# Solutions of the Einstein-Dirac Equation on Riemannian 3-Manifolds with Constant Scalar Curvature. *

Thomas Friedrich

*Humboldt-Universität zu Berlin, Institut für Reine Mathematik, Ziegelstraße 13a, D-10099 Berlin, Germany*

**Abstract**

This paper contains a classification of all 3-dimensional manifolds with constant scalar curvature $S \neq 0$ that carry a non-trivial solution of the Einstein-Dirac equation.

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## 1 Introduction

Consider a Riemannian spin manifold of dimension $n \geq 3$ and denote by $D$ the Dirac operator acting on spinor fields. A solution of the Einstein-Dirac equation is a spinor field $\psi$ solving the equations

$$Ric - \frac{1}{2}\, S \cdot g = \pm \frac{1}{4}\, T_{\psi} \quad , \quad D(\psi) = \lambda \psi .$$

Here $S$ denotes the scalar curvature of the space, $\lambda$ is a real constant and $T_{\psi}$ is the energy-momentum tensor of the spinor field $\psi$ defined by the formula

$$T_{\psi}(X, Y) = (X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, \psi).$$

The scalar curvature $S$ is related to the eigenvalue $\lambda$ and the length of the spinor field $\psi$ by the formula

$$S = \pm \frac{\lambda}{n-2} |\psi|^2 .$$

In [KimF] we introduced the weak Killing equation for a spinor field $\psi^*$:

$$\nabla_X \psi^* = \frac{n}{2(n-1)}\, dS(X)\psi^* + \frac{2\lambda}{(n-2)S}\, Ric(X)\cdot\psi^* - \frac{\lambda}{n-2}\, X\cdot\psi^* + \frac{1}{2(n-1)S}\, X\cdot dS\cdot\psi^*$$

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Any weak Killing spinor $\psi^*$ (WK-spinor) yields a solution $\psi$ of the Einstein-Dirac equation after normalization

$$\psi = \sqrt{\frac{(n-2)|S|}{|\lambda||\psi^*|^2}}\,\psi^*.$$

In fact, in dimension $n = 3$ the Einstein-Dirac equation is essentially equivalent to the weak Killing equation (see [KimF]). Up to now the following 3-dimensional Riemannian manifolds admitting WK-spinors are known:

1. the flat torus $T^3$ with a parallel spinor;
2. the sphere $S^3$ with a Killing spinor;
3. two non-Einstein Sasakian metrics on the sphere $S^3$ admitting WK-spinors. The scalar curvature of these two left-invariant metrics equals $S = 1 \pm \sqrt{5}$.

The aim of this paper is to classify all Riemannian 3-manifolds with constant scalar curvature and admitting a solution of the Einstein-Dirac equation. In particular, we will prove the existence of a one-parameter family of left-invariant metrics on $S^3$ with WK-spinors. This family contains the two non-Einstein Sasakian metrics with WK-spinors on $S^3$, but does not contain the standard sphere $S^3$ with Killing spinors. Moreover, any simply-connected, complete Riemannian manifold $N^3 \neq S^3$ with WK-spinors and constant scalar curvature is isometric to a space of this one-parameter family. In order to formulate the result precisely, we fix real parameters $K, L, M \in \mathbb{R}$ and denote by $N^3(K,L,M)$ the 3-dimensional, simply-connected and oriented Riemannian manifold defined by the following structure equations:

$$\omega_{12} = K\sigma^3 \quad , \quad \omega_{13} = L\sigma^2 \quad , \quad \omega_{23} = M\sigma^1,$$

or, equivalently:

$$d\sigma^1 = (L-K)\sigma^2 \wedge \sigma^3 \quad , \quad d\sigma^2 = (M+K)\sigma^1 \wedge \sigma^3 \quad , \quad d\sigma^3 = (L-M)\sigma^1 \wedge \sigma^2.$$

