta function at γ, then the matrix of any operator $A \in \mathcal{R}_\Gamma$ has the property that all its diagonal elements are equal. Therefore we define a natural trace on $\mathcal{R}_\Gamma$ as the diagonal element, that is, $\operatorname{tr}_\Gamma A = (A\delta_e, \delta_e)$ where e is the neutral element. Now $\mathcal{A}_\Gamma$ is the tensor product of $\mathcal{R}_\Gamma$ and the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$. If Tr is the usual trace on $\mathcal{B}(\mathcal{H})$ then we have a trace on $\mathcal{A}_\Gamma$ by $\operatorname{Tr}_\Gamma = \operatorname{tr}_\Gamma \otimes \operatorname{Tr}$. For any Γ invariant space $L \subset L^2\Gamma \otimes \mathcal{H}$ i.e. for any Γ–module, the projection $P_L \in \mathcal{A}_\Gamma$ and we define $\dim_\Gamma L = \operatorname{Tr}_\Gamma P_L$. Let us just remark for later use that if $L \subset L^2(\widetilde{\Omega}, \widetilde{F})$ is a Γ–module and f_i is an orthonormal basis of L then

$$\dim_\Gamma L = \sum_i \int_U |f_i|^2 \,. \tag{1.1}$$

We denote in the sequel $N_\Gamma(\lambda, \widetilde{P}) = \dim_\Gamma R(E_\lambda)$. Similary we consider the spectral distribution function (counting function) $N(\lambda, P_0) = \dim R(E^0_\lambda)$ where E^0_λ is the spectral family of P_0; it equals the number of eigenvalues $\leqslant \lambda$. We want to compare $N_\Gamma(\lambda, \widetilde{P})$ and $N(\lambda, P_0)$. For this purpose we use essentially the analysis of Shubin [Sh]. However there exist a difference in our method, namely we work at the beginning with model operator P_0 the operator $\widetilde{P}$ itself with Dirichlet boundary conditions on U whereas Shubin considers a direct sum of tangent operators to $\widetilde{P}$. So we do not have to truncate from the outset the eigenfunctions of the model P_0. (See also Remark 1.3 in [Sh] and compare e.g. formulas (2.7), (2.8) or (3.6) from [Sh] with our corresponding formulas.) To begin with we need a variational principle.

Proposition 1.1([Sh]**).** *Let $\widetilde{P}$ be a Γ invariant self-adjoint positive operator on a free Γ–module $L^2\Gamma \otimes \mathcal{H}$ where $\mathcal{H}$ is Hilbert space. Then*

$$N_\Gamma(\lambda, \widetilde{P}) = \sup \Big\{ \dim_\Gamma L \mid L \textit{ is a } \Gamma - \textit{module} \subset \mathrm{Dom}(\widetilde{Q}), \\ \widetilde{Q}(f,f) \leqslant \lambda \|f\|^2, \forall f \in L \Big\} \quad (1.2)$$

where $\widetilde{Q}$ is the quadratic form of $\widetilde{P}$.

Recall that $\widetilde{Q}$ is the closed symmetric quadratic given by $\mathrm{Dom}(\widetilde{Q}) = \mathrm{Dom}(\widetilde{P}^{1/2})$, $\widetilde{Q}(u) = (\widetilde{P}^{1/2}u, \widetilde{P}^{1/2}u)$. From the variational principle we deduce the following.

Proposition 1.2 (Estimate from below). *The counting functions for $\widetilde{P}$ and P_0 satisfy the inequality*

$$N_\Gamma(\lambda, \widetilde{P}) \geqslant N(\lambda, P_0)\,, \quad \lambda \in \mathbb{R} \tag{1.3}$$

Proof. Let us denote by $\lambda_0 \leqslant \lambda_1 \leqslant \dots$ the spectrum of P_0. Let $\{e_i\}_i$ be an orthonormal basis of $L^2(U, \widetilde{F})$ which consists of eigenfunctions of P_0 corresponding to the eigenvalues $\{\lambda_i\}_i$; if we let $\widetilde{e}_i = 0$ on $\widetilde{\Omega} \smallsetminus \overline{U}$ and $\widetilde{e}_i = e_i$ on U, $\widetilde{e}_i \in \mathrm{Dom}(\widetilde{Q})$, $\{L_\gamma \widetilde{e}_i\}_{i,\gamma}$ is an orthonormal basis of $L^2(\widetilde{\Omega}, \widetilde{F})$ and $\widetilde{e}_{i,\gamma} = L_\gamma \widetilde{e}_i \in \mathrm{Dom}(\widetilde{Q})$. We have $\widetilde{Q}(\widetilde{e}_{i,\gamma}, \widetilde{e}_{i',\gamma'}) = \delta_{i,i'}\delta_{\gamma,\gamma'}\lambda_i$. Let Φ^0_λ be the subspace spanned by $\{e_i : \lambda_i \leqslant \lambda\}$ in $L^2(U, \widetilde{F})$ and Φ_λ the closed subspace spanned by $\{\widetilde{e}_{i,\gamma} : \lambda_i \leqslant \lambda\}$ in $L^2(\widetilde{\Omega}, \widetilde{F})$. Then by (1.1)

$$\dim_\Gamma \Phi_\lambda = \sum_{\lambda_i \leqslant \lambda\, \gamma \in \Gamma} \int_U |\widetilde{e}_{i,\gamma}|^2 = \sum_{\lambda_i \leqslant \lambda} \|e_i\|^2_U = \dim \Phi^0_\lambda = N(\lambda, P_0)$$

since $\widetilde{e}_{i,\gamma}|_U$ vanishes unless γ is the identity, and then it equals e_i . If f is a linear combination of $\widetilde{e}_{i,\gamma}$, $\lambda_i \leqslant \lambda$, then $\widetilde{Q}(f,f) \leqslant \|f\|^2$ and, as $\mathrm{Dom}(\widetilde{Q})$ is complete in the graph norm, we obtain that $\Phi_\lambda \subset \mathrm{Dom}(\widetilde{Q})$ and $\widetilde{Q}(f,f) \leqslant \lambda\|f\|^2$, $f \in \Phi_\lambda$. From the variational principle it follows that $N_\Gamma(\lambda, \widetilde{P}) \geqslant N(\lambda, P_0)$. $\square$

The next step is an estimate from above of $N_\Gamma(\lambda, \widetilde{P})$. Before let us say something about Γ–morphisms. If L_1, L_2 are two Γ–modules then an bounded liniar operator $T : L_1 \longrightarrow L_2$ is called a Γ–morphism if it commutes with the action of Γ. As for the usual dimension the following statements are true (see [Ko]). If T is injective then $\dim_\Gamma L_1 \leqslant \dim_\Gamma L_2$ and if T has dense image then $\dim_\Gamma L_1 \geqslant \dim_\Gamma L_2$. We denote by $\mathrm{rank}_\Gamma T = \dim_\Gamma \overline{R(T)}$. For the following we refer to [Sh], Lemma 3.7.

Lemma 1.3. *Let us consider the same setting as in the variational principle. Assume there is* $T : L^2(\widetilde{\Omega}, \widetilde{F}) \to L^2(\widetilde{\Omega}, \widetilde{F})$ *a* Γ*–morphism such that* $((\widetilde{P}+T)f, f) \geqslant \mu\|f\|^2$, $f \in \mathrm{Dom}(\widetilde{P})$ *and* $\mathrm{rank}_\Gamma T \leqslant p$. *Then*

$$N_\Gamma(\mu-\varepsilon, \widetilde{P}) \leqslant p,\ \forall\varepsilon > 0. \tag{1.4}$$

In order to get an estimate from above we have to enlarge a little bit the fundamental domain U and compare the counting function of $\widetilde{P}$ to the counting function of the Friedrichs extension of $\widetilde{P}$ restricted to compactly supported forms in the enlarged domain. For $h > 0$, the enlarged domain is $U_h = \{x \in \widetilde{\Omega} \mid d(x, U) < h\}$ where d is the distance on $\widetilde{M}$ associated to the Riemann metric on $\widetilde{M}$. Then we take the tranlations $U_{h,\gamma} := \gamma U_h$. Next we construct a partition of unity. Let $\varphi^{(h)} \in C^\infty(\widetilde{\Omega})$, $\varphi^{(h)} \geq 0$, $\varphi^{(h)} = 1$ on $\bar{U}$ and $\mathrm{supp}\,\varphi^{(h)} \subset U_h$, $\varphi^{(h)}_\gamma = \varphi^{(h)} \circ \gamma^{-1}$. We define the function $J^{(h)}_\gamma \in C^\infty(\widetilde{\Omega})$ by $J^{(h)}_\gamma = \varphi^{(h)}_\gamma \big(\sum_\gamma (\varphi^{(h)}_\gamma)^2\big)^{-\frac{1}{2}}$ so that $\sum_{\gamma\in\Gamma} (J^{(h)}_\gamma)^2 = 1$. If $\widetilde{P}$ is of *order* 2, which will be assumed throughout the section, then by [Sh,Lemma 3.1] (Shubin's IMS localization formula, see [CFKS]) we know how to recover the operator $\widetilde{P}$ from its localisations $J^{(h)}_\gamma \widetilde{P} J^{(h)}_\gamma$:

$$\widetilde{P} = \sum_{\gamma\in\Gamma} J^{(h)}_\gamma \widetilde{P} J^{(h)}_\gamma - \sum_{\gamma\in\Gamma} \sigma_0(\widetilde{P})(dJ^{(h)}_\gamma) \tag{1.5}$$

where σ_0 is the principal symbol of $\widetilde{P}$. In (1.5) $J^{(h)}_\gamma$ are thought as multiplication operators on $L^2(\widetilde{\Omega}, \widetilde{F})$ – for which $\mathrm{Dom}(\widetilde{P})$ is invariant – while $\sum_{\gamma\in\Gamma} \sigma_0(\widetilde{P})(dJ^{(h)}_\gamma)$ is the multiplication by a bounded function. Since the derivative of $J^{(h)}_\gamma$ is $O(h^{-1})$ and the order of $\widetilde{P}$ is 2 we see that the latter function is bounded by $C\,h^{-2}$ for some constant $C > 0$ (here we use that the symbol is periodic and that $\varphi^{(h)}_\gamma$ are the translates of $\varphi^{(h)}$). Therefore the operatorial norm of the multiplication satisfies the same estimate and we deduce from (1.5) that

$$\widetilde{P} \geqslant \sum_{\gamma\in\Gamma} J^{(h)}_\gamma \widetilde{P} J^{(h)}_\gamma - \frac{C}{h^2}\,\mathrm{Id} \tag{1.6}$$

We consider now the operator $\widetilde{P}$ with domain $\mathcal{D}(U_h, \widetilde{F})$ and take its Friedrichs extension denoted $P^{(h)}_0$. We will compare $N_\Gamma(\lambda, \widetilde{P})$ with the counting function of $P^{(h)}_0$. Let us fix λ. Denote by $(E^{(h)}_\lambda)$ the spectral family of $P^{(h)}_0$ and fix a positive constant $M = M^{(\lambda)}$ such that $M \geqslant \lambda - \inf\mathrm{spectrum}\, P^{(h)}_0$ to the effect that $P^{(h)}_0 + M\,E^{(h)}_\lambda \geqslant \lambda\,\mathrm{Id}$. We define now a localisation of $E^{(h)}_\lambda$ by taking the bounded operators $G^{(h)}_\gamma$ on $L^2(\widetilde{\Omega}, \widetilde{F})$ given by $G^{(h)}_\gamma = J^{(h)}_\gamma\, L_\gamma\, M\, E^{(h)}_\lambda\, L^{-1}_\gamma\, J^{(h)}_\gamma$ and then summing over Γ, $G^{(h)} = \sum_{\gamma\in\Gamma} G^{(h)}_\gamma$. We have

$$\begin{aligned}
\widetilde{P} + G^{(h)} &\geqslant \sum_{\gamma\in\Gamma} \Big(J^{(h)}_\gamma \widetilde{P} J^{(h)}_\gamma + J^{(h)}_\gamma L_\gamma M\, E^{(h)}_\lambda L^{-1}_\gamma J^{(h)}_\gamma \Big) - \frac{C}{h^2}\,\mathrm{Id} \\
&= \sum_{\gamma\in\Gamma} J^{(h)}_\gamma L_\gamma (H^{(h)}_0 + M\, E^{(h)}_\lambda) L^{-1}_\gamma J^{(h)}_\gamma - \frac{C}{h^2}\,\mathrm{Id} \\
&\geqslant \sum_{\gamma\in\Gamma} J^{(h)}_\gamma L_\gamma \lambda L^{-1}_\gamma J^{(h)}_\gamma - \frac{C}{h^2}\,\mathrm{Id} = \Big(\lambda - \frac{C}{h^2}\Big)\,\mathrm{Id}\,.
\end{aligned} \tag{1.7}$$

It is clear that $G^{(h)}$ will play the role of T in Lemma 2.3. We must check one more hypothesis.

