Inverse problems generated by conformal mappings on complex plane with parallel slits

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Abstract

We study the properties of a conformal mapping $z(k,h)$ from $K(h)=\mathbb{C}\setminus\cup\Gamma_n$ where $\Gamma_n=[u_n-i|h_n|,u_n+i|h_n|], n\in\mathbb{Z}$ is a vertical slit and $h=\{h_n\}\in\ell^2_{\mathbb{R}}$, onto the complex plane with horizontal slits $\gamma_n\subset\mathbb{R}, n\in\mathbb{Z}$, with the asymptotics $z(iv,h)=iv+(iQ_0(h)+o(1))/v,\ v\to+\infty$. Here $u_{n+1}-u_n\geqslant 1, n\in\mathbb{Z}$, and the Dirichlet integral $Q_0(h)=\iint_{\mathbb{C}}|z'(k,h)-1|^2dudv/(2\pi)<\infty, k=u+iv$. Introduce the sequences $l=\{l_n\}, J=\{J_n\}$, where $l_n=|\gamma_n|\operatorname{sign}h_n$, and $J_n=|J_n|\operatorname{sign}h_n, J_n^2=\int_{\Gamma_n}|\operatorname{Im}z(k,h)||dk|/\pi$. The following results are obtain: 1) an analytic continuation of the function $z(\cdot,\cdot):K(h)\times\{f:\|f-h\|<r\}\to\mathbb{C}$ onto the domain $K(h)\times\{f:\|f-h\|_C<r\}$ for $h\in\ell^2_{\mathbb{R}}$ and some $r>0$, and the Löwner equation for $z(k,h)$ when the height of some slit h_n is changed, 2) an analytic continuation of the functional $Q_0:\ell^2_{\mathbb{R}}\to\mathbb{R}_+$ in the domain $\{f:\|\operatorname{Im}f\|<r\}$ and the derivatives $\partial Q_0/\partial h_n$, 3) the mappings $l:\ell^2_{\mathbb{R}}\to\ell^2_{\mathbb{R}}$ and $J:\ell^2_{\mathbb{R}}\to\ell^2_{\mathbb{R}}$ are real analytic isomorphisms, and $l(\cdot)$ has the analytic continuation on the domain $\{f:\|\operatorname{Im}f\|<r\}$, 4) the double-sided estimates for $\|h\|,\|l\|,\|J\|,Q_0(h)$, 5) some properties of the functional $E(h)=\sum l_n(h)$, 6) the extension of 1)-5) for the case $h\in\ell^p_\omega$, where ℓ^p_ω is the Banach space with any weight $\omega=\{\omega_n\}_{n\in\mathbb{Z}},\omega_n\geqslant 1$, and $1\leqslant p\leqslant 2$ with the norm $\|h\|^p_{p,\omega}=\sum\omega_n|h_n|^p$.

1 Introduction

Introduce the set $U=\{k:k=u_n,n\in\mathbb{Z}\}$, where u_n is strongly increasing sequence of real numbers such that $u_n\to\pm\infty$ as $n\to\pm\infty$. For some fixed sequence $h\in\ell^\infty_{\mathbb{R}}$ we define the

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following domains

$$K(h) = \mathbb{C} \setminus \cup_{n\in\mathbb{Z}}\Gamma_n, \quad \Gamma_n = [u_n - i|h_n|, u_n + i|h_n|], \qquad K_+(h) = \mathbb{C}_+ \cap K(h).$$

We call $K_+(h)$ the "comb". It is well known that there exists the conformal mapping $z = z(k,h)$ from $K_+(h)$ onto $\mathbb{C}_+$ with the asymptotics:

$$z(iv,h) = iv(1+o(1)), \quad v \to +\infty, \quad \text{where} \quad z = x+iy, \quad k = u+iv,$$

Moreover, the difference of any two such mappings equals a constant, and then the imaginary part $y(k,h) = \operatorname{Im} z(k,h)$ is unique. Introduce the function $\psi(k,h) = v - y(k,h), k \in K_+(h)$ which is the bounded solution of the Dirichlet problem with the following boundary condition $\psi(k,h) = v, k = u+iv \in \partial K_+(h)$. We call such mapping $z(k,h)$ the comb mapping. Define the inverse mapping $k(z,h,U) : \mathbb{C}_+ \to K_+(h)$. It is clear that such function has the continuous extension in the domain $\overline{\mathbb{C}_+}$. It is convenient to introduce "gaps" γ_n, "bands" σ_n and the "spectrum" $\sigma(h,U)$ of the comb mapping by the formulas:

$$\gamma_n = (z_n^-, z_n^+) = (z(u_n - 0, h), z(u_n + 0, h)), \qquad \sigma_n = [z_{n-1}^+, z_n^-], \qquad \sigma(h,U) = \cup\sigma_n.$$

