Long time behavior of one-dimensional stochastic dynamics

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Abstract: In this paper the authors analyze the long time behavior of certain Markov chains, namely jump processes of second order jump range, as the system size is growing. The motivation has its origin in statistical mechanics, where the time evolution of the magnetization in a Glauber dynamic of a mean field type spin system is considered. As a standard example might serve the Curie-Weiß model. The process considered can also be regarded as the space and time discrete analog of a one-dimensional randomly perturbed dynamical system. In leading order as the system size grows the authors derive in terms of the rate function of the reversible distribution transition probabilities and transition times describing metastability in this model. These quantities are characterized by second order difference equations for which the Green's function has a particular simple structure. This approach leads to the classical problem of the asymptotic behavior of sums of Laplace type.

Keywords: Markov chains, Metastability, Reversibility, Curie-Weiß model, Mean field dynamic, Random perturbation of dynamical systems, Second order difference equation, Green's function, Laplace's method.

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1. Introduction

This paper is devoted to the study of the long time behavior of one dimensional mean field dynamics as the system size grows. More precisely, we consider a finite state, irreducible jump process, discrete and homogeneous in time, on the state space $[L, R] \cap (\mathbb{Z}/N)$ with reflecting boundary conditions. Moreover, we restrict the jump range to second order, i.e. every state except the boundary states, which have only one, has precisely two neighbors. We shall derive in terms of the free energy, namely the rate function of the reversible distribution, in leading order transition probabilities and several transition times between local minima including times of so called admissible transitions introduced in [BEGK].

Our main motivation stems from metastable features in Glauber dynamics of mean field type spin systems with only a single order parameter and where spins are allowed to take two values. For an expatiated exposition of the equilibrium situation consult e.g. [BG1]. Rather concerning the microscopic dynamics on the huge state space of spin configurations one should consider the induced mean field dynamic of the single order parameter, also called the magnetization. For different reasons the situation is (of course) special compared to the case, where several order parameters are included. One important difference is that in the latter case the induced dynamic is in general not markovian. Furthermore, the resulting state space for a single macroscopic observable is one-dimensional. Let us mention that there are special cases in higher dimensions in which the mean field dynamic is markovian, too, namely the Random Field Curie-Weiss model being two dimensional (see e.g. [MP], [BEGK]). Another speciality here even in one dimension is that mean field dynamics of two-valued spins have second order jump range. Regarding spin systems with spins that might assume more than two values the jump range of the induced dynamic increases according to the cardinality of the state space for a single spin. The explicit approach developed in this text is inapplicable for those situations and we will consider this case in a future publication with different methods. The interested reader will find the higher-dimensional context in [BGK]. Surprisingly, there are only very few articles in the literature studying this kind of problem. The most popular example of models described above might be the Curie-Weiss model (see e.g. [E] as a standard reference for the equilibrium situation) and we refer the reader to [CGOV], where metastability is studied in this model. According to the fact that this model is under certain choices of the exterior parameters a two-well problem, in the sense that in the thermodynamic limit the free energy possesses only two local minima, it does not reflect the multiple well situation discussed in this paper.

Passing formally to continuous space and time variables one obtains a randomly perturbed dynamical system on $[L, R]$ with reflecting barriers at $\{L, R\}$ whose generator looks like $-\varepsilon d^2/dx^2 + F_0' d/dx$ and where $\varepsilon \downarrow 0$. Those systems are intensively studied especially in the higher-dimensional context. A standard reference of this topic is [FW]. In particular in [Ku] this connection is studied. Hence, although the state space for fixed N is finite, one should rather think of a randomly perturbed dynamical system with conservative drift than of reversible Markov chains with rare transition probabilities and finite state space independent of N occurring also in the Freidlin-Wentzell theory which are used to analyze the long time behavior of the system. A small selection of papers concerning about this class of chains is given by [CC], [GT], [OS1], [OS2].

The main advantage we are gaining from the fact that the jump range is of second order is that an explicit approach by construction of the Green's function is available. Consequently, one can consider quite weak conditions on the asymptotic character of the free energy. In addition, every irreducible jump process with jump range of second order possesses a reversible measure. There are of course alternative methods, which apply to non-reversible higher order jump processes. Large deviation theory could be used as well as WKB methods. But the former approach (see e.g. [BG2]) typically leads to errors living on a sub-exponential scale, while the latter one requires more regularity of the free energy. There is a large amount of papers employing WKB methods and even though a more detailed discussion of the literature in this context will follow as a likely incomplete list of references could serve [DM], [H], [vK], [KMST], [MR], [CoCo], [SP], [ST], [Wi]. Although we will develop the WKB method in a future publication for infinite order recurrence relations, to our point of view it is worthwhile pointing out before all the advantages of the special case considered here. To our knowledge there is no rigorous treatment of the problem to the degree of generality and accuracy achieved in here. An detailed survey on this topic is given in [vK], while for example in [MKST] they consider in a formal way expansions of WKB type. Rigorous results but in rather special situations are obtained in [H] for example. Standard references concerning the continuous case are given by [Wa] and [Ol].

We now continue with the introduction of the model. Fix $L<R$. For each $N\geq 1$ consider the state space[3]

I.1
$$\Gamma=\Gamma(N)=\mathbb{Z}/N\cap[L,R]. \tag{1.1}$$

Let $(p_{xy})=(p_{xy}(N))_{x,y\in\Gamma}$ be a transition matrix of an irreducible time discrete jump process on Γ with reflecting barriers at $\min\Gamma$ and $\max\Gamma$; i.e. fix probabilities $p_x=p_x(N)$ and $q_y=q_y(N)$, $0<p_x,q_y<1$ for $x,x+1/N,y,y-1/N\in\Gamma$ and set

I.2
$$p_{xy}=\begin{cases}p_x & \text{for}\quad y=x+1/N,\\ q_x & \text{for}\quad y=x-1/N,\\ 0 & \text{for}\quad |x-y|>1/N\end{cases}\qquad (x,y\in\Gamma). \tag{1.2}$$

Denote by $\mathbb{Q}=\mathbb{Q}(N)$ the reversible distribution of the chain defined on the algebra $\{I\,|\,I\subset\Gamma\}$ of subsets of the state space, i.e. detailed balance holds

I.4
$$\mathbb{Q}\{x\}p_{xy}=\mathbb{Q}\{y\}p_{yx}\qquad (x,y\in\Gamma). \tag{1.3}$$

Note that of course $\mathbb{Q}$ is the invariant distribution of the chain. Since the jump range is of second order, it is easy to compute

I.4a
$$\mathbb{Q}(N,\{x\})=\left[\sum_{l\in\Gamma}\prod_{l\leq j<\max\Gamma}\frac{q_{j+1/N}}{p_j}\right]^{-1}\prod_{x\leq j<\max\Gamma}\frac{q_{j+1/N}}{p_j} \tag{1.4}$$

We hope that the products are self-explanatory. Define the free energy function $F(N,\cdot)$ to be

I.3
$$F(N,x)=-(1/N)\log(\mathbb{Q}(N,\{x\}))\qquad (x\in\Gamma). \tag{1.5}$$

Throughout the text we stipulate that estimates abbreviated by Landau-symbols are always understood to be uniform in the argument as well as in the large parameter.
In the 'interior' of the state space we demand the following conditions on the transition matrix (p_{xy}) and the free energy $F(N,\cdot)$:

Hypothesis 1.1: *$F(N,\cdot)\in\mathcal{C}[L,R]$ for all $N\geq 1$. There are constants $\mu\in\mathbb{N}\backslash\{0,1\}$, $0<\nu\leq 1$, $\delta>0$, a sequence C_N and functions[4] $F_0\in\mathcal{C}^{\mu+1}(L,R)\cap\mathcal{C}[L,R]$, $F_1(N,\cdot)\in\mathcal{C}[L,R]$ such that locally in (L,R) as $N\to\infty$*

I.8
$$F(N,x)=F_0(x)+N^{\nu/\mu-1}F_1(N,x)+C_N+\mathcal{O}\left(N^{-(1+\delta)}\right). \tag{1.6}$$

[3] In order to simplify the notations we mostly suppress the dependence of the large parameter N. We will keep this dependence, where we think it is helpful.

[4] It is sufficient to require that $F_0\in\mathcal{C}^{0,\delta}(L,R)$ and that at each local maximum or minimum x_0 there is $\mu=\mu(x_0)\in\mathbb{N}\backslash\{0,1\}$ such that locally around x_0 $F_0\in\mathcal{C}^{\mu}$ and obeys $F_0(x)-F_0(x_0)\sim f_0(x_0)(x-x_0)^{\mu}$. Of course, $\nu=\nu(x_0)$ may also depend on the critical point, but $F_1(N,\cdot)$ has at least to be locally Hölder-continuous. Furthermore, another obvious generalization is relaxing the conditions to the case $\mu\in(1,\infty)$.

At each critical point $L < x_0 < R$ of the leading order F_0 we have

I.9
$$F_0(x) - F_0(x_0) \sim \frac{F_0^{(\mu)}(x_0)}{\mu!}(x - x_0)^\mu \qquad (x \to x_0). \tag{1.7}$$

$F_0|\Gamma$ is assumed to be generic in the sense that values of different arguments (x, y, z) of the function $F_0(y) + F_0(z) - F_0(x)$ defined on local maxima $y, z \in \Gamma$ and local minima $x \in \Gamma$ of $F_0|\Gamma$ differ at least by an amount[5] *δ.*
There is a number $\sigma > 0$ and a function $F_{1,1}(N, \cdot)$ on (L, R) which is locally bounded in (L, R) uniformly in N such that locally in $x, y \in (L, R)$

I.10
$$F_1(N, y) = F_1(N, x) + F_{1,1}(N, x)(y - x)^\nu + \mathcal{O}(|y - x|^{\nu+\sigma}). \tag{1.8}$$

In a neighborhood of each critical point x_0 of F_0 the jump probabilities are γ-Hölder-continuous uniformly in N, i.e. there are $\gamma, b > 0$ independent of N such that

I.23
$$|p_x - p_y|, |q_x - q_y| = \mathcal{O}\left(|x - y|^\gamma\right) \qquad (|x - x_0|, |y - y_0| \leq b). \tag{1.9}$$

The jump probabilities are strictly positive in a neighborhood of the critical points of F_0[6]*, more precisely, there is $c > 0$ independent of N such that*

I.24
$$\min_{\operatorname{dist}(x, \pm 1) \geq (1-\delta) \operatorname{dist}(\mathcal{C}_{F_0}, \pm 1)} p_x,\, q_x \geq c, \tag{1.10}$$

where $\mathcal{C}_{F_0}$ denotes the set of critical points of F_0 on (L, R).

Furthermore we assume that the transition matrix (p_{xy}) and the free energy $F(N, \cdot)$ obey the following 'boundary behavior':

Hypothesis 1.2: *The free energy $F(N, \cdot)$ is non-decreasing in direction to the boundary, more precisely $\pm F(N, \cdot)$ is non-decreasing on the set of points $x \in [L, R]$ such that* $\operatorname{dist}(x, \{L, R\}) \leq \rho$ *for suitable $\rho > 0$ independent of N.*
The jump probabilities converge not faster than sub-exponential to zero, i.e. there is $\delta < 1$ satisfying

I.25
$$\min_{x \in \Gamma \setminus \max \Gamma, y \in \Gamma \setminus \min \Gamma} p_x,\, q_y \geq e^{-N^\delta} \tag{1.11}$$

Let $\mathbb{P}$ and $\mathbb{E}$ be the law, respectively, the expectation of the Markov chain given by the transition matrix (p_{xy}). Let X_t, $t \in \mathbb{N}$, be the position of the chain at time t. For $x \in \Gamma$ and $I \subset \Gamma$ let τ_I^x be the time *after*[7] zero of the first visit of I while the walk starts in x

I.5
$$\tau_I^x = \min\{t > 0 \,|\, X_t \in I, X_0 = x\}. \tag{1.12}$$

We abbreviate $\tau_y^x = \tau_{\{y\}}^x$ for the transition time from x to y. For convenience it is stipulated $\tau_\emptyset^x \equiv \infty$. Let us mention that the general theory of irreducible death and birth chains tells us that (in fact every power of) the stopping time τ_I^x has finite mean[8].

[5] There is no problem in considering the generic case, but the reader will notice that the generalizations are quite obvious so as not to burden clarity to much we renounce this point.

[6] Let us stress once more that we do not aim at the most general conditions. For example it would be enough to require that the probabilities are locally bounded below by an amount that converges at most polynomially fast to zero.

[7] This notation is chosen for convenience as we will see in chapter 3.

[8] More precisely, due to the Perron-Frobenius theorem on the largest eigenvalue of positive matrices the distribution function $\mathbb{P}[\tau_y^x > t]$ converges exponentially fast to zero as t tends to infinity.

The structure of the article consists of two chapters and an appendix. In chapter two we construct the Green's function of the operator $(\delta_{xy} - p_{xy})_{xy}$, where δ_{xy} denotes Kronecker's delta, with respect to generic boundary conditions and we compute this function in cases we are interested in, namely for Dirichlet and Neumann-Dirichlet conditions. Transition times and transition probabilities solve due to the Markov property certain inhomogeneous boundary-value-problems for $(\delta_{xy}-p_{xy})$. Combining their representation with detailed balance they are expressed as sums of Laplace type. Their asymptotic behavior is discussed in detail in the appendix. The main reason for writing this appendix is that for the asymptotic behavior of very fast transition times we need uniform control in the domain of summation for sums of Laplace type. In chapter three we exploit the analysis of chapter two and the appendix. The quantities characterizing metastability in this model are derived in precise leading order sharpening the results written in [BEGK].

2. Representations of transition times and transition probabilities

Due to the Markov property transition times and transition probabilities in time-discrete jump processes viewed as functions of their starting point solve certain boundary value problems. Their explicit solution leads to representations of these quantities which are analytically tractable as N tends to infinity. This chapter is devoted to these well-known representations (see e.g. [vK]). We remark here that as we will see below it is crucial to be in the stochastic context of jump processes for the explicit approach we are following.