Claim 1.4.

$$\operatorname{rank}_\Gamma G^{(h)} \leq N(\lambda, P_0^{(h)}) \tag{1.8}$$

Proof. We start with the bounded operator $\bar{G}^{(h)}$ on $L^2(U_s, \widetilde{F})$, given by $\bar{G}^{(h)} = J_e^{(h)} M\, E_\lambda^{(h)} J_e^{(h)}$. It is a finite rank operator, $\operatorname{rank} \bar{G}^{(h)} \leqslant \operatorname{rank} E_\lambda^{(h)} = N(\lambda, P_0^{(h)})$. Next we consider the free Γ–module $L^2\Gamma \otimes L^2(U_h, \widetilde{F})$ and the bounded Γ–invariant operator $\operatorname{Id} \otimes \bar{G}^{(h)}$. Then $R(\operatorname{Id} \otimes \bar{G}^{(h)}) = L^2\Gamma \otimes R(\bar{G}^{(h)})$ so that $\operatorname{rank}_\Gamma \operatorname{Id} \otimes \bar{G}^{(h)} = \operatorname{rank} \bar{G}^{(h)}$. We identify now the space $L^2\Gamma \otimes L^2(U_h, \widetilde{F})$ with $\bigoplus_{\gamma\in\Gamma} L^2(U_{h,\gamma}, \widetilde{F})$ by the unitary transform $K : \sum_\gamma \delta_\gamma \otimes w_\gamma \longmapsto (L_\gamma w_\gamma)_\gamma$. Thus $\bigoplus_{\gamma\in\Gamma} L^2(U_{h,\gamma}, \widetilde{F})$ is naturally a free Γ–module for which K is Γ invariant. We transport $\operatorname{Id} \otimes \bar{G}^{(h)}$ on $\bigoplus_{\gamma\in\Gamma} L^2(U_{h,\gamma}, \widetilde{F})$ by K and we think it as acting on this latter space. We construct then a restriction operator $V : \bigoplus_{\gamma\in\Gamma} L^2(U_{h,\gamma}, \widetilde{F}) \longrightarrow L^2(\widetilde{\Omega}, \widetilde{F})$, $V\left((w_\gamma)_\gamma\right) = \sum_{\gamma\in\Gamma} w_\gamma$ which is a surjective Γ–morphism. We have also the Γ–morphism I from $L^2(\widetilde{\Omega}, \widetilde{F})$ to $\bigoplus_{\gamma\in\Gamma} L^2(U_{h,\gamma}, \widetilde{F})$, $I(u) = (u \restriction_{U_{h,\gamma}})_\gamma$ which is obviously bounded. With our identifications we have $G^{(h)} = V\,(\operatorname{Id} \otimes \bar{G}^{(h)})\, I$. As in the case of usual dimension $\operatorname{rank}_\Gamma V\,(\operatorname{Id} \otimes \bar{G}^{(h)})\, I \leqslant \operatorname{rank}_\Gamma(\operatorname{Id} \otimes \bar{G}^{(h)})$ (see [Sh], Lemma 3.6).Therfore we conclude $\operatorname{rank}_\Gamma G^{(h)} \leqslant \operatorname{rank}_\Gamma(\operatorname{Id} \otimes \bar{G}^{(h)}) = \operatorname{rank} \bar{G}^{(h)} \leqslant N(\lambda, P_0^{(h)})$. □

Proposition 1.5 (Estimate from above). *There is a constant $C \geq 0$ such that*

$$N_\Gamma(\lambda, \widetilde{P}) \leqslant N\left(\lambda + \frac{C}{h^2}, P_0^{(h)}\right) \quad \lambda \in \mathbb{R}, \quad h > 0 \tag{1.9}$$

Proof. The hypothesis of Lemma 2.3 are fulfilled for $T = G^{(h)}$, $\mu = \lambda - C\, h^{-2}$ and $p = N(\lambda, P_0^{(h)})$ as (1.7) and (1.8) show. Thus $N_\Gamma\left(\lambda - \frac{C}{h^2} - \varepsilon, \widetilde{P}\right) \leqslant N(\lambda, P_0^{(h)})$, if $\varepsilon > 0$. Replacing λ with $\lambda + C\, h^{-2} + \varepsilon$, we obtain $N_\Gamma(\lambda, \widetilde{P}) \leqslant N\left(\lambda + \frac{C}{h^2} + \varepsilon, P_0^{(h)}\right)$. When $\varepsilon \longrightarrow 0$ the estimate (1.9) follows since the spectrum distribution function is right continuous by definition. □

The estimates from below and above for $N_\Gamma(\lambda, \widetilde{P})$ enable us to study as a by–product the behaviour for $\lambda \longrightarrow \infty$ to obtain the Weyl asymptotics for periodic operators (Shubin, see [RSS] and the references therein).

Corollary 1.6. *If $\widetilde{P}$ is a periodic, positive, second order elliptic operator as above then*

$$\begin{aligned}\lim_{\lambda\to\infty} \lambda^{-n/2} N_\Gamma(\lambda, \widetilde{P}) &= \lim_{\lambda\to\infty} \lambda^{-n/2} N(\lambda, P_0)\\ &= (2\pi)^{-n} \int_U \int_{T_x^* \widetilde{M}} N(1, \sigma_0(\widetilde{P})(x, \xi))\, d\xi\, dx\end{aligned}$$

where $\sigma_0(\widetilde{P})(x, \xi) \in \operatorname{Herm}(\widetilde{F}, \widetilde{F})$ is the principal symbol of $\widetilde{P}$ and $N(1, \sigma_0(\widetilde{P})(x, \xi))$ is the counting function for the eigenvalues of this hermitian matrix.

Proof. First let us remark that the last equality is the classical Weyl type formula as established by Carleman, Gårding and others, see [RSS], p.72. It is obvious that

$\liminf \lambda^{-n/2} N(\lambda, P_0) \leqslant \liminf \lambda^{-n/2} N_\Gamma(\lambda, \widetilde{P})$ by the estimate from below. On the other hand the estimate from above gives

$$\begin{aligned}\limsup \lambda^{-n/2} N_\Gamma(\lambda, \widetilde{P}) &\leqslant \limsup \left(1+\frac{C}{\lambda\, h^2}\right)^{n/2} \left(\lambda+\frac{C}{h^2}\right)^{-n/2} N\left(\lambda+\frac{C}{h^2}, P_0^{(h)}\right)\\ &\leqslant \limsup \mu^{-n/2} N(\mu, P_0^{(h)}) = (2\pi)^{-n}\int_{U_h}\int_{T^*_x\widetilde{M}} N(1, \sigma_0(\widetilde{P})(x,\xi))d\xi\, dx\end{aligned}$$

for a fixed small h. We make $h \longrightarrow 0$ and obtain the desired formula. $\square$

We are going to apply the above results to the semi-classical asymptotics as $k \longrightarrow \infty$ of the spectral distribution function of the laplacian $k^{-1}\widetilde{\Delta}''$ on $\widetilde{M}$. Let G be a hermitian holomorphic bundle on M and $\widetilde{G} = p^*G$ its pull-back. We define $\mathcal{D}^{(0,q)}(.,.)$ to be the space of smooth compactly supported $(0,q)$ forms. Let $\bar\partial : \mathcal{D}^{0,q}(\widetilde{M}, \widetilde{G}) \longrightarrow \mathcal{D}^{0,q+1}(\widetilde{M}, \widetilde{G})$ be the Cauchy–Riemann operator and $\vartheta : \mathcal{D}^{0,q+1}(\widetilde{M}, \widetilde{G}) \longrightarrow \mathcal{D}^{0,q}(\widetilde{M}, \widetilde{G})$ the formal adjoint of $\bar\partial$ with respect to the given hermitian metrics on $\widetilde{M}$, $\widetilde{G}$. Then $\widetilde{\Delta}'' = \bar\partial\vartheta + \vartheta\bar\partial$ is a formally self-adjoint, strongly elliptic, positive and Γ–invariant differential operator.

We take $\widetilde{E}$ and $\widetilde{G}$ two Γ invariant holomorphic bundles. Let us form the Laplace–Beltrami operator $\widetilde{\Delta}''_k$ on $(0,q)$ forms with values in $\widetilde{E}^k \otimes \widetilde{G}$. Thus we will consider the Γ invariant hermitian bundle $\widetilde{F} = \Lambda^{(0,q)}T^*\widetilde{M} \otimes \widetilde{E}^k \otimes \widetilde{G}$ and apply the previous results for $\widetilde{P} = k^{-1}\widetilde{\Delta}''_k \restriction_{\widetilde{\Omega}}$ where the index $\widetilde{\Omega}$ emphasises that the Friedrichs extension gives the operator of the Dirichlet problem on $\widetilde{\Omega}$. Now we have to make a good choice of the parameter h. We take $h = k^{-\frac14}$ so that the derivative of the cutting off function $J^{(h)}_\gamma$ is just $O(k^{\frac14})$. Then $\sigma_0(k^{-1}\widetilde{\Delta}''_k)(dJ^{(h)}_\gamma) = k^{-1}|\bar\partial J^{(h)}_\gamma|^2 = O(k^{-\frac12})$. Therefore formula (1.6) becomes $\frac1k\widetilde{\Delta}''_k \restriction_{\widetilde{\Omega}} \geqslant \sum_{\gamma\in\Gamma} J^{(h)}_\gamma \frac1k \widetilde{\Delta}''_k \restriction_{\widetilde{\Omega}} J^{(h)}_\gamma - \frac{C}{\sqrt{k}}\,\mathrm{Id}$. We have thus proved the following semi–classical estimate for laplacian.

Proposition 1.7. *There exists a constant $C > 0$ such that for $\lambda \in \mathbb{R}$ and $k > 0$ we have*

$$N\left(\lambda, \frac1k\widetilde{\Delta}''_k \restriction_U\right) \leqslant N_\Gamma\left(\lambda, \frac1k\widetilde{\Delta}''_k \restriction_{\widetilde{\Omega}}\right) \leqslant N\left(\lambda+\frac{C}{\sqrt{k}}, \frac1k\widetilde{\Delta}''_k \restriction_{U_{k^{-1/4}}}\right) \tag{1.10}$$

Demailly has determined the distribution of spectrum for the Dirichlet problem for $\widetilde{\Delta}''_k$ in [De1], Theorem 3.14. For this purpose he introduces ([De1],(1.5)) the function $\nu_E : \widetilde{M}\times\mathbb{R} \longrightarrow \mathbb{R}$ depending on the curvature of $\widetilde{E}$ and then considers the function $\bar\nu_E(x,\lambda) = \lim_{\varepsilon\searrow 0}\nu_E(x, \lambda+\varepsilon)$. The function $\bar\nu_E(x,\lambda)$ is right continuous in λ and bounded above on compacts of $\widetilde{M}$. Denote by $\alpha_1(x), \dots, \alpha_n(x)$ the eigenvalues of of the curvature form $i\mathbf{c}(\widetilde{E})(x)$ with respect to the metric on $\widetilde{M}$. We also denote for a multiindex $J \subset \{1,...,n\}$, $\alpha_J = \sum_{j\in J}\alpha_j$ and $C(J) = \{1,...,n\} \setminus J$. For $V \Subset M$ we introduce

$$I^q(V,\mu) = \sum_{|J|=q}\int_V \bar\nu_E(2\mu + \alpha_{C(J)} - \alpha_J)\, d\sigma$$

Proposition 1.8 (Demailly). *Assume that ∂V has measure zero and that the laplacian acts on $(0,q)$ forms. Then* $\limsup_k k^{-n} N(\lambda\,, \frac{1}{k}\Delta''_k \restriction_V) \leqslant I^q(V,\lambda)$ *Moreover there exists an at most countable set $\mathcal{N} \subset \mathbb{R}$ such that for $\lambda \in \mathbb{R} \smallsetminus \mathcal{N}$ the limit of the left–hand side expression exists and we have equality.*

We return now to the case of a covering manifold and apply Demailly's formula in (1.10). Let us fix $\varepsilon > 0$. For sufficiently large k we have $U_{k^{-\frac{1}{4}}} \subset U_\varepsilon$ so the fact that the counting function is increasing and the variational principle yield $N(\lambda + \frac{C}{\sqrt{k}}\,, \frac{1}{k}\widetilde{\Delta}'' \restriction_{U_{k^{-1/4}}}) \leqslant N(\lambda+\varepsilon, \frac{1}{k}\widetilde{\Delta}'' \restriction_{U_{k^{-1/4}}}) \leqslant N(\lambda+\varepsilon, \frac{1}{k}\widetilde{\Delta}'' \restriction_{U_\varepsilon})$. Hence by (1.10) and Proposition 1.8 (∂U_ε is negligible for small ε),

$$\limsup_k k^{-n} N_\Gamma(\lambda\,, \frac{1}{k}\widetilde{\Delta}''_k \restriction_{\widetilde{\Omega}}) \leqslant I^q(U_\varepsilon, \lambda+\varepsilon).$$

The use of dominated convergence to make $\varepsilon \longrightarrow 0$ in the last integral yield the asymptotic formula for the laplacian on a covering manifold.

Theorem 1.9. *The spectral distribution function of $\frac{1}{k}\widetilde{\Delta}''_k \restriction_{\widetilde{\Omega}}$ on $L^2_{0,q}(\widetilde{\Omega}, \widetilde{E}^k \otimes \widetilde{G})$ with Dirichlet boundary conditions satisfies*

$$\limsup_k \; k^{-n} N_\Gamma \left(\lambda\,, \frac{1}{k}\widetilde{\Delta}''_k \restriction_{\widetilde{\Omega}}\right) \leqslant I^q(U,\lambda). \tag{1.11}$$

Moreover, there exists an at most countable set $\mathcal{N} \subset \mathbb{R}$ such that for $\lambda \in \mathbb{R} \smallsetminus \mathcal{N}$ the limit exits and we have equality in (1.11).

§2 Geometric situations.

In this section we apply the results from the previous section to the study of the L^2 cohomology of coverings of complex manifolds satisfying certain curvature conditions. If M is a complete Kähler manifold and E a positive line bundle on M the L^2 estimates of Andreotti–Vesentini–Hörmander allow to find a lot of sections of $\widetilde{E}$ on a covering $\widetilde{M}$ (see e.g. [NR]). We prove here the following.

Theorem 2.1. *Let (M,ω) be an n–dimensional complete hermitian manifold such that the torsion of ω is bounded and let (E,h) be a holomorphic hermitian line bundle. Let $K \Subset M$ and a constant $C_0 > 0$ such that $\imath\boldsymbol{c}(E,h) \geqslant C_0\omega$ on $M \smallsetminus K$. Let $p : \widetilde{M} \longrightarrow M$ be a Galois covering with group Γ and $\widetilde{E} = p^*E$ and let Ω be any open set with smooth boundary and $K \Subset \Omega \Subset M$. Then*

$$\dim_\Gamma H^{n,0}_{(2)}(\widetilde{M}, \widetilde{E}^k) \geqslant \frac{k^n}{n!} \int_{\Omega(\leqslant 1,h)} \left(\frac{\imath}{2\pi}\boldsymbol{c}(E,h)\right)^n + o(k^n)\,, \quad k >> 0\,,$$

where $H^{n,0}_{(2)}(\widetilde{M}, \widetilde{E}^k)$ is the space of $(n,0)$–forms with values in $\widetilde{E}^k$ which are L^2 with respect to any metric on $\widetilde{M}$ and the pullback of h and $\Omega(\leqslant 1,h)$ is the subset of Ω where $\imath\boldsymbol{c}(E,h)$ is non–degenerate and has at most one negative eigenvalue.

