NULL-GEODESICS IN COMPLEX CONFORMAL MANIFOLDS AND THE LEBRUN CORRESPONDENCE

FLORIN ALEXANDRU BELGUN

Abstract. In the complex-Riemannian framework we show that a conformal manifold containing a compact, simply-connected, null-geodesic is conformally flat. In dimension 3 we use the LeBrun correspondence, that views a conformal 3-manifold as the conformal infinity of a selfdual four-manifolds. We also find a relation between the conformal invariants of the conformal infinity and its ambient.

1. Introduction

On a complex manifold, the existence of a *complex-Riemannian metric* implies, in general, strong topological assumptions, especially if the manifold is compact (*e.g.* the — square of the — canonical bundle has to be trivial). However, any analytic (pseudo-) Riemannian (or conformal) manifold can be complexified, and a natural question is to see to what extent the *global* properties of the real manifold (*e.g.* existence of closed (null-) geodesics) hold for the complexified spaces. This complexification procedure naturally occurs in *twistor theory* (see below), which has been intensively studied for Riemannian space-times (see, *e.g.*, [1], [5], [10], [15]); the complex-Riemannian setting, in which historically the twistor theory was first introduced [12], can provide a link to the Lorentzian geometry.

In *complex conformal geometry* (which implies weaker assumptions on the topology of the manifold), the conformal structure is determined by the set of *null-geodesics*, which can be organized as a complex manifold under some topological conditions [8], [11]. A natural question is which complex conformal manifolds admit compact null-geodesics; for example, if a *self-dual* manifold admits a globally-defined *twistor space*, then application of a twistorial interpretation of the Weyl tensor [2], implies that it is conformally flat, and the compact null-geodesic is simply-connected.

Our main result (section 4, Theorem 4) states that, if a conformal complex n-manifold admits a rational curve as a null-geodesic, then it is conformally flat (see also [17] for the case of a complex projective manifold). The proof uses the properties of Jacobi fields along the considered compact, simply-connected, null-geodesic : namely, we compute the normal bundle of a compact, simply-connected, null-geodesic, and we show that the small deformations of the latter as a compact curve, or as a null-geodesic, coincide (section 4, Proposition 5). In addition to that, we use, for the (more difficult) case of dimension 3, a criterion for conformal flatness from [2], and

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we apply it to a *locally defined*, by the *LeBrun correspondence* (see below), self-dual ambient.

The other topic of this paper uses implicitly another application of twistor theory: It has been shown by LeBrun [8], [9], that any conformal 3-manifold can be locally realized as the *conformal infinity* of a self-dual Einstein (with non-zero scalar curvature) 4-manifold. We have, thus, a local correspondence assigning to a conformal structure in dimension 3 a self-dual Einstein metric in dimension 4, which we call the *LeBrun correspondence.*

As conformal structures of both manifolds are encoded in the complex, resp. CR, structure of their twistor spaces, they are implicitly related, for example if the 3-manifold M is conformally flat, its ambient N equally is. It is, however, difficult to obtain an explicit relation between the conformal invariants of M and those of N by twistorial methods, as there is no simple expression of the *Cotton-York tensor* of M^3 in twistorial terms, and the twistorial interpretation of the *Weyl tensor* of N^4 is highly non-linear [2].

In this paper we find a relation between these two conformal invariants of the manifolds involved in the LeBrun correspondence, or, more generally, of an umbilic submanifold M^3 and of its self-dual ambient N^4. It appears that the Weyl tensor of N^4 identically vanishes along M^3, and thus the Cotton-York tensor of N^4, restricted to M^3, is conformally invariant and can be identified with the Cotton-York tensor of M^3; in this case, it is also equal to the normal derivative of the Weyl tensor of N^4 (section 3, Theorem 1). This gives conditions for an open self-dual 4-manifold to admit a conformal infinity.

The paper is organized as follows : in section 2 we recall a few basic facts about complex- Riemannian and -conformal geometry, in section 3 we relate the conformal invariants of a 3-dimensional conformal infinity to those of its self-dual ambient (arising from the LeBrun correspondence), and in section 4 we state our results about conformal complex manifolds containing compact null-geodesics.

Throughout the paper we use the following conventions: in complex-Riemannian (or -conformal) geometry we use the same terminology as in the real framework (metric, Levi-Civita connection, curvature), and the holomorphic bundles are denoted like the corresponding bundles in real geometry (for example, the holomorphic tangent bundle of $\mathbf{M}$ is denoted simply by $T\mathbf{M}$, rather than the more precise $T^{1,0}\mathbf{M}$) ; manifolds with holomorphic conformal structures are denoted by bold-face letters (except in section 3, where the results hold also in the real framework).

2. Holomorphic conformal geometry

Definition 1. *Let* $\mathbf{M}$ *be a complex manifold, let* n *be its complex dimension. A* complex-Riemannian metric g *on it is a holomorphic section of* $S^2T^*\mathbf{M}$ *which is non-degenerate at any point. A* holomorphic conformal structure *on* $\mathbf{M}$ *is a holomorphic line subbundle* C *in* $S^2T^*\mathbf{M}$ *such that any non-vanishing local section of* C *is a local complex-Riemannian metric.*

During the rest of this section, and of the whole fourth section of this paper, we shall simply denote these structures as *metric* and *conformal structure* (therefore omitting any reference to the complex framework).

A metric on $T\mathbf{M}$ induces metrics (i.e. non-degenerate symmetric bilinear forms) on all tensor bundles, in particular the square of the canonical bundle $\kappa := \Lambda^n T^*\mathbf{M}$ is trivialized. If we can choose *an orientation* (defined as follows), then the canonical bundle itself can be trivialized by a *volume form* of norm 1. There are exactly 2 such forms in each fiber of $\Lambda^n T^*\mathbf{M}$, and an *orientation* is the choice of one of them, depending continously (thus holomorphically) on the base point.

Remark. The notion of *orientation* is generally related to the reduction (when possible) of the structure group of the frame bundle from G to the connected component of the identity G_0; in absence of any structure, the group G is simply the connected $GL(n, \mathbb{C})$, so the notion of orientation has no meaning in "raw" complex geometry. But a Riemannian metric on $\mathbf{M}$ is equivalent to the reduction of its frame bundle to a $O(n, \mathbb{C})$-bundle, where $O(n, \mathbb{C}) := \{A \in GL(n, \mathbb{C}) | A^t A = \mathbf{1}\}$; a further choice of an orientation reduces the structure group of the frame bundle to the connected component of this group, containing the identity: $SO(n, \mathbb{C}) := O(n, \mathbb{C}) \cap SL(n, \mathbb{C})$.

Unlike in the real framework, these reductions, always possible on (small) contractible open sets, are submitted to some topological constraints if we want to define them globally on $\mathbf{M}$.

Weaker constraints are implied by the existence of a conformal structure : the square of the canonical bundle needs to admit a nth order root $L^{-2} \simeq C$, where $L := \kappa^{-1/n}$ is the[1] (respectively, one of the) *weight 1–density bundle* of the manifold $\mathbf{M}$. We can study the conformal structure C using the formalism of density bundles and of Weyl derivatives [4]. From now on, we shall not make use of the weight 1–density bundle, but only of $C = L^{-2}$, which is enough to define the conformal structure.