Let (p_{xy}) denote the transition probabilities of the Markov chain defined in (1.2). Fix $a, b \in \Gamma$, $a < b$ and set $\dot{I} = \{x \in \Gamma \,|\, a < x < b\}$, $I = \{x \in \Gamma \,|\, a \leq x \leq b\}$, etc. Define $L_I : \mathbb{R}^I \to \mathbb{R}^{\dot{I}}$ to be the operator

B.1
$$L_I u(x) = u(x) - \sum_{y \in I} p_{xy} u(y) \qquad (x \in \dot{I}). \tag{2.1}$$

Let $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{R}^2$ be vectors such that $\alpha_1, \beta_1 \neq 0$. For $f \in \mathbb{R}^{\dot{I}}$ and $g \in \mathbb{R}^{\{a,b\}}$ we want to consider the equation

B.2
$$L_I u(x) = f(x) \qquad (x \in \dot{I}), \tag{2.2}$$

with respect to the boundary conditions

B.3
$$Bu = g, \tag{2.3}$$

where $Bu(a) = \alpha_1 u(a) + \alpha_2 u(a + 1/N)$ and $Bu(b) = \beta_2 u(b - 1/N) + \beta_1 u(b)$ for $u \in \mathbb{R}^I$. Clearly, the solution of (2.2) and (2.3) can be written as the sum of a solution solving the homogenous equation $L_I u = 0$ and the boundary condition $Bu = g$ and of a solution solving the inhomogeneous equation $L_I u = f$ with respect to homogenous boundary values $Bu = 0$. We shall proceed with constructing the Green's function of the latter problem due to homogenous boundary values, namely, we shall construct the kernel of the inverse $(L_I^B)^{-1}$, if it exists, of the operator

B.4
$$L_I^B = L|\ker B \to R^{\dot{I}}, \quad \text{where} \quad \ker B = \{u \in \mathbb{R}^I \,| Bu = 0\}. \tag{2.4}$$

The procedure is parallel to the continuous situation. Since later on we will use a fundamental, i.e. linearly independent set of solutions of the homogenous equation $L_I u = 0$ for the construction, as a side effect we already know how to solve the homogenous equation with respect to inhomogeneous boundary values. The structure of solutions of the homogenous equation (2.2) is well understood when writing it as a first order system. More precisely, associating to a solution $u(x)$ of the homogenous equation the vector $\vec{u}(x) = (u(x), u(x + 1/N))^t$ for $x \in I \backslash b$ this vector is given by $\vec{u}(x) = A_x \ldots A_{a+1/N} \vec{u}(a)$ or equivalently by $\vec{u}(x) = B_{x+1/N} \ldots B_{b-1/N} \vec{u}(b - 1/N)$. Here we have defined

B.7
$$A_x = \begin{pmatrix} 0 & 1 \\ -\frac{q_x}{p_x} & 1 + \frac{q_x}{p_x} \end{pmatrix}, \qquad B_x = (A_x)^{-1} = \begin{pmatrix} 1 + \frac{p_x}{q_x} & -\frac{p_x}{q_x} \\ 1 & 0 \end{pmatrix} \qquad (x \in \Gamma \backslash \{\min \Gamma, \max \Gamma\}). \tag{2.5}$$

For convenience let us introduce $x_< = \min(x, y)$ and $x_> = \max(x, y)$ for two numbers $x, y \in \mathbb{R}$. We then have

Lemma 2.1: *Assume that $\alpha, \beta \in \mathbb{R}^2$ in (2.3) are chosen in a way that there exists a fundamental pair of solutions (w_1, w_2) satisfying $Bw_1(a) = 0$, $Bw_2(b) = 0$. Then, the Green's function $G_I^B(y, x)$ of the operator L_I^B is given by*

B.5
$$G_I^B(y, x) = \frac{w_1(x_<) w_2(x_>)}{p_y W(y)} \qquad (x \in I, y \in \dot{I}), \tag{2.6}$$

where W denotes the Wronski determinant with respect to the pair (w_1, w_2), i.e. with the notation introduced before (2.5)

B.6
$$W(x) = \det(\vec{w}_2(x), \vec{w}_1(x)) \qquad (x \in I \backslash b). \tag{2.7}$$

Proof: Note that the Wronskian solves the equation $W(x) = \det A_x W(x-1/N)$ for $x \in I\backslash a$ and $W(x-1/N) = \det B_x W(x)$ for $x \in I\backslash b$, where A_x and B_x are defined in (2.5). Hence, since (w_1, w_2) are linearly independent, the Wronski determinant has no zero and $G = G_I^B$ is welldefined. That G satisfies the following conditions is straightforward:
(i) For fixed $y \in \dot{I}$ as a function of $x \in I$ $G(y,x)$ solves the homogenous equation $L_I G(y,\cdot)(x) = 0$ for $x \in \dot{I}\backslash y$.
(ii) $G(y,\cdot) = 0$ satisfies the homogenous boundary conditions $BG(y,\cdot) = 0$ for each $y \in \dot{I}$.
(iii) We have the following jump condition for first differences $p_y(G(y,y+1/N)-G(y,y))+q_y(G(y,y-1/N)-G(y,y)) = -1$ for $y \in \dot{I}$.
Obviously, (i), (ii), (iii) imply that for $f \in \mathbb{R}^{\dot{I}}$ the function

B.6a
$$u(x) = \sum_{y\in\dot{I}} f(y)G(y,x) \qquad (x \in I) \tag{2.8}$$

solves $L_I^B u = f$. Dimension counting shows that L_I^B is invertible and G is the kernel of its inverse. $\Diamond$

The reason for regarding death and birth chains only is that it is quite easy to compute explicit solutions of the homogenous equation $L_I u = 0$, while in higher order recurrences this is not the case and one has to introduce other methods. The helpful fact in here is that 1 is a common eigenvalue with common eigenvector $(1,1)^t$ for all matrices A_x, B_x defined in (2.5). Hence, of course $u(x) \equiv$constant are solutions of the homogenous equation $L_I u = 0$. Furthermore, starting with the vector $\vec{u}(a) = (0,1)^t$ at the point a we compute for $x \in \dot{I}$

B.8
$$(A_x \ldots A_{a+1/N}\vec{u}(a))_1 = 1 + \frac{q_{a+1/N}}{p_{a+1/N}} + \ldots + \frac{q_{a+1/N}}{p_{a+1/N}} \cdots \frac{q_{x-1/N}}{p_{x-1/N}}. \tag{2.9}$$

Starting with $\vec{u}(b-1/N) = (1,0)^t$ at b we obtain

B.9
$$(B_{x+1/N} \ldots B_{b-1/N}\vec{u}(b-1/N))_1 = 1 + \frac{p_{b-1/N}}{q_{b-1/N}} + \ldots + \frac{p_{b-1/N}}{q_{b-1/N}} \cdots \frac{p_{x+1/N}}{q_{x+1/N}}. \tag{2.10}$$

Here $(v)_i$ denotes the i-th component of a vector $v \in \mathbb{R}^2$ with respect to the standard basis and we use the convention that empty products are interpreted as one. As we shall see later on we are only interested in Dirichlet and mixed Dirichlet Neumann conditions. Let us remark here that in terms of the process a point carrying a Neumann condition indicates a reflecting barrier, while a Dirichlet condition stands for an absorbing barrier. In the following the superscript D indicates Dirichlet conditions at a and b, i.e $\alpha = \beta = (1,0)$, while the superscript ND stands for Neumann condition at a and Dirichlet condition at b, i.e. $\alpha = (1,-1)$ and $\beta = (1,0)$. As it is documented in (2.9) and (2.10) the functions $w_I^D(\cdot;c)$, $c = a,b$, defined in (2.11) form a fundamental set for the operator L_I satisfying the conditions of Lemma 2.1 with respect to Dirichlet boundary conditions.

B.10
$$w_I^D(x;c) = \begin{cases} \left[\sum_{a<l\le b}\prod_{a<k<l}\frac{q_k}{p_k}\right]^{-1} \sum_{a<l\le x}\prod_{a<k<l}\frac{q_k}{p_k} & \text{for} \quad c = a \\ \left[\sum_{a\le l<b}\prod_{l<k<b}\frac{p_k}{q_k}\right]^{-1} \sum_{x\le l<b}\prod_{l<k<b}\frac{p_k}{q_k} & \text{for} \quad c = b \end{cases} \qquad (x \in I). \tag{2.11}$$

Here we stipulate that sums over the empty set are understood to be zero and products to be one. Furthermore, we observe that the functions $w_I^{ND}(x;a) \equiv 1$ and $w_I^{ND}(x;b) \equiv w_I^D(x;b)/(ND)w_I^D(a;b)$ may serve as a fundamental set of solutions for constructing the Green's function due to Dirichlet condition at b and Neumann condition at a. Note that $(ND)w_I^D(a;b) = w_I^D(a;b) - w_I^D(a+1/N;b) \neq 0$. The transition probability $\mathbb{P}[\tau_b^x < \tau_a^x]$ when starting in $x \in (a,b)$ to reach b before a solves as a function of x the homogenous problem (2.2) with Dirichlet boundary values $g(b) = 1$ at b and $g(a) = 0$ at a. This follows immediately from (2.12) in the following Lemma in the case $y = b$, $K = \partial\{x\} \cup x$, $J = a$ summing over $t \in \mathbb{N}$. Here the boundary of a set is the set of communicating points in the complement. From the strong Markov-property we obtain the fundamental (see also [BEGK])

Lemma 2.2: *Let $x, y \in \Gamma$ be arbitrary points and J, K arbitrary subsets of Γ. Then for every $t \in \mathbb{N}$ we have*

B.24
$$\begin{aligned}\mathbb{P}[\tau_y^x = t, \tau_y^x \le \tau_J^x] = \mathbb{P}[\tau_y^x = t, \tau_y^x \le \tau_{J\cup K}^x] \\ + \sum_{0<s<t} \sum_{z \in K\setminus(J\cup y)} \mathbb{P}[\tau_z^x = s, \tau_z^x = \tau_{K\cup J\cup y}^x]\mathbb{P}[\tau_y^z = t-s, \tau_y^z \le \tau_J^z]\end{aligned} \tag{2.12}$$

Proof: (2.12) follows by construction writing out the definitions. $\diamondsuit$

Similarly, the expected time $\mathbb{E}[\tau_b^x]$ to reach b from x solves problem (2.2) with inhomogeneity $f(x) = 1$ for $x \in (\min\Gamma, b)$, Dirichlet boundary value $g(b) = 0$ at b and Neumann boundary value $g(\min\Gamma) = 1/p_{\min\Gamma}$ at $\min\Gamma$. Furthermore, the expected time $\mathbb{E}[\tau_b^x, \tau_b^x < \tau_a^x] = \mathbb{E}[\tau_b^x \mathbb{1}_{\{\tau_b^x < \tau_a^x\}}]$ to reach b before a from x is the solution of (2.1) with inhomogeneity $f(x) = \mathbb{P}[\tau_b^x < \tau_a^x]$ for $x \in (a,b)$ and Dirichlet boundary values $g(a) = g(b) = 0$ at a and b. Thus we may apply Lemma 2.1 to obtain the representations:

Corollary 2.3: *Fix $a, b, b_1, b_2 \in \Gamma$ such that $a < b$ and $b_1 < a < b_2$. We have*

B.12
$$\mathbb{P}[\tau_b^a < \tau_a^a] = \left(\sum_{a<x\le b} q_x^{-1} e^{N(F(N,x)-F(N,a))}\right)^{-1}, \tag{2.13}$$

B.13
$$\mathbb{E}[\tau_b^a | \tau_b^a < \tau_a^a] = \frac{\sum_{a<l\le j\le r<b} (q_l p_r)^{-1} e^{N(F(N,l)-F(N,j)+F(N,r))}}{\sum_{a<l\le b} q_l^{-1} e^{NF(N,l)}}, \tag{2.14}$$

B.23
$$\begin{aligned}\mathbb{E}[\tau_a^a, \tau_a^a < \tau_{b_1\cup b_2}^a] = e^{NF(N,a)} \Bigg(\frac{\sum_{a<j\le l, r<b_2} (p_l p_r)^{-1} e^{N(-F(N,j)+F(N,l)+F(N,r))}}{\left(\sum_{a\le l<b_2} q_l^{-1} e^{NF(N,l)}\right)^2} \\ + \frac{\sum_{b_1<l,r\le j<a} (q_l q_r)^{-1} e^{N(F(N,l)+F(N,r)-F(N,j))}}{\left(\sum_{b_1<l\le a} p_l^{-1} e^{NF(N,l)}\right)^2} \Bigg),\end{aligned} \tag{2.15}$$

B.14
$$\mathbb{E}[\tau_b^a] = \sum_{\substack{a\le l<b \\ \min\Gamma<k\le l}} p_l^{-1} e^{N(F(N,l)-F(N,k))} \tag{2.16}$$

Proof: The considerations preceding the Corollary show that the functions on both sides of the following equation solve the same Dirichlet problem. Hence, we have

B.15
$$P[\tau_b^x < \tau_a^x] = w_I^D(x;a) \qquad (x \in \dot{I}), \tag{2.17}$$

where we have set $I = [a,b]$. Invoking the Markov property for the step form a to $a+1/N$ (2.17) gives

B.16
$$\mathbb{P}[\tau_b^a < \tau_a^a] = p_a w_I^D(a+1/N;a). \tag{2.18}$$

Combining (2.11) with the following elementary consequence of the detailed balance condition (1.3) and the definition (1.12)

B.17
$$\prod_{k=i}^{j} \frac{p_k}{q_k} = \frac{p_j e^{-NF(N,j/N)}}{q_i e^{-NF(N,i/N)}} \qquad (i,j \in \Gamma, i \leq j), \tag{2.19}$$

we are led to

B.18
$$w_I^D(x;c) = \begin{cases} \left[\sum_{a<l\leq b} q_l^{-1} e^{NF(N,l)}\right]^{-1} \sum_{a<l\leq x} q_l^{-1} e^{NF(N,l)} & \text{for} \quad c=a \\ \left[\sum_{a\leq l<b} p_l^{-1} e^{NF(N,l)}\right]^{-1} \sum_{x\leq l<b} p_l^{-1} e^{NF(N,l)} & \text{for} \quad c=b \end{cases} \qquad (x \in I). \tag{2.20}$$

Hence combination of (2.18) with (2.20) and using (1.3) again proves (2.13).
The proof of (2.14) is similar. Set $I = [a,b]$. The representation of the solution of the boundary value problem solved by $I \ni x \mapsto \mathbb{1}_{\dot{I}}(x)\mathbb{E}[\tau_b^x, \tau_b^x < \tau_a^x]$ via the corresponding Green's function in combination with (2.17) and the Markov property for one step give

B.19
$$\mathbb{E}[\tau_b^a, \tau_b^a < \tau_a^a] = p_a \sum_{a<j<b} w_I^D(j;a) G_I^D(j, a+1/N). \tag{2.21}$$

Combining (2.21) with (2.6) for Dirichlet conditions and (2.17) yields

B.20
$$\mathbb{E}[\tau_b^a | \tau_b^a < \tau_a^a] = \sum_{a<j<b} \frac{w_I^D(j;b) w_I^D(j;a)}{p_j W_I^D(j)}, \tag{2.22}$$

where W_I^B denotes the Wronski determinant with respect to the pair $(w_I^B(\cdot;a), w_I^B(\cdot;b))$ defined in (2.7) for $B = D, ND$. The equations solved by the Wronskian written in the beginning of the proof of Lemma 2.1 imply

B.11
$$W_I^B(x) = W_I^B(a) \prod_{a<k\leq x} \frac{q_k}{p_k} = W_I^B(b-1/N) \prod_{x<k<b} \frac{p_k}{q_k} \qquad (x \in \dot{I}), \tag{2.23}$$

which in view of (2.19) and (1.3) equals

B.11a
$$W_I^B(x) = W_I^B(a)(p_a/p_x) e^{N(F(N,x)-F(N,a))} = W_I^B(b-1/N)(q_b/p_x) e^{N(F(N,x)-F(N,b))} \qquad (x \in \dot{I}). \tag{2.24}$$

Combining the latter equality in (2.24) with the equation $W_I^D(b-1/N) = w_I^D(b-1/N;b)$, inserting the result into (2.22) and invoking (2.20) it is straightforward to arrive at (2.14).
In order to prove (2.15) the Markov property for a single step implies

B.22a
$$\mathbb{E}[\tau_a^a, \tau_a^a < \tau_{b_1\cup b_2}^a] = p_a \mathbb{E}[\tau_a^{a+1/N}, \tau_a^{a+1/N} < \tau_{b_2}^{a+1/N}] + q_a \mathbb{E}[\tau_a^{a-1/N}, \tau_a^{a-1/N} < \tau_{b_1}^{a-1/N}]. \tag{2.25}$$

Replacing b by b_2 the same arguments leading to (2.22) show

B.22
$$\mathbb{E}[\tau_a^{a+1/N}, \tau_a^{a+1/N} < \tau_{b_2}^{a+1/N}] = \sum_{a<j<b_2} \frac{w_I^D(j;b_2)^2 w_I^D(a+1/N;a)}{p_j W_I^D(j)}. \tag{2.26}$$

As before combining the first equality in (2.24) with the equation $W_I^D(a) = w_I^D(a+1/N;a)$, inserting the result into (2.26) and invoking (2.20) yields the first term in the sum on the right-hand side of

(2.15). Repeating the procedure for the second term in the sum on the right-hand side of (2.25) proves (2.15).