Proof. We endow $\widetilde{M}$ with the metric $\widetilde{\omega} = p^*\omega$ and $\widetilde{E}$ with $\widetilde{h} = p^*h$. All the norms, Laplace–Beltrami operators, spaces of harmonic forms and L^2–cohomology groups are with respect to $\widetilde{\omega}$ and $\widetilde{h}$. In particular the operators $\bar{\partial}$ and Lapalce–Beltrami are Γ–invariant. It is standard to see that $\widetilde{\omega}$ is also complete. To justify this let us first take a compact set $K \Subset X$ and consider $\widetilde{K} = p^{-1}K$. The metric $\widetilde{\omega}_\varepsilon$ is complete on $\widetilde{K}$ in the following sense. There exist functions $\varphi_\varepsilon \in \mathcal{C}^\infty(\widetilde{K})$ with values in $[0,1]$ such that $\operatorname{supp}\varphi_\varepsilon$ is compact in $\widetilde{K}$, the sets $\{z \in \widetilde{K} \,:\, \varphi_\varepsilon(z) = 1\}$ form an exhaustion of $\widetilde{K}$ and $\sup|d\varphi_\varepsilon| = O(\varepsilon)$ as $\varepsilon \longrightarrow 0$. This is seen as usual by observing that the balls are relatively compact in $\widetilde{K}$ and then taking cut–off functions. Since M is complete there exist an exhaustion K_ν with compacts and functions $\psi_\nu \in \mathcal{C}^\infty(M)$ with values in $[0,1]$ and $\operatorname{supp}\psi_\nu \Subset K_{\nu+1}$ such that $K_\nu \subset \{z \in M \,:\, \psi_\nu(z) = 1\}$ and $\sup|d\psi_\nu| \leqslant 2^{-\nu}$. Let us choose now a point $z_0 \in \widetilde{K}_0$ and fix fundamental domains U_ν for the action of Γ on $\widetilde{K}_\nu$ such that $z_0 \in U_\nu$. We also choose an exhaustion by finite sets $I_0 \subset I_1 \subset \cdots \subset I_\nu \subset \cdots \subset \Gamma$ of Γ. Indeed, since $\widetilde{M}$ is paracompact Γ is countable. For each ν let us take $\varphi_\nu \in \mathcal{C}^\infty(\widetilde{K}_{\nu+1})$ such that $\varphi_\nu = 1$ on $\cup\{\gamma U_{\nu+1} \,:\, \gamma \in I_{\nu+1}\}$ and $\sup|d\varphi_\nu| \leqslant 2^{-\nu}$. We consider also the function $\widetilde{\psi}_\nu = \psi_\nu \circ p$. Then the functions $\widetilde{\psi}_\nu\varphi_\nu$ have compact support in $\widetilde{M}$, the sets where they equal 1 exhaust $\widetilde{M}$ and their derivative is $O(2^{-\nu-1})$, which proves that $\widetilde{M}$ is complete. We remark here that $\mathcal{U} = \cup_\nu U_\nu$ is a fundamental domain for the action of Γ on $\widetilde{M}$ and that if $\widetilde{G}$ is a Γ–invariant bundle on $\widetilde{M}$ then $L^2(\widetilde{M}, \widetilde{G})$ is a free Γ–module.

We take Ω as in the hypothesis and let U be a fundamental domain of $\widetilde{\Omega}$ as in §1. Since p is locally biholomorphic we see that $\imath\mathbf{c}(\widetilde{E}, \widetilde{h}) \geqslant C_0\widetilde{\omega}$ on $\widetilde{M} \smallsetminus \widetilde{K}$. Let u be a smooth $(n,1)$ form on $\widetilde{M}$ with values in $\widetilde{E}^k$ and compactly supported outside $\widetilde{K}$. We apply now the Bochner–Kodaira–Nakano formula for u:

$$3\left(\widetilde{\Delta}''_k u, u\right) \geqslant 2\left(\left[\imath\boldsymbol{c}(\widetilde{E}^k), \widetilde{\Lambda}\right] u, u\right) - \left(\|\tau\, u\|^2 + \|\bar{\tau}\, u\|^2 + \|\tau^*\, u\|^2 + \|\bar{\tau}^*\, u\|^2\right)\,,$$

where Λ is the operator of taking the interior product with $\widetilde{\omega}$ and the τ's are the torsion operators of the metric $\widetilde{\omega}$. More precisely $\tau = [\Lambda, \partial\widetilde{\omega}]$. Therefore there exists a constant $C_1 > 0$ (depending just on the metric ω) such that

$$3\left(\widetilde{\Delta}''_k u, u\right) \geqslant 2C_0\, k\, \|u\|^2 - C_1\, \|u\|^2\,,$$

and hence

$$\left(\widetilde{\Delta}''_k u, u\right) \geqslant \frac{C_0\, k}{2}\, \|u\|^2\,, \quad k \geqslant \frac{C_1}{2C_0}\,. \tag{2.1}$$

Indeed, by hypothesis the torsion operators are pointwise bounded. Moreover $([\imath\boldsymbol{c}(\widetilde{E}^k, \widetilde{h}^k), \Lambda]u, u) \geqslant k\,\alpha_1\,|u|^2$ where $\alpha_1 \leqslant \cdots \leqslant \alpha_n$ are the eigenvalues of $\imath\boldsymbol{c}(\widetilde{E}, \widetilde{h})$ with respect to $\widetilde{\omega}$.

Let $\rho \in \mathcal{C}^\infty(M)$ such that $\rho = 0$ on L and $\rho = 1$ on $M \smallsetminus \Omega$, where L is a neighbourhood of K in Ω. We put $\widetilde{\rho} = \rho \circ p$. Let $u \in \mathcal{D}^{n,1}(\widetilde{M}, \widetilde{E}^k)$, so that $\widetilde{\rho}\,u$ has support outside $\widetilde{K}$. We use now the elementary estimate:

$$\left(\widetilde{\Delta}''_k(\widetilde{\rho}\,u), \widetilde{\rho}\,u\right) \leqslant \frac{3}{2}\left(\widetilde{\Delta}''_k u, u\right) + 6\sup|d\widetilde{\rho}\,|^2\|u\|^2\,. \tag{2.2}$$

Obviously $C_2 = 6\sup|d\widetilde{\rho}\,|^2 < \infty$. Estimates (2.1) and (2.2) yield

$$\|u\|^2 \leqslant \frac{12}{C_0\, k}\left(\widetilde{\Delta}''_k u, u\right) + 4\int_{\widetilde{\Omega}} \left|(1-\widetilde{\rho}\,)u\right|^2, \quad k \geqslant \frac{\max\{C_1, 16C_2\}}{2C_0} \tag{2.3}$$

for any compactly supported u. Since the metric $\widetilde{\omega}$ is complete the density lemma of Andreotti and Vesentini [AV] shows that $\widetilde{\Delta}''_k$ is essentially self–adjoint. Thus (2.3) is true for any u in the domain of the quadratic form $\widetilde{Q}_k$ of the self–adjoint extension of $k^{-1}\widetilde{\Delta}''_k$. From relation (2.3) we infer that the spectral spaces corresponding to the lower part of the spectrum of $k^{-1}\widetilde{\Delta}''_k$ on $(n,1)$–forms can be injected into the spectral spaces of the Γ–invariant operator $k^{-1}\widetilde{\Delta}''_k\restriction_{\widetilde{\Omega}}$ which correspond to the Dirichlet problem on $\widetilde{\Omega}$ for $k^{-1}\widetilde{\Delta}''_k$. The latter operator was studied in §1. This idea appears in Witten's proof (see Henniart [He]) and in [Bou] in the context of q–convex manifolds in the sense of Andreotti–Grauert. We claim that for $\lambda < C_0/24$,

$$L^1_k(\lambda) \longrightarrow L^1_{k,\widetilde{\Omega}}(12\lambda + C_3k^{-1})\,, \quad u \longmapsto E_{12\lambda+C_3k^{-1}}(k^{-1}\widetilde{\Delta}''_k\restriction_{\widetilde{\Omega}})(1-\widetilde{\rho})u\,, \tag{2.4}$$

is an injective Γ–morphism, where $L^1_k(\lambda) = \operatorname{Range}\big(E_\lambda(k^{-1}\widetilde{\Delta}''_k\restriction_{\widetilde{\Omega}})\big)$ is the spectral space of $k^{-1}\widetilde{\Delta}''_k$ on $(n,1)$–forms, $L^1_{k,\widetilde{\Omega}}(\mu) = \operatorname{Range} E_\mu(k^{-1}\widetilde{\Delta}''_k\restriction_{\widetilde{\Omega}})$, the spectral spaces of $k^{-1}\widetilde{\Delta}''_k\restriction_{\widetilde{\Omega}}$ and $C_3 = 8\,C_2$. To prove the claim let us remark that the map (2.4) is the restriction of an operator on $L^2_{0,1}(\widetilde{M},\widetilde{E}^k\otimes K_{\widetilde{M}})$ of the same form; this is continuous and Γ–invariant being a composition of a multiplication with a bounded Γ–invariant function and a Γ–invariant projection. To prove the injectivity we choose $u\in L^1_k(\lambda)$, $\lambda < C_0/24$ to the effect that $\widetilde{Q}_k(u)\leqslant \lambda\|u\|^2\leqslant (C_0/24)\|u\|^2$. Plugging this relation in (2.3) we get

$$\|u\|^2 \leqslant 8\int_{\widetilde{\Omega}}\big|(1-\widetilde{\rho})u\big|^2\,, \quad u\in L^1_k(\lambda)\,, \quad \lambda < C_0/24\,. \tag{2.5}$$

Let us denote by $\widetilde{Q}_{k,\widetilde{\Omega}}$ the quadratic form of $k^{-1}\widetilde{\Delta}''_k\restriction_{\widetilde{\Omega}}$. Then by (2.2) and (2.5), $\widetilde{Q}_{k,\widetilde{\Omega}}\big((1-\widetilde{\rho})u\big)\leqslant \frac{3}{2}\widetilde{Q}_k(u)+\frac{C_2}{k}\|u\|^2\leqslant \big(12\,\lambda+\frac{8\,C_2}{k}\big)\int_{\widetilde{\Omega}}\big|(1-\widetilde{\rho})u\big|^2$ which shows that if $E(12\lambda + C_3k^{-1}, k^{-1}\widetilde{\Delta}''_k\restriction_{\widetilde{\Omega}})\,(1-\widetilde{\rho})u = 0$ then $(1-\widetilde{\rho})u=0$ so that $u=0$ by (2.5). Therefore (2.4) is injective and hence

$$N^1_\Gamma\left(\lambda,\frac{1}{k}\widetilde{\Delta}''_k\right)\leqslant N^1_\Gamma\left(12\lambda+\frac{C_3}{k},\widetilde{\Delta}''_k\restriction_{\widetilde{\Omega}}\right)\,, \quad \lambda<(C_0/24)\,, \tag{2.6}$$

and thus the spectral spaces $L^1_k(\lambda)$, $\lambda<C_0/24$, are of finite Γ–dimension.

Now we can apply Theorem 1.9 for $k^{-1}\widetilde{\Delta}''_k\restriction_{\widetilde{\Omega}}$ on $\widetilde{\Omega}$ (with $\widetilde{G}=K_{\widetilde{M}}$). By the variational principle we have that $N^0_\Gamma(\lambda,\frac{1}{k}\widetilde{\Delta}''_k)\geqslant N^0_\Gamma(\lambda,\frac{1}{k}\widetilde{\Delta}''_k\restriction_{\widetilde{\Omega}})$ and by Theorem 1.9 for $q=0$

$$\liminf_k k^{-n}N^0_\Gamma\left(\lambda,\frac{1}{k}\widetilde{\Delta}''_k\right)\geqslant I^0(U,\lambda)\,, \quad \lambda<C_0/24\,, \quad \lambda\in\mathbb{R}\smallsetminus\mathcal{N} \tag{2.7}$$

We find now an upper bound. Fix an arbitrary $\delta>0$. For $k>C_3/\delta$ we have $N^1(\lambda,k^{-1}\widetilde{\Delta}''_k)\leqslant N^1_\Gamma(12\lambda+C_3\,k^{-1}\widetilde{\Delta}''_k\restriction_{\widetilde{\Omega}})\leqslant N^1_\Gamma(12\lambda+\delta,\widetilde{\Delta}''_k\restriction_{\widetilde{\Omega}})$ hence by (1.11) $\limsup_k k^{-n}N^1_\Gamma(\lambda,\frac{1}{k}\widetilde{\Delta}''_k)\leqslant I^1(U,12\lambda+\delta)$. We can let $\delta\longrightarrow 0$ so that

$$\limsup_k k^{-n}N^1_\Gamma\left(\lambda,\frac{1}{k}\widetilde{\Delta}''_k\right)\leqslant I^1(U,12\lambda)\,, \quad \lambda<C_0/24\,. \tag{2.8}$$