Remark. A conformal structure is equivalent to the reduction of the structure group of the frame bundle to $CO(n, \mathbb{C}) := O(n, \mathbb{C}) \times \mathbb{C}^* / \{\pm\mathbf{1}\}$ (the quotient is due to the fact that $-\mathbf{1} \in O(n, \mathbb{C})$). This group is disconnected if n is even, and the connected component of $\mathbf{1}$ is $CO_0(n, \mathbb{C}) = SO(n, \mathbb{C}) \times \mathbb{C}^* / \{\pm\mathbf{1}\}$, but it is connected if n is odd (and in this case the right hand side of the previous equality coincides with $CO(n, \mathbb{C})$). Therefore, although a complex-Riemannian 3-manifold admits, locally, 2 possible orientations, they are conformally equivalent, fact that makes impossible a canonical way to associate an orientation to a metric in the conformal class.

Remark. In complex-, as in real-Riemannian geometry, the orientation determines (and is determined by) a family of compatible oriented orthonormal basis in $T\mathbf{M}$; if $\dim \mathbf{M}$ is even, by multiplying all the vectors of such a basis by a non-zero complex number we obtain an oriented orthonormal basis compatible with another metric in the conformal class (if $\dim \mathbf{M}$ is odd, multiplication by -1 yields a basis compatible with the same metric, but with the opposite orientation).

[1] the dual of L is a square root of C; the choice of such a square root is implied, if $n = 2m + 1$, by the conformal structure C as $\kappa = C^m \otimes L^{-1}$; if n is even, neither C nor an orientation — see below — imply the choice of L.

The notion of orientation, in four-dimensional conformal geometry, is important for the definition of *anti–*, resp. *self-duality* (see below). More generally, the *Hodge $*$ operator* (defined as in real Riemannian geometry) is conformally invariant, and gives an explicit expression for the splitting of the bundle of 2-forms and of the curvature tensor [14], see also next section.

Maybe the simplest way to view a conformal structure is as an equivalence class c of *local* metrics, two such local representants g and h satisfying, on the open set where both are defined, to $g = fh$, with f a non-vanishing holomorphic function. Unlike in the real framework, global representants may not exist in general. From now on we shall consider a conformal structure on $\mathbf{M}$ as being given by the conformal class c rather than by the line bundle C.

Geometrically, a conformal structure is given by its *isotropy cone* $\mathcal{C} \subset T\mathbf{M}$ of vectors of norm 0. Because of the non-degeneracy of any local metric in c, the projective isotropy cone $\mathbb{P}(\mathcal{C})$ is a non-degenerate hyperquadric in $\mathbb{P}(T\mathbf{M})$. In dimension 3, $\mathbb{P}(\mathcal{C})$ is a conic (curve) in $\mathbb{CP}^2$, and in dimension 4 it is a ruled surface in $\mathbb{CP}^3$; therefore, in this latter case, there are 2 families — each of which can be characterized with respect to a given orientation [2] — of isotropic planes in $T\mathbf{M}$ called α-, resp. *β-planes.*

For any local metric we have a *Levi-Civita connection*, whose curvature has the same components as in the real Riemannian geometry (see next section). In particular, the *Weyl tensor* is independent of the metric in the conformal class.

The geodesics for which the tangent direction at a point (and thus, at any point) is isotropic are called *null-geodesics*, and they are locally independent (up to a reparametrization) of the metric in the conformal class. The same is true for higher-dimensional totally geodesic and isotropic submanifolds — if they exist —, called null-submanifolds. In dimension 4 they are α-, resp. *β-surfaces* (tangent to α-, resp. β-planes, see above), and they exist if and only if the (oriented) conformal structure is *anti-*, resp. *self-dual*, i.e. the component W^+, resp. W^-, of the Weyl tensor W of $(\mathbf{M}, c)$ vanishes identically [1],[12],[2].

If the manifold is self-dual, one considers locally the twistor space Z of $(\mathbf{M}^4, c)$ as the set of β-surfaces[2]. It is a 3-dimensional complex manifold [1],[12] containing rational curves whose normal bundle is isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$ (called *twistor lines*) (where $\mathcal{O}(1)$ is the dual of the *tautological bundle* $\mathcal{O}(-1)$ of $\mathbb{CP}^1$). If, in addition, we can choose an Einstein metric g in the conformal class c, we get an extra structure on Z, namely a distribution of 2-planes, which is totally integrable (and yields a foliation) if the scalar curvature of g vanishes, otherwise it is a contact structure [15],[5],[8]. Conversely, from a manifold Z containing twistor lines as above (called a *twistor space*), plus — possibly — the additional 2-planes distribution, one can recover — at least locally —, via the *reverse Penrose construction* [1], the self-dual manifold $(\mathbf{M}^4, c)$.

[2]see [8], [11], [2] and section 4 for an explanation of the difficulties of a global definition of the twistor space.

In all generality, one can always consider locally (on a geodesically convex open set, for example, see section 4) the space of null-geodesics of a conformal manifold $(\mathbf{M}^n, c)$, and the key point in the *LeBrun correspondence* (defined below) is that the space of null-geodesics of a 3-dimensional conformal manifold $(\mathbf{M}^3, c)$ (also called the *twistor space* of $\mathbf{M}$) is a twistor space endowed with a contact structure, therefore we get (locally again) a self-dual manifold $(\mathbf{N}^4, c)$, in which $(\mathbf{M}^3, c)$ is umbilic, and it is the *conformal infinity* of an Einstein metric g on $\mathbf{N}$ with non-zero scalar curvature [8], [9].

Definition 2. *Let* $(\mathbf{M}, c)$ *be a conformal 3-manifold, that we shall suppose* civilized *(*e.g. *geodesically connected for some metric in the conformal class). The* LeBrun correspondence *associates to* $\mathbf{M}$ *the (germ-unique) self-dual Einstein 4-manifold* $\mathbf{N}$ *such that the twistor spaces of* $\mathbf{M}$ *and* $\mathbf{N}$ *coincide.*

Proposition 1. [8], [9] *In the LeBrun correspondence,* $(\mathbf{M}^3, c)$ *is an umbilic hypersurface of* $(\mathbf{N}^4, c)$ *(and has the induced conformal structure) and the Einstein metric of* $\mathbf{N}^4$ *has a second order pole at* $\mathbf{M}^3$ *(conformal infinity). Conversely, in such a geometric setting, the twistor spaces of the manifolds* $\mathbf{M}$ *and* $\mathbf{N}$ *coincide.*

Remark. There is no *a priori* definition of a conformal infinity of an open (real- or complex-) Riemannian manifold X, even if the metric is complete (in the real framework). Here we consider uniquely the case when this infinity is an (umbilic) submanifold (or boundary) of a conformal extension of X, $\bar{X} \supset X$; the conformal structure extends smoothly beyond the infinity. In other cases, in which the notion of conformal infinity can still be defined, the conformal structure is singular at infinity, which, in these cases, is no longer conformal, but admits instead a CR [6],[3] or a quaternionic contact structure [3].