Again, application of (2.6) to the solution of the boundary value problem solved by $I \ni x \mapsto \mathbb{1}_{\dot{I}}(x)\mathbb{E}[\tau_b^x]$, where $I = [\min\Gamma, b]$, leads for $a \in \dot{I}$ to the representation

B.21

$$\begin{aligned}\mathbb{E}[\tau_b^a] &= \sum_{k\in\dot{I}} G_I^{ND}(k,a) + (1/p_{\min\Gamma})w_I^{ND}(a;b) \\ &= \sum_{a\leq k<b} \frac{w_I^{ND}(a,\min\Gamma)w_I^{ND}(k,b)}{p_k W_I^{ND}(k)} + \sum_{\min\Gamma\leq k<a} \frac{w_I^{ND}(k,\min\Gamma)w_I^{ND}(a,b)}{p_k W_I^{ND}(k)}.\end{aligned} \tag{2.27}$$

Recall that we have chosen $w_I^{ND}(\cdot;b) = w_I^D(\cdot;b)/(ND)w_I^D(\min\Gamma;b)$ and $w_I^{ND}(\cdot;\min\Gamma) = 1$. In particular, we have $W_I^{ND}(\min\Gamma) = 1$, which was used in the second equality on the right-hand side of (2.27). Combining the latter equality in (2.24) with $W_I^{ND}(b-1/N) = w_I^{ND}(b-1/N;b)$ and inserting (2.20) once more we obtain (2.16) from (2.27). $\diamondsuit$

3. Transition probabilities and transition times

In this chapter several probabilities and expected times associated to a transition from one local minimum to another are investigated in leading order. The asymptotic formulas are determined by the structure of the free energy $F(N,\cdot)$ and we have to introduce some notation concerning the energy landscape of the leading part F_0. The notation is made in a way that it easily ranges in those in [BEGK]. Naturally the notation appears sometimes to be slightly long winded. But still, we think it is worthwhile for the reader, who is interested in the higher dimensional problem.
Let $\mathcal{M}=\mathcal{M}(N)$ be the set of local minima of F_0 on Γ, i.e.

M.1
$$\mathcal{M}=\{x\in\Gamma\,|\,x \quad \text{is a local minimum of} \quad F_0|\Gamma\}. \tag{3.1}$$

With slight abuse of notation we refer to points in $\mathcal{M}$ as to local minima of F_0. Actually, using any other point in a vicinity of order $N^{-1/\mu-\delta}$, $\delta>0$, of a local minimum x of F_0 will not alter the asymptotics presented in this chapter.
With respect to the process let $V(z)$ for $z\in\Gamma$ be the (maximal) irreducible component of level set $\{\tilde{x}\in\Gamma\,|\,F_0(\tilde{x})\le F_0(z)\}$ containing z. A point z is called a saddle if and only if the set $V(z)\backslash z$ is decomposed into two irreducible valleys $V^{\pm}(z)$ with the convention that $\min F_0(V^-(z))\le\min F_0(V^+(z))$. Of course, being in the generic situation the definition is equivalent in saying that z is a local maximum of $F_0|\Gamma$. Let $\mathcal{E}=\mathcal{E}(N)$ denote the set of saddles. The choice of the letter $\mathcal{E}$ has its origin in the d-dimensional context, where so called essential saddles $\mathcal{E}$ with signature $d-2$ are distinguished from others, while for $d=1$ every saddle is essential. For $I\subset\mathcal{M}\backslash x$ we define $z(x,I)\in\mathcal{E}$ to be the unique saddle with minimal energy such that the event of visiting I before x without exceeding the energy $F_0(z(x,I))$ is not absurd, i.e. with $L_z=\{(F_0|\Gamma)>F_0(z)\}$ set

M.2
$$z(x,I)=\arg\min_{z\in\mathcal{E}:\mathbb{P}[\tau_I^x<\tau_{L_z}^x]>0}F_0(z). \tag{3.2}$$

Genericness of $F_0|\Gamma$ guarantees that $z(x,I)$ is well defined. Let us remark here that the concepts are given in a way that they can naturally be extended to the non-generic case replacing points by sets. We abbreviate $z(x,y)=z(x,\{y\})$ and stipulate $z(x,I)=x$ if $x\in I$.
In order to extract precise asymptotic formulae for the quantities in question via the representation presented in chapter 2 using detailed balance one is reduced to evaluate several sums of Laplace type. In the following Lemma we summarize their common structure and consider the obtained formula in the so called Gaussian case. Let us remark here that the results of Lemma 3.1 do not require the control of Laplace sums uniform in the domain of summation which we have paid a great deal of attention to in the appendix. As a rule the asymptotic features of transition probabilities and those transition times, during which the particle might visit other wells, are of local nature. This is meant in the sense that the limiting properties of the transition matrix and of the free energy at suitable minima and saddles, which of course depend on the global character of the energy landscape, already determine the leading behavior. They can all be treated by means of Lemma 3.1. There is one exception, where the sum of Laplace type has to be controlled uniform in the argument, namely the transition time of the transition from one minimum to another under the condition that the particle does not visit any other minimum on its way. The asymptotic of this time is characterized only by the whole structure of the transition matrix and the free energy on the state space between both minima.
By $f(N,x)\asymp 1$ we abbreviate the property of the existence of constants $C>c>0$ independent of N such that $c\le f(N,x)\le C$ for all x and we say that the order of magnitude of the function f is one. We write $f\asymp g$ if and only if the order of magnitude of f/g is one. Furthermore, we make the general assumption that every sequence of minima and every sequence of regions are chosen in a way that they converge to a limiting object modulo additional errors of order $\mathcal{O}(1/N)$. For example for the sequence of regions $\Delta=\Delta_N$ chosen in the next Lemma it is implicitly assumed that there is a suitable region $D\subset\mathbb{R}^k$ independent of N sufficing $\Delta_N=D+\mathcal{O}(1/N)$.

Lemma 3.1: *Assume that Hypothesis 1.1 and 1.2 hold. Fix* $k = 1, 2, 3$, *a subset* $\Delta \subset \Gamma^k$ *and locally uniform* γ*-Hölder-continuous functions* $g_i(N, \cdot)$ *satisfying* $g_i(N, x) \asymp 1$ *locally uniform in* $x \in (L, R)$ *and* $|g_i(N, x)| \asymp e^{-N^\delta}$ *on* $[L, R]$ *for suitable* $\delta < 1$. *Consider the sum*

M.6a

$$S(N, \Delta) = \sum_{x \in \Delta} g_1(N, x_1) \dots g_k(N, x_k) e^{N\left(-F(N, x_k) + \sum_{i=1}^{k-1} F(N, x_i)\right)}. \tag{3.3}$$

Define $x^* = (x_1^*, \dots, x_k^*) \in \mathcal{E}^{k-1} \times \mathcal{M}$ *by*

M.6

$$x^* = \arg\max_{x \in \Delta} -F(N, x_k) + \sum_{i=1}^{k-1} F(N, x_i). \tag{3.4}$$

Assume in addition that x^* *is well in the 'interior' of the region* Δ, *i.e.* $\operatorname{dist}(x^*, \Gamma \backslash \Delta) \geq c > 0$ *for suitable* $c > 0$ *independent of* N. *Then, there is* $\varepsilon > 0$ *such that*

M.7

$$\begin{aligned} S(N, \Delta) = \left(1 + \mathcal{O}\left(N^{-\varepsilon}\right)\right) & g_1(N, x_1^*) \dots g_k(N, x_k^*) \Sigma(N, x_1^*) \dots \Sigma(N, x_k^*) \\ & N^{k - k/\mu} e^{-N\left(F(N, x_k^*) - \sum_{i=1}^{k-1} F(N, x_i^*)\right)}, \end{aligned} \tag{3.5}$$

where for each critical point $w \in \mathcal{E} \cup \mathcal{M}$ *we have defined*

M.3

$$\Sigma(N, w) = 2(\mu!/|F_0^{(\mu)}(w)|)^{1/\mu} \operatorname{Fi}(\nu/\mu, 1/\mu; \pm\mu^{-\nu} F_{1,1}(N, w)(\mu!/|F_0^{(\mu)}(w)|)^{\nu/\mu}), \tag{3.6}$$

where the sign is negative in the case that w *is a local minimum and positive if* w *is a saddle. Here Fi denotes Faxen's integral defined as* $\operatorname{Fi}(\alpha, \beta, s) = \int_0^\infty e^{w + s w^\alpha} w^{\beta - 1} dw$. *It follows for every critical point* w *that* $\Sigma(N, w) \asymp 1$ *as* N *tends to infinity .*
Furthermore, assume that the asymptotic development $F(N, \cdot) = F_0 + N^{-1/2} F_1 + N^{-1} F_2 + \mathcal{O}(N^{-3/2}) + C_N$ *holds locally uniform on* (L, R), *where* $F_0 \in \mathcal{C}^2(L, R)$, $F_1, F_2 \in \mathcal{C}^{1,1}(L, R)$. *Suppose that the functions* $g_i(N, \cdot)$ *are locally uniformly Lipschitz-continuous. It follows that we may choose* $\varepsilon = 1/2$ *in (3.5), while*

M.4

$$\Sigma(N, w) = \left(1 + \mathcal{O}\left(N^{-1/2}\right)\right) 4 |F_0''(w)|^{-1/2} e^{\pm(1/4) F_1'(w)^2 / |F_0''(w)|} \int_{\mp(1/2) F_1'(w)/|F_0''(w)|^{1/2}}^{\infty} e^{u^2/2} du \tag{3.7}$$

with the same convention for the sign as in (3.6). In particular in the Gaussian case ,namely if $F_1(\cdot) \equiv$*constant, we have* $\Sigma(N, w) = \left(1 + \mathcal{O}\left(N^{-1/2}\right)\right) 2^{3/2} (\pi / |F_0''(w)|)^{1/2}$.

Proof: Since $F_{1,1}(N, w)$ is bounded, obviously the order of magnitude of the sequence $S(N, w)$ is one.
By means of the following equation (3.8) in combination with the continuity result obtained in (4.35) the cases summarized in (3.7) follow directly from the definitions using the variable $u = (2w)^{1/2}$ and from evaluation of a standard Gauss integral. Under the stronger assumptions on the free energy it holds that

M.4a

$$\mu^{-\nu} F_{1,1}(N, w)(\mu!/|F_0^{(\mu)}(w)|)^{\nu/\mu} = 2^{-1/2} F_1'(w) / |F_0''(w)| + \mathcal{O}(N^{-1/2}) \tag{3.8}$$

In order to prove (3.5) we remark that due to genericness of $F_0|\Gamma$ x^* is well defined. Fix $\sqrt{k}c > 0$ smaller than the distance of x^* to the region $\Gamma \backslash \Delta$. Invoking the lower bounds on the amplitudes g_i, the quadratic behavior of F_0 at $\lim x_i^* \in \mathcal{C}_{F_0}$, $i = 1, \dots, k$, written in (1.7), uniform H'older continuity of the subleading term $F_1(N,\cdot)$ from the form of the free energy of $F(N,\cdot)$ given in (1.6) one easily concludes that for suitable $\delta > 0$

M.8
$$S(N,\Delta) = \left(1+\mathcal{O}\left(e^{-N\delta}\right)\right) \sum_{|x-x_k^*|\le c} g_k(N,x)e^{-NF(N,x)} \prod_{i=1}^{k-1} \sum_{|x-x_i^*|\le c} g_i(N,x)e^{NF(N,x)}. \tag{3.9}$$

Splitting the sum to the index i on the right-hand side of (3.9) at x_i^* (4.39) applies to all $2k$ sums yielding (3.5) and the refined statement under the stronger condition concerning the amplitudes and the free energy. $\diamondsuit$

We now proceed in combining the results of chapter two and the previous Lemma. The first Proposition tells us that the rate of a transition probability is given by the energy difference between the saddle associated with the transition and the starting minimum. Moreover, the polynomial order of the amplitude in the large parameter is determined by the diffusive character connected with the degree of degeneracy of F_0 at the saddle, while the constant in the amplitude is given by the curvature of F_0 and the subleading part of the free energy at the saddle.

Proposition 3.2: *Assume that Hypothesis 1.1 and 1.2 are satisfied. Let $a,b \in \mathcal{M}$ be different. Let $z = z(a,b) \in \mathcal{E}$ be the saddle between a and b defined in (3.2). Then there is $\varepsilon > 0$ such that the probability of reaching b before a when starting in a behaves like*

M.9
$$\mathbb{P}[\tau_b^a < \tau_a^a] = \left(1+\mathcal{O}\left(N^{-\varepsilon}\right)\right)(q_z p_z)^{1/2}\Sigma(N,z)^{-1}N^{1/\mu-1}e^{-N(F(N,z)-F(N,a))}, \tag{3.10}$$

where $\Sigma(N,z)$ is defined in (3.6). In particular, in the Gaussian case we obtain

M.10
$$\mathbb{P}[\tau_b^a < \tau_a^a] = \left(1+\mathcal{O}\left(N^{-1/2}\right)\right) 2^{3/2}(q_z p_z)^{1/2}(|F_0''(z)|/\pi)^{1/2}N^{-1/2}e^{-N(F(N,z)-F(N,a))}. \tag{3.11}$$

Proof: Without loss we may assume that $a < b$. Applying Lemma 3.1 to the sum on the right-hand side of (2.13) proves (3.10), where we have used that the assumptions ensure $q_w \equiv p_w$ modulo factors of required order. (3.11) follows from the refinement of (3.5) written in Lemma 3.1. $\diamondsuit$

Given a local minimum $x \in \mathcal{M}$ and a set $I \subset \mathcal{M}$ denote by $C_x(I) \subset \mathcal{M}\backslash I$ the set of local minima in the irreducible component of $\Gamma\backslash(I\cup x)$ that contains x. In higher-dimensional lattices usually $C_x(I)$ coincides with the set $\mathcal{M}\backslash I$. Again we abbreviate $C_x(y) = C_x(\{y\})$. As a Corollary using the renewal structure of the process we obtain

Corollary 3.3: *Fix three different minima $a, b_1, b_2 \in \mathcal{M}$ such that a lies between b_1 and b_2, i.e. $C_a(b_1 \cup b_2) \neq \emptyset$. Let $z_1 = z(a,b_1)$ and $z_2 = z(a,b_2)$ be the saddles between a and b_1, b_2 respectively defined in (3.2). Suppose the saddle energy of the transition from a to b_1 is higher than that of the transition from a to b_2, i.e. $F_0(z_1) > F_0(z_2)$. Under the assumptions of the previous Lemma it follows the existence of $\varepsilon > 0$ such that*

M.12
$$\mathbb{P}[\tau_{b_1}^a < \tau_{b_2}^a] = \left(1+\mathcal{O}\left(N^{-\varepsilon}\right)\right)\left(\frac{q_{z_1}p_{z_1}}{q_{z_2}p_{z_2}}\right)^{1/2}\frac{\Sigma(N,z_2)}{\Sigma(N,z_1)}e^{-N(F(N,z_1)-F(N,z_2))}, \tag{3.12}$$

where $\Sigma(N,z_1)$ and $\Sigma(N,z_2)$ are defined in (3.6). In particular, in the Gaussian case we obtain

M.13
$$\mathbb{P}[\tau_{b_1}^a < \tau_{b_2}^a] = \left(1+\mathcal{O}\left(N^{-1/2}\right)\right)\left(\frac{q_{z_1}p_{z_1}}{q_{z_2}p_{z_2}}\right)^{1/2}\left(\frac{|F_0''(z_2)|}{|F_0''(z_2)|}\right)^{1/2} e^{-N(F(N,z_1)-F(N,z_2))}, \tag{3.13}$$

Proof: Dividing the paths contributing to the probability on the left-hand side of (3.12) wether they return to a before visiting b_1 from (2.12) we obtain

M.14
$$\mathbb{P}[\tau^a_{b_1} < \tau^a_{b_2}] = \mathbb{P}[\tau^a_{b_1} < \tau^a_{\{b_2,a\}}] + \mathbb{P}[\tau^a_a < \tau^a_{\{b_2,b_1\}}]\mathbb{P}[\tau^a_{b_1} < \tau^a_{b_2}]. \tag{3.14}$$

Equivalently we may write

M.15
$$\mathbb{P}[\tau^a_{b_1} < \tau^a_{b_2}] = \frac{\mathbb{P}[\tau^a_{b_1} < \tau^a_{\{b_2,a\}}]}{\mathbb{P}[\tau^a_{\{b_2,b_1\}} < \tau^a_a]}. \tag{3.15}$$

Being one dimensional from (3.10) and (3.15) it follows (3.12), while (3.11) shows (3.13). $\diamondsuit$

We now turn to the various expected times of a transition. The exponential rate in terms of the free energy of these times is always determined by a common mechanism, which can be interpreted nicely (see also [BEGK]). For, fix a local minimum $x \in \mathcal{M}$ and a subset $I \subset \mathcal{M} \backslash x$. We define the effective depth as the depth of the valley $V = V^{\pm}(z(x,I))$ containing x, i.e.