We consider the group $H^{n,0}_{(2)}(\widetilde{M},\widetilde{E}^k) = \{u \in L^2_{n,0}(\widetilde{M},\widetilde{E}^k,\widetilde{\omega},\widetilde{h}) \,:\, \bar\partial u = 0\}$ which is a Γ–module and we find a lower bound for its Γ–dimension. We know that the L^2 norm doesn't actually depend on the metric on $\widetilde{M}$. We consider also the operator $\widetilde{\Delta}''_k$ defined on $L^2_{n,0}(\widetilde{M},\widetilde{E}^k)$ and denote by $L^0_k(\lambda)$ its spectral spaces. Since $\widetilde{\Delta}''_k$ commutes with $\bar\partial$ it follows that the spectral projections of $\widetilde{\Delta}''_k$ commute with $\bar\partial$ too, showing thus $\bar\partial L^0_k(\lambda) \subset L^1_k(\lambda)$ and therefore we have the Γ–morphism $L^0_k(\lambda) \xrightarrow{\bar\partial_\lambda} L^1_k(\lambda)$ where $\bar\partial_\lambda$ denotes the restriction of $\bar\partial$ (by the definition of $L^0_k(\lambda)$, $\bar\partial_\lambda$ is bounded by $k\lambda$). Since for any Γ–morphism A we have $\dim_\Gamma \overline{R(A)} = \dim_\Gamma \ker(A)^\perp$ we see that $\dim_\Gamma \ker\bar\partial_\lambda + \dim_\Gamma \overline{R(\bar\partial_\lambda)} = \dim_\Gamma L^0_k(\lambda)$. Moreover $\dim_\Gamma \overline{R(\bar\partial_\lambda)} \leqslant \dim_\Gamma L^1_k(\lambda)$ and they are finite. Therefore by (2.7) and (2.8), $\dim_\Gamma H^{n,0}_{(2)}(M,\widetilde{E}^k) \geqslant \dim_\Gamma \ker\bar\partial_\lambda \geqslant k^n\Big[I^0(U,2\lambda) - I^1(U,12\lambda)\Big]$ for $\lambda < C_0/24$ and $\lambda \in \mathbb{R}\smallsetminus\mathcal{N}$. We can now let λ go to zero through these values. The limits $I^0(U,0)$ and $I^1(U,0)$ are calculated in [De1] and if we identify the fundamental domain U with Ω the result is exactly the integral from the conclusion. □

To state the following result let us remind that by the definition of Andreotti and Grauert [AG] a manifold is called 1–concave if there exists a smooth function $\varphi : X \longrightarrow (a,b\,]$ where $a \in \{-\infty\}\cup\mathbb{R}$, $b \in \mathbb{R}$, such that $X_c := \{\varphi > c\} \Subset X$ for all $c \in (a,b\,]$ and φ is strictly plurisubharmonic outside a compact set. Let E be a holomorphic line bundle on X. In [Oh], [Ma] one constructs a function $\chi : (-\infty,0) \longrightarrow \mathbb{R}$ such that $\int_{-1}^0 \chi(t)^{1/2}dt = \infty$, $\chi'(t)^2 \leqslant 4\chi(t)^3$, $\chi(t) \geqslant 4$ and a hermitian metric ω which equals $\frac{1}{3}\partial\bar\partial\varphi$ near bX_c. For convenience we denote $\psi = c-\varphi$. We define $\omega_0 = \omega + \chi(\psi)\partial\varphi\wedge\bar\partial\varphi$, a complete metric on X_c and a hermitian metric $h_0 = h\exp(-A\int_{\inf\psi}^{\psi}\chi(t)dt)$ on E over X_c.

Theorem 2.2. *Let X be a 1–concave manifold of dimension $n \geqslant 3$ and let X_c be a sublevel set such that the exhaustion function φ is strictly plurisubharmonic near bX_c. Let $p : \widetilde{X}_c \longrightarrow X_c$ be a Galois covering of group Γ. Assume that $\widetilde{X}_c$ and $\widetilde{E}$ are endowed with the lifts of the metrics ω_0 and h_0. Then*

$$\dim_\Gamma H^0_{(2)}(\widetilde{X}_c,\widetilde{E}^k) \geqslant \frac{k^n}{n!}\int_{\Omega(\leqslant 1,h_0)}\Big(\frac{\imath}{2\pi}\boldsymbol{c}(E,h_0)\Big)^n + o(k^n)\,,\quad k >> 0\,. \qquad 2.9$$

for any sufficiently large open set $\Omega \Subset X_c$.

Proof. The metrics ω_0 and h_0 satisfy the following conditions:

(i) Denoting by γ_i the eigenvalues of $\imath\chi(\psi)\partial\bar\partial\psi + \imath\chi'(\psi)\partial\psi\wedge\bar\partial\psi$ with respect to ω_0 we have $\gamma_1 \leqslant \cdots \leqslant \gamma_{n-1} \leqslant -2\chi(\psi)$ and $\gamma_n \leqslant \chi(\psi)$ so that $\gamma_n + \cdots + \gamma_2 \leqslant (5-2n)\chi(\psi) \leqslant -\chi(\psi)$ for $n \geqslant 3$ outside a compact set $K := X_e \Subset X_c$.

(ii) The torsion operators of the metric ω_0 are pointwise bounded by $C_2\chi(\varphi)^{1/2}$ outside K.

(iii) The eigenvalues of $\imath\boldsymbol{c}(E,h_0)$ with respect to ω_0 are bounded above on X_c by $C_1 > 0$.

Let us take the lifts $\widetilde{\omega}_0$, h_0 and $\widetilde{\psi} = c-\varphi\circ p$. It is easy to see that properties (i), (ii) and (iii) are still valid for $\widetilde{\omega}_0$ and $\widetilde{h}_0$ and $\widetilde{\psi}$ on $\widetilde{X}_c\smallsetminus\widetilde{K}$. For $u \in \mathcal{D}^{(0,1)}(\widetilde{X}_c\smallsetminus\widetilde{K},\widetilde{E}^k)$ we apply the Bochner–Kodaira–Nakano inequality and take into account the formula $([\imath\boldsymbol{c}(\widetilde{E}^k,\widetilde{h}^k),\Lambda]u,u) \geqslant -k(\alpha_n+\ldots+\alpha_2)|u|^2$. Then

$$3\left(\widetilde{\Delta}''_k u,u\right) \geqslant \int\left(-knC_1 + kA\chi(\psi) - 4C_2\chi(\psi)\right)|u|^2\,.$$

For sufficiently A and since $\chi \geqslant 4$ we derive easily an estimate analogous to (2.1). From this point the proof of Theorem 2.1 applies whith just notational changes. □

§3 Coverings of some strongly pseudoconcave manifolds.

Let us recall the solution of the Grauert-Riemenschneider conjecture ([GR], p. 277) as given by Siu [Si] and Demailly [De1]. Namely the Siu–Demailly criterion says that if X be a compact complex manifold and E a line bundle over X. and either E is semi-positive and positive at one point (Siu's criterion), or

$$\int_{X(\leqslant 1)} \big(\imath \boldsymbol{c}(E)\big)^n > 0 \tag{D}$$

(Demailly's criterion) then $\dim H^0(X, E^k) \approx k^n$, for large k and X is Moishezon. Our aim is to extend this result in two directions. We allow X to belong to certain classes of strongly pseudoconcave manifolds and we study (directly) Galois coverings of such manifolds.

For 1–concave and compact manifolds (all which are pseudoconcave in the sense of Andreotti [An]) the transcendence degree of the meromorphic function field is less than or equal to the dimension of X. In the latter case we say that the manifold is Moishezon by extending the terminology from compact manifolds.

If, in the Andreotti–Grauert definition, the function φ can be taken such that $a = \inf \varphi = -\infty$, we say that X is *hyper* 1*–concave.* Let us note that not all 1–concave manifolds are hyper 1–concave. Indeed, the complement of $S^1 \subset \mathbb{C} \subset \mathbb{P}^1$ in $\mathbb{P}^1$ is 1–concave but cannot possibly be hyper 1–concave since S^1 is not a polar set in $\mathbb{C}$ (I have learnt this example from M. Colţoiu and V. Vâjâitu).

Let us describe some examples. Let Y be a compact complex manifold, S a complete pluripolar set (the set where a strictly psh function takes the value $-\infty$). Then $M = Y \setminus S$ is hyper 1–concave. Conversely, if $\dim M \geqslant 3$ any hyper 1–concave manifold M is biholomorphic to a complement of a pluripolar set in a compact manifold as a consequence of Rossi's compactification theorem. Another example of hyper 1–concave manifold is $\operatorname{Reg}(X)$ where M is a compact complex space with isolated singularities. Suppose that p is an isolated singular point and that the germ (X, p) is embedded in the germ $(\mathbb{C}^N, 0)$ and $z = (z_1, \dots, z_N)$ are local coordinates in the ambient space $\mathbb{C}^N$. The function φ is then obtained by cutting-off functions of the type $-\log(|z|^2)$. If M is a complete Kähler manifold of finite volume and bounded negative sectional curvature, M is hyper 1–concave. This is shown by Siu–Yau in [SY] by using Buseman functions. Moreover, if $\dim M \geqslant 3$, this example falls in the previous case since by [Nad] M can be compactified to an algebraic space by adding finitely many points.

Theorem 3.1. *Let M be a hyper 1–concave manifold carrying a line bundle (E, h) which is semi-positive outside a compact set. Let $\widetilde{M}$ be a Galois covering of group Γ and $\widetilde{E}$ the lifting of E. Then*

$$\dim_\Gamma H^{n,0}_{(2)}(\widetilde{M}, \widetilde{E}^k) \geqslant \frac{k^n}{n!} \int_{M(\leqslant 1, h)} \Big(\frac{\imath}{2\pi} \boldsymbol{c}(E, h)\Big)^n + o(k^n)\,, \quad k \longrightarrow \infty\,,$$

where the L^2 condition is with respect to $\widetilde{h}$ and any metric on $\widetilde{M}$.

Proof. Let us consider a proper function $\varphi : M \longrightarrow (-\infty, 0)$ which is strictly plurisubharmonic outside a compact set. The fact that φ goes to $-\infty$ to the ideal boundary of M allows to construct a complete hermitian metric on M which has moreover the feature of being Kähler outside a compact set. Namely we consider the function $\chi = -\log(-\varphi)$ so that $\partial\bar{\partial}\chi = \varphi^{-2}\,\partial\varphi \wedge \bar{\partial}\varphi - \varphi^{-1}\,\partial\bar{\partial}\varphi$ which is obviously positive definite on the set where $\partial\bar{\partial}\varphi$ is. We can now patch $\partial\bar{\partial}\chi$ and an arbitrary hermitian metric on M by using a smooth partition of unity to get a metric ω_0 on M such that $\omega_0 = \partial\bar{\partial}\chi \quad \text{on} \quad M \setminus K,\ K \Subset M$. It is easy to verify that ω_0 is complete since the function $-\chi$ is an exhaustion function and $\omega_0 = \omega + \partial(-\chi) \wedge \bar{\partial}(-\chi)$ where $\omega = -\varphi^{-1}\partial\bar{\partial}\varphi$ is a metric on $M \setminus K$, so that $d(-\chi)$ is bounded in the metric ω_0. Note that ω_0 is obviously Kähler on $M \setminus K$.

Let us consider a holomorphic hermitian line bundle E endowed with a metric h such that $\imath\boldsymbol{c}(E,h) \geqslant 0$ on $M \setminus K$ (we stretch K if necessary). We equip E with the metric $h_\varepsilon = h\exp(-\varepsilon\chi)$ and the curvature relative to the new metric satisfies $\imath\boldsymbol{c}(E,h_\varepsilon) \geqslant \varepsilon\,\omega_0$ on $M \smallsetminus K$. We are therefore in the conditions of Theorem 2.1. First observe that $h_\varepsilon \gtrsim h$ so that $H^{n,0}_{(2)}(\widetilde{M},\widetilde{E}^k,\widetilde{\omega}_0,\widetilde{h}_\varepsilon) \subset H^{n,0}_{(2)}(\widetilde{M},\widetilde{E}^k,\widetilde{\omega}_0,\widetilde{h})$ which is an injective Γ–morphism. By Theorem 2.1

$$\liminf_k k^{-n}\dim_\Gamma H^{n,0}_{(2)}(\widetilde{M},\widetilde{E}^k,\widetilde{\omega}_0,\widetilde{h}_\varepsilon) \geqslant \frac{1}{n!}\int_{\Omega(\leqslant 1,h_\varepsilon)}\Big(\frac{\imath}{2\pi}\boldsymbol{c}(E,h_\varepsilon)\Big)^n$$

so that

$$\liminf_k k^{-n}\dim_\Gamma H^{n,0}_{(2)}(\widetilde{M},\widetilde{E}^k) \geqslant \frac{1}{n!}\int_{\Omega(\leqslant 1,h_\varepsilon)}\Big(\frac{\imath}{2\pi}\boldsymbol{c}(E,h_\varepsilon)\Big)^n \tag{4.1}$$

We let now $\varepsilon \searrow 0$ in (4.1); since h_ε converges uniformly together with its derivatives to h on compact sets we see that we can replace h_ε with h in the right-hand side of (4.1). Let $M(q,h)$ be the set where $\imath\boldsymbol{c}(E,h)$ is non-degenerate and has exactly q negative eigenvalues. By hypothesis $M(1,h) \subset K$ and on $M(0,h) = M(\leqslant 1,h) \smallsetminus M(1,h)$ the integrand is positive. Hence we can let Ω exhaust X and we get the inequality from the statement of the theorem. □

We prove now that Siu's criterion extends tale quale for hyper 1–concave manifolds.

Corollary 3.2. *Let M be a hyper 1– concave manifold carrying a line bundle which is semi-positive outside a compact set and satisfies Demailly's condition* (D). *Then X is Moishezon. In particular the conclusion holds true if E is semi-positive and positive at one point.*

Proof. By Theorem 3.1 (for $\Gamma = \{\mathrm{Id}\}$) we have

$$\dim H^0(M,E^k\otimes K_M) \geqslant \dim H^{n,0}_{(2)}(M,E^k) \geqslant C\,k^n$$

with $C > 0$ for large k, by condition (D). We note that the first space is finite dimensional since M is 1–concave. By the Siegel–Serre Lemma (Proposition 5.7 from [Ma]), $\dim H^0(M,E^k\otimes K_M) \leqslant C\,k^{\varkappa(E)}$, $(k>0)$, where $\varkappa(E)$ is the supremum over k of the generic rank of the canonical meromorphic mapping from M to $\mathbb{P}\big(H^0(M,E^k\otimes K_M)^*\big)$. We obtain that $\varkappa(E) = n$, that is, the line bundle $E^k\otimes K_M$ gives local coordinates on an open dense set of M for sufficiently large k. This clearly implies M Moishezon and thereby concludes the proof. □

Remark 3.1.