3. Conformal infinity of a self-dual manifold

The object of this section is to find a relationship between the conformal invariants of a conformal infinity and of its self-dual ambient arising from the LeBrun correspondence. The results are local, and they hold in the complex as well as in the real Riemannian or in the signature (2,2) pseudo-Riemannian framework. We begin by recalling a few facts about the conformally invariant tensors in Riemannian geometry.

For a n-dimensional Riemannian manifold (M^n, g), the curvature has the following expression:

$$R^M(X,Y) = (h \wedge \boldsymbol{I})(X,Y) + W(X,Y), \text{ where} \tag{1}$$

$$(h \wedge \boldsymbol{I})(X,Y) := h(X,\cdot) \wedge Y - h(Y,\cdot) \wedge X, \ \forall X, Y \in TM, \tag{2}$$

is *the suspension* by the identity $\boldsymbol{I}$ of the *normalized Ricci tensor*

$$h = \frac{1}{2n(n-1)} Scal^g \cdot g + \frac{1}{n-2} Ric_0; \tag{3}$$

$Scal$ and Ric_0 are the scalar curvature, resp. the trace-free Ricci tensor, and, together with the Weyl tensor W, they are the irreducible components of the curvature under the orthogonal group if $n \geq 5$. If $n = 3$, W vanishes identically, and if $n = 4$ it further decomposes in two irreducible components

W^+, resp. W^-, called the *self-dual*, resp. *anti-self-dual* (or *positive*, resp. *negative*) Weyl tensor.

The Weyl tensor, viewed as a section in $\mathrm{Hom}(\Lambda^2 TM \otimes TM, TM)$ (as a (3,1)–tensor), is conformally invariant, and, if $n \geq 4$, it completely determines, locally, the conformal structure of $(M, [g])$ (for a self-dual manifold, $W^- \equiv 0$, thus the Weyl tensor actually coincides with W^+). In dimension $n = 3$, this function is fulfilled by *the Cotton-York tensor*, which can be defined in all dimensions by

$$C(X,Y)(Z) := (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z),\ \forall X,Y,Z \in TM, \tag{4}$$

and it can be shown that, for another metric $g' := e^{2\varphi} g$ in the same conformal class, the corresponding Cotton-York tensor C' is related to C by the formula [4]:

$$C'(X,Y)(Z) = C(X,Y)(Z) - \mathrm{d}\varphi(W(X,Y)Z). \tag{5}$$

In particular C is conformally invariant along the zero set of W, thus everywhere if $\dim M = 3$.

Remark. The Cotton-York tensor C of M is a 2-form with values in T^*M, and it satisfies a *first* Bianchi identity, as h is a symmetric tensor, and also a *contracted* (second) Bianchi identity, coming from the second Bianchi identity in Riemannian geometry, [4] :

$$\sum C(X,Y)(Z) \quad = \quad 0 \ \text{ circular sum;} \tag{6}$$

$$\sum C(X,e_i)(e_i) \quad = \quad 0 \ \text{ trace over an orthonormal basis.} \tag{7}$$

This means that C is an irreducible tensor if $n = 3$ or $n > 4$, and, if $n = 4$, C has two irreducible components, the *self-dual*, resp. the *anti-self-dual* Cotton-York tensor

$$C^+ \in \Lambda^+ M \otimes \Lambda^1 M, \text{ resp. } C^- \in \Lambda^- M \otimes \Lambda^1 M.$$

They both satisfy (6) and (7) (note that these two relations are equivalent in their case).

The Cotton-York tensor is related to the Weyl tensor of M by the formula [4]:

$$\delta W = C, \tag{8}$$

where $\delta : \Lambda^2 M \otimes \Lambda^2 M \to \Lambda^2 M \otimes \Lambda^1 M$ is induced by the codifferential on the second factor, and by the Levi-Civita connection ∇. Then, again if $n = 4$, C^+ has to be the component of δW in $\Lambda^+ M \otimes \Lambda^1 M$, and we know that the restriction of W^- to $\Lambda^+ M \otimes \Lambda^2 M$ is identically zero. This means that

$$\delta W^+ \quad = \quad C^+, \text{ and also} \tag{9}$$

$$\delta W^- \quad = \quad C^-. \tag{10}$$

We have thus:

Lemma 1. *On a self-dual manifold, C^- vanishes identically.*

We consider now the situation in the LeBrun correspondence : Let (M, c) be a 3-dimensional conformal manifold, and we suppose, without any local loss of generality, that it is the conformal infinity of the self-dual manifold (N, c) (no use will be made of the Einstein metric on it); $M \subset N$ is, thus, an umbilic hypersurface, such that the restriction of the conformal structure c of N to M is non-degenerate (equivalently, TM is nowhere tangent to an isotropic cone) and coincides with the conformal structure, still denoted by c, on M.

If we introduce the Hodge operator $*^M : \Lambda^2 M \to \Lambda^1 M$, then the curvature tensor R^M is equivalent to the symmetric 2-tensor $*^M \circ R^M \circ *^M$. A straightforward application of the above formula yields

$$*R^M := *^M \circ R^M \circ *^M = -h + (\mathrm{tr} h)I. \tag{11}$$

For the 4-dimensional manifold N, the components of the Riemannian curvature can also be expressed as eigenspaces of $*$-type operators. Namely, considering $R := R^N$ as a symmetric endomorphism of $\Lambda^2 N = \Lambda^+ N \oplus \Lambda^- N$, W^+ is the trace-free component of R in $\mathrm{End}(\Lambda^+ N)$, and W^- is the trace-free component of R in $\mathrm{End}(\Lambda^- N)$ [14]. (The scalar curvature is four times the trace of $R|_{\Lambda^+}$ or of $R|_{\Lambda^-}$, and the trace-free Ricci tensor is identified to the component of R sending Λ^+ into Λ^- [14].)

We can canonically identify $\Lambda^+ N$ and $\Lambda^- N$, restricted to $M \subset N$, to $\Lambda^2 M$, by:

$$\begin{array}{rcl} \Lambda^2 M \ni \alpha & \mapsto & \alpha + *^N \alpha \in \Lambda^+ N \\ \Lambda^2 M \ni \alpha & \mapsto & \alpha - *^N \alpha \in \Lambda^- N. \end{array} \tag{12}$$

Our first result is:

Theorem 1. *Let M be an umbilic hypersurface of a self-dual manifold N. Then:*

(i) The Weyl tensor of N vanishes along M :

$$W^+|_M \equiv 0;$$

(ii) The Cotton-York tensor of M is related to the self-dual Weyl tensor of N by the formula:

$$g(\nabla_\nu W^+(A), B)_x = -C(A)(*^M B)_x, \ \forall x \in M$$

where $A, B \in \Lambda^2 T_x M$, $\nu \perp T_x M$ is unitary for the metric g, and the Hodge operator $*^M$ is induced by g and the orientation on M admitting ν as an exterior normal vector.

(iii) The restriction to M of the (self-dual) Cotton-York tensor of N is equal to the Cotton-York tensor of M :

$$C^+(X,Y)(Z) = C^M(X,Y)(Z), \ \forall X, Y, Z \in TM.$$

Proof. The claimed identities are conformally invariant : for *(i)* it is obvious, and the conformal invariance of *(iii)* follows from *(i)* and (5); to see that for *(ii)*, let X, Y, Z, ν be a g-orthonormal oriented basis of N, such that X, Y, Z is a g-orthonormal basis on M giving the orientation as above. Then $*^M(Z \wedge X) = Y$, and, if we take $A := X \wedge Y$, $B := Z \wedge X$, the identity *(ii)* becomes

$$\langle \nabla_\nu W^+(X,Y)Z, X \rangle = -C(X,Y)(Y), \tag{13}$$

where angle brackets denote the scalar product induced by g.