M.16
$$d_0(x,I) = F_0(z(x,I)) - \min F_0(\mathcal{M} \cap V). \tag{3.16}$$

Let $(c)_+ = (1/2)(|c| + c)$ denote the positive part of a number in $c \in \mathbb{R}$. For local minima $x, y \in \mathcal{M}$ and a subset $I \subset \mathcal{M} \backslash x$ we denote by $m(x,y,I) \subset \mathcal{M}$ the set of effective minima of the transition from x to y under the condition of avoiding I, i.e.

M.17
$$\begin{aligned} m(x,y,I) = \arg \max_{x' \in C_x(I \cup y)} & - (F_0(z(x,x')) - F_0(z(x, I \cup y)))_+ \\ & - (F_0(z(x',y)) - F_0(z(x',I)))_+ \\ & + (F_0(z(x,y)) - F_0(z(x,I)))_+ + d_0(x', I \cup y). \end{aligned} \tag{3.17}$$

Let us abbreviate $m(x,y) = m(x,y,\emptyset)$. Later on we shall see that under our assumptions concerning genericness of F_0 the set of effective minima consists of a single point called the effective minimum.
The maximized function on the left-hand side of (3.17) should be read in the following way. Starting the process in x the particle might visit the local minimum x' on its way to y. The exponential rate of the probability of visiting x' before y under the condition of avoiding I is given by the sum of the first three terms of the sum on the right-hand side of (3.17). Now, once the particle has met x' it stays in the valley associated with it and the saddle $z(x', I \cup y)$ for a long time. The exponential rate of this survival time in the valley is given by the effective depth $d_0(x', I \cup y)$.
For convenience let us introduce the effective time $T(x,y,I)$ to be

M.17a
$$\begin{aligned} (1/N)\log(T(x,y,I)) = \sum_{m \in m(x,y,I)} & - (F(N,z(x,m)) - F(N,z(x,I \cup y)))_+ \\ & - (F(N,z(m,y)) - F(N,z(m,I)))_+ \\ & + (F(N,z(x,y)) - F(N,z(x,I)))_+ + d(m, I \cup y), \end{aligned} \tag{3.18}$$

where $d(m, I \cup y) = F(N, z(m, I \cup y)) - F(N,m)$ for all $m \in m(x,y,I)$. As before we abbreviate $T(x,y) = T(x,y,\emptyset)$ and $T(x,y,z) = T(x,y,\{z\})$. We mention here that this definition does not match with those given in [BEGK] concerning $t_I(x,y)$. We will frequently use the rule that due to genericness of F_0 we have

M.18
$$F_0(z(u,v)) < F_0(z(v,w)) \Longleftrightarrow z(u,w) = z(v,w) \qquad (u,v,w \in \mathcal{M}, u \neq v) \tag{3.19}$$

In the case $y = x$ we may write in view of (3.19) again for N sufficiently large

M.19a
$$(1/N)\log(T(x,x,I)) = \sum_{m\in m(x,x,I)} -(F(N,z(x,m)) - F(N,x)) - (F(N,z(m,x)) - F(N,z(m,I)))_+ \\ + d(m, I\cup x). \tag{3.20}$$

We shall first treat the unconditioned mean time $\mathbb{E}[\tau_b^a]$ for a transition from $a \in \mathcal{M}$ to $b \in \mathcal{M}\backslash a$.

Proposition 3.4: *Assume that Hypothesis 1.1 and 1.2 are satisfied. Let $a, b \in \mathcal{M}$. It follows that the set of effective minima $m(a,b)$ of the transition from a to b consists of a single point and we write $m = m(a,b)$ in slight abuse of notation. Let $T(a,b)$ be the effective time of the transition defined after (3.18). Let $z = z(b, a\cup m)$ be the saddle between b and $\{a, m\}$ defined in (3.2). Then there exists $\varepsilon > 0$ such that*

M.20
$$\mathbb{E}[\tau_b^a] = \left(1 + \mathcal{O}\left(N^{-\varepsilon}\right)\right)(q_z p_z)^{-1/2}\Sigma(N,z)\Sigma(N,m)N^{2-2/\mu}T(a,b). \tag{3.21}$$

Here the terms $\Sigma(N,z)$ and $\Sigma(N,m)$ involved in the amplitude are defined in (3.6). In particular, in the Gaussian case we obtain

M.21
$$\mathbb{E}[\tau_b^a] = \left(1 + \mathcal{O}\left(N^{-?}\right)\right) 4\pi(q_z p_z)^{-1/2}(|F_0''(z)F_0''(m)|)^{1/2} N T(a,b). \tag{3.22}$$

Proof: Invoking (3.19) it is easily shown that

M.23
$$-(F_0(z(a,x')) - F_0(z(a,b)))_+ + d(x',b) = F_0(z(b,a\cup x')) - F_0(x') \qquad (x' \in C_a(b)) \tag{3.23}$$

Hence, it follows that the set of effective minima satisfies

M.23a
$$m(a,b) = \arg\max_{x'\in C_a(b)} F_0(z(b,a\cup x') - F_0(x') \tag{3.24}$$

Due to genericness of F_0 $m(a,b) = m$ is singleton. Without loss it may be assumed that $a < b$. The proof now follows the line of that of (3.10). Combination of (2.16), (3.5) and (3.24) yields (3.21) except the difference that p_z has to be replaced by $(p_z q_z)^{1/2}$. This remaining difference can be handled in the same manner as in the proof of (3.10). (3.22) follows easily from the refinement of (3.5) written in Lemma 3.1. ♢

Fix two minima $a, b \in \mathcal{M}$. The asymptotic nature of the time $\mathbb{E}[\tau_b^a | \tau_b^a < \tau_a^a]$ differs whether there is or is not a local minimum, which the particle can visit on its way form a to b without returning. While in the former case the leading, exponential order is determined by an effective minimum, the corresponding saddle and the curvature of F_0 at this critical points, in the latter case the time behaves polynomially and the whole behavior of the free energy between a and b accounts to the asymptotics. This is the content of the following Proposition. Let us stress again that only the proof of (3.27) requires the whole strength of the appendix, wherein sums of Laplace type are controlled uniformly in their domain of summation.
For convenience let us define $\Phi(s,t) = e^t \int_z^\infty e^{-w+s(w^{1/\mu}-t^{1/\mu})} w^{1/\mu-1} dw$. The main properties of this function are written in (4.28) and (4.29). Let $\Psi(t) = te^t(e^t - 1)$. For two different minima $a, b \in \mathcal{M}$ and a the associated saddle $z = z(a,b) \in \mathcal{E}$ let $\varphi(x) = \min_{w\in\lim\{a,z,b\}} \varphi_w(x)$ for x between a and b, where $\varphi_w(x)^\mu(x) = |F_0(x) - F_0(w)|$. These function are investigated in Lemma 4.2 in the appendix.

Proposition 3.5: *Assume Hypothesis 1.1 and 1.2 hold. Fix two different minima $a, b \in \mathcal{M}$ such that there is a local minimum between a and b. It follows that the set of effective minima $m(a,b,a)$ of the transition from a to b under the condition of no return is singleton and we write $m = m(a,b,a)$ in slight abuse of notation. Let $T(a,b,a)$ be the effective time defined in (3.18). Let $z = z(m, a\cup b) \in \mathcal{E}$ be the saddle between y and $\{a, m\}$ defined in (3.2). Then, there exists $\varepsilon > 0$ such that*

M.24
$$\mathbb{E}[\tau_b^a | \tau_b^a < \tau_a^a] = \left(1 + \mathcal{O}\left(N^{-\varepsilon}\right)\right) (p_z q_z)^{-1/2}\, \Sigma(N,z)\Sigma(N,m) N^{2-2/\mu} T(a,b,a). \tag{3.25}$$

Here $\Sigma(N,z)$ and $\Sigma(N,m)$ are defined in (3.6). In particular, in the Gaussian case we obtain

M.25
$$\mathbb{E}[\tau_b^a | \tau_b^a < \tau_a^a] = \left(1 + \mathcal{O}\left(N^{-1/2}\right)\right) 4\pi (q_z p_z)^{-1/2} (|F_0''(z) F_0''(m)|)^{1/2} N T(a,b,a). \tag{3.26}$$

Suppose there is no local minimum between a and b. Assume in addition that $F_{1,1}(N,\cdot)$ is uniformly Hölder-continuous. Then, there is $\varepsilon > 0$ such that

M.26
$$\mathbb{E}[\tau_b^a | \tau_b^a < \tau_a^a] = \left(1 + \mathcal{O}\left(N^{-\varepsilon}\right)\right) N^{2-1/\mu} \int_{\lim a}^{\lim b} \frac{\Psi(|F_0'(x)|)}{p_x \mu \varphi'(x)} \Phi\left(F_{1,1}(N,x)/\varphi'(x)^\nu, N\varphi(x)^\mu\right) dx, \tag{3.27}$$

where φ, $\Psi(u)$ and Φ are defined preceding the Proposition and where the jump probability p_x is regarded as a uniformly Hölder-continuous function by linear interpolation. Consequently, the order of magnitude of the transition time is $N^{2-1/\mu}$.

Proof of (3.25) and (3.26): Since under the condition of visiting b from a the particle visits all minima in $C_a(a \cup b)$ surely, we have

M.29
$$-(F_0(z(a,x')) - F_0(z(a,b)))_+ = 0 \qquad (x' \in C_a(a \cup b)), \tag{3.28}$$

which can be proved using (3.19). This leads directly to

M.30
$$m(a,b,a) = \arg \max_{x' \in C_a(a \cup b)} d(x', a \cup b), \tag{3.29}$$

showing in particular that $m(a,b,a) = m$ is singleton by genericness of F_0. The procedure is analogous to the proof of (3.21). Assume $a < b$. Combination of (2.14) with (3.5), (3.29) with (3.4) and invoking $p_w \equiv q_w$ at points $w \in \mathcal{E} \cup \mathcal{M}$ modulo factors of required order gives (3.26) as asserted. Again, the proof of (3.27) is an immediate consequence of the refinement of (3.5) in the Gaussian case. $\diamondsuit$

Proof of (3.27): Let the sum $S(N)$ in the nominator on the right-hand side of (2.14) be decomposed into $S(N) = S_1(N) + S_2(N)$, where $z = z(a,b)$ and

M.31
$$\begin{aligned} S_1(N) &= \sum_{\substack{a<l\leq j\leq r<b \\ j>z}} (q_l p_r)^{-1} e^{N(F(N,l) - F(N,j) + F(N,r))}, \\ S_2(N) &= \sum_{\substack{a<l\leq j\leq r<b \\ j\leq z}} (q_l p_r)^{-1} e^{N(F(N,l) - F(N,j) + F(N,r))} \end{aligned} \tag{3.30}$$

In order to evaluate $S_1(N)$ asymptotically define $\tilde{g}(N,\cdot)$ by

M.32
$$\tilde{g}(N, l/N) = N^{1/\mu - 2} \sum_{\max(z+1/N, l) \leq j \leq r < b} p_r^{-1} e^{N(F(N,r/N) - F(N,j/N))} \qquad (l \in [a,b]). \tag{3.31}$$

We recall that the sum is understood to be taken over points in Γ only. Since F_0 is monotonically decreasing on $[z + 1/N, b]$, the subleading part F_1 is uniformly Hölder-continuous to the parameter

ν and since the order of magnitude of the jump probabilities is one, we have the a priori bound $cN^{1/\mu-1} < \tilde{g}(N,l) < CN^{1/\mu}$ for $l \in [a,b]$. Invoking in addition that F_0 attains its maximum on (a,b) at z for fixed small $0 < c < b-z$ and suitable $\delta > 0$ it follows that

M.33
$$S_1(N) = \left(1 + \mathcal{O}\left(e^{-N\delta}\right)\right) N^{2-1/\mu} \sum_{|l-z|<c} (\tilde{g}(N,l)/q_l) e^{NF(N,l)}. \tag{3.32}$$

We now split the sum in (3.31) at $j = z+c$. For, fix $c < \tilde{c} < b-z$ and define the continuous functions $h_1(N,\cdot)$, $h_2(N,\cdot)$ and $g(N,\cdot)$ by

M.34
$$\begin{aligned} h_1(N,j) &= N^{1/\mu-2} p_j e^{-NF(N,j)} \sum_{j \le r < z+\tilde{c}} (1/p_r) e^{NF(N,r)} \qquad (j \in (z, z+c]), \\ h_2(N,r) &= N^{1/\mu-2} (1/p_r) e^{NF(N,r)} \sum_{z+c<j\le r} p_j e^{-NF(N,j)} \qquad (r \in (z+c, b]), \\ g(N,l) &= \sum_{(z+1/N)\vee l \le j \le z+c} h_1(N,j) + \sum_{z+c<r<b} h_2(N,r) \qquad (l \in (z-c, z+c)). \end{aligned} \tag{3.33}$$

The difference between g and $\tilde{g}$ is that we have to replace $z+\tilde{c}$ by b. But on $[z, z+c]$ the function obtained by inserting b instead of $z+\tilde{c}$ in the definition of $h_1(N,\cdot)$ differs only by an factor, which is exponentially close to one. Hence, it follows $g/\tilde{g} = 1 + \mathcal{O}(e^{-N\delta}$ for suitable $\delta > 0$ on $(z-c, z+c)$. Thus combination with (3.32) and (3.31) yields

M.35
$$S_1(N) = \left(1 + \mathcal{O}\left(e^{-N\delta}\right)\right) N^{2-1/\mu} \sum_{|l-z|<c} g(N, l/N) q_l^{-1} e^{NF(N,l)}. \tag{3.34}$$

The function $g(N,\cdot)$ is uniformly Lipschitz-continuous on $(z-c, z+c)$. For, obviously $\Psi(t)$ is a bounded function. (4.29) ensures that Φ is bounded on $s \ge 0, t \in K$, where $K \subset \mathbb{R}$ is an arbitrary but fixed compact set. Furthermore, application of (4.62) to $h_1(N,\cdot)$ gives for $j \in (z+c]$

M.36
$$h_1(N,j) = \left(1 + \mathcal{O}\left(N^{-\varepsilon}\right)\right) N^{-1} \frac{\Psi(-F_0'(j))}{p_j \mu \varphi'_{\lim z}(j)} \Phi\left(\frac{F_{1,1}(N,j)}{\varphi_z'(j)^\nu, N\varphi_z(j)^\mu}\right), \tag{3.35}$$

where we have used that $\Psi(N(F_0([z+\tilde{c}]) - F_0(j)), -F_0'(j)) \equiv \Psi(-F_0'(j)$ modulo a factor which is exponentially close to one. Thus combination of the predications after (3.34) with $F_{1,1}(N,\cdot)/(p.\varphi_z' \asymp 1$ on $[z-c, z+c]$ implies for $k, l \in (z-c, z+c)$

M.37
$$|g(N,k) - g(N,l)| = \sum_{k \le j < l} h_1(N,j) = \mathcal{O}\left(|k-l|\right). \tag{3.36}$$

Denote by $\tilde{S}(N)$ the sum in the denominator on the right-hand side of (2.14). Combination with the leading order of the sum on the right-hand side of (3.34), which in view of (3.36) follows from (4.41), gives for some $\varepsilon > 0$

M.38
$$S_1(N)/\tilde{S}(N) = \left(1 + \mathcal{O}\left(N^{-\varepsilon}\right)\right) N^{2-1/\mu} \left(g(N,z) + \mathcal{O}\left(N^{-1/\mu}\right)\right). \tag{3.37}$$