Here and below the union, the sum and the integral without the limits means the union, the sum and the integral from $-\infty$ until ∞. All others cases well be denoted precisely. The function $u(z,h) = \operatorname{Re} k(z,h)$ is strongly increasing on each band and equals u_n on the interval $[z_n^-, z_n^+]$, $n \in \mathbb{Z}$; the function $v(z,h) = \operatorname{Im} k(z,h)$ equals zero on each band and is strongly convex on each gap γ_n and has the maximum at some point z_n, and such that $|h_n| = v(z_n)$. If the gap is empty we set $z_n = z_n^{\pm}$. These and others properties of the comb mappings it is possible to find in the papers of Levin [Le].

For the weight $\omega = \{\omega_n\}_{n\in\mathbb{Z}}$, where $\omega_n \geqslant 1$, and the number $p \geqslant 1$ introduce the real spaces

$$\ell^p_\omega = \{f = \{f_n, n \in \mathbb{Z}\}, \ \ \|f\|_{p,\omega} < \infty\}, \quad \|f\|^p_{p,\omega} = \sum_{n\in\mathbb{Z}} \omega_n f_n^p < \infty.$$

If the weight has the form $\omega_n = (2u_n)^{2m}, m \in \mathbb{R}$, then we will write ℓ^p_m with the norm $\|\cdot\|_{p,m}$, if the weight $\omega_n = 1, n \in \mathbb{Z}$, then we will write $\ell^p_0 = \ell^p$ with the norm $\|\cdot\|_p$, $\ \|\cdot\| = \|\cdot\|_2$. Let $\ell^p_{\omega,\mathbb{C}}$ be the complexification of the space ℓ^p_ω. In the complex space $\ell^p_{\omega,\mathbb{C}}$ introduce the domain (a layer)

$$\mathcal{J}^p_\omega(\rho) = \{\eta \in \ell^p_{\omega,\mathbb{C}} : \|\operatorname{Im}\eta\|_{p,\omega} < \rho\}, \quad \rho > 0, \quad p \geqslant 1.$$

For $h \in \ell^\infty_\mathbb{R}$ we define the singed gap length by the following formula:

$$l_n(h) = (z_n^+ - z_n^-)\operatorname{sign} h_n, \quad n \in \mathbb{Z},$$

and let $l(h) = \{l_n(h)\}_{n\in\mathbb{Z}}$. Below we will use very often the following estimate:

$$|\gamma_n(h)| \leq 2|h_n|, \quad n \in \mathbb{Z}, \tag{1.1}$$

see [MO1]. Hence if $h \in \ell^p_\omega$ then $l(h) \in \ell^p_\omega$ and we have the mapping $l(\cdot) : \ell^p_\omega \to \ell^p_\omega$ for any $p \geqslant 1$ and ω. Introduce two maps

$$A : h \to A = \{A_n\}_{-\infty}^{\infty}, \qquad J : h \to J = \{J_n\}_{-\infty}^{\infty},$$

acting from ℓ^p_ω into ℓ^p_ω for any $p \geqslant 1$ and the weight ω, where the components are defined by the formulas

$$A_n = \frac{2}{\pi} \int_{\gamma_n} v(z,h) dz, \quad J_n = |A_n|^{1/2} \operatorname{sign} h_n. \tag{1.2}$$

Define now the functional $Q_r(\cdot) : \ell^p_\omega \to \mathbb{R}$ by the formula:

$$Q_r(h) = \frac{1}{\pi} \int_{\mathbb{R}} x^r v(x,h)\, dx, \quad r \in \mathbb{Z},$$

and the Dirichlet integral

$$I_D(h) = \frac{1}{\pi} \iint_{\mathbb{C}} |z'(k,h) - 1|^2 \, du\, dv = \frac{1}{\pi} \iint_{\mathbb{C}} |k'(z,h) - 1|^2 \, dx\, dy,$$

the last identity holds since the Dirichlet integral is invariant under the conformal mappings. For $h \in \ell^\infty_{\mathbb{R}}$ the following estimates and identities were proved

$$\frac{1}{2} \sum_{n \in \mathbb{Z}} |\gamma_n(h)||h_n| \leq \pi Q_0(h) \leq \sum_{n \in \mathbb{Z}} |\gamma_n(h)||h_n|, \tag{1.3}$$

$$2Q_0(h) = I_D(h) = \sum A_n = \|J\|^2, \tag{1.4}$$

see [KK1]. Relations (1.3-4) show that functional Q_0 is bounded for $h \in \ell^2_{\mathbb{R}}$.