The tensors W^+, C, in the above form, are independent of the chosen metric g [4], which depends on the normal vector ν, supposed to be g-unitary. If $\nu' := \lambda \nu$, for $\lambda \in \mathbb{C}^*$, then the corresponding metric $g' = \lambda^{-2} g$, and also $*^{M'} = \lambda^{-1} *^M$, thus the identity (13) for ν', g' is equivalent to the one for ν, g.

Remark. As W^+ is the trace-free component of the Riemannian curvature contained in $End(\Lambda^+ N)$, and is symmetric, it is enough to evaluate it on pairs $A, B \in \Lambda^2 M \simeq \Lambda^+ N$ which are unitary and orthogonal for the metric g, therefore the check of the equation (13) will prove the Theorem.

As $W^\pm$ are $*^N$-eigenvectors in $End_0(\Lambda^2 N)$ (the space of trace-free endomorphisms of $\Lambda^2 N$), they are determined by the following formulas, where X, Y, Z is any oriented orthonormal basis of TM:

$$\begin{aligned}(14)\qquad \langle W^+(X,Y)Z, X\rangle &= \frac{1}{4}(\langle R(X,Y)Z,X\rangle + \langle R(Z,\nu)Y,\nu\rangle + \\ &\quad + \langle R(X,Y)Y,\nu\rangle + \langle R(Z,\nu)Z,X\rangle) \\ (15)\qquad \langle W^-(X,Y)Z, X\rangle &= \frac{1}{4}(\langle R(X,Y)Z,X\rangle + \langle R(Z,\nu)Y,\nu\rangle - \\ &\quad - \langle R(X,Y)Y,\nu\rangle - \langle R(Z,\nu)Z,X\rangle),\end{aligned}$$

where X, Y, Z, ν is supposed to be a local extension, around a region of M, of the g-orthonormal frame used in (13). As N is self-dual, W^- is identically zero, thus, in the points $x \in M$, we have

$$(16)\qquad \langle W^+(X,Y)Z,X\rangle_x = \frac{1}{2}(\langle R(X,Y)Y,\nu\rangle + \langle R(Z,X)Z,\nu\rangle)_x.$$

It is a standard fact that, if M is umbilic, there is a local metric g in the conformal class c of N, such that, for g, M is totally geodesic. Without any loss of generality, because of the conformal invariance of the claimed identities (see above), we fix such a metric. Then we have

$$(17)\qquad R(X',Y')Z' = R^M(X',Y')Z',\ \forall X',Y',Z' \in TM,$$

which, together with (16), implies that $W^+|_M \equiv 0$, and thus proves the point *(i)* in the Theorem.

On the other hand, (17), together with (16) and (15), yield

$$(18)\qquad \langle R(X,Y)Z,X\rangle_x + \langle R(Z,\nu)Y,\nu\rangle_x = 0, \forall x \in M.$$

Let us compute now the normal derivative of W^+ in a point $x \in M$; we suppose that X, Y, Z, ν are locally extended by an orthonormal frame, and that they are parallel at x (we omit, for simplicity of notation, the point x in the following lines:

$$\langle \nabla_\nu W^+(X,Y)Z,X\rangle = \frac{1}{2}(\langle \nabla_\nu R(X,Y)Y,\nu\rangle + \langle \nabla_\nu R(Z,X)Z,\nu\rangle),$$

from (16). This is then equal to:

$$\begin{aligned}\langle \nabla_\nu W^+(X,Y)Z,X\rangle &= -\tfrac{1}{2}(\langle \nabla_X R(Y,\nu)Y,\nu\rangle + \langle \nabla_Y R(\nu,X)Y,\nu\rangle + \\ &\quad + \langle \nabla_Z R(X,\nu)Z,\nu\rangle + \langle \nabla_X R(\nu,Z)Z,\nu\rangle),\end{aligned}$$

from the second Bianchi identity. Then we have

$$\begin{aligned}\langle \nabla_\nu W^+(X,Y)Z,X\rangle &= \tfrac{1}{2}(\langle \nabla_X R(Z,X)Z,X\rangle + \langle \nabla_Y R(Z,Y)Z,X\rangle + \\ &\quad + \langle \nabla_Z R(Y,Z)X,Y\rangle + \langle \nabla_X R(Y,X)X,Y\rangle)\end{aligned}$$

from analogs of (18). Then

$$\begin{aligned}\langle\nabla_\nu W^+(X,Y)Z,X\rangle = \tfrac{1}{2}(\langle\nabla_X R^M(Z,X)Z,X\rangle + \langle\nabla_Y R^M(Z,Y)Z,X\rangle +\\ +\langle\nabla_Z R^M(Y,Z)X,Y\rangle + \langle\nabla_X R^M(Y,X)X,Y\rangle)\end{aligned}$$

from (17)

$$\begin{aligned}\langle\nabla_\nu W^+(X,Y)Z,X\rangle &= \tfrac{1}{2}(\nabla_X h(Z,Z) + \nabla_X h(X,X) + \nabla_Y h(Y,X) -\\ &\quad -\nabla_Z h(Z,X) - \nabla_X h(X,X) - \nabla_X h(Y,Y)),\end{aligned}$$

from (1). Finally, from (7), we get

$$\langle\nabla_\nu W^+(X,Y)Z,X\rangle = \frac{1}{2}(C(X,Z)(Z) - C(X,Y)(Y)) = -C(X,Y)(Y)$$

This proves equation (13) and the point *(ii)* in the Theorem.

For the point *(iii)*, we use (13) and (9); from the codifferential of W^+, only the derivative along the normal vector, ν, can be non-zero, as W^+ vanishes along M. This proves the Theorem. □

Remark. The point *(i)* gives a condition for a self-dual manifold to admit an umbilic hypersurface : W^+ has to vanish along it, generically at order 0 (following *(ii)*), thus such a hypersurface, if it exists, is locally defined as the zero set of W^+.

Considering the umbilic hypersurfaces which arise from the LeBrun correspondence, the point *(i)* gives a condition for an open self-dual manifold to admit such a conformal infinity, namely it has to be asymptotically conformally flat.

4. Null-geodesics of complex conformal manifolds

4.1. **Properties of the twistor space of a 3-dimensional conformal manifold.** Consider $(\mathbf{M},c)$ a complex 3-dimensional conformal manifold. In some topological conditions ($\mathbf{M}$ has to be *civilized* [8] ; as a geodesically convex set is always of this type [11], any point has a basis of civilized neighbourhoods), the *twistor space* of $\mathbf{M}$ is defined as the space Z of null-geodesics of $\mathbf{M}$, and it is a complex 3-manifold, containing *twistor lines* (*i.e.* rational curves with normal bundle isomorphic to $\mathcal{O}(1)\oplus\mathcal{O}(1)$), and endowed with a distribution of 2-planes $F_{\bar\gamma} \subset T_{\bar\gamma}Z$ which is a contact structure [8].