The 1-forms $\sigma^1, \sigma^2, \sigma^3$ are the dual forms of an orthonormal frame of vector fields. Using this frame the Ricci tensor of $N^3(K,L,M)$ is given by the matrix

$$Ric = \begin{pmatrix} -2KL & 0 & 0 \\ 0 & 2KM & 0 \\ 0 & 0 & -2LM \end{pmatrix}.$$

**Main Theorem:** *Let $N^3 \neq S^3$ be a complete, simply-connected Riemannian manifold with constant scalar curvature $S \neq 0$. If $N^3$ admits a WK-spinor, then $N^3$ is isometric to $N^3(K,L,M)$ and the parameters are a solution of the equation*

$$-K^2L(L-M)^2M + L^3M^3 + KL^2M^2(M-L) + K^3(L-M)(L+M)^2 = 0 \qquad (*)$$

*Conversely, any space $N^3(K,L,M)$ such that $(K,L,M) \neq (0,0,0)$ is a solution of $(*)$ admits two WK-spinors for one and only one WK-number $\lambda$. With respect to the fixed orientation of $N^3(K,L,M)$ we have the two cases:*

$$\lambda = +\frac{S}{2\sqrt{2}}\sqrt{\frac{S}{S^2 - 2|Ric|^2}} \qquad \textit{if} \quad -K < M$$

$$\lambda = -\frac{S}{2\sqrt{2}} \sqrt{\frac{S}{S^2 - 2|Ric|^2}} \qquad \textit{if} \quad M < -K.$$

*The spaces $N^3(K, L, M)$ are isometric to $S^3$ equipped with a left-invariant metric.*

**Remark:** If the parameters $K = M$ coincide, the solution of the equation $(*)$ is given by

$$L = \frac{1}{4}K(1 - \sqrt{5}) \quad , \quad L = \frac{1}{4}K(1 + \sqrt{5})$$

and we obtain the Ricci tensors

$$Ric = \begin{pmatrix} \frac{1}{2}K^2(\sqrt{5} - 1) & 0 & 0 \\ 0 & 2K^2 & 0 \\ 0 & 0 & \frac{1}{2}K^2(\sqrt{5} - 1) \end{pmatrix}$$

or

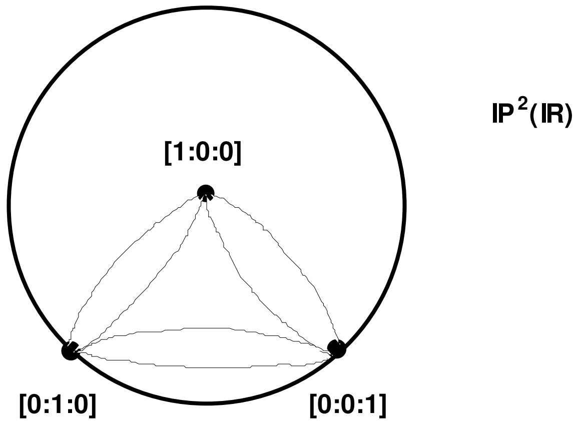
$$Ric = \begin{pmatrix} -\frac{1}{2}K^2(1 + \sqrt{5}) & 0 & 0 \\ 0 & 2K^2 & 0 \\ 0 & 0 & -\frac{1}{2}K^2(1 + \sqrt{5}) \end{pmatrix}.$$

The non-Einstein-Sasakian metrics on $S^3$ occur for the parameter $K = 1$ (see [KimF]).

**Remark:** Using the standard basis of the Lie algebra $\mathfrak{so}(3)$ we can write the left-invariant metric of the space $N^3(K, L, M)$ in the following way:

$$\begin{pmatrix} \frac{1}{|M-L||K+M|} & 0 & 0 \\ 0 & \frac{1}{|K-L||M-L|} & 0 \\ 0 & 0 & \frac{1}{|K-L||K+M|} \end{pmatrix}.$$

The equation $(*)$ is a homogeneous equation of order six. The transformation $(K, L, M) \to (\mu K, \mu L, \mu M)$ corresponds to a homothety of the metric. Therefore - up to a homothety - the moduli space of solutions is a subset of the real projective space $\mathbb{P}^2(\mathbb{R})$ given by the equation $(*)$. This subset is a configuration of six curves in $\mathbb{P}^2(\mathbb{R})$ connecting the three points $[K : L : M] = [1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$ corresponding to flat metrics.