Note that the comb mappings are used in various fields of mathematics. We enumerate the more important directions:

1) the conformal mapping theory,
2) the Löwner equation and the quadratic differentials,
3) the electrostatic problems on the plane,
4) analytic capacity,
5) the spectral theory of the operators with periodic coefficients,
6) inverse problems for the Hill operator and the Dirac operator,
7) KDV equation and NLS equation with periodic initial value problem.

We need some results from the conformal mapping theory. Recall the Hilbert Theorem about the conformal mappings from a multiply connected domains onto a domain with parallel slits. Let $S_1, S_2, \dots, S_N$ be disjoint continua in the plane $\mathbb{C}$; $D = \mathbb{C} \setminus \cup_{n=1}^N S_n$. Introduce the class $\Sigma'(D)$ of the conformal mapping w from the domain D onto $\mathbb{C}$ with the following asymptotics: $w(k) = k + [Q(w) + o(1)]/k, \quad k \to \infty$. We formulate the well known Theorem (see [G], [J]).

Theorem. (Hilbert) . *Let $S_1, S_2, \dots, S_N$ be disjoint continua in the plane $\mathbb{C}$; $D = \mathbb{C} \setminus \cup_{n=1}^N S_n$. Then for each $t \in [0, \pi]$ there exists a unique function $w_t \in \Sigma'(D)$ which maps*

D onto a domain with parallel slits and the angles of intersection between the parallel slits and the real axis are all equal to t. Moreover, for each function $f \in \Sigma'(D)$, $f \neq w_0$, the following estimate is fulfilled $\operatorname{Re} Q(f) < \operatorname{Re} Q(w_0)$.

Levin [Le] studied general comb mappings for the case $h \in \ell^2_m, m = -1$, and proved the existence of the mapping for a very general case. Marchenko and Ostrovski [MO1-2] considered comb mappings as $m \geqslant 1, u_n = \pi n, n \in \mathbb{Z}$. The authors of the present paper [KK1] found new estimates of the various parameters, the identities (see (1.4)), the Poisson integral (2.39) (convenient for the application) and used the Dirichlet integral. New identities and various estimates were obtained by one of the authors (E.K.) in the papers [K1-4], [K7-11]. Remark that first the Dirichlet integral for the comb mappings at $m = -1$, was used in the paper [Ka1] of another author (P.K.) of the present paper.

Estimates are important in the conformal mapping theory. Some double-sided inequalities for $\|h\|_{2,1}$ and Q_2 were found in [MO2] only for the following case :

$$u_n = \pi n, n \in \mathbb{Z}, \quad \sqrt{Q_0} \leqslant C_1(1 + \|h\|_\infty)(1 + \|h\|_{2,1}), \quad \|h\|_{2,1} \leqslant C_2\sqrt{Q_0}\exp(C_3\sqrt{Q_0}),$$

for some constants C_1, C_2, C_3. Note that these estimates are overstated since the Bernstein inequality was used. In this consideration the condition $u_n = \pi n, n \in \mathbb{Z}$, is very important, note that the estimates in terms of the gap lengths were absent. Later on various estimates were showed in [KK1]. First double-sided inequalities (very rough) for the gap lengths (for for $\|l\|_{2,1}$ and Q_2) were proved in [KK2]. Some double-sided estimates of various parameters were found in [K4] for the case when on any unit interval there are finite number of vertical slits and results were applied to the Dirac operator and the weighted operator [K10-11]. Various inequalities for some values of the Hill operator were proved in [K1, 8] and remark that the proof of the estimates for the Hill operator is more complicated than for the Dirac operator.

Recall the well known results about the Löwner equation [L] and quadratic differentials. The Löwner equation usually is used for a simply connected domain, such that zero (the norming point) lies inside this domain and ∞ does not belongs to one. Moreover, there exists a slit inside this domain such that one of the ends lies on the boundary of the domain (see [G]). The Löwner equation for the comb is not studied since in this case the norming point, ∞, lies on the boundary. Maybe with a unique exception, there exists a paper of Garnet and Trubowitz [GT1] where the authors considered such problems.