(a) We can use this criterion in the Nadel compactification theorem [Nad]. It asserts that if M is a connected manifold of dimension $\geqslant 3$ satisfying: (i) M is hyper 1–concave, (ii) M is Moishezon, (iii) M can be covered by Zariski-open sets which are uniformized by Stein manifolds, then M is biholomorphic to $M^* \smallsetminus S$ where M^* is a compact Moishezon space and S is finite. We see thus that condition (ii) in Nadel's theorem may be replaced with the analytic condition: M possesses a line bundle which is semi-positive outside a compact set and satisfies Demailly's condition (D).
(b) In general, if M is a hyper 1–convave manifold of dimension $n \geqslant 3$ possesing a semi–positive line bundle satistying (D) then (by a theorem of Rossi) it can be compactified so that M is biholomorphic to an open set of a compact Moishezon manifold which is the complement of a complete pluripolar set. Therefore there exist a meromorphic mapping defined on X with values in a projective space which is an embedding outside a proper analytic set of X. To see this we have to apply the corresponding statement for compact Moishezon manifolds, a result due to Moishezon. The difficulty in Nadel's theorem is to show that under additional hypothesis the pluripolar set is actually a finite set.
(c) The argument in the proof of Corollary 3.2 shows that the integral appearing in Theorem 3.1 is finite. Thus, if E is positive outside a compact set K then $M \smallsetminus K$ has finite volume with respect to the metric $\imath\boldsymbol{c}(E)$ (this observation stems from [NT]).
(d) If M possesses a positive line bundle E then $\imath\boldsymbol{c}(E) + \imath\partial\bar{\partial}\chi$ is a complete Kähler metric and Hörmander's L^2 estimates and Andreotti–Tomassini's theorem [AT] show that E is ample and M can be embedded in the projective space. So even in dimension 2 we can compactify M (by [An]).
(f) Let X is a compact complex space of dimension $n \geqslant 2$ and with isolated singularities. Suppose that we have a line bundle E on $\operatorname{Reg}(X)$ which is semi-positive in a deleted neighbourhood of $\operatorname{Sing}(X)$ and satisfies (D). Then X is Moishezon. Indeed, by the previous result we find $n = \dim X$ independent meromorphic functions on $\operatorname{Reg}(X)$ which extend to X by the Levi extension theorem. This is a generalization of Takayama's criterion [Ta] in the case of isolated singularities. We allow weaker hypothesis, that is E is defined just on $\operatorname{Reg}(X)$ and the curvature condition is just semi-positivity. The reason is the good exhaustion function we have at hand. In the general case one has to use the Poincaré metric and the strict positivity near $\operatorname{Sing}(X)$ is essential. Note however that the method of Takayama gives that the line bundle who forms local coordinates is E^k, while in our proof is $E^k \otimes K_X$.

We want now to study the following type of stongly pseudoconcave manifold. Let X be an irreducible compact complex space with isolated singularities and of dimension $\geqslant 2$. We know that $\operatorname{Reg}(X)$ is hyper 1–concave and we denote by $\varphi : \operatorname{Reg}(X) \longrightarrow \mathbb{R}$ the exhaustion function. Since φ is strictly plurisubharmonic outside a compact set we have that the sub–level sets $X_c = \{\varphi > c\}$ are 1–concave manifolds i.e. stongly pseudoconcave domains. In our previous paper [Ma] we have shown that in general if M is a 1–concave manifold of dimension $\geqslant 3$ which carries a hermitian line bundle E which semi-negative near the boundary and satisfies (D) then the Kodaira dimension of E is maximal and M is Moishezon. The assumption about the change of curvature sign (i.e. semi-negativity) near the boundary is imposed by the construction of complete hermitian metrics ω_0 and h_0 as in Theorem 2.2 which give the L^2 estimate and preserve condition (D) for h_0; the negativity of the Levi form of the sublevel sets of M requires as a natural curvature condition for E the semi–negativity. The restriction on dimension comes from the fact that we need an L^2 estimate in bi–degree $(0,1)$. Of course, usually we are given an overall positive bundle E on M. We show that for manifolds X_c as before we can also apply the criterion in [Ma] alluded to by modifying the metric.

We recall at the outset some terminology. Let us consider a covering $\{U_\alpha\}$ of X and embeddings $\iota_\alpha : U_\alpha \hookrightarrow \mathbb{C}^{N_\alpha}$ such that $E|_{U_\alpha}$ is the inverse image by ι_α of the trivial line bundle $\mathbb{C}_\alpha$ on $\mathbb{C}^{N_\alpha}$. Moreover we consider hermitian metrics $h_\alpha = e^{-\varphi_\alpha}$ on $\mathbb{C}_\alpha$ such that $\iota_\alpha^* h_\alpha = \iota_\beta^* h_\beta$ on $U_\alpha \cap U_\beta \cap \operatorname{Reg}(X)$. The system $h = \{\iota_\alpha^* h_\alpha\}$ is called a hermitian metric on E over X. It clearly induces a hermitian metric on E over $\operatorname{Reg}(X)$. The curvature current $\imath\boldsymbol{c}(E)$ is given in U_α by $\iota_\alpha^*(\imath\partial\bar\partial\varphi_\alpha)$ which on $\operatorname{Reg}(X)$ agrees with the curvature of the induced metric.

Theorem 3.3. *Let X be an irreducible compact complex space with isolated singularities and let X_c be the sublevel sets of the hyper 1–concave manifold $\operatorname{Reg}(X)$. Assume that there exists a holomorphic line bundle $E \longrightarrow X$ with a smooth hermitian metric such that condition* (D) *is fulfilled on* $\operatorname{Reg}(X)$*. Then for sufficiently small c there exists a metric on E such that E is negative in the neighbourhood of bX_c and $\int_{X_c(\leqslant 1)} \big(\imath\boldsymbol{c}(E)\big)^n > 0$.*

Proof. Let $\pi : \widetilde{X} \longrightarrow X$ be a resolution of singularities of X. Let us denote by D_i the components of the exceptional divisor. Then there exist positive integers n_i such that $D := \sum n_i\, D_i$ admits a smooth hermitian metric such that the induced line bundle $[D]$ is negative in a neighbourhood $\widetilde{U}$ of D (cf. [Sa]). Let us consider a canonical section s of $[D]$, i.e. $D = (s)$, and denote by $|s|^2$ the pointwise norm of s with respect to the above metric. By Lelong-Poincaré equation $\varphi = \log|s|^2$ is strictly plurisubharmonic on $\widetilde{U} \setminus D$. By using a smooth function on $\widetilde{X}$ with compact support in $\widetilde{U}$ which equals one near D we construct a smooth function χ on $\widetilde{X} \smallsetminus D \simeq \operatorname{Reg}(X)$ such that $\chi = -\log(-\log|s|^2)$ on $\widetilde{U} \setminus D$.
Since $\log|s|^2$ goes to $-\infty$ on D, this is the analogue of the function constructed in the proof of Theorem 3.1 . As there we show that $\imath\partial\chi\wedge\bar\partial\chi \leqslant \imath\partial\bar\partial\chi$. Let us consider a metric ω on $\operatorname{Reg}(X)$ which on every open set U_α as above is the pullback of a hermitian metric on the ambient space $\mathbb{C}^{N_\alpha}$, $\omega = \iota_\alpha^*\,\omega_\alpha$. We consider then the metric (Kähler near $\operatorname{Sing}(X)$) $\omega_0 = A\omega + \partial\bar\partial\chi$ where $A > 0$ is chosen sufficiently large (to ensure that ω_0 is a metric away from the open set where $\partial\bar\partial\chi$ is positive definite). It is easily seen that ω_0 is complete by the same argument as in the proof of Theorem 3.1 . This kind of metrics were introduced by Saper in [Sa]. They have finite volume.

Let us consider now a neighbourhood U of the singular set. We assume that U is small enough so that there are well defined on U a potential ρ for ω and a potential η for the curvature $\imath\boldsymbol{c}(E)$ (they are restrictions from ambient spaces). By suitably cutting-off we may define a function $\psi \in \mathcal{C}^\infty(\operatorname{Reg}(X))$ such that $\psi = -\chi - \eta - A\,\rho$ near $\operatorname{Sing}(X)$. Remark that since $\imath\boldsymbol{c}(E)$ is bounded above by a continuous $(1,1)$ form near $\operatorname{Sing}(X)$ the potential $-\eta$ is bounded above near the singular set. This holds true for ρ too (it is smooth) so that ψ tends to ∞ at the singular set $\operatorname{Sing}(X)$.
Let us consider a smooth function $\gamma : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\gamma(t) = \begin{cases} 0 & \text{if} \quad t \leqslant 0\,, \\ t & \text{if} \quad t \geqslant 1\,. \end{cases}$$

and the functions $\gamma_\nu : \mathbb{R} \longrightarrow \mathbb{R}$ given by $\gamma_\nu(t) = \gamma(t-\nu)$ for all positive integers ν. Let us denote the hermitian metric on E by h and let us consider the following metric on E: $h_\nu = h \exp\big(-\gamma_\nu(\psi)\big)$, with curvature

$$\imath\boldsymbol{c}(E,h_\nu) = \imath\boldsymbol{c}(E,h) + \gamma_\nu'(\psi)\partial\bar\partial\psi + \gamma_\nu''(\psi)\partial\psi\wedge\bar\partial\psi\,.$$

On the set $\{\psi \geqslant \nu+1\}$ we have $\gamma_\nu(\psi) = \psi - \nu$ so that $\gamma_\nu'(\psi) = 1$ and $\gamma_\nu''(\psi) = 0$ and therefore $\imath\boldsymbol{c}(E,h_\nu) = \imath\boldsymbol{c}(E,h) + \partial\bar\partial\psi$. Since ψ goes to ∞ when we approach the singular set we may choose ν_0 such that for $\nu \geqslant \nu_0$ we have $\{\psi \geqslant \nu+1\} \subset U$ where U is a sufficiently small neighbourhood of $\operatorname{Sing}(X)$. Bearing in mind the meaning of η and ρ together with the definition of ω_0 it is straightforward that $\imath\boldsymbol{c}(E,h_\nu) = -\omega_0$ on $\{\psi \leqslant \nu+1\}$, that is (E,h_ν) is negative on this set. We denote Ω_ν the compact set $\{\psi \leqslant \nu+2\}$. We decompose this set in $\Omega_\nu' = \{\psi \leqslant \nu\}$ and $\Omega_\nu'' = \{\nu \leqslant \psi \leqslant \nu+2\}$. On Ω_ν' we have $\gamma_\nu(\psi) = 0$ and $\imath\boldsymbol{c}(E,h_\nu) = \imath\boldsymbol{c}(E,h)$. We infer that

$$\begin{aligned}\int_{\Omega_\nu'(\leqslant 1,h_\nu)} \big(\imath\boldsymbol{c}(E,h_\nu)\big)^n &= \int_{\Omega_\nu'(\leqslant 1,h)} \big(\imath\boldsymbol{c}(E,h)\big)^n \\ &= \int_{\operatorname{Reg}(X)(\leqslant 1,h)} \mathbf{1}_{\Omega_\nu'}\alpha_1\cdots\alpha_n\,dV_0 \end{aligned} \tag{4.2}$$

where $\alpha_1,\dots,\alpha_n$ are the eigenvalues of $\imath\boldsymbol{c}(E,h)$ with respect to ω_0 and dV_0 is the volume form of the same metric. Since $\imath\boldsymbol{c}(E,h)$ is dominated by the euclidian metric near $\operatorname{Sing}(X)$, $\imath\boldsymbol{c}(E,h)$ is dominated by ω and by ω_0. Hence the product $\alpha_1\cdots\alpha_n$ is bounded on $\operatorname{Reg}(X)$. Since $\operatorname{Reg}(X)(\leqslant 1)$ has finite volume with respect to ω_0 the functions $|\mathbf{1}_{\Omega_\nu'}\alpha_1\cdots\alpha_n|$ are bounded by an integrable function. On the other hand $\mathbf{1}_{\Omega_\nu'} \longrightarrow 1$ when $\nu \longrightarrow \infty$ so that the integrals in (4.2) tend to $\int_{\operatorname{Reg}(X)(\leqslant 1,h)}\big(\imath\boldsymbol{c}(E,h)\big)^n$ which is assumed to be positive. Thus it suffices to show that the integral on the set Ω_ν'' i.e. $\int_{\Omega_\nu''(\leqslant 1,h_\nu)}\big(\imath\boldsymbol{c}(E,h_\nu)\big)^n$ tends to zero as $\nu \longrightarrow \infty$. For this purpose we use the obvious bound

$$\int_{\Omega_\nu''(\leqslant 1,h_\nu)} \Big(\frac{\imath}{2\pi}\boldsymbol{c}(E,h_\nu)\Big)^n \leqslant \sup|\,\delta_1\cdots\,\delta_n|\cdot\operatorname{vol}(\Omega_\nu'')$$

where $\delta_1,\dots,\delta_n$ are the eigenvalues of $\imath\boldsymbol{c}(E,h_\nu)$ with respect to ω_0 and the volume is taken in the same metric. We use now the minimum-maximum principle to see that: (i) δ_1 is bounded below and $\delta_2,\dots,\delta_n$ are bounded above on the set of integration $\Omega_\nu''(1,h_\nu)$ and (ii) $\delta_1,\dots,\delta_n$ are upper bounded on $\Omega_\nu''(0,h_\nu)$. For this we need the domination of $\imath\boldsymbol{c}(E,h)$ by ω and the boundedness of γ_ν' and γ_ν''. Since $\operatorname{vol}(\Omega_\nu'') \longrightarrow 0$ as $\nu \longrightarrow \infty$ our contention follows. Hence for large ν the metric h_ν does the required job. □

Remark 3.2. We have seen that Siu's criterion generalizes to compact complex spaces with isolated singularities. Demailly's criterion extends too. Let X be an irreducible compact complex space of dimension $n \geqslant 3$ with isolated singularities and E a smooth hermitian line bundle over X. Assume that condition (D) is fulfilled on $\operatorname{Reg}(X)$. Then X is Moishezon. Indeed, for small c the sets X_c are Moishezon by Corollary 4.3 of [Ma] and the meromorphic functions from X_c extend to X. In fact the result holds also for $n = 2$ with a proof very similar to that of Theorem 4.4. We note also that we can allow the metric h of E to be singular at $\operatorname{Sing}(X)$ but the cuvature current $\imath\boldsymbol{c}(E)$ should be dominated (above ans below) by the euclidian metric near $\operatorname{Sing}(X)$. The proof of Theorem 4.4 goes through with minor changes.