In particular, we have constructed two paths of solutions of the Einstein-Dirac equation deforming the non-Einstein Sasakian metrics on $S^3$.

## 2 The integrability condition for the Einstein-Dirac equation in dimension $n = 3$.

The spinor bundle of a 3-dimensional Riemannian manifold is a complex vector bundle of dimension two. Moreover, there exists a quaternionic structure commuting with the Clifford multiplication by real vectors (see [F]). Consequently, in case of a real WK-number $\lambda$, the corresponding space of WK-spinors is a quaternionic vector space. In the spinor bundle let us introduce the metric connection $\nabla^\lambda$ given by the formula

$$\nabla^\lambda_X \psi := \nabla_X \psi - \frac{3}{4} dS(X)\psi - \lambda \Big\{ \frac{2}{S} Ric(X) - X \Big\} \cdot \psi - \frac{1}{4S} X \cdot dS \cdot \psi$$

and denote by $\Omega^\lambda$ its curvature form. Then we obtain the following

**Proposition 1:** *Let $N^3$ be a simply-connected 3-dimensional Riemannian manifold and suppose that the scalar curvature $S \neq 0$ does not vanish. Then the following conditions are equivalent:*

1. *$N^3$ is a non-trivial solution of the Einstein-Dirac equation with real eigenvalue $\lambda$;*
2. *$N^3$ admits a WK-spinor with real WK-number $\lambda$;*
3. *$N^3$ admits two WK-spinors with real WK-number $\lambda$;*
4. *The curvature $\Omega^\lambda \equiv 0$ vanishes identically.*

If the scalar curvature $S \neq 0$ is constant, the condition $\Omega^\lambda \equiv 0$ has been investigated and yields algebraic equations involving the Ricci tensor and its covariant derivative (see [KimF], Theorem 8.3.). In order to formulate the integrability condition, we denote by $X \times Y$ the vector cross product of two vectors $X, Y \in T(N^3)$. For brevity, let us introduce the endomorphism $T : T(N^3) \to T(N^3)$ given by the formula

$$T(X) = \sum_{i=1}^{3} e_i \times (\nabla_{e_i} Ric)(X),$$

which will be used in the proof of the main Theorem.

**Theorem 1 (see [KimF]):** *Let $N^3$ be a simply-connected 3-dimensional Riemannian manifold with constant scalar curvature $S \neq 0$. $N^3$ admits a solution of the Einstein-Dirac equation with real eigenvalue $\lambda$ if and only if the following three conditions are satisfied:*

1. $8\lambda^2\{S^2 - 2|Ric|^2\} = S^3$;
2. $8\lambda^2\{SRic(X) - 2Ric \circ Ric(X)\} - 4\lambda ST(X) - S^2 Ric(X) = 0$;
3. $8\lambda^2\{2Ric(X) - SX\} \times \{2Ric(Y) - SY\} + 8\lambda S\{(\nabla_X Ric)(Y) - (\nabla_Y Ric)(X)\}$
   $+S^3 X \times Y = 2S^2 \sum_{i<j} \{R_{jY}\delta_{iX} + R_{iX}\delta_{jY}\} e_i \times e_j$.