In our paper we do not study the quadratic differential, but some applications of our results with this subject will are discussed after Theorem 2.1.

Consider the Hill operator $T = -d^2/dx^2 + q(x)$, acting on $L^2(\mathbb{R})$, where a potential q is 1-periodic and belongs to the space of even functions $\mathcal{H}_e = \{q \in L^2(0,1), q(x) = q(1-x), x \in [0,1]\}$. It is well known, that the spectrum of T is absolutely continuous and consists of intervals $[\lambda^+_{n-1}, \lambda^-_n]$, where $\lambda^+_{n-1} < \lambda^-_n \leqslant \lambda^+_n$, $n \geqslant 1$ and let $\lambda^+_0 = 0$. These intervals are separated by the gap $\widetilde{\gamma}_n = (\lambda^-_n, \lambda^+_n)$. If a gap degenerates, that is $\widetilde{\gamma}_n = \emptyset$, then the corresponding segments merge. Let $\Delta(z,q), z \in \mathbb{C}$, be the Lyapunov function for the Hill operator. Define the quasimomentum by the formula

$$k(z) = \arccos \Delta(z,q), \quad z \in \mathcal{Z} = \mathbb{C} \setminus \cup\gamma_n,$$

where the slit γ_n has the form

$$\gamma_n = (z_n^-, z_n^+) = -\gamma_{-n}, \qquad z_n^\pm = \sqrt{\lambda_n^\pm} > 0, \quad n \geqslant 1.$$

Note that for each $n \geqslant 1$ there exists a unique point $z_n \in [z_n^-, z_n^+]$ such that $\Delta_z(z_n, q) = 0$. The function $k(z)$ is a conformal mapping from $\mathcal{Z}$ onto $K(h)$ where $u_n = \pi n, n \in \mathbb{Z}$ and $|h_n|$ is defined by the equation $\cosh|h_n| = (-1)^n \Delta(z_n, q)$. Let $\mu_n(q), n \geqslant 1$, be an eigenvalue of the Dirichlet problem $-y'' + qy = \lambda y$, $y(0) = y(1) = 0$. It is well known that $\mu_n \in [\lambda_n^-, \lambda_n^+], n \geqslant 1$.

Construct the height slit mapping $h : q \to h(q) = \{h_n\}_{-\infty}^{\infty}$ where $h_n = |h_n| \operatorname{sign}(z_n^2 - \mu_n)$, from $\mathcal{H}_e$ into ℓ_1^2.

Define the gap length mapping $G : q \to G(q) = \{G_n\}_1^\infty$ from $\mathcal{H}_e$ into ℓ^2, where the components have the form $G_n = \lambda_n^+ + \lambda_n^- - \mu_n$,

Consider the Zakharov-Shabat (or Dirac) operator T_{zs} in the Hilbert space $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, where

$$T_{zs} = \widetilde{J}\frac{d}{dx} + \begin{pmatrix} q_1(x) & q_2(x) \\ q_2(x) & -q_1(x) \end{pmatrix}, \quad \widetilde{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.5}$$

and $q_1, q_2 \in L^2(1,0)$ are real 1-periodic functions in $x \in \mathbb{R}$. The spectrum of T_{zs} is purely absolutely continuous and is given by the set $\cup \sigma_n$, where an interval $\sigma_n = [z_{n-1}^+, z_n^-]$, where $\ldots < z_{2n-1}^- \leqslant z_{2n-1}^+ < z_{2n}^- \leqslant z_{2n}^+ < \ldots$, and $z_n^\pm = n(\pi + o(1))$, as $|n| \to \infty$. These intervals are separated by gaps $\gamma_n = (z_n^-, z_n^+)$ with the length $l_n = |g_n|$. If a gap g_n is degenerate, i.e., $l_n = 0$, then the corresponding segments σ_n, σ_{n+1} merge. For the Dirac operator T_D we are able to define the quasimomentum, some analog of the height slit mapping and the gap length mapping.