Since the manifold $\overline{X}_c$ is compact Theorem 4.4 can be used to prove some stability results for certain perturbation of the complex structure of $\overline{X}_c$. Since our approach relies on the use of a sufficiently positive line bundle E we need to consider perturbations of the complex structure which lift to a perturbation of E. This kind of sitiuation was studied by L. Lempert in [Le].

Proposition 3.4. *Let X be a Moishezon variety with isolated singularities and dimension $n \geqslant 3$. Let $\mathcal{J}$ denote the complex structure of* $\operatorname{Reg}(X)$ *and let* $Z \subset \operatorname{Reg}(X)$ *be a non–singular hypersurface such that the line bundle* $E = [Z]$ *satisfies* (D)*. Then for sufficiently small c and any complex structure $\mathcal{J}'$ on $\overline{X}_c$ such that*

(1) *$T(Z)$ is $\mathcal{J}'$ invariant and*
(2) *$\mathcal{J}'$ is sufficiently close to $\mathcal{J}$ in the $\mathcal{C}^\infty$ topology*

there exists a $\mathcal{J}'$–holomorphic line bundle E' on $\overline{X}_c$ which is negative near bX_c and satisfies (D)*. In particular $(\overline{X}_c, \mathcal{J}')$ is a Moishezon pseudoconcave manifold and any compactification of $(\overline{X}_c, \mathcal{J}')$ is Moishezon.*

Proof. Let us first choose c_0 such that for $c < c_0$ there exists a 'good' hermitian metric h on E over a neighbourhood of X_c, that is, whith negative curvature near the boundary and satisfying (D). We use now the description of the lifting of $\mathcal{J}'$ with properties (1) and (2) as given in [Le]. Namely, Z determines a new $\mathcal{J}'$ holomorphic line bundle $E' \longrightarrow (\overline{X}_c, \mathcal{J}')$. There exists a finite open covering $\mathcal{U} = \{U\}$ of $\overline{X}_c$ such that E and E' are trivial on each U and they are defined by multiplicative cocycles $\{g_{UV}\, \mathcal{J} \text{ holomorphic on } \overline{U} \cap \overline{V} \ :\ U, V \in \mathcal{U}\}$ and $\{g'_{UV}\, \mathcal{J}' \text{ holomorphic on } \overline{U} \cap \overline{V} \ :\ U, V \in \mathcal{U}\}$. Moreover g_{UV} and g'_{UV} are as close as we please assuming $\mathcal{J}$ and $\mathcal{J}'$ are sufficiently close. (By 'close' we always understand close in the $\mathcal{C}^\infty$ topology.) Next we can define a smooth bundle isomorphism $E \longrightarrow E'$ by resolving the smooth additive cocycle $\log(g'_{UV}/g_{UV})$ in order to find smooth functions f_U, close to 1 on a neighbourhood of $\overline{U}$ such that $g'_{UV} = f_U\, g_{UV}\, f_V^{-1}$. Then the isomorphism between E and E' is given by $f = \{f_U\}$. The metric h is given in terms of the covering $\mathcal{U}$ by a collection $h = \{h_U\}$ of smooth strictly positive functions satisfying the relation $h_V = h_U\, |g_{UV}|$. We define a hermitian metric $h' = \{h'_U\}$ on E' by $h'_U = h_U\, |f_U^{-1}|$; h'_U is close to h_U. The curvatures forms of E and E' are given by

$$\frac{\imath}{2\pi}\mathbf{c}(E) = \frac{1}{4\pi}\, d \circ \mathcal{J} \circ d\,(\log h_U)\,, \quad \frac{\imath}{2\pi}\mathbf{c}(E') = \frac{1}{4\pi}\, d \circ \mathcal{J}' \circ d\,(\log h'_U)\,.$$

Therefore, when $\mathcal{J}'$ is sufficiently close to $\mathcal{J}$, $\frac{\imath}{2\pi}\mathbf{c}(E')$ is negative near the boundary of $\overline{X}_c$ and, since the eigenvalues of $\frac{\imath}{2\pi}\mathbf{c}(E')$ are close to those of $\frac{\imath}{2\pi}\mathbf{c}(E)$, E' satisfy the condition (D) i.e. $\int_{X_c(\leqslant 1)} \big(\imath\mathbf{c}(E')\big)^n > 0$. We can apply thus the Corrolary 4.3 of [Ma] to the strongly pseudoconcave manifold $(\overline{X}_c, \mathcal{J}')$ to conclude that $(\overline{X}_c, \mathcal{J}')$ is Moishezon. □

Remark 3.3. If $[Z]$ is positive, part of the stability property follows from the rigidity of embeddings with positive normal bundle. Indeed, assume $N_Z = [Z]\restriction_Z$ is positive in $(\overline{X}_c, \mathcal{J}')$ (for any c such that this manifold is still pseudoconcave). Then Ph. Griffiths [Gri1] has shown that there exists a neighbourhood W of Z such that the mapping $\Phi : (X_c, \mathcal{J}') - - \to \mathbb{P}^N$ given by $[mZ]$ is an embedding of W for large m . Thus $(X_c, \mathcal{J}')$ is Moishezon. Our result deals with the slightly more general situation of a 'big' embedding i.e. when $[Z]$ is not ample but satisfies condition (D). Moreover we have a useful quantitative way of measuring whether the perturbed structure is Moishezon.

Corollary 3.5. *Let $(\overline{X}_c, \mathcal{J}')$ and E' be as in Proposition 4.6. Then there exists hermitian metrics on X_c and E' and a positive constant C such that for any Galois covering $\widetilde{X}_c \longrightarrow X_c$ of group Γ we have*

$$\dim_\Gamma H^0_{(2)}(\widetilde{X}_c\,, \widetilde{E'}^k) \geqslant C\, k^n + o(k^n)\,, \quad k \longrightarrow \infty\,.$$

the L^2 condition being with respect to lifts of the hermitian metrics on X_c and E'.

Proof. We know that we have on E' a metric h satisfying the conclusion of Theorem 4.4. Then, as in Theorem 2.2, we can construct metrics ω_0 and h_0 in order to obtain (2.9). Note that the integral in (2.9) depends on the modified metric h_0 so we cannot always infer that it is positive even if (E', h) satisfies (D). But under the assumption of semi–negativity of h near the boundary we can construct an h_0 such that the integral in (2.9) is positive (cf. Corollary 4.3 of [Ma]). Thus by applying Theorem 2.2 we get the conclusion. □

§4 L^2 generalization of a theorem of Takayama.

In this section we study the L^2 cohomology of coverings of Zariski open sets in compact complex spaces. For compact spaces with singularities Takayama [Ta] generalized Siu–Demailly criterion if $E \longrightarrow X$ is a line bundle endowed with a singular hermitian metric which is smooth outside a proper analytic set $Z \supset \operatorname{Sing}(X)$ and defines a strictly positive current near Z.

Using the setting of Takayama's theorem we shall study coverings of Zariski open sets in compact complex spaces.

Proposition 4.1. *Let X be an n–dimensional compact manifold and let E be a holomorphic line bundle with a singular hermitian metric h. We assume that:*

(1) *$\imath\boldsymbol{c}(E,h)$ is smooth on $M = X \smallsetminus Z$ where Z is a divizor with only simple normal crossings;*
(2) *$\imath\boldsymbol{c}(E,h)$ is a strictly positive current in a neighbourhood of Z.*

Let $p : \widetilde{M} \longrightarrow M$ be a Galois covering with group Γ and $\widetilde{E} = p^*E$. Then,

$$\dim_\Gamma H^0_{(2)}(\widetilde{M}, \widetilde{E}^k) \geqslant \frac{k^n}{n!}\int_{M(\leqslant 1,h)} \Big(\frac{\imath}{2\pi}\boldsymbol{c}(E,h)\Big)^n + o(k^n)\,, \quad k >> 0\,,$$

where $H^0_{(2)}(\widetilde{M}, \widetilde{E}^k)$ is the space of sections of $\widetilde{E}^k$ which are L^2 with respect to the pullbacks of the restrictions to M and $E\restriction_M$ of smooth metrics on X and E.

Proof. This is an equivariant form of Takayama's main technical result in [Ta]. Namely we construct the Poincaré metric ω_ε on M (for details see [Zu]) and h_ε as in [Ta] and remark that the hypothesis of Theorem 2.1 are satisfied. Moreover we can work with $(0,1)$–forms since the Ricci curvature of the Poincaré metric is bounded below.

More specifically, we write $Z = \sum Z_j$ and consider a section σ_j of the line bundle $[Z_j]$ which vanishes to first order on Z_j. Then we endow $[Z_j]$ with a hermitian metric such that the norm of σ_j satisfies $|\sigma_j| < 1$. Take then an arbitrary smooth metric ω' on X and define $\omega_\varepsilon = \omega' - \varepsilon\, \imath \sum \partial\overline{\partial}(-\log|\sigma_j|^2)^2$ on $M = X \smallsetminus Z$ which for small $\varepsilon > 0$ is a complete metric on M. Then we consider the following family of metrics on $E\restriction_M$: $h_\varepsilon = h\prod_j(-\log|\sigma_j|^2)^\varepsilon$, $\varepsilon > 0$. We check now the hypotheses of Theorem 2.1 is satisfied. First we remark that the torsion operators of the Poincaré metric are pointwise bounded with respect to the Poincaré metric since $d\omega_\varepsilon = d\omega'$ and $\omega_\varepsilon > \omega'$. Also the Ricci curvature $\boldsymbol{c}(K^*_{\widetilde{M}})$ of $\widetilde{\omega}_\varepsilon$ is bounded below with respect to $\widetilde{\omega}_\varepsilon$ by a constant independent of ε (since this is true for ω_ε). Since E is strictly positive in the neighbourhood of Z condition (A) is satisfied for a compact K outside which E is positive (and it doesn't depend on ε).

Let h' be a smooth hermitian metric on E over X. Near Z the metric h is locally represented by a strictly plurisubharmonic weight. Thus h is locally bounded below near Z and thus $h \geqslant C\,h'$ on X for some positive constant C. We remark now that $h_\varepsilon > h \geqslant C\,h'$ and $\omega_\varepsilon > \omega'$ near Z so that we have the inclusion $H^0_{(2)}(\widetilde{M}, \widetilde{E}^k)_\varepsilon \subset H^0_{(2)}(\widetilde{M}, \widetilde{E}^k)$, (which is an injective Γ–morphism) in the last group the L^2 condition being taken with respect to $\widetilde{h}'$ and $\widetilde{\omega}'$. By Theorem 2.1 for $K \Subset \Omega \Subset M$

$$\dim_\Gamma H^0_{(2)}(\widetilde{M}, \widetilde{E}^k) \geqslant \dim_\Gamma H^0_{(2)}(\widetilde{M}, \widetilde{E}^k)_\varepsilon \geqslant \int_{\Omega(\leqslant 1, h_\varepsilon)} \Big(\frac{\imath}{2\pi}\boldsymbol{c}(E, h_\varepsilon)\Big)^n + o(k^n)\,.$$

We can let $\varepsilon \longrightarrow 0$ in the right–hand side in order to replace h_ε with h. Then we can let Ω exhaust X to get the inequality from the statement. □

Theorem 4.2. *Let X be an irreducible reduced compact Moishezon space and let $M \subset \operatorname{Reg}(X)$ be a Zariski open set. There exists a holomorphic line bundle $E \longrightarrow \operatorname{Reg}(X)$ endowed with a singular hermitian metric whose curvature current $\imath\boldsymbol{c}(E)$ is positive and such that for any Galois covering $\widetilde{M} \xrightarrow{p} M$ of group Γ we have*

$$\dim_\Gamma H^0_{(2)}(\widetilde{M}, \widetilde{E}^k) \geqslant \frac{k^n}{n!}\int_M \Big(\frac{\imath}{2\pi}\boldsymbol{c}(E)\Big)^n + o(k^n)\,, \quad k \longrightarrow \infty$$

where the integration takes place outside $\operatorname{Sing\,supp}\boldsymbol{c}(E)$. *The L^2 condition is taken with respect to liftings of smooth hermitian metrics on M and E induced from a resolution of singularities of X.*

Proof. Step 1. Let X be a Moishezon manifold and $M = X \setminus Z$ a Zariski open set, where Z is a proper analytic set. Thanks to Moishezon X admits a projective modification. Therefore there exists a strictly positive integral Kähler current T on X. Equivalently there exists a holomorphic line bundle E on X possesing a singular hermitian metric such that the curvature current $T = \imath\boldsymbol{c}(E)$ is strictly positive (bounded below by a smooth hermitian metric). Assume that $\operatorname{Sing\,supp} T \subset Z$. Then M is biholomorphic to a Zariski open set as in the statement of Proposition 3.1. Indeed, we can blow up Z to make it a divisor with only simple normal crossings. By replacing E with higher tensor powers and twisting it with the dual of the exceptional divisor at each step of the blowing up process we can ensure that on the blow–up we still have a positive line bundle with singular metric along Z. Thus in this case we can apply Proposition 3.1.