# 3 Proof of the Main Theorem

We fix an orthonormal frame $e_1, e_2, e_3$ of vector fields on $N^3$ consisting of eigenvectors of the Ricci tensor:

$$Ric = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}.$$

Denote by $\sigma^1, \sigma^2, \sigma^3$ the dual frame and consider the connection forms $\omega_{ij} = \langle \nabla e_i, e_j \rangle$ of the Levi-Civita connection. The structure equations of the Riemannian manifold $N^3$ are

$$d\omega_{12} = \omega_{13} \wedge \omega_{32} + \frac{C - A - B}{2}\, \sigma^1 \wedge \sigma^2$$
$$d\omega_{13} = \omega_{12} \wedge \omega_{23} + \frac{B - A - C}{2}\, \sigma^1 \wedge \sigma^3$$
$$d\omega_{23} = \omega_{21} \wedge \omega_{13} + \frac{A - B - C}{2}\, \sigma^2 \wedge \sigma^3$$

and the covariant derivative $\nabla Ric$ is given by the matrix of 1-forms

$$\nabla Ric = \begin{pmatrix} dA & (A-B)\omega_{12} & (A-C)\omega_{13} \\ (A-B)\omega_{12} & dB & (B-C)\omega_{23} \\ (A-C)\omega_{13} & (B-C)\omega_{23} & dC \end{pmatrix}.$$

Using the third equation of Theorem 1 we obtain

$$\langle (\nabla_{e_i} Ric)(e_j) - (\nabla_{e_j} Ric)(e_i), e_i \rangle = 0$$

and, consequently,

$$dA(e_1) = dA(e_2) = dB(e_1) = dB(e_3) = dC(e_2) = dC(e_3) = 0.$$

Since $A+B+C=S$ is constant, we conclude that any eigenvalue $A,B,C$ of the Ricci tensor is constant, too. The second equation of Theorem 1 yields the condition that all elements outside the diagonal of the (1,1)-tensor $T$ are zero:

$$(A-B)\omega_{12}(e_1) = 0 = (A-B)\omega_{12}(e_2)$$

$$(C-A)\omega_{13}(e_1) = 0 = (C-A)\omega_{13}(e_3)$$

$$(B-C)\omega_{23}(e_2) = 0 = (B-C)\omega_{23}(e_3).$$

First, we discuss the generic case that $A,B,C$ are pairwise different. Then there exist numbers $K,L,M$ such that

$$\omega_{12} = K\sigma^3 \quad , \quad \omega_{13} = L\sigma^2 \quad , \quad \omega_{23} = M\sigma^1.$$

The parameter triples $\{A,B,C\}$ and $\{K,L,M\}$ are related via the structure equations by the formulas

$$A = -2KL \quad , \quad B = 2KM \quad , \quad C = -2LM.$$

The first and second equation of Theorem 1 become equivalent to the following system of algebraic equations:

1'. $\lambda = \pm \dfrac{S}{2\sqrt{2}}\sqrt{\dfrac{S}{S^2 - 2|Ric|^2}}$ ;

2'.
$$\begin{aligned}
2S(S^2-2|Ric|^2)\{(A-C)L+(B-A)K\}^2 &= S(SA-2A^2) - A(S^2-2|Ric|^2)\\
2S(S^2-2|Ric|^2)\{(C-B)M+(A-B)K\}^2 &= S(SB-2B^2) - B(S^2-2|Ric|^2)\\
2S(S^2-2|Ric|^2)\{(B-C)M+(C-A)L\}^2 &= S(SC-2C^2) - C(S^2-2|Ric|^2).
\end{aligned}$$

We solve this system of algebraic equations with respect to the parameters $K,L,M$. It turns out that the equations 2'. can be written in the form

$$P_i(K,L,M)\cdot Q(K,L,M) = 0,$$

$(1 \le i \le 3)$, where the polynomials $P_1, P_2, P_3$ and $Q$ are given by the formulas

$$\begin{aligned}
P_1(K,L,M) &= (-KL^2 + L^2M + K^2(L+M))^2\\
P_2(K,L,M) &= (KM^2 + LM^2 + K^2(L+M))^2\\
P_3(K,L,M) &= (LM(-L+M) + K(L^2+M^2))^2\\
Q\,(K,L,M) &= -K^2L(L-M)^2M + L^3M^3 + KL^2M^2(M-L) + K^3(L-M)(L+M)^2.
\end{aligned}$$