Consider the defocussing cubic non-linear Schrödinger equation (NLS)

$$-\widetilde{J}\frac{d\psi}{dt} = -\psi_{xx} + 2|\psi|^2\psi, \quad \psi = \psi(t,x) = \begin{pmatrix} \psi_1(t,x) \\ \psi_2(t,x) \end{pmatrix}, \quad \psi(0,x) = q(x).$$

It is well known that NLS is a completely integrable infinite dimensional Hamiltonian system. The periodic spectrum of (1.5) is invariant under the flow of NLS. The Hamiltonian has the following form

$$H(q) = \frac{1}{2}\int_0^1 [|q'(x)|^2 + |q(x)|^4] dx = 4Q_2(h);$$

Moreover, there are two functionals. First, the number of particles I_N has the forms

$$I_N(q) = \frac{1}{2}\|q\|^2 = \frac{1}{2}\int_0^1 |q(x)|^2 dx = Q_0(h) = \frac{1}{2\pi}\int\int |z'(k) - 1|^2 du dv,$$

and the second, the momentum I_M,

$$I_M(q) = \frac{1}{2}(\widetilde{J}q', q) = \frac{1}{2}\int_0^1 (q_2'(x)q_1(x) - q_1'(x)q_2(x)) dx = 2Q_1(h).$$

The NLS equation has the Hamiltonian T_D and the Poisson bracket has the form

$$\{F, G\} = i\int_0^1 [\frac{\partial F}{\partial q_1(x)}\frac{\partial G}{\partial q_2(x)} - \frac{\partial G}{\partial q_1(x)}\frac{\partial F}{\partial q_2(x)}] dx.$$

The NLS equation admits globally defined real analytic action-angle variables (see [Ar]). Define the action variable A_n by the formula (1.2) (see [FM]). Then the Hamiltonian H, the number of particles I_N and the momentum I_M depend only on the action-variables $A_n, n \in \mathbb{Z}$. Introduce the frequencies $\omega_n = \frac{\partial H}{\partial A_n}$, $n \in \mathbb{Z}$. The parameters ω_n are very important since the angle variables has the form $\phi_n(t) = \omega_n t + \phi_n(0), t \geqslant 0, n \in \mathbb{Z}$.

A great many papers are devoted to the inverse problem for the Hill operator. Marchenko and Ostrovski [MO1], [MO2] constructed a "global quasimomentum" and proved the continuous isomorphism of the mapping H ("cut height mapping"). Note that the global quasimomentum was introduced into the spectral theory of the Hill operator by Firsova [F], and by Marchenko-Ostrovski [MO1] sumiltaneously. Dubrovin [D], Its and Matveev [IM], Moser [Mos], Novikov [N], Trubowitz [T] considered the inverse problem for finite band potentials (potentials were more general in [T]). Garnett and Trubowitz [GT1] proved the real analytic isomorphism both of H and of G for the case of even potentials. In the next paper [GT2], Garnett and Trubowitz proposed a new approach to solve the inverse problem. This approach is based on functional analysis (see Theorem 8.1) and gives the direct proof that the mapping G is an isomorphism. Note that here the apriori estimate (similarly (1.6)) is important. But the proof is not complete since the needed estimates were absent, and note that later on such estimates was proved in [K7]. Kargaev and Korotyaev [KK] reproved the results of Garnett and Trubowitz [GT1] by the direct method. Moreover, they considered other mappings, for example, they solved the inverse problems for the gap length mapping of the operator $\sqrt{T - \lambda_0^+} \geqslant 0$.

Kappeler [Ka] proved that the "gap length mapping" for the Hill operator is a real analytic isomorphism. But in his paper there exists a mistake connected with real analyticity, since in fact, analyticity seems only to have been proved for each single component of the gap length mapping Remark that the definition of this mapping is not explicit and differs from [K6].

In the paper [BGGK] the authors proved that the "gap length mapping" for the Zakharov-Shabat operator (for the case of $q, q' \in H$) is a real analytic isomorphism. We feel, however, that there is a gap in the proof of real analyticity in [Ka] (analyticity seems only to have been proved for each single component of the gap length mapping), and this proof was referred to in the subsequent work on the Dirac operator. Furthermore, it is not clear to us how in [BGGK] the gaps are labeled without using the quasimomentum.

In the paper [BBGK] the symplectomorphism for "the action-angle variable mapping" was proved. Korotyaev [K1], [K6] proved that two mappings for the Hill operator are the real analytic isomorphisms by the direct method both for the "gap length mapping" and the "cut height mapping". Later on the same results were extended for the weighted periodic operator [KKo], [K10].

Double-side estimates for various parameters of the Hill operator (the norm of a periodic potential, effective masses, gap lengths, height of slits $|h_n|$ and so on) were obtained in [K2-4] and for the Dirac operator in [KK2], [K2], [K5]. The precise double-sided estimates (see (1.6)) for gap lengths was found in [K7]. Pöschel and Trubowitz [PT] wrote a nice book concerning the inverse Dirichlet problem.