So as to identify $g(N,z)$ fix $0 < \alpha < 1/\mu$ and denote $\hat{z} = \lim z$. Let p_j also denote the γ-Hölder-continuous function obtained by linear interpolation on $a - 1/N \le j/leqb + 1/N$. (3.35) then implies that modulo factors of order $1 + \mathcal{O}(N^{-\varepsilon})$

M.39
$$\sum_{z<j\le z+c} h_1(N,j) \equiv \Sigma_1(N) + \Delta_0(N) + \Delta_1(N) + \Delta_2(N) + \Delta_3(N), \tag{3.38}$$

where the leading order is given by

M.40
$$\Sigma_1 = \int_{\hat{z}}^{\hat{z}+c} \frac{\Psi(-F_0'(t))}{p_t \mu \varphi_{\hat{z}}(t)} \Phi\left(\frac{F_{1,1}(N,t)}{\varphi_z'(t)^\nu}, N\varphi_z(t)^\mu\right) dt, \tag{3.39}$$

while the error terms are defined as

M.41
$$\begin{aligned}
\Delta_0 &= \sum_{z<j<z+c} \int_j^{j+1/N} \left(\frac{\Psi(-F_0'(j))}{p_t \mu \varphi_{\hat{z}}(j)} - \frac{\Psi(-F_0'(t))}{p_t \mu \varphi_{\hat{z}}(t)}\right) \Phi\left(\frac{F_{1,1}(j)}{\varphi_z'(j)^\nu}, N\varphi_z(j)^\mu\right) dt,\\
\Delta_1 &= \sum_{z<j<z+N^{-\alpha}} \int_j^{j+1/N} \frac{\Psi(-F_0'(t))}{p_t \mu \varphi_{\hat{z}}(t)} \left(\Phi\left(\frac{F_{1,1}(j)}{\varphi_z'(j)^\nu}, N\varphi_z(j)^\mu\right) - \Phi\left(\frac{F_{1,1}(t)}{\varphi_z'(t)^\nu}, N\varphi_z(t)^\mu\right)\right) dt,\\
\Delta_2 &= \sum_{z+N^{-\alpha}\le j<z+c} \int_j^{j+1/N} \frac{\Psi(-F_0'(t))}{p_t \mu \varphi_{\hat{z}}(t)} \left(\Phi\left(\frac{F_{1,1}(j)}{\varphi_z'(j)^\nu}, N\varphi_z(j)^\mu\right) - \Phi\left(\frac{F_{1,1}(t)}{\varphi_z'(t)^\nu}, N\varphi_z(t)^\mu\right)\right) dt.\\
\Delta_3 &= \left(\int_z^{\hat{z}} + \int_{\hat{z}+c}^{z+c}\right) \Psi(-F_0'(t)) \Phi\left(\frac{F_{1,1}(t)}{\varphi_z'(t)^\nu}, N\varphi_z(t)^\mu\right) dt,
\end{aligned} \tag{3.40}$$

Recalling that the integrand defining Δ_3 is bounded uniformly in N it follows that $\Delta_3 = \mathcal{O}(N^{-1})$. It is straightforward to show that $\Psi\prime(t)$ is bounded. Recalling that the function involving Φ in the integrand defining Δ_0 is bounded uniformly in N it follows that $\Delta_0 = \mathcal{O}(N^{-1})$. Invoking the degenerate behavior of F_0 at z for $j \in [z, z+N^{-\alpha})$ and $0 \le t-j \le 1/N$ the following inequality holds

M.42
$$N|\varphi_z(t)^\mu - \varphi_z(j)^\mu| = |F_0(t) - F_0(j)| \le CN^{-\alpha(\mu-1)}. \tag{3.41}$$

Furthermore $F_{1,1}(N,\cdot)/(\varphi_{\hat{z}}')^\nu$ is uniformly δ-Hölder-continuous for suitable $\delta > 0$. Since we know that the integrand on the right-hand side of (3.39) is bounded, combining (4.28) with (3.41) leads to $\Delta_1 = \mathcal{O}(N^{-\alpha\mu} + N^{-\alpha(\delta\nu/\mu-1)})$. Combination of (4.29) with Hölder continuity of $F_{1,1}(N,\cdot)/(\varphi_{\hat{z}}')^\nu$ and with the inequality $N\varphi_z(t)^\mu \ge cN^{1-\alpha\mu} > 0$ for $z+N^{-\alpha} \le t \le z+c$, yields $\Delta_2 = \mathcal{O}(N^{-(1-\alpha\mu)(1-1/\mu)})$. Obviously $\Psi(t)$ is locally bounded away form zero by a positive amount. Applying (4.29) once more to the integrand of $\Sigma_1(N)$ and using the variable $u = N\varphi(t)^\mu$ there exists a generic number $c' > 0$ such that

M.43
$$\Sigma_1(N) \ge c' N^{-1/\mu} \int_{N\varphi_z(z+N^{-1/\mu})^\mu}^{N\varphi_z(z+c/2)^\mu} u^{1/\mu-1} du \ge c'. \tag{3.42}$$

The last inequality in (3.42) is assured by the fact that $\varphi_z(z+t) \asymp t$ in $z < z \le z+c/2$. Combining (3.42) with (3.38) and with the error estimates for Δ_i there is $\varepsilon > 0$ such that

M.44
$$\sum_{z<j\le z+c} h_1(N,j) = \left(1 + \mathcal{O}\left(N^{-\varepsilon}\right)\right) \Sigma_1(N). \tag{3.43}$$

The second sum in the definition of $g(N,z)$ in (3.33) is treated analogously except the difference that we have to be more careful with the replacement of $\Psi(s,t)$ by $\Psi(t)$ in (4.62). Paying attention to this feature in the term Δ_4 we arrive at

M.45
$$g(N,z) = \left(1 + \mathcal{O}\left(N^{-\varepsilon}\right)\right)(\Sigma_1(N) + \Sigma_2(N)) + \Delta_4(N), \tag{3.44}$$

where we have defined $\hat{b} = \lim b$ and

M.46
$$\Sigma_2(N) = \int_{\hat{z}+c}^{\hat{b}} \frac{\Psi(-F_0'(t))}{p_t \mu \varphi_{\hat{b}}(t)} \Phi\left(\frac{F_{1,1}(N,t)}{\varphi_{\hat{b}}'(t)^\nu}, N\varphi_{\hat{b}}(t)^\mu\right) dt, \tag{3.45}$$

and where the error term Δ_4 is given by

M.47
$$\Delta_4(N) = \frac{1}{N}\left(\sum_{z+c<j\le z+c+N^{-\alpha}} + \sum_{z+c+N^{-\alpha}<j<b}\right)\frac{-F_0'(j)e^{-N(F_0(z+c)-F_0(j))}}{p_j\mu\varphi_{\hat b}'(j)(e^{-F_0'(j)}-1)}\Phi\left(\frac{F_{1,1}(N,j)}{\varphi_{\hat b}'(j)^\nu}N\varphi_{\hat b}(j)^\mu\right), \tag{3.46}$$

where $0<\alpha<1$. Using that for $z+c+N^{-\alpha}\le t\le b$ it follows $N(F_0(z+c)-F_0(t))\ge c'N^{1-\alpha}>0$ and that the terms in the sum defining Δ_4 are uniformly bounded we conclude that $\Delta_4=\mathcal{O}(N^{-\alpha}+N^{-\infty})$. Inserting (3.44) and the error estimate for Δ_4 into (3.37) and using that (3.42) guarantees the existence of $c'>0$ such that $g(N,z)\ge c'$ it follows for some $\varepsilon>0$

M.48
$$S_1(N)/\tilde S(N) = \left(1+\mathcal{O}\left(N^{-\varepsilon}\right)\right)(\Sigma_1(N)+\Sigma_2(N)). \tag{3.47}$$

The sum $S_2(N)$ defined in (3.30) is evaluated in the same manner as $S_1(N)$ yielding

M.49
$$S(N)/\tilde S(N) = (S_1(N)+S_2(N))/\tilde S(N) = \left(1+\mathcal{O}\left(N^{-\varepsilon}\right)\right)(\Sigma_1(N)+\Sigma_2(N)+\Sigma_3(N)+\Sigma_4(N)), \tag{3.48}$$

where $\hat a=\lim a$ and for fixed $0<\hat c<z-a$

M.50
$$\begin{aligned}\Sigma_3(N) &= \int_{\hat z-\hat c}^{\hat z}\frac{\Psi(-F_0'(t))}{p_t\mu\varphi_{\hat z}(t)}\Phi\left(\frac{F_{1,1}(N,t)}{\varphi_{\hat z}'(t)^\nu}N\varphi_{\hat z}(t)\right)dt,\\ \Sigma_4(N) &= \int_{\hat a}^{\hat z-\hat c}\frac{\Psi(-F_0'(t))}{p_t\mu\varphi_{\hat a}(t)}\Phi\left(\frac{F_{1,1}(N,t)}{\varphi_{\hat a}'(t)^\nu}N\varphi_a(t)\right)dt.\end{aligned} \tag{3.49}$$

(3.48) completes the proof of (3.27) since choosing $0<c<b-z$, $0<\hat c<z-a$ as the boundary points of the set on which the function $\varphi(t)$ equals $\varphi_{\hat z}$ we may write

M.51
$$\sum_{i=1}^4\Sigma_i(N) = \int_a^b\frac{\Psi(|F_0'(t))|}{p_t\mu\varphi(t)}\Phi\left(F_{1,1}(N,t)/\varphi'(t)^\nu, N\varphi(t)^\mu\right)dt. \tag{3.50}$$

◇◇

We continue the discussion with the return time to a minimum under the condition of avoiding two other minima.

Lemma 3.6: *Assume Hypothesis 1.1 and 1.2 hold. Fix three minima $a,b_1,b_2\in\mathcal{M}$ such that a lies between b_1 and b_2. It follows that the set of effective minima $m=m(a,a,b_1\cup b_2)$ of the return to a under the condition of avoiding $b_1\cup b_2$ is a singleton. Let $T(a,a,b_1\cup b_2)$ be the effective time of returning defined in (3.20). In the case $m\ne a$ let $z_<=z(m,a\cup b_1\cup b_2)\in\mathcal{E}$ be the saddle between m and the set $a\cup b_1\cup b_2$ and let $z_>=z(m,a)$ be the saddle between m and a defined in (3.2). In the case $m=a$ we set $z_<=z_>=a$. It follows the existence of $\varepsilon>0$ such that*

M.52
$$\mathbb{E}[\tau_a^a|\tau_a^a<\tau_{b_1\cup b_2}^a] = \left(1+\mathcal{O}\left(N^{-\varepsilon}\right)\right)\frac{p_{z_>}q_{z_>}}{p_{z_<}q_{z_<}}\left(\frac{\Sigma(N,z_<)}{\Sigma(N,z_>)}\right)^2\Sigma(N,m)N^{1-1/\mu}T(a,a,b_1\cup b_2) \tag{3.51}$$

In the Gaussian case it follows

M.53
$$\mathbb{E}[\tau_a^a|\tau_a^a<\tau_{b_1\cup b_2}^a] = \left(1+\mathcal{O}\left(N^{-1/2}\right)\right)\frac{p_{z_>}q_{z_>}}{p_{z_<}q_{z_<}}4(\pi)^{1/2}\frac{|F_0''(z_>)|}{|F_0''(z_<)F_0''(m)|}N^{1/2}T(a,a,b_1\cup b_2). \tag{3.52}$$

Proof: Without loss we assume that $b_1 < a < b_2$. Combination of (2.15) with (3.5) and (3.4) and invoking that $p_w \equiv q_w$ for critical points $w \in \mathcal{E} \cup \mathcal{M}$ modulo factors of required order gives

$$\begin{aligned}&\mathbb{E}[\tau_a^a | \tau_a^a < \tau_{b_1 \cup b_2}^a]\\ \equiv & \frac{p_{z(a,b_1)} q_{z(a,b_1)}}{p_{z(x_1^*,b_1)} q_{z(x_1^*,b_1)}} \left(\frac{\Sigma(N, z(x_1^*, b_1)) \Sigma(N, x_1^*)}{\Sigma(N, z(a, b_1))} \right)^2 N^{1-1/\mu} e^{N(2(F(N,z(x_1^*,b_1)-F(N,z(a,b_1))+F(N,a)-F(N,x_1^*))}\\ + & \frac{p_{z(a,b_2)} q_{z(a,b_2)}}{p_{z(x_2^*,b_2)} q_{z(x_2^*,b_2)}} \left(\frac{\Sigma(N, z(x_2^*, b_2)) \Sigma(N, x_2^*)}{\Sigma(N, z(a, b_2))} \right)^2 N^{1-1/\mu} e^{N(2(F(N,z(x_2^*,b_2)-F(N,z(a,b_2))+F(N,a)-F(N,x_2^*))},\end{aligned} \tag{3.53}$$

M.54 where by means of genericness of F_0 x_i^* is well defined by

M.55
$$x_i^* = \arg \max_{x' \in C_a(a \cup b_i)} 2(F_0(z(x', b_i)) - F_0(z(a, b_i))) + F_0(a) - F_0(x'). \tag{3.54}$$

In view of (3.19) we observe that $z(a, b_i) = z(x_i^*, b_i)$ is equivalent to the case $z(x^*, a) = z(x^*, a \cup b_i)$ or simultaneously $z(a, b_i) = z(x^*, a)$ and $z(x^*, a \cup b_i) = z(x_i^*, b_i)$. Invoking (3.19) again one observes for $x' \in C_a(a \cup b_i)$

M.56
$$\begin{aligned}F_0(z(x', b_i)) - F_0(z(a, b_i)) =& F_0(z(x', a \cup b_i)) - F_0(z(a, x'))\\ =& F_0(z(x', a \cup b_1 \cup b_2)) - F_0(z(a, x')),\end{aligned} \tag{3.55}$$

where in the last step we have used that being in the one-dimensional situation $z(x', a \cup b_i) = z(x', a \cup b_1 \cup b_2)$. Inserting (3.55) into (3.54) shows that the minima x_i^* are defined as 'critical points' of the same function over different regions. Hence since the amplitudes on the right-hand side of (3.53) do not change the exponential rate of both terms we may write

M.57
$$\begin{aligned}&\mathbb{E}[\tau_a^a | \tau_a^a < \tau_{b_1 \cup b_2}^a]\\ \equiv & \frac{p_{z(x^*,a\cup b_1\cup b_2)} q_{z(x^*,a\cup b_1 \cup b_2)}}{p_{z(a,x^*)} q_{z(a,x^*)}} \left(\frac{\Sigma(N, z(x^*, a \cup b_1 \cup b_2))}{\Sigma(N, z(a, x^*))} \Sigma(N, x^*) \right)^2\\ & N^{1-1/\mu} e^{N(2(F(N,z(x^*,a\cup b_1\cup b_2)-F(N,z(a,x^*))+F(N,a)-F(N,x^*))},\end{aligned} \tag{3.56}$$

where

M.58
$$x^* = \arg \max_{x' \in C_a(b_1 \cup b_2)} 2(F(N, z(x', a \cup b_1 \cup b_2)) - F(N, z(a, x'))) + F(N, a) - F(N, x'). \tag{3.57}$$

Invoking (3.19) once more it is straightforward to prove for $x' \in C_a(a \cup b_i)$

$$\begin{aligned}2(F_0(z(x', b_i)) - F_0(z(a, b_i))) + F_0(a) - F_0(x') =& -(F_0(z(a, x') - F_0(a)) + d_0(x', a \cup b_1 \cup b_2)\\ & -(F_0(z(x', a)) - F_0(z(x', b_1 \cup b_2)))_+.\end{aligned} \tag{3.58}$$

M.58a Thus, combination of (3.58) and (3.55) with (3.57) leads to $m(a, a, b_1 \cup b_2) = x^*$. In particular we may replace the exponential factor on the right-hand side of (3.56) by the effective time $T(a, a, b_1 \cup b_2)$. In view of (3.56) the proof of (3.51) is complete. Again, (3.52) is an immediate consequence of the refined estimate of the Gaussian case written in Lemma 3.1. $\diamondsuit$