The real solutions of $P_1 = P_2 = P_3 = 0$ are the pairs $\{K=0, L=0\}$ (the flat metric) and $\{K=M, L=-M\}$ (the space of positive constant curvature). Therefore, we proved that a 3-dimensional complete, simply-connected manifold $N^3$ with constant scalar curvature $S \neq 0$ and different eigenvalues of the Ricci tensor is isometric to one of the spaces $N^3(K,L,M)$, where the parameters $K,L,M$ are solutions of the equation $Q(K,L,M) = 0$. These spaces satisfy the conditions 1. and 2. of Theorem 1 and, moreover, a simple computation yields the result that condition 3. of Theorem 1 is satisfied, too. We next discuss the case that two of the eigenvalues $A, B, C$ coincide, for example, $A = C \neq B$. Then we obtain again

$$\omega_{12} = K\sigma^3 \quad , \quad \omega_{23} = M\sigma^1,$$

but there is no condition for the connection form $\omega_{13}$. We compute the matrix of the $(1,1)$-tensor $T$:

$$T = \begin{pmatrix} (B-C)K & 0 & 0 \\ 0 & (C-B)(K+M) & 0 \\ 0 & 0 & (B-C)M \end{pmatrix}.$$

Since the scalar curvature $S$ as well as the eigenvalues $A = C, B$ of the Ricci tensor are constant, the second equation of Theorem 1 yields that $K$ and $M$ are constant and, moreover, coincide:

$$K = M = \text{const.}$$

In case of $K = M = 0$ we have $\omega_{12} = \omega_{23} = 0$ and $A = C$. In particular, the Ricci tensor is parallel, $\nabla Ric = 0$. Therefore, in this case $N^3$ is a Ricci-parallel 3-dimensional manifold admitting a WK-spinor. Then $N^3$ is either flat or a space of constant positive curvature (see [KimF], Theorem 8.2.). Finally, we consider that the case of $K = M = 1$, i.e., $\omega_{12} = \sigma^3$ and $\omega_{23} = \sigma^1$. Differentiating the equation $\omega_{12} = \sigma^3$, we obtain

$$\omega_{13} \wedge \omega_{32} - \frac{B}{2}\sigma^1 \wedge \sigma^2 = d\omega_{12} = d\omega^3 = \omega_{31} \wedge \sigma^1 + \omega_{32} \wedge \sigma^2$$

$$-\frac{B}{2}\sigma^1 \wedge \sigma^2 = -\sigma^1 \wedge \sigma^2.$$

Consequently, $B = 2$ and the tensors $T$ and $Ric$ are given by the matrices

$$T = \begin{pmatrix} 2-C & 0 & 0 \\ 0 & 2(C-2) & 0 \\ 0 & 0 & 2-C \end{pmatrix} \quad , \quad Ric = \begin{pmatrix} C & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & C \end{pmatrix}.$$

The second condition of Theorem 1 yields the equations ($S = 2 + 2C$):

$$\begin{aligned} 8\lambda^2(SC - 2C^2) - 4\lambda S(2-C) - S^2 C &= 0 \\ 8\lambda^2(2S - 8) + 8\lambda S(2-C) - 2S^2 &= 0 \end{aligned}$$

Solving these equations with respect to $\lambda$ and $C$ we obtain the three solutions:

1. $C = 2$ and $\lambda = \pm\frac{3}{2}$. Then $N^3$ is isometric to $S^3$.
2. $C = -1$ and $\lambda = 0$. Then the scalar curvature $S = 0$ is zero.
3. $C = \frac{1}{2}(-1 \pm \sqrt{5})$ and $\lambda = 1 \pm \frac{\sqrt{5}}{2}$. These metrics are the non-Einstein Sasakian metrics on $S^3$ admitting WK-spinors (see [KimF]). The corresponding space is contained in the family $N^3(K, L, M)$.

We have discussed all possibilities and, therefore, we have finished the proof of the main Theorem.