Recall that for a compact subset $K \subset \mathbb{C}$ the analytic capacity is called the following value

$$\mathcal{C} = \mathcal{C}(K) = \sup\{|f'(\infty)| : f \text{ analytic in the domain } \mathbb{C}\setminus K; \quad |f(k)| \le 1,\ k \in \mathbb{C}\setminus K\}$$

where $f'(\infty) = \lim_{k\to\infty} k(f(k) - f(\infty))$. We will use well known Theorem of Ivanov-Pommerenke (see [Iv], [Po])

Theorem *(Ivanov-Pommerenke) Let $E \subset \mathbb{R}$ be compact. Then the analytic capacity $\mathcal{C}(E) = |E|/4$, where $|E|$ is the Lebesgue measure (the length) of the set E. Moreover, the Ahlfors function f_K (the unique function, which gives* sup *in the definition of the analytic capacity) has the following form:*

$$f_K(z) = \frac{\exp\left(\frac{1}{2}\phi_K(z)\right) - 1}{\exp\left(\frac{1}{2}\phi_K(z)\right) + 1}, \qquad \phi_K(z) = \int_E \frac{dt}{z-t}; \quad z \in \mathbb{C}\setminus E.$$

We will use the following simple remark: if $K \subset \mathbb{C}$ is compact ; $D = \mathbb{C}\setminus K,\ g \in \Sigma'(D)$, then $\mathcal{C}(K) = \mathcal{C}(\mathbb{C}\setminus g(D))$. It follows immediately from the definition of the analytic capacity.

Let $\Phi_+ \subset \ell^\infty_{\mathbb{R}}$ be the subset of finite sequences of non negative numbers. Then, using the Ivanov-Pommerenke Theorem and the last remark we obtain for $h \in \Phi_+$

$$\|l(h)\|_1 = A(h) = \mathcal{C}(\Gamma(h)), \quad \text{where } \Gamma(h) = \cup_{n\in\mathbb{Z}}[u_n - i|h_n|, u_n + |h_n|];$$

$$\gamma(\sigma(k(\cdot, h, u))) = \gamma(\Gamma(h)).$$

Moreover, using Theorem 2.4 we have for $h \in \Phi_+$

$$\partial_n F(h) = \phi'_{\sigma(k(\cdot,h,u))}(\lambda_n), \quad \text{where } \ F(\xi) = \gamma(E(\xi)),\ \xi \in \Phi_+.$$

Similarly, we are able to get other results devoted to $\|\cdot\|_1$ and to the functional E.

The main gaol of our paper is to prove the following points

1) to study the function $z(k,h)$ as a function of two variables $k \in K(h), h \in \ell^p_\omega$,
2) to get an analytic continuation of the functional $Q_0 : \ell^2_{\mathbb{R}} \to \mathbb{R}_+$ into the domain $\mathcal{J}^2(\rho)$ and and to find the derivatives $\partial Q_0/\partial h_n$,
3) to show that the mappings $l : \ell^p_\omega \to \ell^p_\omega$ and $J : \ell^p_\omega \to \ell^p_\omega$ are real analytic isomorphisms for any $1 \leqslant p \leqslant 2$ and any weight $\omega_n \geqslant 1$,
4) to find the double-sided estimates for $\|h\|_{p,\omega}$ and $\|l\|_{p,\omega},\ 1 \leqslant p \leqslant 2$.
5) to get the double-sided estimates for $\|h\|_{p,\omega}$ and $I_D(h),\ 1 \leqslant p \leqslant 2$.
6) to study the functional $E(h) = \sum l_n(h)$.

2 Main results

Define the following constants

$$\rho = \frac{u_* h_*^2}{32}, \quad \alpha_p = \frac{1}{\pi}\Big(\frac{2^{p+2}(2+\pi)}{u_*}\Big)^p, \quad C_A = \frac{48\cdot 6^{1/6}}{(1-\delta)u_* h_*}, \quad C_H = \frac{32}{h_*^2}, \quad C_L = C_H 3\sqrt{2}.$$

where $p \geqslant 1$. We formulate the basic result of our paper. We prove the following properties of the non-linear mapping $l(h) = \{l_n(h)\}_{n\in\mathbb{Z}}$. Recall $l_n(h) = (z_n^+ - z_n^-)\operatorname{sign} h_n, \quad n \in \mathbb{Z},$.