We denote by $\bar\gamma$ the point of Z corresponding, in the following way, to the null-geodesic γ : some twistor lines tangent to $F_{\bar\gamma}$ (actually a non-empty open set in the space of curves of Z, tangent to F) correspond to the points of the null-geodesic γ [3].

The first question we raise is whether there exist twistor lines tangent to any direction of a given 2-plane $F_{\bar\gamma}$; the answer is:

Theorem 2. *Let Z be a twistor space of a conformal civilized 3-manifold $\mathbf{M}$; let $F_{\bar\gamma} \subset T_{\bar\gamma}Z$ be its contact structure. Suppose there is a point $\bar\gamma \in Z$ such that there are twistor lines tangent to any direction in $F_{\bar\gamma}$. Then Z is projectively flat, and $\mathbf{M}$ is conformally flat.*

[3] deformations of these twistor lines are, in general, not tangent to the distribution F; they correspond to points in the self-dual ambient N (and outside of M) arising from the LeBrun correspondence.

This follows directly from [2], Theorem 3, which has a similar statement referring to the twistor space of a self-dual manifold; we simply apply it to the ambient $\mathbf{N}$ from the LeBrun correspondence; its conformal flatness implies the flatness of $\mathbf{M}$ (Theorem 1).

Remark. The key point in the above cited Theorem is a *twistorial interpretation* of the Weyl tensor of a self-dual manifold $\mathbf{N}$ [2], Theorem 2, together with the remark that, for a given 2-plane F in $T_{\bar{\beta}}Z$, the union of all twistor lines tangent to it (supposing there exists one pointing in any direction of F) is a complex surface which is *smooth* at $\bar{\beta} \in Z$. The above cited Theorem and the following one (Theorem 3$'$ from [2]) show that this situation implies the vanishing of the Weyl tensor of $\mathbf{N}$, W^+, in certain directions:

Theorem 3. [2] *Let Z be the twistor space of the (civilized) self-dual manifold $\mathbf{N}$, and let $\bar{\beta} \in Z$ be a point in Z, corresponding to the β-surface $\beta \subset \mathbf{N}$; let $F^\gamma \subset T_{\bar{\beta}}Z$ be a 2-plane, corresponding to the null-geodesic $\gamma \subset \beta$ [10]. Suppose that, for each direction $\sigma \in \mathbb{P}(F^\gamma) \subset \mathbb{P}(T_{\bar{\beta}}Z)$, there is a smooth (non-necessarily compact) curve Z_σ tangent to σ, such that:*
(i) if σ is tangent to a twistor line Z_x at $\bar{\beta}$, then $Z_\sigma = Z_x$;
(ii) Z_σ varies smoothly with $\sigma \in \mathbb{P}(F^\gamma)$.
Then

$$\bar{Z}^\gamma_{\bar{\beta}} := \bigcup_{\sigma \in \mathbb{P}(F^\gamma)} Z_\sigma$$

is a smooth surface around β, and $W^+(F^\gamma_x) = 0$, $\forall x \in \gamma$, where $F^\gamma_x \subset T_x\mathbf{M}$ is the α-plane containing $\dot{\gamma}$.

In other words, if the *integral α-cone* corresponding to the 2-plane $F \subset T_{\bar{\gamma}}Z$ — defined as the union of all twistor lines tangent F [2] — can be completed to a surface, smooth around its "vertex" $\bar{\gamma}$, then W^+ vanishes along the null-geodesic γ (whose points correspond to the twistor lines that constitute the α-cone [2]).

4.2. **Compact null-geodesics and conformal flatness.** Our main result is:

Theorem 4. *Let $\mathbf{M}$ be a conformal n-manifold containing an immersed rational curve as null-geodesic. Then $\mathbf{M}$ is conformally flat.*

This fact has been proven by Ye [17] for complex projective manifolds — we are grateful to B. Klingler for having brought this paper into our attention. In the above Theorem, we do not make any assumption on the topology of the manifold, but only of one null-geodesic contained in it.

Proof. The proof is different in the cases $n > 3$ and $n = 3$; one of the reasons is that conformal flatness reduces, in higher dimensions, to the vanishing of the Weyl tensor, while in dimension 3 it is a higher-order condition more difficult to handle. The first step (common to all cases) is to prove that a small deformation (seen just as a compact submanifold of $\mathbf{M}$, [7]) of such a compact null-geodesic γ is still a compact null-geodesic, and to characterize the global sections of the normal bundle of γ as locally determined by *Jacobi fields.*

Lemma 2. *Let γ be a (immersed) null-geodesic in $(\mathbf{M}, c)$. Let J be a vector field along γ. Then the condition $\dot{J} \perp \dot{\gamma}$ (where $\dot{J} := \nabla^g_{\dot{\gamma}} J$) is independent of the metric g with respect to which we take the derivative ∇^g.*

Proof. The relation between two Levi-Civita connections (or, more generally, *Weyl structures*) of metrics in the same conformal class, is given by [4]:

$$\nabla'_X Y - \nabla_X Y = \theta(X)Y + \theta(Y)X - \theta^\sharp g(X,Y), \tag{19}$$

where θ is a 1-form, and the rising of indices in $\theta^\sharp$ is made using the same (arbitrary) metric $g \in c$ as in the scalar product $g(X,Y)$. The Lemma immediately follows. □

Denote by $N(\gamma)$ the normal bundle of γ in $\mathbf{M}$, and by $N^\perp(\gamma)$ its subbundle represented by vectors orthogonal to $\dot{\gamma}$. Fix a metric g in the conformal class c. Let J be a Jacobi field along γ, satisfying to the Jacobi equation

$$\ddot{J} = R^g(\dot{\gamma}, J)\dot{\gamma}.$$

It represents an infinitesimal deformation of γ through null-geodesics if and only if $\dot{J} \perp \dot{\gamma}$. J induces a section in the normal bundle $N(\gamma)$, or $N^\perp(\gamma)$ if $J \perp \dot{\gamma}$ in a point, hence everywhere. We want to prove that this section is independent of the connection ∇^g:

Proposition 2. *The Jacobi equations on a null-geodesic $\gamma \subset \mathbf{M}$ induce a second order linear differential operator P on $N(\gamma)$, which, restricted to the sections J such that $\dot{J} \perp \dot{\gamma}$, depends only on the conformal structure c of $\mathbf{M}$. In particular, P restricted to $N^\perp(\gamma)$ is conformally invariant.*

Proof. For a Levi-Civita connection ∇ of a local metric on $\mathbf{M}$, we locally define the following differential operator on the sections of $T\mathbf{M}|_\gamma$:

$$P : \Gamma(T\mathbf{M}|_\gamma \otimes S^2(T\gamma)) \to \Gamma(T\mathbf{M}|_\gamma),$$

by $P(Y; X, X) := \nabla_X \nabla_X Y - \nabla_{\nabla_X X} Y - R(X,Y)X$. Because γ is a null-geodesic, P induces a (local) differential operator on $N(\gamma)$, and we need to relate P to the corresponding operator P' induced by another connection ∇'. First we write

$$P(Y, X, X) = \nabla_X [X, Y] + \nabla_{[X,Y]} X - [\nabla_X X, Y],$$

then we recall that another Levi-Civita connection ∇' is related to ∇ by the formula (19), such that we get

$$P'(Y; X, X) - P(Y; X, X) = 2[Y.(\theta(X)) - \theta([X,Y])]X - g(\nabla_X Y, X)\theta^\sharp,$$

which is identically zero modulo $T\gamma$, provided that $\nabla_X Y \perp X$ (the latter condition being independent of the Levi-Civita connection, according to the previous Lemma). □