Step 2. To go further let M be a Zariski open set in a Moishezon manifold X. By a theorem of Demailly [De2] we know that there exists a strictly positive integral Kähler current T with analytic singularities. As a consequence $\operatorname{Sing\,supp} T \subset S$, where S is a proper analytic set. As before we can suppose that $S \cup Z$ is a divisor with only simple normal crossings. Let E be a line bundle with singular hermitian metric such that $T = \imath\boldsymbol{c}(E)$. Denote by $M_1 = X \setminus (S \cup Z) = M \setminus S$: M_1 and E are as in Proposition 3.1. Let $p : \widetilde{M} \longrightarrow M$ be a Galois covering of group Γ.

Setting $\widetilde{M_1} = p^{-1}M_1$ we have a Galois covering $\widetilde{M_1} \longrightarrow M_1$ of group Γ. Hence, $\dim_\Gamma H^0_{(2)}(\widetilde{M_1}, \widetilde{E}^k) \geqslant \frac{k^n}{n!}\int_{M_1}\left(\frac{\imath}{2\pi}\boldsymbol{c}(E)\right)^n + o(k^n)$, for $k \longrightarrow \infty$. The L^2 condition on $\widetilde{M_1}$ is with respect to liftings of smooth hermitian metrics on X and E. But a holomorphic section defined outside the analytic set $\widetilde{S} = p^{-1}S$ which is square integrable with respect to a smooth metric on $\widetilde{M}$ extends past $\widetilde{S}$ as a holomorphic section on $\widetilde{M}$. We infer $\dim_\Gamma H^0_{(2)}(\widetilde{M}, \widetilde{E}^k) \geqslant \frac{k^n}{n!}\int_M\left(\frac{\imath}{2\pi}\boldsymbol{c}(E)\right)^n + o(k^n)$ the integral being taken on the smooth locus of $\imath\boldsymbol{c}(E)$ The L^2 condition is taken with respect to pullbacks of smooth metrics on X and E.

Step 3. Finally let X and M as in hypothesis. By a resolution of singularities M is biholomorphic to a Zariski open set of a Moishezon manifold. By the preceding remarks we can conclude. □

The following Proposition is a consequence of Theorem 4.2 in the case of Galois coverings (taking into acount that the number of sheets of such a covering equals the cardinal of Γ). However, using Theorem 2.2 of Napier and Ramachandran [NR], we can prove it for any unramified covering.

Proposition 4.3. *Let X be an irreducible reduced compact Moishezon space and let $M \subset \mathrm{Reg}(X)$ be a Zariski open set. There exists a holomorphic line bundle $E \longrightarrow \mathrm{Reg}\,(X)$ such that for any unramified covering $p : \widetilde{M} \longrightarrow M$ we have*

$$\dim H^0_{(2)}(\widetilde{M}, \widetilde{E}^k) \geqslant C\,k^n\,d\,, \quad k >> 0 \tag{4.1}$$

where d is the number of sheets of the covering and $C > 0$.

Proof. In the situation of Step 1 of the preceding proof we see that the Poincaré metric on M is a complete Kähler metric since M has the Kähler metric $\imath\mathbf{c}(E)$. Therefore $\widetilde{M}$ possesses a complete Kähler metric and a positive line bundle $\widetilde{E}$. By applying the L^2 estimates of Hörmander as in [NR, Theorem 2.2] we get the result for the L^2 cohomology with respect to the metrics $\widetilde{\omega}_\varepsilon$ and $\widetilde{h}_\varepsilon$ (notations of Proposition 4.1). As in the proof of Proposition 4.1 we see that we can actually use pull-backs of smooth metrics on X. Steps 2 and 3 go through as before. □

§5 Further remarks.

We will apply Theorem 2.1 to the case of a complete Kähler manifold M with positive canonical bundle K_M. The case $\Gamma = \{\mathrm{Id}\}$ is due to Nadel and Tsuji [NT]. If D is a bounded domain of holomorphy in $\mathbb{C}^n$ we know by a theorem of Bremermann that the Bergman metric $\omega = \omega_B$ is complete. On the other hand the Bergman metric is invariant under analytic automorphisms. Thus this metric descends to a complete Kähler metric on any quotient of the domain by a properly discontinuous discrete group $\Gamma \subset \mathrm{Aut}(D)$. We denote $M = D/\Gamma$ and ω_* the induced Bergman metric on $M = D/\Gamma$. If we denote by $B(z, \overline{z})$ the Bergman kernel of D we know that B^{-1} can be considered as a hermitian Γ–invariant metric on K_D. Since $\omega = \partial\bar\partial \log B(z, \overline{z})$ there exists a hermitian metric on K_M such that $\boldsymbol{c}(K_M) = \omega_*$. We have thus the following.

Proposition 5.1. *Let D is a bounded domain of holomorphy in $\mathbb{C}^n$, $\Gamma \subset \mathrm{Aut}\,D$ a discrete group acting properly discontinuously on D and $M = D/\Gamma$. Then*

$$\dim_\Gamma H^0_2(D, K^k_D) \geqslant \left(\frac{k}{2\pi}\right)^n \int_M \frac{\omega_*^n}{n!} + o(k^n)\,, \quad k \longrightarrow \infty$$

where the L^2 condition is taken with respect to the Bergman metric on D and the metric B^{-1} on K_D.

Note that the space $H^0_2(D, K^k_D)$ is a space of square integrable functions with respect to the Bergman metric and to the weight B^{-k}. An immediate consequence is the following.

Corollary 5.2. *Assume that the Bergman metric on M has infinite volume. Then* $\dim_\Gamma H^0_2(D, K^k_D) = \infty$ *for k large enough.*

We remark that the last conclusion is stronger than the results coming from the L^2 method which gives just $\dim H^0_2(D, K^k_D) \geqslant C\,|\Gamma|\,k^n$ for some positive constant $C \in \mathbb{R}$.

Let us see what become our results in the simplest case of the unit disk $D \subset \mathbb{C}$. Then the Bergman metric equals the hyperbolic metric $(1-|z|^2)^{-2} dz \wedge d\overline{z}$. If Γ is a Fuchsian group, we have the following possibilities for large k:

(a) If $M = D/\Gamma$ is compact, $\dim_\Gamma H^0_2(D, K^k_D) = k \operatorname{vol}(M) + o(k)$.

(b) If M is non–compact and has a finite number of cusps, the hyperbolic volume $\operatorname{vol}(M)$ is finite and $\dim_\Gamma H^0_2(D, K^k_D) \geqslant k \operatorname{vol}(M) + o(k)$,

(c) If M is non–compact and the discontinuity set $\Omega \subset S^1$ is a union of intervals, $\dim_\Gamma H^0_2(D, K^k_D) = \infty$ (since $\operatorname{vol}(M) = \infty$).

According to a conjecture of Griffiths [Gri2, p.50], if D is a bounded domain in $\mathbb{C}^n$ which is topologically a cell and D/Γ is quasi–projective then (i) the Bergman metric on D/Γ is complete and (ii) the volume of D/Γ with respect to this metric is finite. In the sequel we discuss the conjecture without the topological restriction. If D is a domain of holomorphy and $M = D/\Gamma$ is pseudoconcave (e.g. $\operatorname{codim}(\overline{M} \smallsetminus M) \geqslant 2$), the answer is yes. Indeed, this follows from the Riemann–Roch inequalities for $\Gamma = \{\mathrm{Id}\}$ in Proposition 5.1. If D is not necessarily a domain of holomorphy but D/Γ can be compactified by adding a finite number of points we can show that the answer to (ii) is affirmative. We do not assume D/Γ quasi–projective.

Proposition 5.3. *Let $D \Subset \mathbb{C}^n$ be an open set having a properly discontinous group $\Gamma \subset \operatorname{Aut} D$ such that there exists a compact complex space Y with $D/\Gamma \subset \operatorname{Reg} Y$ and $D/\Gamma = Y \smallsetminus S$, where S is a finite set. Then the volume of D/Γ in the induced Bergman metric is finite.*

Proof. Since $M = D/\Gamma$ is hyper 1–concave and possesses a positive canonical bundle, we may apply Theorem 3.1 for Γ trivial and $E = K_M$. As Remark 3.1 (c) shows this gives an upper bound for $\operatorname{vol}(M) = \int_M \omega^n_*/n!\,$. $\square$

Remark 5.1. We can prove a complete generalization of the asymptotic Morse inequalities of Demailly [De1] for the L^2 cohomology of the covering of a compact manifold X. For this purpose we elabotate the proof of Theorem 2.1. As there we exploit the idea of Witten–Demailly of constructing a family of subcomplexes of the L^2–Dolbeault complex having the same cohomology. First let us introduce cohomology. Let us denote by $N^q(\bar\partial)$ the kernel and by $R^q(\bar\partial)$ the range of $\bar\partial$, by $N^q(\bar\partial^*)$ the kernel of $\bar\partial^*$ and by $N^q(k^{-1}\widetilde{\Delta}''_k)$ the kernel of $k^{-1}\widetilde{\Delta}''_k$, all acting on $L^2_{0,q}(\widetilde{X}, \widetilde{E}^k \otimes \widetilde{F})$ where $\widetilde{F}$ is a Γ–invariant holomorphic vector bundle of rank r. We have $\mathcal{H}^{0,q}_{(2)}(\widetilde{X}, \widetilde{E}^k \otimes \widetilde{F}) := N^q(k^{-1}\widetilde{\Delta}''_k) = N^q(\bar\partial) \cap N^q(\bar\partial^*)$, where the first equality is the definition of the space of harmonic forms and the second is a consequence of the completeness of the metric. If $q=0$ then $\mathcal{H}^{0,0}_{(2)}(\widetilde{X},\widetilde{E}^k\otimes\widetilde{F})$ coincides to the space of holomorphic L^2 sections of $\widetilde{E}^k\otimes\widetilde{F}$. We note also the orthogonal decomposition $N^q(\bar\partial)=N^q(k^{-1}\widetilde{\Delta}''_k)\oplus\overline{R^{q-1}(\bar\partial)}$ so that

$$\mathcal{H}^{0,q}_{(2)}(\widetilde{X},\widetilde{E}^k\otimes\widetilde{F})=N^q(\bar\partial)/\overline{R^{q-1}(\bar\partial)}=:H^{0,q}_{(2)}(\widetilde{X},\widetilde{E}^k\otimes\widetilde{F})$$

the last group being the (reduced) L^2 cohomology.

We apply the results of §1 in the following form. Since X is compact we can take the set $\Omega\Subset X$ to be X so that $\widetilde{\Omega}=\widetilde{X}$. We do not use any special metric but take an arbitrary metric on X and its pull–back on $\widetilde{X}$. Moreover we have $\widetilde{\Delta}''_k=\widetilde{\Delta}''_k\restriction_{\widetilde{\Omega}}$. Since $k^{-1}\widetilde{\Delta}''_k$ commutes with $\bar\partial$ it follows that the spectral projections of $k^{-1}\widetilde{\Delta}''_k$ commute with $\bar\partial$ too, showing thus $\bar\partial L^q_k(\lambda)\subset L^{q+1}_k(\lambda)$ and therefore we have a complex of Γ–modules of finite Γ–dimension:

$$0\longrightarrow L^0_k(\lambda)\xrightarrow{\bar\partial_\lambda}L^1_k(\lambda)\xrightarrow{\bar\partial_\lambda}\cdots\xrightarrow{\bar\partial_\lambda}L^n_k(\lambda)\longrightarrow 0\,. \tag{5.1}$$

$k^{-1}\widetilde{\Delta}''_k$ commutes also with $\bar\partial^*$ and $\left(\bar\partial_\lambda\right)^*$ equals the restriction of $\bar\partial^*$ to $L^q_k(\lambda)$. Keeping this in mind it is easy to see that

$$N^q(\bar\partial_\lambda)/\overline{R^{q-1}(\bar\partial_\lambda)}=\Big\{u\in L^q_k(\lambda)\,:\,\bar\partial_\lambda u=0\,,\ \left(\bar\partial_\lambda\right)^* u=0\Big\}=\mathcal{H}^{0,q}_{(2)}(\widetilde{X},\widetilde{E}^k\otimes\widetilde{F})\,. \tag{5.2}$$

We can now apply the following lemma (see [Sh]).

Algebraic Lemma. *Let $0\longrightarrow L_0\xrightarrow{d_0}L_1\xrightarrow{d_1}\cdots\xrightarrow{d_n}L_n\longrightarrow 0$ be a complex of Γ–modules (d_q commutes with the action of Γ and $d_{q+1}d_q=0$). If $l_q=\dim_\Gamma L_q$ is finite and $h_q=\dim_\Gamma H^q(L)$ where $H_q(L)=N(d_q)/\overline{R(d_{q-1})}$,*

$$\sum_{j=0}^{q}(-1)^{q-j}h_j\leqslant\sum_{j=0}^{q}(-1)^{q-j}l_j$$

for every $q=0,1,...,n$ and for $q=n$ the inequality becomes equality.

The Algebraic Lemma for the complex (5.1) and relation (5.2) yield

$$\sum_{j=0}^{q}(-1)^{q-j}\dim_\Gamma H^{0,j}_{(2)}(\widetilde{X},\widetilde{E}^k\otimes\widetilde{F})\leqslant\sum_{j=0}^{q}(-1)^{q-j}N^j_\Gamma\left(\lambda,\frac{1}{k}\widetilde{\Delta}''_k\right)$$

for $q=0,1,\dots,n$ and for $q=n$ the inequality becomes equality. We apply now (1.11):

$$\sum_{j=0}^{q}(-1)^{q-j}\dim_\Gamma H^{0,j}_{(2)}(\widetilde{X},\widetilde{E}^k\otimes\widetilde{F})\leqslant k^n\big(I^q(U,\lambda)-I^{q-1}(U,\lambda)+\cdots$$
$$+(-1)^qI^0(U,\lambda)\big)+o(k^n)\,,$$

for $k\longrightarrow\infty$. We can now let λ go to zero through values $\lambda\in\mathbb{R}\setminus\mathcal{N}$. We have thus proved the following.