Theorem 2.1. *Let $u_{n+1} - u_n \geq u_* > 0,\ n \in \mathbb{Z}$. Then for each $1 \leq p \leq 2$ and the weight $\omega_n \geqslant 1, n \in \mathbb{Z}$ the mapping $l:\ \ell^p_\omega \to \ell^p_\omega$ is a real analytic isomorphism. Moreover, this mapping has an analytic continuation into the domain $\mathcal{J}^p_\omega(\rho)$ for some $\rho > 0$ and the following estimates are fulfilled:*

$$|l_n(h)| \leqslant C_L|h_n|(1 + C_A^2\|h\|^2),\ h \in \mathcal{J}^p_\omega(\rho). \tag{2.1}$$

$$\|l(h)\|_{p,\omega} \leqslant C_L\|h\|_{p,\omega}(1 + C_A^2\|h\|^2), \qquad h \in \mathcal{J}^p_\omega(\rho), \tag{2.2}$$

$$\|l(h)\|_{p,\omega} \leqslant 2\|h\|_{p,\omega} \leqslant \|l(h)\|_{p,\omega}\exp(9\|h\|_\infty/u_*), \quad h \in \ell^p_\omega, \tag{2.3}$$

$$\frac{1}{2}\|l(h)\|_p \leqslant \|h\|_p \leqslant 2\|l(h)\|_p(1 + \alpha_p\ \|l(h)\|_p^p), \qquad h \in \ell^p. \tag{2.4}$$

The real analytic isomorphism of some mappings were proved in the papers [GT], [KK2], [K1-2]. In the present paper there exists two absolutely new points: first, this mapping has an analytic continuation into the domain $\mathcal{J}^p_\omega(\rho)$ for some $\rho > 0$ and the second, the estimates both for the real vectors and the complex one (see (2.1-3)). Estimates (2.2-3) are new, but in the case of the real space ℓ^2_1 inequality (2.2) was obtained in the paper [KK2], where the method and the estimates were very rough. One of the author (E.K. [K4], [K8]) found the double - sided estimates for the space $\ell^2_m, m \geqslant 0$.

Define the square differential in the domain $D = K(h) \cup \{\infty\}$ on the Riemann sphere, which is considered as Riemann surface with hyperbolic boundary components by the following formula:

$$w = (k'(z,h) - 1)^2 dz^2 = (z'(k,h) - 1)^2 dk^2.$$

Then w is the analytic square differential on D (in particular, analytic at any boundary point in terms of a corresponding uniformizing parameter). In the present paper w-metric is important to get the needed estimates. Moreover, these estimates have the following geometry interpretation.

The invariant length L_n of the cut γ_n has the following form

$$L_n = 2\int_{z_n^-}^{z_n^+} |k'(x) - 1|dx = 2\int_{z_n^-}^{z_n^+} \sqrt{v'(x)^2 + 1}dx.$$

Then using (1.1) we obtain

$$2|h_n| \leq L_n \leq 2(|h_n| + |l_n|) \leq 6|h_n|$$

Moreover, the invariant area S of the Riemann surface D has the form

$$S = \iint_{\mathbb{C}} |k'(z) - 1|^2 dxdy = 2\pi Q_0(h).$$

It is possible to consider Theorem 2.1 and corollaries as a statement which gives the double-sided estimate for S if we know the sequence of the boundary component lengths and the points where the vertical slits cross the real line.

We show now the corollary asociated with the notion of the extremal length. Let $h = \{h_n\}_{n=1}^N$ be a finite sequence of positive numbers. For any $1 \le n \le N$ we introduce the set G_n of rectifiable curves which connect the slit $[u_n + ih_n/2,\ u_n + ih_n]$ with the real line and lying in $\mathbb{C}_+$. Let P be the set of metrics $\rho(z)|dz|$ on $\mathbb{C}_+$, such that $\rho^2(z)$ is integrable and

$$\int_{\gamma_n} \rho(z)|dz| \ge h_n/2, \quad \gamma_n \in G_n,\ n = 1, \dots, N$$

It is well known that in this case (see [J]) the following value

$$M = \inf_{\rho \in P} \int\int_{\mathbb{C}_+} \rho^2(z) dx dy$$

is called the extremal length for the set G_n and numbers $h_n/2$.