By means of the renewal structure such as in (3.64) in principle one could proceed considering general transition times of type $\mathbb{E}[\tau_{b_2}^a | \tau_{b_2}^a < \tau_{b_1}^a]$. The exponential rate of these times would be the effective time $T(a, b_2, b_1)$ defined in (3.18). But, since every transition $\{\tau_b^a < \infty\}$ possesses a natural decomposition into a deterministic sequence of so called admissible transitions modulo an event of

probability $\mathcal{O}(e^{-N^\delta})$, $\delta>0$, every transition time $\mathbb{E}[\tau_b^a]$ essentially is a sum of admissible transition times. The following discussion is restricted to this case and we refer the reader who is interested in the decomposition procedure to [BEGK]. The main advantage we are gaining from being in an admissible situation is that the effective time is of pleasant simplicity as we shall see in a moment.
Let us now briefly recall the main concepts introduced in [BEGK]. We construct a tree representing the structure of the energy landscape in the following way. For each local minimum $x\in\mathcal{M}$ draw a link to the saddle $z=\arg\min\{F_0(z'):\exists y\in\mathcal{M}\backslash x : z'=z(x,y)\}$ with lowest energy connecting x to other minima. For each saddle $z\in\mathcal{E}$ draw a link to the saddle $\tilde{z}$ with minimal energy connecting z to other minima. Let $\mathcal{T}=\mathcal{T}_N$ denote the tree constructed in this way. We remark here that being one-dimensional in contrast to [BEGK] there is no need to introduce certain additional edges called yellow arrows, which reflects the gradient flow of the free energy; here they are automatically incorporated in the tree. In this context yellow arrows are trivially those edges containing a minimum.
For $x\in\mathcal{M}$, $z\in\mathcal{E}$ connected by a strictly increasing path in the tree $\mathcal{T}$ let $\mathcal{T}_{x,z}$ be the branch of the tree containing x and emanating from z. Let $x,y\in\mathcal{M}$ be different local minima and $z=z(x,y)$ be the corresponding saddle. Set $\mathcal{T}^c_{x,z}=\mathcal{T}\backslash\mathcal{T}_{x,z}$ and call the transition

M.59
$$\mathcal{F}(x,z,y)=\left\{\tau_y^x<\tau^x_{\mathcal{T}^c_{x,z}\cap\mathcal{M}}\right\}\tag{3.59}$$

admissible if and only if x is the absolute minimum in the branch $\mathcal{T}_{x,z}$ and y is the first descendant of z in $\mathcal{T}_{x,z}$. Observe that in view of (3.10) we have $\mathbb{P}[\mathcal{F}(x,z,y)]=1-\mathcal{O}(e^{-N\delta})$, where δ is arbitrarily close to the energy difference between z and its first ancestor. In this situation we have

Proposition 3.7: *Assume Hypothesis 1.1 and 1.2 hold. Fix $a,b\in\mathcal{M}$ such that with $z=z(a,b)$ defined in (3.2) the transition $\mathcal{F}(a,z,b)$ defined in (3.59) is admissible. Abbreviate $I=\mathcal{T}^c_{a,z}\cap\mathcal{M}$ and let $T(a,b,I)$ be the effective time of the transition defined in (3.18). Then there is $\varepsilon>0$ such that*

M.60
$$\mathbb{E}[\tau_b^a|\mathcal{F}(a,z,b)]=\left(1+\mathcal{O}\left(N^{-\varepsilon}\right)\right)\left(\frac{p_aq_a}{p_zq_z}\right)\Sigma(N,z)\Sigma(N,a)N^{2-2/\mu}T(a,b,I),\tag{3.60}$$

where the rate of the effective time is given by $(1/N)\log(T(a,b,I))=F(N,z)-F(N,a)$. In particular, in the Gaussian case we obtain

M.61
$$\mathbb{E}[\tau_b^a|\mathcal{F}(a,z,b)]=\left(1+\mathcal{O}\left(N^{-?}\right)\right)\left(\frac{p_aq_a}{p_zq_z}\right)^{1/2}8\pi\left(|F_0''(z)||F_0''(a)|\right)^{1/2}NT(a,b,I),\tag{3.61}$$

Proof: Splitting the process wether or not the particle returns to a and stopping the process in the former case at its return to a from (2.12) we obtain

M.62
$$\mathbb{E}[\tau_b^a,\tau_b^a<\tau_I^a]=\mathbb{E}[\tau_b^a,\tau_b^a<\tau_{I\cup a}^a]+\mathbb{E}[\tau_a^a,\tau_a^a<\tau_{I\cup b}^a]\mathbb{P}[\tau_b^a<\tau_I^a]+\mathbb{P}[\tau_a^a<\tau_{I\cup b}^a]\mathbb{E}[\tau_b^a,\tau_b^a<\tau_I^a].\tag{3.62}$$

Similarly as in (3.15) by means of (2.12) again it follows

M.63
$$\mathbb{P}[\tau_b^a<\tau_I^a]=\frac{\mathbb{P}[\tau_b^a<\tau_{I\cup a}^a]}{\mathbb{P}[\tau_{I\cup b}^a<\tau_a^a]}.\tag{3.63}$$

In view of (3.63) (3.62) rearranges to

M.64
$$\mathbb{E}[\tau_b^a|\tau_b^a<\tau_I^a]=\mathbb{E}[\tau_b^a|\tau_b^a<\tau_{I\cup a}^a]+\frac{\mathbb{E}[\tau_a^a,\tau_a^a<\tau_{I\cup b}^a]}{\mathbb{P}[\tau_{I\cup b}^a<\tau_a^a]}.\tag{3.64}$$

Concerning first the second term on the right-hand side of (3.64) we observe that $T(a,a,I\cup b)=1$ and that $m(a,a,I\cup b)=a$ since a is the deepest minimum in the branch $\mathcal{T}_{a,z}$. Furthermore, since due to admissibility either the saddle $z=z(a,b)$ has strictly lower energy than the saddle $z(a,I)$ or $z=z(a,I)$, being one-dimensional by means of (3.10) for every $0<\delta<F_0(z(a,I))-F_0(z)$ in every case it holds

M.65
$$\mathbb{P}[\tau^a_{I\cup b}<\tau^a_a]=\mathbb{P}[\tau^a_b<\tau^a_a]-\mathbb{P}[\tau^a_I<\tau^a_{b\cup a}]\geq \mathbb{P}[\tau^a_b<\tau^a_a]-\mathbb{P}[\tau^a_I<\tau^a_a]\geq \left(1-e^{-N\delta}\right)\mathbb{P}[\tau^a_b<\tau^a_a]. \quad (3.65)$$

Obviously we have $z_<=z_>=a$, where $z_<,z_>$ are defined in Lemma 3.6. Combining (3.10) with (3.65) and the result with (3.51) for $b_1\cup b_2=\partial(I\cup b)$ we conclude

M.66
$$\frac{\mathbb{E}[\tau^a_a,\tau^a_a<\tau^a_{I\cup b}]}{\mathbb{P}[\tau^a_{I\cup b}<\tau^a_a]}=\left(1+\left(N^{-\varepsilon}\right)\right)(p_zq_z)^{-1/2}\Sigma(N,a)\Sigma(N,z)N^{2-2/\mu}e^{N(F(N,z)-F(N,a))}. \quad (3.66)$$

The first term on the right-hand side of (3.64) is exponentially smaller than the second one. This follows from (3.25) or (3.27) since obviously being one-dimensional $\{\tau^a_b<\tau^a_{I\cup a}\}=\{\tau^a_b<\tau^a_a\}$ and since in an admissible situation we have $(1/N)\log(T(a,b,a))\leq F(N,z)-F(N,a)-\delta$ for suitable $\delta>0$ and N sufficiently large. Hence, to complete the proof it remains to identify the rate of the effective time $T(a,b,I)$. But clearly, from the definition (3.18) of this time and being in an admissible situation we conclude

M.67
$$(1/N)\log(T(a,b,I))=F(N,z)-F(N,a). \quad (3.67)$$

(3.61) follows from inserting (3.11) and (3.52) into (3.64). $\diamondsuit$

4. Appendix: Asymptotic behavior of sums of Laplace type

In order to extract asymptotic information from Corollary 2.2 about the quantities describing transitions which leads to the results of chapter 3 we have to understand the behavior of following sums of 'Laplace type' uniformly in the variable $x \in I = [x_0, b] \cap \mathbb{Z}/N$ for N large.

A.1
$$S(N,x) = \sum_{x \le k \le b} e^{-NF(N,k)} g(N,k). \tag{4.1}$$

Here $x_0 < b$ are independent of N and in contrast to the notation used in chapter two $[\alpha, \beta]$ denotes the usual interval in $\mathbb{R}$ for real numbers α, β. As a general assumption during the appendix we demand that the following Hypothesis holds as long as nothing else is said.

Hypothesis 4.1: *Let $g(N,\cdot) \in \mathcal{C}[x_0, b]$, $N \geq 1$, be uniformly Hölder-continuous to the parameter $\gamma > 0$. Furthermore, there is $c > 0$ such that $g(N,x) \geq c > 0$, $x \in [x_0, b]$.*
$F(N,\cdot)$ satisfies Hypothesis 1.1 while $L < x_0 < b < R$ holds. The point x_0 is the absolute minimum of the leading order F_0 of the free energy $[x_0, b]$. In addition, x_0 is the only critical point of F_0 between x_0 and b, i.e. $\mathcal{C}_{F_0} \cap [x_0, b] = x_0$.

Thus $e^{-NF(N,\cdot)}$ is a highly peaked function at x for large N. The situation is of course parallel to the situation in the theory of Laplace's integrals. Actually, in regions where $F(N,\cdot)$ is sufficiently flat the corresponding Laplace's integral approximates the sum in leading order. This is the content of Corollary 4.5 of Lemma 4.3. We mention [Ol] as a reference, wherein the case for fixed regions of integration is treated. In the following analysis the emphasis lies on the uniformity in the argument x in $S(N,x)$.
The approximation fails to be right in domains, where $F(N,\cdot)$ is strongly varying. As the simplest example may serve $g(N,x) \equiv 1$, $F(N,x) \equiv x$:

A.2
$$\begin{aligned} \sum_{k=n}^{K-1} e^{-N(k/N)} &= e^{-n}(1 - e^{-(K-n)})e/(e-1) \\ N \int_{n/N}^{K/N} e^{-Ns} ds &= e^{-n}(1 - e^{-(K-n)}). \end{aligned} \tag{4.2}$$

The example above already provides us how to handle the case of rapidly varying $F(N,\cdot)$. This feature allows us to restrict the domain of summation to a small vicinity of the minimum x in $S(N,x)$. In this neighborhood the affine part of the leading order F_0 approximates $F(N,\cdot)$ sufficiently well without affecting the leading order of the sum. Hence one is left evaluating a geometric sum. This will be done in Lemma 4.6.
Since the different regions of validity of both approximation procedures have nontrivial intersection, in Corollary 4.7 one finds the interpolating function giving the leading order uniformly in I.

Let us first analyze the continuous situation, namely where the sum in (4.1) is replaced by an integral. In order to understand the form of the subleading term in (1.6) given in (1.8) notice that it is the borderline case, where this term deserves its name. This is meant in the sense that for fixed $0 \leq \alpha < 1$ functions of order of magnitude $N^{-\alpha} x^{\nu}$ with $\alpha < \nu/\mu$ take over to be the dominant part in the exponential behavior of the Laplace's sum.
As an auxiliary function to work out the degeneracy of F_0 at x_0 let us define φ by the relation

A.3
$$F_0(x) - F_0(x_0) = \varphi(x)^{\mu} \qquad (x_0 \leq x \leq b). \tag{4.3}$$

From the conditions that the free energy $F(N,\cdot)$ suffices we know that $F_0^{(\mu)}(x_0) > 0$ and that $F_0^{(\mu)}$ is continuously differentiable. Hence, it is elementary to prove that φ is twice continuously differentiable and strictly increasing:

Lemma 4.1: *The function $\varphi \in \mathcal{C}[x_0, b]$ defined in (4.3) is twice continuously differentiable. In particular, it follows that*

A.4
$$\varphi'(x) = \left(\frac{F_0^{(\mu)}(x_0)}{\mu!}\right)^{1/\mu} + \mathcal{O}(|x - x_0|) \qquad (x \downarrow x_0). \tag{4.4}$$

Proof: Without loss of generality we assume $x_0 = 0$ and $F_0(0) = 0$. Obviously, by definition we have $\varphi \in \mathcal{C}^{(\mu+1)}(0, b]$ and

A.5
$$\varphi'(x) = (1/\mu)F_0(x)^{1/\mu - 1}F_0'(x) \qquad (0 < x \leq b). \tag{4.5}$$

Taylor's formula applied to $F_0^{(k)}$ for $k = 0, 1, 2$ gives

A.6
$$F_0^{(k)}(x) = \frac{F_0^{(\mu)}(0)}{(\mu - k)!}x^{\mu - k} + \mathcal{O}\big(x^{\mu - k + 1}\big) \qquad (x \downarrow 0). \tag{4.6}$$

Combining (4.6) with (4.5) we obtain (4.4). Hence φ' extends to a continuous function on $[0, b]$. Furthermore, using (4.6) and (4.5) a short computation modulo $o(1)$ as $x \downarrow 0$ leads to

A.7
$$(1/x)(\varphi'(x) - \varphi'(0)) \equiv \varphi''(x) = (((1/\mu) - 1)F_0'(x)/F_0(x) + F_0''(x)/F_0'(x))\varphi'(x). \tag{4.7}$$

Applying Taylor's formula once more to $F_0^{(k)}$, $k = 0, 1, 2$, the right-hand side of (4.7) equals modulo $o(1)$

A.8
$$\begin{aligned}\varphi'(x)\Bigg(&\left(\frac{1}{\mu} - 1\right) \frac{F_0^{(\mu)}(0)x^{\mu-1}/(\mu-1)! + F_0^{(\mu+1)}(0)x^{\mu}/\mu!}{F_0^{(\mu)}(0)x^{\mu}/\mu! + F_0^{(\mu+1)}(0)x^{\mu+1}/(\mu+1)!} \\ &+ \frac{F_0^{(\mu)}(0)x^{\mu-2}/(\mu-2)! + F_0^{(\mu+1)}(0)x^{\mu-1}/(\mu-1)!}{F_0^{(\mu)}(0)x^{\mu-1}/(\mu-1)! + F_0^{(\mu+1)}(0)x^{\mu}/\mu!}\Bigg).\end{aligned} \tag{4.8}$$

In view of (4.7) computing (4.8) modulo $o(1)$ gives the assertion:

A.9
$$\varphi''(0) = \lim_{x \downarrow 0} \varphi''(x) = \varphi'(0)(1/\mu)F_0^{(\mu+1)}(0)/F_0^{(\mu)}(0). \tag{4.9}$$

$\diamondsuit$

In terms of the function φ and its derivative we have:

Lemma 4.2: *Assume that $F(N, x)$ and $(x - x_0)^{-\beta}g(N, x)$ satisfy Hypothesis 4.1 for fixed $\beta \geq 0$. Define for $x_0 \leq x, y \leq b$ Laplace's integral to be*

A.10
$$I(N, x, y) = \int_x^y e^{-NF(N,u)}g(N, u)du. \tag{4.10}$$

Then, for $\varepsilon = \min(\delta, \sigma/\mu, \gamma/\mu)$ it holds uniformly in $x_0 \leq x \leq y \leq b$

A.11
$$\begin{aligned}I(N, x, y) = \big(1 + \mathcal{O}\left(N^{-\varepsilon}\right)\big)\, &\frac{1}{\mu}\frac{g(N, x)}{\varphi(x)^{\beta}\varphi'(x)}N^{-(\beta+1)/\mu}e^{N(\varphi(x)^{\mu} - F(N,x))} \\ &\int_{N\varphi(x)^{\mu}}^{N\varphi(y)^{\mu}} e^{-w - F_{1,1}(N,x)/\varphi'(x)^{\nu}(w^{1/\mu} - N^{1/\mu}\varphi(x))^{\nu}} w^{(\beta+1)/\mu - 1}dw.\end{aligned} \tag{4.11}$$