$\blacksquare$

# 4 The moduli space of solutions

The moduli space of all 3-dimensional Riemannian manifolds with constant scalar curvature $S \neq 0$ and WK-spinors is given by the triples $\{K, L, M\}$ of real numbers satisfying the equation of order six $Q(K, L, M) = 0$. The polynomial $Q$ is symmetric in $\{K, -L, M\}$. Denote by

$$\gamma_1 = K - L + M \quad , \quad \gamma_2 = -KL + KM - LM \quad , \quad \gamma_3 = -KLM$$

the elementary symmetric functions of these variables. Then we have

$$Q = 4\gamma_1\gamma_2\gamma_3 - \gamma_2^3 - 4\gamma_3^2.$$

Consider the projective variety $V_{\mathbb{C}} \subset \mathbb{P}^2(\mathbb{C})$ defined by the homogeneous polynomial $Q$:

$$V_{\mathbb{C}} = \Big\{[K : L : M] \in \mathbb{P}^2(\mathbb{C}) : Q(K, L, M) = 0\Big\}.$$

$V_{\mathbb{C}}$ has three singular points:

$$V_{\mathbb{C}}^{\text{sing}} = \Big\{[1:0:0],\ [0:1:0],\ [0:0:1]\Big\}$$

and these points correspond to the flat metric. We will now parametrize the variety $V_{\mathbb{C}}$ by two meromorphic functions defined on a smooth Riemann surface. $V_{\mathbb{C}}$ is given by the equation ($K = 1$):

$$Q(1, L, M) = L^3(M-1)^2(M+1) + L^2M(1+M)^2 - LM^2(1+M) - M^3 = 0.$$

Let us introduce the variables

$$a = M - L - LM \quad , \quad b = (L - M)LM.$$

Then we obtain $Q(1, L, M) = -a^3 + 4b(1+a)$ and the equation defining the variety $V_{\mathbb{C}}$ becomes much simpler:

$$b = \frac{1}{4}\,\frac{a^3}{1+a}.$$

Next we consider a square root of $a+1$ and we solve the equations

$$z^2 - 1 = a = M - L - LM \quad , \quad \frac{1}{4}\,\frac{(z^2-1)^3}{z^2} = b = (L - M)LM$$

with respect to $L$ and $M$. Then we obtain four solution pairs $\{L, M\}$ depending on the variable $z$. For example,

$$L(z) = \frac{-(1+z)(1 - 2z + z^2 + \sqrt{(1+z)(1+3z-5z^2+z^3)})}{4z}$$

$$M(z) = \frac{(1+z)(1 - 2z + z^2 + \sqrt{(1+z)(1+3z-5z^2+z^3)})}{4z}.$$

The polynomial

$$(z+1)(1+3z-5z^2+z^3) = (z+1)(z-1)(z+(2+\sqrt{5}))(z+(2-\sqrt{5}))$$

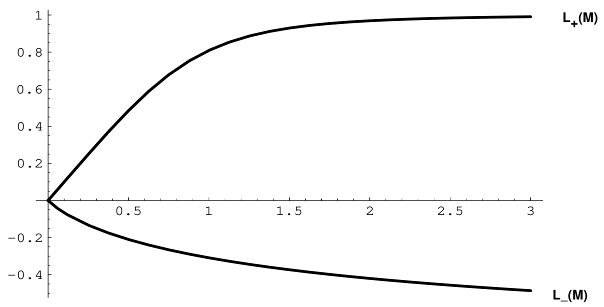
has four different zeros. The square root $\sqrt{(1+z)(1+3z-5z^2+z^3)}$ is a meromorphic function on the compact Riemann surface of genus $g=1$. Consequently, there exists a torus $\mathbb{C}/\Gamma$ and elliptic functions $L, M : \mathbb{C}/\Gamma \to \mathbb{P}^1(\mathbb{C})$ such that the components of the variety $V_{\mathbb{C}} \backslash V_{\mathbb{C}}^{\mathrm{sing}}$ are parametrized by $L$ and $M$:

$$V_{\mathbb{C}} = \Big\{ [1 : L(z) : M(z)] : \ z \in \mathbb{C}/\Gamma \Big\}.$$