Theorem 5.4. *Let $\widetilde{X}$ be a Galois covering of group Γ of a compact manifold X. As $k \to \infty$, the following strong Morse inequalities hold for every $q = 0, 1, \dots, n$:*

$$\sum_{j=0}^{q}(-1)^{q-j}\dim_\Gamma H^{0,j}_{(2)}(\widetilde{X}, \widetilde{E}^k \otimes \widetilde{F}) \le r\frac{k^n}{n!}\int_{X(\leqslant q)}(-1)^q\left(\frac{\imath}{2\pi}\boldsymbol{c}(E)\right)^n + o(k^n).$$

with equality for $q = n$ (asymptotic L^2 Riemann-Roch formula).

In particular $\dim_\Gamma H^0_{(2)}(\widetilde{X}, \widetilde{E}^k \otimes \widetilde{F}) \geqslant \frac{k^n}{n!}\int_{X(\leqslant 1)}\left(\frac{\imath}{2\pi}\boldsymbol{c}(E)\right)^n + o(k^n)$. It follows that if E satisfies (D) then for $k \longrightarrow \infty$

$$\begin{aligned}&\dim_\Gamma H^0_{(2)}(\widetilde{X}, \widetilde{E}^k \otimes \widetilde{F}) \approx k^n\,,\\&\dim_\Gamma H^q_{(2)}(\widetilde{X}, \widetilde{E}^k \otimes F) = o(k^n)\,, \quad q \geqslant 1\,.\end{aligned}$$

Hence the usual dimension of the space of holomorphic L^2 sections has the same cardinal as $|\Gamma|$ for large k. This is a generalization of the result of Napier [Nap] that $\widetilde{X}$ is holomorphically convex with respect to $\widetilde{E}^k$ for large k if X is projective and E is positive. If the canonical bundle K_X satisfies condition (D), i.e. if there exists a metric ω on M such that $\int_{X(\leqslant 1)}(-\operatorname{Ric}\omega)^n > 0$ where $X(\leqslant 1)$ is the set of points where $-\operatorname{Ric}\omega$ is nondegenerate and has at most one negative eigenvalue, then $\dim_\Gamma H^0_{(2)}(\widetilde{X}, K^{\otimes k}_{\widetilde{X}}) \approx k^n$.

Remark 5.2. In Proposition 4.1 we have treated the case of a singular hermitian line bundle (E, h) over a compact manifold X. The condition on the singularities were that they are concentrated on an analytic set and moreover the curvature is positive near this analytic set. Then we can work on the complement of the analytic set and by means of the basic estimate study its coverings. If we are interested only in the coverings of X then we can rule out the condition of positivity near the singularities. Namely, when the singularities of the metric are algebraic (cf. [De2]), Bonavero [Bon] shows that the Morse inequlities are true for the cohomology of E^k twisted with the corresponding sequence of Nadel's multiplier ideal sheaves. Given a Galois covering as above we can adapt his proof to estimate the von Neuman dimension of the space $H^0_{(2)}(\widetilde{X}, \widetilde{E}^k \otimes \mathcal{I}_k(\widetilde{h}))$ of L^2 holomorphic sections in $\widetilde{E}^k$ twisted with the Nadel's multiplier ideal sheaf coming from the singularities of the Γ–invariant metric $\widetilde{h}$ on $\widetilde{E}^k$ (which is the pull–back of a Nadel multiplier ideal sheaf on X). The conclusion is that when (D) is true, the integral being taken over the regular set of the curvature current, then the von Neuman dimension of $H^0_{(2)}(\widetilde{X}, \widetilde{E}^k \otimes \mathcal{I}_k(\widetilde{h}))$ grows as k^n for large k.

Remark 5.3. Using the approach of this section we can study the growth of the cohomology groups of coverings of q–convex and q–concave manifolds. We can either use complete metrics or follow [GHS] and use the $\bar\partial$–Neumann problem setting. Let us give the statements in the latter set-up. Consider a q–convex manifold X in the sense of [AG], i.e. there exists a smooth exhausting function $\varphi : X \longrightarrow \mathbb{R}$ such that $\imath\partial\bar\partial\varphi$ has at least $n - q + 1$ positive eigenvalues outside a compact set K ($n = \dim X$, $1 \leqslant q \leqslant n - 1$). Consider $X_c = \{\varphi < c\} \supset K$ with smooth boundary. Then the Levi form of bX_c has at least $n - q$ positive eigenvalues. Let us consider a Galois covering $\widetilde{X}_d$ of a bigger sublevel set $X_d \supset X_c$ and denote by $\widetilde{X}_c$ the induced covering of X_c. As usual we denote by Γ the group of deck transformations. Let us consider also a line bundle E over X and denote by $\widetilde{E}$ its lifting to $\widetilde{X}_d$. Both $\widetilde{X}_c$ and $\widetilde{E}$ come with the liftings of metrics defined on X_d. We define the (reduced) L^2 cohomology groups $H^j_{(2)}(\widetilde{X}_c, \widetilde{E}^k)$ with respect to these metrics. By [GHS] we know that $\dim_\Gamma H^j_{(2)}(\widetilde{X}_c, \widetilde{E}^k) < \infty$ for $j \geqslant q$. With the method used in this paper we can prove that for $j \geqslant q$ and $k \longrightarrow \infty$:

(1) $\dim_\Gamma H^j_{(2)}(\widetilde{X}_c, \widetilde{E}^k) = O(k^n)$.

(2) If E is q–positive outside K (its curvature has at least $n-q+1$ positive eigenvalues) we have an explicit bound,

$$\dim_\Gamma H^j_{(2)}(\widetilde{X}_c, \widetilde{E}^k) \leqslant \frac{k^n}{n!}\int_{X(j)} (-1)^j \left(\frac{\imath}{2\pi}\boldsymbol{c}(E)\right)^n + o(k^n).$$

The proof consists of showing that the basic estimate holds in bidegree $(0, j)$ on $\widetilde{X}_c \subset \widetilde{L}$, where L is a compact set of X_c, for forms satisfying the $\bar\partial$–Neumann conditions on $b\widetilde{X}_c$. This is achieved using the liftings of the metrics constructed in [AV] where the case Γ trivial is treated. Then we can apply again the analysis from §1. If E is q–positive outside K then the leading term in (1) simplifies as shown in (2). These estimates were obtained in the case $\Gamma = \{\mathrm{Id}\}$ in [Bou] for certain complete metrics on X_c which permit to prove the same inequalities for the full cohomology group $H^j(X_c, E^k)$. For the case of coverings we have to restrict ourselves to L^2 cohomology groups. As for coverings of q–concave manifolds we get the same conclusion as in (1) for $j \leqslant n-q-1$. The nice simplification of the leading term holds if we impose a negativity condition outside a compact set. However there are cases of concave manifolds and positive bundles for which we have an effective estimate of $\dim_\Gamma H^0_{(2)}(\widetilde{X}_c, \widetilde{E}^k)$, see §3.

§6 Weak Lefschetz theorems.

Nori [No] generalized the Lefschetz hypersurface theorem. Assume X and Y are smooth connected projective manifolds and Y is a hypersurface in X with positive normal bundle and $\dim Y \geqslant 1$. Then the image of $\pi_1(Y)$ in $\pi_1(X)$ is of finite index. Recently, Napier and Ramachandran [NR] proposed an analytic approach and generalized Nori's theorem showing that Y may have arbitrary codimension (but $\dim Y \geqslant 1$). They use the $\bar\partial$–method on complete Kähler manifolds to separate the sheets of appropriate coverings. In the sequel we use the Riemann–Roch inequalities to study non–necessarily Kähler manifolds. However our method requires that the image group is normal since we can deal only with Galois coverings. First we introduce the notion of formal completion. Let Y be a complex analytic subspace of the manifold U and denote by $\mathcal{I}_Y$ the ideal sheaf of Y. The formal completion $\widehat{U}$ of U with respect to Y is the ringed space $(\widehat{U}, \mathcal{O}_{\widehat{U}}) = (Y, \operatorname{proj}\lim \mathcal{O}_U/\mathcal{I}^\nu_Y)$. If $\mathcal{F}$ is an analytic sheaf on U we denote by $\widehat{\mathcal{F}}$ the sheaf $\widehat{\mathcal{F}} = \operatorname{proj}\lim \mathcal{F} \otimes (\mathcal{O}/\mathcal{I}^\nu_Y)$. If $\mathcal{F}$ is coherent then $\widehat{\mathcal{F}}$ is too. Moreover by Proposition VI.2.7 of [BS] the kernel of the mapping $H^0(U, \mathcal{F}) \longrightarrow H^0(\widehat{U}, \widehat{\mathcal{F}})$ consists of the sections of $\mathcal{F}$ which vanish on a neighbourhood of Y. Hence for locally free $\mathcal{F}$ the map is injective.

Theorem 6.1. *Let M be a hyper 1–concave manifold carrying a line bundle E which satisfies* (D) *and is semi-positive outside a compact set. Let Y be a connected compact complex subspace of M satisfying:* (i) *for any k, $\dim H^0(\widehat{M}, \widehat{\mathcal{F}}_k) < \infty$,* *where* $\mathcal{F}_k = \mathcal{O}(E^k \otimes K_M)$, (ii) *the image* G *of* $\pi_1(Y)$ *in* $\pi_1(X)$ *is normal in* $\pi_1(X)$. *Then* G *is of finite index in* $\pi_1(X)$.

Proof. We follow the proof given in [NR]. Since G is normal there exists a connected Galois covering $\pi : \widetilde{M} \longrightarrow M$ such that the group of deck transformations is $\Gamma = \pi_1(M)/G$. The cardinal $|\Gamma|$ equals the index of G in $\pi_1(M)$. Let $\widetilde{E} = \pi^{-1}E$. By Theorem 3.1, there exists $C > 0$ such that for large k, $\dim_\Gamma H^{n,0}_{(2)}(\widetilde{M}, \widetilde{E}^k) \geqslant C\, k^n$. Let us choose a small open neighbourhood V of Y such that $\pi_1(Y) \longrightarrow \pi_1(V)$ is an isomorphism; so the image of $\pi_1(V)$ in $\pi_1(M)$ is G. Hence, if we denote by $\jmath$ the inclusion of V in M, there exists a holomorphic lifting $\widetilde{\jmath} : V \longrightarrow \widetilde{M}$, $\pi \circ \widetilde{\jmath} = \jmath$. Since $\widetilde{\jmath}$ is locally biholomorphic the pull–back map $\widetilde{\jmath}^* : H^{n,0}_{(2)}(\widetilde{M}, \widetilde{E}^k) \longrightarrow H^{n,0}(V, E^k)$ is injective. On the other hand $H^0(V, \mathcal{F}_k) \hookrightarrow H^0(\widehat{V}, \widehat{\mathcal{F}}_k) = H^0(\widehat{M}, \widehat{\mathcal{F}}_k)$. By (i) the latter space is finite dimensional so $\dim H^{n,0}_{(2)}(\widetilde{M}, \widetilde{E}^k) < \infty$. We know that $\dim_\Gamma H^0_{(2)}(\widetilde{M}, \widetilde{E}^k \otimes K_{\widetilde{M}}) > 0$ for $k > C^{-1/n}$. If Γ were infinite this would yield $\dim H^{n,0}_{(2)}(\widetilde{M}, \widetilde{E}^k) = \infty$ which is a contradiction. Therefore $|\Gamma| < \infty$ and $\dim H^{n,0}_{(2)}(\widetilde{M}, \widetilde{E}^k) \geqslant C\,|\Gamma|\, k^n \geqslant |\Gamma|$ for $k > C^{-1/n}$. Thus $|\Gamma| \leqslant \dim H^0(\widehat{M}, \widehat{\mathcal{F}}_k)$ for large k. □

Remark 6.2.
(a) By a theorem of Grothendieck [Gro], condition (i) is fulfilled if Y is locally a complete intersection with ample normal bundle N_Y (or k–ample in the sense of Sommese, $k = \dim Y - 1$).
(b) We can replace condition (i) with the requirement that Y has a fundamental system of pseudoconcave neighbourhoods $\{V\}$. Then $\dim H^0(V, \mathcal{F}_k)$ is finite by [An]. This happens for example if Y is a smooth hypersurface and N_Y has at least one positive eigenvalue or, if Y has arbitrary codimension, if N_Y is sufficiently positive in the sense of Griffiths [Gri1].
(c) Condition (ii) is trivially satisfied if $\pi_1(Y) = 0$. Thus, if M contains a simply connected subvariety satisfying either (a) or (b), $\pi_1(M)$ is finite.
(d) By Corollary 3.6, Theorem 6.1 can also be applied to the pertubed structures considered there.

Using Proposition 4.3 we can can show that Nori's theorem holds for all Moishezon spaces X.

Theorem 6.2. *Let* X *be an irreducible reduced normal Moishezon compact complex space and let* E *be the (positive in the sense of currents) line bundle given by Theorem* 4.2. *Suppose that* M *is a Zariski open set of* X *and* $Y \subset \mathrm{Reg}(M)$ *be a connected compact complex subspace such that for any* k, $\dim H^0(\widehat{M}, \widehat{\mathcal{E}}_k) < \infty$, *where* $\mathcal{E}_k = \mathcal{O}(E^k)$. *Then the image* G *of* $\pi_1(Y)$ *in* $\pi_1(M)$ *is of finite index in* $\pi_1(M)$.

Proof. Since X is normal we have an isomorphism $\pi_1(\mathrm{Reg}\, M) \longrightarrow \pi_1(M)$, so that we may assume $M \subset \mathrm{Reg}\,(X)$. We find a connected unramified covering $p : \widetilde{M} \longrightarrow M$ such that $p_*\pi_1(\widetilde{M}) = G$. If d is the number of sheets, $d = |\pi_1(M)/G|$, the index of G in $\pi_1(M)$. The preceding proof applies by using Proposition 4.3 instead of Theorem 3.1. and the usual dimension instead of the Γ–dimension. □

Note that Napier and Ramachandran also considered cases when X is not necessarily projective, but their result does not imply diectly Theorem 6.2.

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