Proof: Without loss of generality we assume $x_0 = 0$, $F_0(0) = 0$ and $\nu + \sigma \leq \mu$ in (1.6). Combining (1.6), (1.7), (1.8) with (4.3) and denoting $\psi = \varphi^{-1}$ one obtains for $\varphi(x) = r \leq v = \varphi(u) \leq s = \varphi(y)$

A.12
$$\begin{aligned} F(N,u) - F(N,x) + F_0(x) = v^{\mu} &+ N^{\nu/\mu-1} F_{1,1}(N,x)\psi'(r)^{\nu}(v-r)^{\nu} \\ &+ N^{\nu/\mu-1}\mathcal{O}\left(|v-r|^{\nu+\sigma}\right) + \mathcal{O}\left(N^{-1-\delta}\right). \end{aligned} \tag{4.12}$$

Defining the function

A.13
$$h(N,v) = v^{-\beta} g(N,u)\psi'(v) \qquad (0 \leq u \leq b), \tag{4.13}$$

we get by means of (4.10), (4.12), (4.13) and the substitution $v = \varphi(u)$

A.14
$$\begin{aligned} I(N,x,y) = &\left(1 + \mathcal{O}\left(N^{-\delta}\right)\right) e^{-N(F(N,x)-F_0(x))} \\ &\int_r^s e^{-Nv^{\mu} - N^{\nu/\mu} F_{1,1}(N,x)\psi'(r)^{\nu}(v-r)^{\nu}} e^{\mathcal{O}\left(N^{\nu/\mu}|v-r|^{\nu+\sigma}\right)} h(N,v) v^{\beta} dv. \end{aligned} \tag{4.14}$$

Substitution $w = Nv^{\mu}$ in (4.14) then leads to

A.15
$$\begin{aligned} I(N,x,y) = &\left(1 + \mathcal{O}\left(N^{-\delta}\right)\right) e^{-N(F(N,x)-F_0(x))} N^{-(\beta+1)/\mu} h(N,r)/\mu \\ &(I_1(N,x,y) + \Delta_1(N,x,y) + \Delta_2(N,x,y)). \end{aligned} \tag{4.15}$$

Here we have defined

A.16
$$I_1(N,x,y) = \int_{Nr^{\mu}}^{Ns^{\mu}} e^{-w - F_{1,1}(N,x)\psi'(r)^{\nu}(w^{1/\mu} - (Nr^{\mu})^{1/\mu})^{\nu}} w^{(\beta+1)/\mu - 1} dw, \tag{4.16}$$

while the error terms are given by

A.17
$$\begin{aligned} \Delta_1(N,x,y) = &\int_{Nr^{\mu}}^{Ns^{\mu}} e^{-w - F_{1,1}(N,x)\psi'(r)^{\nu}(w^{1/\mu} - (Nr^{\mu})^{1/\mu})^{\nu}} w^{(\beta+1)/\mu - 1} \\ &(h(N,v)/h(N,r) - 1) dw, \end{aligned} \tag{4.17}$$

A.18
$$\begin{aligned} \Delta_2(N,x,y) = &\int_{Nr^{\mu}}^{Ns^{\mu}} e^{-w - F_{1,1}(N,x)\psi'(r)^{\nu}(w^{1/\mu} - (Nr^{\mu})^{1/\mu})^{\nu}} w^{(\beta+1)/\mu - 1} \\ &\left(e^{\mathcal{O}(N^{\nu/\mu}|v-r|^{\nu+\sigma})} - 1\right) h(N,v)/h(N,r) dw. \end{aligned} \tag{4.18}$$

We now claim that the following equalities hold

A.19
$$\Delta_1(N,x,y) = \mathcal{O}\left(N^{-\gamma/\mu}\right) I_1(N,x,y), \qquad \Delta_2(N,x,y) = \mathcal{O}\left(N^{-\sigma/\mu}\right) I_1(N,x,y). \tag{4.19}$$

To prove (4.19) one observes that in view of definition (4.13) uniform γ-Hölder continuity of $v^{-\beta}g(N,u)$, Lipschitz continuity of ψ', the existence of $c > 0$ with $v^{-\beta}g(N,u), \psi'(v) \geq c > 0$ and $1/\mu$-Hölder continuity of $t^{1/\mu}$, $t \geq 0$, imply

A.20
$$\begin{aligned} h(N,v)/h(N,r) - 1 &= \mathcal{O}\left(N^{-\gamma/\mu}|w^{1/\mu} - (Nr^{\mu})^{1/\mu}|^{\gamma}\right) \\ &= \mathcal{O}\left(N^{-\gamma/\mu}|w - (Nr^{\mu})|^{\gamma/\mu}\right). \end{aligned} \tag{4.20}$$

Furthermore, invoking in addition the elementary estimate $|e^x - 1| \leq |x|e^{|x|}$ one clearly can bound

A.21
$$\begin{aligned} \left(e^{\mathcal{O}(N^{\nu/\mu}|v-r|^{\nu+\sigma})} - 1\right) \frac{h(N,v)}{h(N,r)} &= \mathcal{O}\left(N^{\nu/\mu}|v-r|^{\nu+\sigma} e^{\mathcal{O}\left(N^{\nu/\mu}|v-r|^{\nu+\sigma}\right)}\right) \\ &= \mathcal{O}\left(N^{-\sigma/\mu}|w - (Nr^{\mu})|^{(\nu+\sigma)/\mu} e^{\mathcal{O}\left(N^{-\sigma/\mu}|w-(Nr^{\mu})|^{(\nu+\sigma)/\mu}\right)}\right). \end{aligned} \tag{4.21}$$

Hence, inserting (4.20) into (4.17) and (4.21) into (4.18) the following estimate proves (4.19).

A.22
$$\frac{\int_{Nr^\mu}^{Ns^\mu} e^{-w+t(w^{1/\mu}-N^{1/\mu}r)^\nu} w^{(\beta+1)/\mu-1} e^{N^{-\sigma/\mu}(w-Nr^\mu)^{(\nu+\sigma)/\mu}} (w-Nr^\mu)^\rho dw}{\int_{Nr^\mu}^{Ns^\mu} e^{-w+t(w^{1/\mu}-N^{1/\mu}r)^\nu} w^{(\beta+1)/\mu-1} dw} = \mathcal{O}(1), \tag{4.22}$$

Here we have set $\rho = \gamma/\mu, (\nu+\sigma)/\mu$, while $t = -F_{1,1}(x)\psi'(x)$ varies only in a compact set $K \subset \mathbb{R}$. (4.22) is understood for $N \geq 1$ sufficiently large to be uniform in $r > 0$, $s > 0$, $t \in K$. Since $\psi'(r) \equiv 1/\varphi'(x)$ and $v = \varphi(u)$ the assertion now follows from combining (4.19) and (4.16) with (4.15).

Fix $0 < \alpha < 1$, $0 \leq \beta, \delta \leq 1$, $\gamma, \varepsilon > 0$ and $0 < c < 1$. Let $K \subset \mathbb{R}$ be a compact set. For $s \geq r \geq 0$ and $t \in K$ let

A.23
$$\begin{aligned} I_1(r,s,t) &= \int_r^s e^{-w+t(w^\alpha - r^\alpha)^\beta} w^{\gamma-1} e^{c(w-r)^\delta} (w-r)^\varepsilon dw \\ I_2(r,s,t) &= \int_r^s e^{-w+t(w^\alpha - r^\alpha)^\beta} w^{\gamma-1} dw. \end{aligned} \tag{4.23}$$

In order to prove (4.22) it suffices to prove that the function I_1/I_2 is bounded with respect to r, s and t. Denote by $k_{r,t}(u,v)$ the symmetric, positive function

A.24
$$k_{r,t}(u,v) = e^{-(u+v)+t((u^\alpha - r^\alpha)^\beta + (v^\alpha - r^\alpha)^\beta)} (uv)^{\gamma-1} \qquad (r \leq u, v \leq s) \tag{4.24}$$

Taking the derivative with respect to t, rewriting products of integrals as double integrals and invoking symmetry of $k_{r,t}(u,v)$ yields

A.25
$$\begin{aligned} \partial_t(I_1/I_2) = (I_2)^{-2} \int_{\substack{[r,s]^2 \\ u>v}} k_{r,t}(u,v) \left(e^{c(v-r)^\delta}(v-r)^\varepsilon - e^{c(u-r)^\delta}(u-r)^\varepsilon\right) \\ \left((v^\alpha - r^\alpha)^\beta - (u^\alpha - r^\alpha)^\beta\right) dudv \geq 0. \end{aligned} \tag{4.25}$$

Hence I_1/I_2 is increasing in t. Differentiation with respect to s gives

A.26
$$\partial_s(I_1/I_2) = (I_2)^{-2} \int_r^s k_t(w,s) \left(e^{c(s-r)^\delta}(s-r)^\varepsilon - e^{c(w-r)^\delta}(w-r)^\varepsilon\right) dw \geq 0. \tag{4.26}$$

Thus I_1/I_2 is increasing in s, too. From (4.25) and (4.26) it follows that $I_1/I_2 \leq J_1/J_2$, where we have defined $T = \max(\max K, 0)$ and $J_i(r) = \lim_{s\to\infty} I_i(r,s,T)$. The assertion is proven observing that $\lim_{r\to\infty} J_1(r)/J_2(r) = 0$. Actually it can be shown that $J_1(r)/J_2(r) = \mathcal{O}(r^{-\varepsilon})$ as r grows. This for example follows using that $-(w-r) \leq -(w-r) + T(w^\alpha - r^\alpha)^\beta + c(w-r)^\delta \leq -C((w-r)-1)$ for some $C > 0$ and $w \geq r \geq 1$ via the scaling $v = (w-r)/r$ and application of Watson's Lemma (see [Ol]). $\diamondsuit$

Intending to have a convenient notation let us introduce the function $\Phi(s,t) = e^t \,\mathrm{Fi}(1/\mu, \nu, 1/\mu; s, t)$, where we have defined

A.27
$$\mathrm{Fi}(\alpha,\beta,\gamma;s,t) = \int_t^\infty e^{-w+s(w^\alpha - t^\alpha)^\beta} w^{\gamma-1} dw \qquad (\alpha,\beta,\gamma > 0, \alpha\beta < 1, s \in \mathbb{R}, t \geq 0). \tag{4.27}$$

We refer to Fi as the Faxen's complementary, incomplete integral in generalization of the Faxen's integral (see (4.40)) and the complementary, incomplete Gamma function $\Gamma(\gamma,t) = \mathrm{Fi}(1,1,\gamma;0,t)$. As a technical Lemma for later purposes we isolate the following continuity property of Φ

Lemma 4.4: *On every compact set $K \subset \mathbb{R}$ we have*

A.28
$$\Phi(s+\delta, t+\varepsilon) = \left(1 + \mathcal{O}\left(|\varepsilon|^{\nu/\mu} + |\varepsilon| + |\delta|\right)\right) \Phi(s,t) \qquad (\delta, \varepsilon, s \in K, t, t+\varepsilon \geq 0) \tag{4.28}$$

A.48
$$t^{1-1/\mu}\Phi(s,t) = 1 + \mathcal{O}(t^{\nu(1/\mu-1)}) \qquad (s\in K, t\to\infty). \tag{4.29}$$

Proof of (4.28): Fix $\alpha,\beta,\gamma>0$, $\alpha\beta,\gamma<1$, $\beta\le 1$. We prove (4.28) in two steps. The first one consists in showing that

A.29
$$\frac{\mathrm{Fi}(\alpha,\beta,\gamma;s,t+\varepsilon)}{\mathrm{Fi}(\alpha,\beta,\gamma;s,t)} = \left(1+\mathcal{O}\left(|\varepsilon|^{\alpha\beta}+|\varepsilon|+|\varepsilon|^{\gamma}\right)\right) \qquad (\varepsilon, s\in K, t, t+\varepsilon>0). \tag{4.30}$$

For, we abbreviate $I(s,t) = \mathrm{Fi}(\alpha,\beta,\gamma;s,t)$ and write

A.30
$$I(s,t+\varepsilon) - I(s,t) = \Delta_1(s,t,t+\varepsilon) + \Delta_2(s,t,t+\varepsilon), \tag{4.31}$$

where Δ_i are defined by

A.31
$$\begin{aligned}\Delta_1(s,t,t+\varepsilon) &= \int_{t+\varepsilon}^{t} e^{-r+s(r^\alpha-(t+\varepsilon)^\alpha)^\beta} r^{\gamma-1}dr,\\ \Delta_2(s,t,t+\varepsilon) &= \int_{t}^{\infty} e^{-r+s(r^\alpha-t^\alpha)^\beta} r^{\gamma-1} e^{s\left((r^\alpha-(t+\varepsilon)^\alpha)^\beta-(r^\alpha-t^\alpha)^\beta)\right)}dr.\end{aligned} \tag{4.32}$$

Hölder-continuity to the parameter ρ of the function r^ρ, $0<\rho\le 1$, gives $|(r^\alpha-(t+\varepsilon)^\alpha)^\beta-(r^\alpha-t^\alpha)^\beta| \le |\varepsilon|^{\alpha\beta}$. Invoking in addition the estimate $|e^z-1|\le|z|e^{|z|}$ we obtain $\Delta_2(s,t,t+\varepsilon)/I(s,t) = \mathcal{O}(|\varepsilon|^{\alpha\beta})$. For the proof of $\Delta_1(s,t,t+\varepsilon)/I(s,t) = \mathcal{O}(|\varepsilon|^\gamma+|\varepsilon|)$ we distinguish the case $t\le 1$ from $t>1$. In the former case obviously $I(s,t)\ge c>0$, while by γ-Hölder continuity of t^γ, $t\ge 0$,

A.32
$$\Delta_1(s,t,t+\varepsilon) = \mathcal{O}\left(\int_{t+\varepsilon}^{t} r^{\gamma-1}dr\right) = \mathcal{O}\left(|\varepsilon|^\gamma\right). \tag{4.33}$$

In the latter case one can bound $I(s,t)\ge ce^{-t}t^{\gamma-1}>0$ which in view of

A.33
$$\begin{aligned}&\Delta_1(s,t,t+\varepsilon)\\ &= e^{-t}t^{\gamma-1}\int_{\varepsilon}^{0} e^{-r+s((t+r)^\alpha-(t+\varepsilon)^\alpha)^\beta}(1+(r/t))^{\gamma-1}dr\\ &= \mathcal{O}(|\varepsilon|)\end{aligned} \tag{4.34}$$

gives the assertion. Hence we arrive at (4.30). The second step consists in showing that

A.36
$$\frac{\mathrm{Fi}(\alpha,\beta,\gamma;s+\delta,t)}{\mathrm{Fi}(\alpha,\beta,\gamma;s,t)} = 1+\mathcal{O}(|\delta|) \qquad (s, s+\delta\in K, t\ge 0). \tag{4.35}$$

Since the proof of (4.22) written above tells us that $I_1(t,\infty,s)/I_2(t,\infty,s)$ defined in (4.23) is bounded on $s\in K$, $t\ge 0$, (4.35) follows easily from the observation that

A.37
$$\mathrm{Fi}(\alpha,\beta,\gamma;s+\delta,t) = \mathrm{Fi}(\alpha,\beta,\gamma;s,t) + \mathcal{O}\left(|\delta|\int_t^\infty e^{-r+s(r^\alpha-t^\alpha)^\beta}r^{\gamma-1}(r-t)^{\alpha\beta}e^{|\delta|(r-t)^{\alpha\beta}}dr\right). \tag{4.36}$$

Again which is proved using $|e^z-1|\le|z|e^{|z|}$. (4.30) and (4.35) it follows (4.28). $\diamondsuit$

Proof of (4.29): Integration by parts applied to Faxen's (complementary, incomplete) integral shows for $t\ge 1$

A.48a
$$\begin{aligned}e^t t^{1-1/\mu}\,\mathrm{Fi}(1/\mu,\nu,1/\mu;s,t) =& 1 + t^{1-1/\mu}\\ &\left(\int_0^\infty e^{-r+s((r+t)^{1/\mu}-t^{1/\mu})^\nu}(1/\mu-1)(r+t)^{1/\mu-2}dr\right.\\ &\left.+\int_0^\infty e^{-r+s((r+t)^{1/\mu}-t^{1/\mu})^\nu}(\nu/\mu)s((r+t)^{1/\mu}-t^{1/\mu})^{\nu-1}(r+t)^{2/\mu-2}dr\right).\end{aligned} \tag{4.37}$$

Invoking that $|(r+t)^{1/\mu}-t^{1/\mu}|\leq r^{1/\mu}$ and that $|(r+t)^{1/\mu}-t^{1/\mu}|\geq(1/\mu)t^{1/\mu-1}r$ for $t\geq1$ (4.37) gives (4.29). ◇◇

In small vicinities of the critical point, where F_0 is sufficiently flat, $S(N,\cdot)$ behaves as the corresponding Laplace's integral:

Corollary 4.5: *Define $S(N,x)$, $x\in I$ as in (4.1) with respect to $F(N,\cdot)$ and $g(N,\cdot)$. Let Φ be the function defined before (4.27) and φ be the function defined in (4.3). Then, for every $\alpha>0$ and $x\in I$, $|x-x_0|\leq N^{-\alpha}$ it follows*

A.38
$$S(N,x)=\left(1+\mathcal{O}\left(N^{-\varepsilon}\right)\right)\frac{g(N,x)}{\mu\varphi'(x)}\Phi\left(\frac{-F_{1,1}(N,x)}{\varphi'(x)^{\nu}},N\varphi(x)^{\mu}\right)N^{1-1/\mu}e^{-NF(N,x)},\tag{4.38}$$

where $\varepsilon=\min((\mu-1)\alpha,(\mu-1)\nu/\mu,\delta,\sigma/\mu,\gamma/\mu)$.