The functions $L - M$ and $L \cdot M$ are given by the formulas:

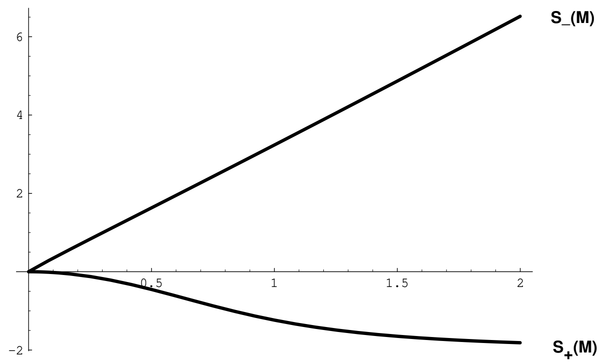
$$L - M = -\frac{(1+z)(z-1)^2}{2z} \quad , \quad L \cdot M = -\frac{(1+z)^2(z-1)}{2z}$$

The moduli space we are interested in coincides with the real points of the projective variety $V_{\mathbb{C}}$. If $K=0$, the only solutions of the equation $Q(0,L,M)=0$ are $L=0$ or $M=0$, i.e., the points $[0:1:0]$ and $[0:0:1]$. Therefore we can parametrize the moduli space by the parameter $M \in \mathbb{R}$ solving the equation $Q(1,L,M)=0$ with respect to $L = L(M)$. In this way we obtain a configuration of six curves in $\mathbb{P}^2(\mathbb{R})$ connecting the three singular points of $V_{\mathbb{C}}$ (see the figure in the Introduction). However, we obtain geometrically different metrics on $S^3$ only for two curves parametrized by the real parameter $0 \le M \le \infty$. The graphs of the function $L_{\pm}(M)$ are given in the following figure:



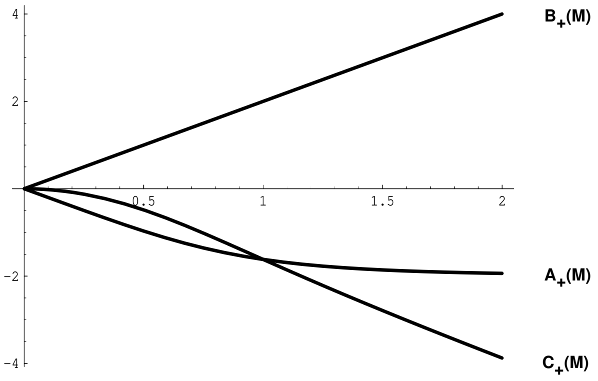

(Figure 1)

The functions $L_{\pm}(M)$ are monotone and tend to $\pm 1$ in case that $M$ tends to infinity. Let us discuss the geometric invariants of these metrics. The graph of the scalar curvatures $S_{\pm}(M)$ depending on $M$ is given by the next figure:

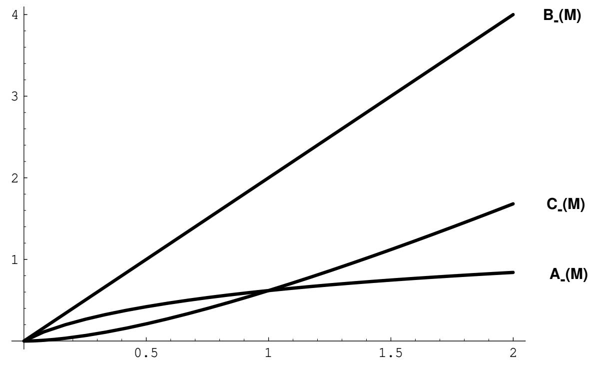



(Figure 2: The scalar curvatures)

Next we plot the eigenvalues $A_{\pm}(M), B_{\pm}(M), C_{\pm}(M)$ of the Ricci tensor for both families of metrics:




(Figure 3: The eigenvalues of the Ricci tensor for $L_{+}(M)$)




(Figure 4: The eigenvalues of the Ricci tensor for $L_{-}(M)$)