Using the fact that $\mathbb{CP}^1$ is the union of two contractible sets $U_1 \cup U_2$ (on each of which the Jacobi equation, with any initial condition — the same for U_1 and for U_2 — in $x_0 \in U_1 \cap U_2$, has a unique solution — and these solutions necessarily coincide on the connected intersection $U_1 \cap U_2$), we immediately get:

Proposition 3. *Let γ be an immersed null-geodesic, diffeomorphic to a projective line $\mathbb{CP}^1$. Then any local Jacobi field J with $\dot{J} \perp \dot{\gamma}$ induces a global normal field ν^J on γ.*

This has important consequences about the normal bundle of γ in $\mathbf{M}$, as Jacobi fields provide it with global sections; in particular, $N(\gamma)/N^{\perp}(\gamma)$ is a line bundle admitting nowhere-vanishing sections, hence it is trivial; on the other hand, $N^{\perp}(\gamma)$ is a $(n-2)$–rank bundle over $\mathbb{CP}^1$, admitting sections with any prescribed 1-jet (induced, again, by some appropriate Jacobi fields), hence

$$N^{\perp}(\gamma) \simeq \bigoplus_{k=1}^{n-2} \mathcal{O}(a_k);\ N(\gamma) \simeq \bigoplus_{k=1}^{n-2} \mathcal{O}(a_k) \oplus \mathcal{O}(0);\ a_k \in \mathbb{N}^*. \tag{20}$$

For $a_k \in \mathbb{Z}$, this is the general form of a vector bundle over $\mathbb{CP}^1$, according to a theorem of Grothendieck; the condition $a_k \geq 1$ arises from the existence of sections of $N^{\perp}(\gamma)$ with prescribed 1-jet. We are going to show later that all a_k are equal to 1. First we prove:

Proposition 4. *Null-geodesics close to a compact, simply-connected one are also compact and simply-connected, and they are generically embedded.*

Proof. We consider the projectivized bundle $\mathbb{P}(\mathcal{C})$ of the isotropic cone $\mathcal{C} \subset T\mathbf{M}$. It is a standard fact [8] that any null-geodesic $\gamma \subset \mathbf{M}$ has a canonical *horizontal* lift $\tilde{\gamma} \subset \mathbb{P}(\mathcal{C})$ (depending only on the conformal structure), such that $\pi_*(T_s\tilde{\gamma}) = T_x\gamma$, where $\pi : \mathbb{P}(\mathcal{C}) \longrightarrow \mathbf{M}$ is the projection, and $s \in \pi^{-1}(x)$.

Note that $\tilde{\gamma}$ is always embedded, even if γ may have self-intersections (it is always immersed).

The lifts of the null-geodesics of $\mathbf{M}$ consist in a foliation of $\mathbb{P}(\mathcal{C})$, which has a compact, simply-connected leaf, namely the lift $\tilde{\gamma}$ of our compact, simply-connected null-geodesic γ. By Reeb's stability Theorem [13], then there is a *saturated* tubular neighbourhood of $\tilde{\gamma}$, diffeomorphic to $\tilde{\gamma} \times D$ (where $D \subset \mathbb{C}^{n-2}$ is a polydisc), such that the leaves close to $\tilde{\gamma}$ are identified, via the above diffeomorphism, to the (compact and simply-connected) curves $\tilde{\gamma} \times \{\mathrm{z}\},\ z \in D$.

So all null-geodesics close to γ are compact and simply connected. If γ has self-intersections at the points $x_1, \ldots, x_k$, we blow-up $\mathbf{M}$ at those points, and the lift of γ is now embedded. So must be then the lifts of the null-geodesics close to γ, as they are now deformations of the lifted (hence, embedded) curve. But, generically, such curves avoid the finite set of points $x_1, \ldots, x_k$; the corresponding null-geodesics must have been embedded from the beginning. ☐

From now on, according to the previous Proposition, we may suppose that γ is a compact, simply-connected, *embedded* null-geodesic.

We compute the normal bundle of γ in $\mathbf{M}$, using the relation (20) and the projection $\pi : \mathbb{P}(\mathcal{C}) \longrightarrow \mathbf{M}$, as follows: We have the following exact sequence of bundles:

$$0 \to N^{\pi}(\tilde{\gamma}) \to N(\tilde{\gamma}) \to \pi^* N(\gamma) \to 0, \tag{21}$$

where $N^{\pi}(\tilde{\gamma})$ is the normal subbundle of $\tilde{\gamma}$ represented by vectors tangent to the fibers of π and $N(\tilde{\gamma})$ is the normal bundle of $\tilde{\gamma}$ in $\mathbb{P}(\mathcal{C})$. In a point

$T_x\gamma \in \tilde{\gamma} \subset \mathbb{P}(\mathcal{C})$, the fiber of π is equal to $\mathbb{P}(\mathcal{C})_x$, so the tangent space to it is isomorphic to $\mathrm{Hom}(T_x\gamma, N_x^{\perp}(\gamma))$, for the projective variety $\mathbb{P}(\mathcal{C})_x \subset \mathbb{P}(T_x\mathbf{M})$. Thus

$$N^{\pi}(\tilde{\gamma}) \simeq \mathrm{Hom}(T\gamma, N^{\perp}(\gamma)) \simeq \mathcal{O}(-2) \otimes N^{\perp}(\gamma),$$

as $T\gamma \simeq T\mathbb{CP}^1 \simeq \mathcal{O}(-2)$.

The central bundle in the exact sequence (21) is trivial, because $\tilde{\gamma}$ is a leaf of a foliation. (20) and (21) imply that the Chern numbers $a_1, \dots, a_{n-2}$ are subject to the following constraint:

$$\sum_{k=1}^{n-2}(2a_k - 2) = 0,$$

thus, as $a_k \geq 1$, we have $a_k = 1$, $\forall k = \overline{1, n-2}$. We have then:

Proposition 5. *The normal bundle of a compact, simply-connected, null-geodesic γ in $\mathbf{M}$ is isomorphic to*

$$N(\gamma) \simeq \big(\mathbb{C}^{n-2} \otimes \mathcal{O}(1)\big) \oplus \mathcal{O}(0),$$

and all its global sections are induced by Jacobi fields J such that $\dot{J} \perp \dot{\gamma}$. Moreover, the deformations of γ as a compact curve coincide with the null-geodesics close to γ.

The last statement follows from the expression of the normal bundle, and a Theorem of Kodaira [7]: the normal bundle satisfies $H^1(N(\gamma)) = 0$, thus the space of deformations of γ as a compact curve has dimension equal to $\dim H^0(N(\gamma)) = \dim \Gamma(N(\gamma)) = 2n-3$, which is precisely the dimension of the space of null-geodesics, defined locally, over a geodesically convex open set, as the space of the leaves of the horizontal foliation of $\mathbb{P}(\mathcal{C})$ [11]. We conclude using the fact that all null-geodesics close to γ are deformations of this one (as a compact, and simply-connected, curve).