Remark: Inserting $I\ni x=x_0+\mathcal{O}(1/N)$ and (4.4) into (4.38) and invoking (4.28) yields the in the continuous context well known formula (see [Ol])

A.39
$$\begin{aligned}S(N,x)=\left(1+\mathcal{O}\left(N^{-\varepsilon}\right)\right)\frac{g(N,x)}{\mu}&\left(\frac{\mu!}{F_0^{(\mu)}(x)}\right)^{1/\mu}\\ &\mathrm{Fi}\left(\frac{\nu}{\mu},\frac{1}{\mu};-\mu^{-\nu}F_{1,1}(N,x)\left(\frac{\mu!}{F_0^{(\mu)}(x)}\right)^{\nu/\mu}\right)N^{1-1/\mu}e^{-NF(N,x)}.\end{aligned}\tag{4.39}$$

Here ε is as in (4.38) and Fi denotes the Faxen's integral

A.40
$$\mathrm{Fi}(\alpha,\beta;s)=\mathrm{Fi}(\alpha,1,\beta;s,0).\tag{4.40}$$

Remark: Dropping the condition that the amplitude $g(N,\cdot)$ is bounded below but still assuming Hölder-continuity along the lines of the proof of (4.39) it is readily verified that for $x=x_0+\mathcal{O}(1/N)$

A.42
$$\begin{aligned}S(N,x)=N^{1-1/\mu}e^{-NF(N,x)}\Bigg(g(N,x)&\left(\frac{\mu!}{F_0^{(\mu)}(x)}\right)^{1/\mu}\\ &\mathrm{Fi}\left(\frac{\nu}{\mu},\frac{1}{\mu};-\mu^{-\nu}F_{1,1}(N,x)\left(\frac{\mu!}{F_0^{(\mu)}(x)}\right)^{\nu/\mu}\right)+\mathcal{O}\left(N^{-\varepsilon}\right)\Bigg),\end{aligned}\tag{4.41}$$

again where ε is defined as in (4.38).

Proof of (4.38): From (1.7) and (1.8) in Hypothesis 1.1 and the property of F_0 being $\mathcal{C}^2$ one easily reads off for $k-1/N\leq x\leq k$ and $k\in I$

A.43
$$|F(N,x)-F(N,k)|=\mathcal{O}\left(N^{-1}F_0'(x)+N^{\nu/\mu-1-\nu}+N^{\nu/\mu-1-\nu-\delta}+N^{-1-\delta}\right).\tag{4.42}$$

Hence, it follows from γ-Hölder continuity of g for $\varepsilon=\min((\mu-1)\nu/\mu,\delta,\gamma)$

A.44
$$\left|g(N,x)e^{-NF(N,x)}-g(N,k)e^{-NF(N,k)}\right|=\mathcal{O}\left(g(N,x)e^{-NF(N,x)}(N^{-\varepsilon}+F_0'(x))\right).\tag{4.43}$$

In view of (4.43) we obtain for $x \in I$

A.45
$$S(N,x) = \left(1 + \mathcal{O}\left(N^{-\varepsilon}\right)\right) NI(N,x,[b]) + \mathcal{O}\left(N \int_x^{[b]} g(N,t) F_0'(t) e^{-NF(N,t)} dt\right), \tag{4.44}$$

where the Laplace integral $I(N,\cdot,\cdot)$ is defined in (4.10). Application of (4.11) to $I(N,x,[b])$ and combination with (4.29) we obtain for suitable $\delta > 0$

A.46
$$I(N,x,[b]) = \left(1 + \mathcal{O}\left(e^{-N\delta}\right)\right) \frac{g(N,x)}{\mu\varphi'(x)} \Phi\left(-\frac{F_{1,1}(N,x)}{\varphi'(x)^\nu}, N\varphi(x)\mu\right) N^{-1/\mu} e^{-NF(N,x)}. \tag{4.45}$$

Let us denote by $\Delta(N,x)$ the second integral on the right-hand side of (4.44). Applying (4.11) to $\Delta(N,x)$ gives a suitable constant $C > 0$

A.47
$$\Delta(N,x) = \mathcal{O}\left(N^{-1} e^{N(\varphi(x)^\mu - F(N,x))} \operatorname{Fi}(1/\mu, \nu, 1; C, N\varphi(x)^\mu)\right). \tag{4.46}$$

It is elementary to prove that (4.46) implies $\Delta(N,x) = \mathcal{O}(N^{-1}e^{-NF(N,x)})$. Invoking (4.29) we conclude that for some generic $c > 0$

A.47a
$$\begin{aligned}\Phi(-F_{1,1}(N,x)/\varphi'(x)^\nu, N\varphi(x)\mu) &\geq c\min(1, N\varphi(x)^\mu)^{1/\mu - 1}) \\ &\geq c\min(1, N^{1/\mu}|x - x_0|)^{1-\mu}) \qquad (x \in I, |x - x_0| \leq N^{-\alpha}). \\ &\geq c\min(1, N^{1/\mu - 1} N^{(1-\mu)\alpha})\end{aligned} \tag{4.47}$$

Combining (4.44), (4.45), (4.46) and (4.47) proves (4.38). $\diamondsuit$

We now evaluate the sum in (4.1) of Laplace type in regions, where the derivative of F_0 does not vanish (to fast) for large N. This allows us to cut the sum at a scale $k \leq N^{-\delta}$ without changing the leading order. In this small domain of summation $F(N,\cdot)$ is approximated sufficiently well by the linearization of F_0 at x. Consequently, we are left computing a geometric sum. We already know from the previous Corollary that the 'effective' variable behaves like $x \approx N^{-1/\mu}$. To be able to match the asymptotic in the case of non vanishing derivative with those obtained in Corollary 4.4, we compute the sum in the domain $x \geq N^{-\alpha}$ for arbitrary fixed $\alpha < 1/\mu$. If we neglect the subleading part of $F(N,\cdot)$, it would be enough to ensure that the derivative doesn't vanish faster than N grows. This only means $\alpha < 1/(\mu - 1)$. But to incorporate the subleading part which should only produce a negligible factor in the sum we have to choose $\alpha < 1/\mu$. Before we formulate the Lemma, let us introduce another auxiliary function for convenience. Define

A.50
$$\Psi(s,t) = t(e^t - e^{-s})/(e^t - 1) \qquad (0 \leq s \leq \infty, t \geq 0). \tag{4.48}$$

In terms of the function Ψ we have:

Lemma 4.6: *Define $S(N,x)$, $x \in I$, as in (4.1). Fix $\alpha < 1/\mu$. Then, there exists a number $\varepsilon > 0$ such that uniformly in $N^{-\alpha} \leq x \leq b$*

A.51
$$S(N,x) = \left(1 + \mathcal{O}\left(N^{-\varepsilon}\right)\right) \frac{g(N,x)}{F_0'(x)} \Psi(N(F_0([b]) - F_0(x)), F_0'(x)) e^{-NF(N,x)}. \tag{4.49}$$

Proof: Since $\alpha < 1/\mu$, it is possible to choose $1/2 + (1 - \mu/2)\alpha < \rho < 1 + (1 - \mu)\alpha$. Since the asymptotic (1.7) in Hypothesis 1.1 is twice differentiable, we obtain the existence of generic constants $c > 0$ and C such that for $N^{-\alpha} \leq x \leq b$

A.52
$$N|F_0(x) - F_0(x + [N^{-\rho}])| \geq c|x|^{\mu-1} N^{1-\rho} \geq cN^{(1-\mu)\alpha + 1 - \rho} \tag{4.50}$$

A.53
$$N|F_0(x)+F_0'(x)([N^{-\rho}]-x)-F_0(x+[N^{-\rho}])| \le C|x|^{\mu-2}N^{1-2\rho} \\ \le CN^{1-2\rho+(2-\mu)\alpha}. \tag{4.51}$$

and invoking (1.6) and (1.8) of Hypothesis 1.1

A.54
$$N|(F(N,x)-F_0(x))-(F(N,x+[N^{-\rho}])-F_0(x+[N^{-\rho}]))| \\ \le C\left(N^{\nu(1/\mu-\rho)}+N^{\nu/\mu-(\nu+\sigma)\rho}+N^{-\delta}\right). \tag{4.52}$$

As in the proof of Lemma 4.2 we divide the sum S into three terms

A.55
$$S(N,x)=g(N,x)e^{-NF(N,x)}\left(S_1(N,x)+\Delta_1(N,x)+\Delta_2(N,x)\right), \tag{4.53}$$

where we have defined

A.56
$$S_1(N,x)=\sum_{x\le k\le x+[N^{-\rho}]} e^{-N(F(N,k)-F(N,x))}, \tag{4.54}$$

A.57
$$\Delta_1(N,x)=\sum_{x\le k\le x+[N^{-\rho}]}\left(\frac{g(N,k)}{g(N,x)}-1\right)e^{-N(F(N,k)-F(N,x))}, \tag{4.55}$$

A.58
$$\Delta_2(N,x)=\sum_{x+[N^{-\rho}]\le k\le [b]}\frac{g(N,k)}{g(N,x)}e^{-N(F(N,k)-F(N,x))}. \tag{4.56}$$

We first discuss the case $x+[N^{-\rho}]<[b]$. Due to γ-Hölder-continuity and uniform boundness from below of $g(N,\cdot)$ we know that $\Delta_1(N,x)=\mathcal{O}(N^{-\gamma})S_1(N,x)$.
Invoking (4.44), (4.11) and uniform boundedness of $F_{1,1}(N,\cdot)$ and again that $g(N,\cdot)\asymp 1$ we conclude

A.59
$$\Delta_2(N,x)=\mathcal{O}\left(N\int_{[N^{-\rho}]}^{[b]}\left|\frac{g(N,u)}{g(N,x)}\right|e^{-N(F(N,u)-F(N,x))}du\right) \\ =\mathcal{O}\left(N^{1-1/\mu}e^{-N(F(N,[N^{-\rho}])-F(N,x))}\right). \tag{4.57}$$

Inserting (4.50) and (4.52) on the right-hand side of (4.57) yields $\Delta_2(N,x)=\mathcal{O}(N^{-\infty})$.
From (4.51) and (4.52) it follows that there is a generic number $\varepsilon>0$ such that

A.60
$$S_1(N,x)=\left(1+\mathcal{O}\left(N^{-\varepsilon}\right)\right)\sum_{x\le k\le x+[N^{-\rho}]}e^{-NF_0'(x)(k-x)}. \tag{4.58}$$

Observing that (4.6) for $\nu=1$ gives the existence of a number $c>0$ such that $NF_0'(x)[N^{-\rho}]\ge cN^{-\rho+1+(1-\mu)\alpha}$, we obtain from (4.58)

A.61
$$S_1(N,x)=\left(1+\mathcal{O}\left(N^{-\varepsilon}\right)\right)\frac{e^{F_0'(x)}-\mathcal{O}\left(N^{-\infty}\right)}{e^{F_0'(x)}-1}. \tag{4.59}$$

Since we are treating the case $x+[N^{-\rho}]<[b]$, using Taylor's formula and once more the equation (4.6) for $\nu=1$ there exists a generic, positive $c>0$ such that $F_0([b])-F_0(x)\ge cN^{-\rho+(1-\mu)\alpha}$. This leads to

A.62
$$e^{-N(F_0([b])-F_0(x))}=\mathcal{O}\left(N^{-\infty}\right). \tag{4.60}$$

Combining the last equality with (4.59) and with $F_0'\ge 0$ we conclude

A.63
$$S_1(N,x)=\left(1+\mathcal{O}\left(N^{-\varepsilon}\right)\right)\frac{e^{F_0'(x)}-e^{-N(F_0([b])-F_0(x))}}{e^{F_0'(x)}-1} \tag{4.61}$$

as asserted. In view of the error estimates for Δ_i, $i = 1, 2$, and (4.61) from (4.53) it follows (4.49). Repeating the procedure for $S(N, x)$ in the case $x + [N^{-\rho}] \geq [b]$ again gives (4.49). The only difference occurring is that we have to replace $x + [N^{-\rho}]$ by $[b]$ and that already $\Delta_2 \equiv 0$. ♢

As a Corollary of Corollary 4.5 and Lemma 4.6 we obtain the leading order of the asymptotic of the sum of Laplace type uniform in the argument x:

Corollary 4.7: *Define $S(N, x)$ as in (4.1). Let φ, Φ and Ψ be the functions defined in (4.3), before (4.27) and in (4.48), respectively. Then, there is $\varepsilon > 0$ such that uniformly in $x \in I$*

A.64
$$\begin{aligned} S(N,x) = \left(1 + \mathcal{O}\left(N^{-\varepsilon}\right)\right) \frac{g(N,x)}{\mu\varphi'(x)} \Psi(N(F_0(x) - F_0([Nb]/N)), F_0'(x))) \\ \Phi\left(\frac{F_{1,1}(N,x)}{\varphi'(x)^{\nu}}, N\varphi(x)^{\mu}\right) N^{1-1/\mu} e^{-NF(N,x)}. \end{aligned} \tag{4.62}$$

Proof: Fix $0 < \alpha < 1/\mu$. Via Corollary 4.5 and Lemma 4.6 the proof of (4.62) is based on

A.65
$$(\mu\varphi'(x))^{-1} \Phi\left(\frac{F_{1,1}(N,x)}{\varphi'(x)^{\nu}}, N\varphi(x)^{\mu}\right) = \left(1 + \mathcal{O}\left(N^{-\varepsilon}\right)\right)(1/F_0'(x)) \qquad (x \geq x_0 + N^{-\alpha}) \tag{4.63}$$

A.66
$$\Psi(N(F_0([b]) - F_0(x)), F_0'(x)) = \left(1 + \mathcal{O}\left(N^{-\varepsilon}\right)\right) \qquad (0 \leq x \leq x_0 + N^{-\alpha}). \tag{4.64}$$

(4.63) is proven by inserting $t = N\varphi(x)^{\mu}$ for $x \geq x_0 + N^{-\alpha}$ into (4.29) and using (4.5), while (4.64) follows from (4.60) and $\Psi(s,t) = (1 + \mathcal{O}(t))(1 + e^{-s})$ as t tends to zero. ♢

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