In dimension $n = 3$ the number

$$\lambda^2(D) \cdot [\mathrm{vol}\,(N^3)]^{\frac{2}{3}}$$

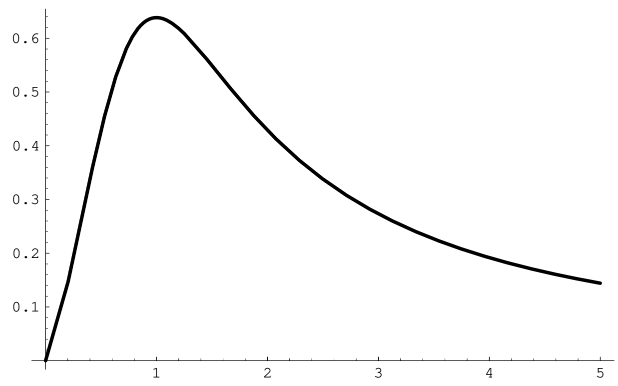
is a homothety invariant, where $\lambda(D)$ is an eigenvalue of the Dirac operator. In case of a WK-spinor we have

$$\lambda^2 = \frac{1}{8}\ \frac{S^3}{S^2 - 2|Ric|^2}$$

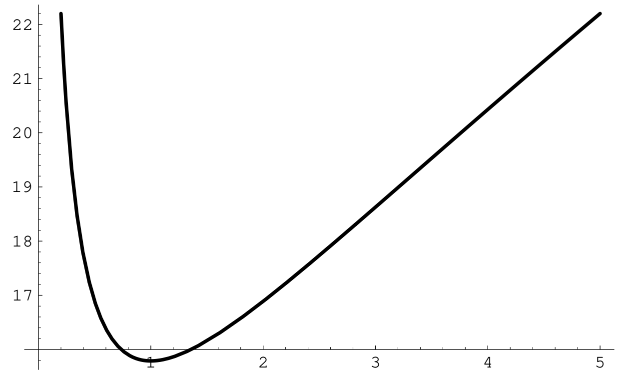
and, therefore, we obtain the formula

$$\lambda^2 \cdot \mathrm{vol}^{\frac{2}{3}} = \frac{1}{8}(2\pi^2)^{\frac{2}{3}} \frac{S^3}{S^2 - 2|Ric|^2}\ \frac{1}{\{|K-L||M-L||K+M|\}^{\frac{2}{3}}}.$$

The next figures contain the graph of $\lambda^2 \mathrm{vol}^{\frac{2}{3}}(M)$ depending on the parameter $M$ for both families of metrics.




(Figure 5: $\lambda^2 \mathrm{vol}^{\frac{2}{3}}$ in case of $L_+(M)$)




(Figure 6: $\lambda^2 \mathrm{vol}^{\frac{2}{3}}$ in case of $L_-(M)$)

Finally, let us discuss the behaviour of the rational function

$$\Psi = \frac{L^2}{KM}$$

on the variety $V_{\mathbb{C}} \subset \mathbb{P}^2(\mathbb{C})$. It turns out that $\Psi$ has simple zeros at the singular points $[1:0:0]$ and $[0:0:1]$. Indeed, solving the equation defining $V_{\mathbb{C}}$ with respect to $L = L(M)$ $(K = 1)$ we obtain

$$\lim_{M\to 0} \frac{L^2(M)}{M} = 0 \quad , \quad \lim_{M\to 0} \frac{d}{dM}\Big(\frac{L^2(M)}{M}\Big) = 1.$$

The third singular point $[0:1:0]$ is a pole of order two. In the regular part of $V_{\mathbb{C}}$ the function $\Psi$ has 12 ramification points. Among them 10 points are first order ramification points. The ramification points of order two are the points

$$[K:L:M] = [1 : \frac{1}{4}(1 \pm \sqrt{5}) : 1].$$

These parameters correspond precisely to the non-Einstein Sasakian metrics on $S^3$ admitting solutions of the Einstein-Dirac equation.

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