We return to the proof of Theorem 4. Consider first the case when the dimension of $\mathbf{M}$, $n > 3$. We are going to show that the Weyl tensor of $\mathbf{M}$, W, is identically zero (a special sub-case is $n = 4$, when $W = W^+ + W^-$). For simplicity, suppose first that $n > 4$, and consider the fiber of $N^{\perp}(\gamma)$ at an arbitrary point $x \in \gamma$: it has a non-degenerate conformal structure, induced from $\mathbf{M}$, and the isotropy cone spans the whole fiber $N^{\perp}(\gamma)_x$ (this still holds for $n = 4$, but not for $n = 3$). Let $L \subset N^{\perp}(\gamma)$ be an isotropic line. We have:

Lemma 3. *Let γ^U be an open set of the null-geodesic γ, on which local metrics $g, g' \in c$ are well defined. If a (locally defined, over γ^U) line subbundle $L \subset N^{\perp}(\gamma)$ is parallel (or stable) for ∇^g, then it is parallel for $\nabla^{g'}$ as well.*

The proof is a straightforward application of (19).

Let $(L_1)_x, \dots, (L_{n-2})_x$ be linearly independent isotropic lines in $N^{\perp}(\gamma)_x$. According to the previous Lemma, and to the fact that γ is simply-connected, their parallel transport over γ does not depend on any Levi-Civita connection of a metric in the conformal class. We get, thus, a *global* splitting

$$N^{\perp}(\gamma) = L_1 \oplus \cdots \oplus L_{n-2}, \tag{22}$$

where the line bundles L_i, $i = \overline{1, n-2}$ are all isotropic and parallel.

All these bundles are isomorphic to $\mathcal{O}(b_i),\ b_i \in \mathbb{Z}$. As their sections are also sections of $N^{\perp}(\gamma) \simeq \mathbb{C}^{n-2} \otimes \mathcal{O}(1)$, they cannot vanish at more that 1 point, for each L_i, thus $b_i \leq 1,\ i = \overline{1, n-2}$. On the other hand, the sum of all b_i's has to be $n-2$, thus $b_i = 1,\ \forall i = \overline{1, n-2}$.

Let ϕ_i be a section of L_i; from Proposition 5, it is locally represented by a Jacobi field J_i, for the metric $g \in c$. From the Jacobi equation, by taking the scalar product with J, we get:

$$g(R(\dot{\gamma}, J)\dot{\gamma}, J) = 0, \tag{23}$$

and it is easy to see that, because of the fact that all scalar products involving $\dot{\gamma}$ and J are 0, the term $h \wedge \boldsymbol{I}$ of the curvature (1) satisfies the above relation identically. This equation holds, at x, for any isotropic vector $J_x \perp \dot{\gamma}_x$, but we may consider also other compact, simply-connected, null-geodesics γ', containing x, and close to γ (namely, small deformations of the compact curve γ).

For any 2-plane $F \subset T_x\mathbf{M}$, we denote by R^F the *sectional curvature* of F:

$$R^F : S^2(\Lambda^2 F) \to \mathbb{C},\ R^F(X \wedge Y, X \wedge Y) := \langle R(X \wedge Y), X \wedge Y \rangle,\ \forall X \wedge Y \in \Lambda^2 F,$$

and we have seen that, if F is totally isotropic, R^F depends only on W (and on the metric g only via the scalar product $\langle .,. \rangle$).

Lemma 4. *If $\dim \mathbf{M} > 4$, the Weyl tensor at $x \in \mathbf{M}$, W_x, is determined by the sectional curvatures $\{R^F,\ F \in U(F_0)\}$, for $U(F_0)$ a small neighbourhood of the totally isotropic arbitrary 2-plane $F_0 \subset T_x\mathbf{M}$ in the Grassmanian of totally isotropic 2-planes at x.*

Remark. A similar statement holds in dimension 4, but in that case, the Grassmanian of totally isotropic 2-planes has 2 connected components; as a consequence, W^+ is determined by the sectional curvatures of α-planes, and W^- by the sectional curvatures of the β-planes [2], Proposition 2.

Proof. This reduces to the claim that $W_x = 0$ if and only if

$$R^F = 0,\ \forall F \in U(F_0), \tag{24}$$

which is a problem of linear algebra. If we consider the space $\mathcal{K}$ of all curvature tensors R' satisfying $(R')^F = 0,\ \forall F \in U(F_0)$, then this is a vector space, which is invariant to the action of $\mathfrak{so}(n, \mathbb{C})$ (which is the Lie algebra infinitesimal action corresponding to the action of $SO(n, \mathbb{C})$ — note that the Grassmanian of totally isotropic planes is preserved by this action). But there are only 3 $\mathfrak{so}(n, \mathbb{C})$-irreducible components of the space of curvature tensors, and we have seen that for the Ricci-like tensors $h \wedge \boldsymbol{I}$, the totally isotropic planes always have zero sectional curvature. Then either any Weyl tensor satisfying (24) is zero at x, or $\mathcal{K}$ contains the whole space of curvature tensors. The latter possibility can easily be excluded by an example of a curvature tensor K satisfying:

$$K(X_0, Y_0)X_0 = A_0,\ \text{where } \langle Y_0, A_0 \rangle = 1,\ \text{and}$$
$$\langle X_0, X_0 \rangle = \langle X_0, Y_0 \rangle = \langle Y_0, Y_0 \rangle = \langle X_0, A_0 \rangle = 0.$$

□

From (23) and the previous Lemma we conclude that $W_x = 0$ for any x contained in a compact, simply-connected, null-geodesic; but we know from Proposition 5 that the set of such points contains a neighbourhood of γ, thus, by analyticity of W, it vanishes identically.

The proof is similar in dimension 4 (note that, in the self-dual case, we can retrieve the result by applying Theorem 3; this is how we shall proceed for the case of dimension 3, using the LeBrun correspondence); the difference with the higher-dimensional case is that the splitting (22) is canonical, L_1 corresponding, say, to the α-plane F^α containing $\dot\gamma$, and L_2 to the β-plane F^β containing $\dot\gamma$. It is important now that *each of* L_1, L_2 is isomorphic to $\mathcal{O}(1)$, because the vanishing of R^{F^α} implies $W^+ \equiv 0$, and the vanishing of R^{F^β} implies $W^- \equiv 0$ [2], Proposition 2. Thus the manifold $(\mathbf{M}^4, c)$ is conformally flat.

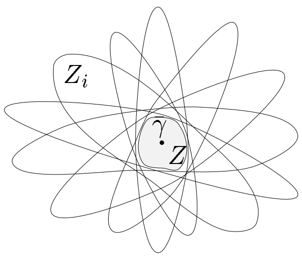
Consider now the particular case where $n = 3$. We are going to use the LeBrun correspondence, then Theorem 3, to prove that $\mathbf{M}$ is then conformally flat. Note that we cannot use directly Theorem 1 and the above proven result for self-dual manifolds, as the "ambient" self-dual manifold N can only be defined for a *civilized* (e.g. geodesically convex) 3-manifold.

We cover γ with geodesically convex open sets U_i, $i = \overline{1, n}$, such that:

$$\forall i \neq j \text{ such that } U_i \cap U_j \cap \gamma \neq \emptyset,\ \exists U_{ij} \supset (U_i \cup U_j), \tag{25}$$

where U_{ij} is still geodesically convex (with respect to some particular Levi-Civita connection). This is possible by choosing U_i, $i = \overline{1, n}$, small enough [16]. Then we choose a relatively compact tubular neighbourhood $N(r_0)$ of γ, such that its closure is covered by the U_i's. We may choose this tubular neighbourhood small enough to be contained in the projection U, from $\mathbb{P}(\mathcal{C})$, of a saturated neighbourhood (see Proposition 4) of the lift $\tilde\gamma$.

We consider then the twistor spaces Z_i, the spaces of null-geodesics of U_i. The compact, simply-connected, null-geodesics close to γ identify (diffeomorphically) the neighbourhoods of $\bar\gamma^i \in Z_i$ with the space Z of the deformations (contained in U) of γ as a compact curve. We can see then (a small open set of) Z as an open set common to all the Z_i's:



Following LeBrun, we define the self-dual manifolds $\mathbf{N}_i$ as the spaces of projective lines in Z_i, with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Then U_i is an umbilic hypersurface in $\mathbf{N}_i$.

The local twistor spaces Z_i admit contact structures, which coincide on Z, and contain projective lines Z_x^i corresponding to points $x \in \gamma \cap U_i$. If we denote by Z_{ij} the twistor space of U_{ij}, then Z_{ij} is identified to an open set in Z_i and, at the same time, to an open set in Z_j, in particular the twistor lines $Z^i_x \subset Z^i$ and $Z^j_x \subset Z^j$ (corresponding to the same point $x \in U_{ij}$) are identified. Thus $Z^i_x \cap Z$ and $Z^j_x \cap Z$ coincide and we denote by Z_x this (non-compact) curve in Z, and by F the canonical contact structure of Z (restricted from the ones of Z_i).

Remark. We already have obtained that the integral α-cone (*i.e.* the union of twistor lines passing through $\bar\gamma$ and tangent to $F_{\bar\gamma}$, see the comment after Theorem 3) corresponding to $F_{\bar\gamma}$ is a part of a smooth surface: the union of the lines Z_x, $x \in \gamma$. Thus, from Theorem 3, the Weyl tensor W^+_i of the self-dual manifold $\mathbf{N}_i$ vanishes on the α-planes generated by $T\gamma$. But this is nothing new: we know, from Theorem 1, that W^+_i vanishes on U_i.

We intend to apply Theorem 3 to prove that W^+_i vanishes on points close to U_i, but generically in $\mathbf{N}_i \setminus U_i$. We do that by showing that the integral α-cones corresponding to planes $F' \subset T_{\bar\gamma}Z$ close to F are parts of smooth surfaces, then we conclude using Theorem 3.

First we choose Hermitian metrics h_i on Z_i, such that they coincide (with h) on Z. We have a diffeomorphism between γ and $\mathbb{P}(F_{\bar\gamma})$, so we choose relatively compact open sets in $\mathbb{P}(T_{\bar\gamma}Z)$, covering $\mathbb{P}(F_{\bar\gamma})$, with the following properties: As the metrics h_i induce metrics on $\mathbf{N}_i$, we first choose a small enough distance $r_1 > 0$ such that

1. $\forall i$, there is a sub-covering $V_i \Subset U_i$ of γ such that the "tubular neighbourhoods" $Q_i := \{y \in \mathbf{N}_i \,|\, \mathrm{d}(y, \bar V_i) \le r_1\ \pi_i(y) \in \bar V_i \cap \gamma\}$ are compact ($\mathrm{d}(y, \bar V_i)$ is the distance between y and $\bar V_i$, and π_i is the "orthogonal projection" — for the Hermitian metric — from $\mathbf{N}_i$ to $\gamma \cap U_i$; it is well defined because of the condition below);
2. r_1 is less than the bijectivity radius of the (Hermitian) exponentials for the points of $\bar V_i$ in $\mathbf{N}_i$, and for the points of $\overline{V_i \cup V_j}$ in $\mathbf{N}_{ij}$ (if $U_i \cap U_j \cap \gamma \neq \emptyset$.).

We have then

Lemma 5. *For any $y_i \in Q_i \subset \mathbf{N}_i$, $y_j \in Q_j \subset \mathbf{N}_j$ such that the curves $Z_{y_i} := Z^i_{y_i} \cap Z$, $Z_{y_j} := Z^j_{y_j} \cap Z$ are tangent to the same direction in $\bar\gamma \in Z$, the respective curves Z_{y_i}, Z_{y_j} coincide.*

Proof. We first note that the projection π_i from $\mathbf{N}_i$ to $\gamma \cap U_i$ is equivalent to the h–orthogonal projection of the direction of $T_{\bar\gamma}Z_{y_i}$ to a direction in $F_{\bar\gamma}$, so $\pi_i(y_i) = \pi_j(y_j) =: y \in \gamma$; thus y belongs to both U_i and U_j, and we use again the twistor space Z_{ij} to conclude that Z_{y_i} and Z_{y_j} are "restrictions" to Z of the same projective line (as they both have the same tangent space at $\bar\gamma$) $Z^{ij}_{y_{ij}}$, for a point $y_{ij} \in \mathbf{N}_{ij}$. □

Now we have a tubular neighbourhood $S \subset \mathbb{P}(T_{\bar\gamma}Z)$ of $\mathbb{P}(F_{\bar\gamma})$, of radius $r_1/2$, such that, for any 2-plane $F' \subset S$, the conditions in Theorem 3 are satisfied (considering any of the local twistor spaces Z_i).

We recall that, via the LeBrun correspondence, a point $\bar\gamma_0$ in the twistor space of $\mathbf{M}_0$, Z_0, is identified to the point $\bar\beta_0$ in the twistor space of $\mathbf{N}_0$, still denoted by Z_0. They correspond to the null-geodesic $\gamma_0 \subset \mathbf{M}_0$, resp. to the β-surface $\beta_0 \subset \mathbf{N}_0$, such that $\gamma_0 \subset \beta_0$ [8], [2]. The planes F' above are included in $T_{\bar\gamma}Z \equiv T_{\bar\beta^i}Z_i$, and they correspond to null-geodesics in $\mathbf{N}_i$ contained in β^i [11], [2].

By Theorem 3, we conclude that the Weyl tensor W_i^+ of $\mathbf{N}_i$ vanishes along all null-geodesics of $\mathbf{N}_i$, close (in Q_i) to γ and included in the β-surface β^i, determined by γ. This means that W^+ vanishes everywhere on β^i. By deforming γ, we obtain that W_i^+ vanishes on a neighbourhood of U_i in $\mathbf{N}_i$, hence $\mathbf{N}_i$ is conformally flat.

It follows from Theorem 1 that U_i, hence $\mathbf{M}$, is conformally flat. $\square$

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Mathematisches Institut
Humboldt Universität zu Berlin
Unter den Linden 6, 10099 Berlin
Germany
e-mail: `belgun@mathematik.hu-berlin.de`