It is found the metric $|z'(k) - 1||dk|$ belongs to the set P and by Theorem 2.1, there exists an absolute constant C such that

$$CM \le \int\int_{\mathbb{C}_+} |z'(k) - 1|^2 dp dq \le M.$$

We formulate the basic result about the mapping $J(h) = \{J_n(h)\}_{n\in\mathbb{Z}}$. and recall $J_n = |A_n|^{1/2} \operatorname{sign} h_n,\ A_n = \frac{2}{\pi}\int_{\gamma_n} v(z,h)dz$.
Theorem 2.2. *Let $u_{n+1} - u_n \ge u_* > 0,\ n \in \mathbb{Z}$. Then for each $1 \le p \le 2$ and the weight $\omega_n \geqslant 1, n \in \mathbb{Z}$ the mapping $J:\ \ell^p_\omega \to \ell^p_\omega$ is a real analytic isomorphism. Moreover, the following estimates are fulfilled:*

$$\|J(h)\|_{p,\omega} \leqslant 2\|h\|_{p,\omega} \leqslant 2\|J(h)\|_{p,\omega} \exp\left(5\|J(h)\|_{p,\omega}/u_*\right), \quad h \in \ell^p_\omega, \tag{2.5}$$

$$\frac{\|l\|_p}{2} \leqslant \|J\|_p \leqslant \frac{2}{\sqrt{\pi}}\|l\|_p(1 + \alpha_p\|l\|_p^p)^{1/2}, \quad h \in \ell^p, \tag{2.6}$$

$$\frac{\sqrt{\pi}}{2}\|J\|_p \leqslant \|h\|_p \leqslant 4\|J\|_p(1 + \alpha_p 2^p\|J\|_p^p), \quad h \in \ell^p. \tag{2.7}$$

The real analytic isomorphism of some such mapping was proved in the papers [DBGK] for the Hill operator. In the present paper there exists two absolutely new points: first, we consider this mapping in the Banach space ℓ^p_ω for any $\omega > 0$ and the second, the estimates are proved for any $1 \le p \le 2, \omega_n \geqslant 1$.

Let $\partial_n = \partial/\partial h_n, n \in \mathbb{Z}$. Introduce the constants

$$\nu_n = \begin{cases} \operatorname{sign} h_n/|k''(\lambda_n, h)|, & \text{if} \quad h_n \ne 0, \\ 0 & \text{if} \quad h_n = 0. \end{cases}$$

We formulate the basic result concerning the functional Q_0. Recall $Q_0(h) = \frac{1}{\pi}\int_{\mathbb{R}} v(x,h)\,dx$.
Theorem 2.3.*Assume that $u_{n+1} - u_n \ge u_* > 0,\ n \in \mathbb{Z}$. Then the functional $Q_0 : \ell^p_\omega \to \mathbb{R}_+$ has analytic continuation in the domain $\mathcal{J}^p_\omega(\rho)$ for some $\rho > 0$ and the following estimates are fulfilled:*

$$|Q_0(h)| \le \frac{1}{2}C_H^2\|h\|^2_{p,\omega}\Big(1 + C_A^2\|h\|^2\Big), \qquad h \in \mathcal{J}^p_\omega(\rho), \tag{2.8}$$

$$\frac{\pi}{2}\,Q_0(h) \leqslant \|h\|^2 \leqslant \pi^2 \max\Big\{1,\ \frac{\sqrt{2Q_0(h)}}{u_*}\Big\}\,Q_0(h), \qquad h \in \ell^2_{\mathbb{R}}, \tag{2.9}$$

and the derivative has the form:

$$\partial_n Q_0(h) = \nu_n(h), \quad n \in \mathbb{Z}, \quad h \in \ell^2_{\mathbb{R}}. \tag{2.10}$$

In the present paper there exists three absolutely new points: first, this function $Q_0(h)$ has an analytic continuation into the domain $\mathcal{J}^p_\omega(\rho)$ for some $\rho > 0$, the second, the estimates both for the real vectors and the complex one (see (2.8-9)) and the derivatives (2.10) is found for the first time.

For each $h \in l^1_{\mathbb{R}}$ we define the functional

$$E(h) = \sum_{n\in\mathbb{Z}} l_n(h).$$

It is possible since there exists estimate (1.1). Recall $E(h)/4$ is the analytic capacity of the set $\cup\gamma_n$. It the our paper the following properties were proved.

Theorem 2.4. *For each sequence U and $h \in \ell^1_{\mathbb{R}}$, such that $h_n \geq 0, n \in \mathbb{Z}$ the following estimates are fulfilled:*

$$\pi Q_0(h) \leq \|h\|_\infty E(h) \leq \frac{2}{\pi} E(h)^2, \qquad E(h) \leq 2\|h\|_1. \quad h \in \ell^1_{\mathbb{R}}, \tag{2.11}$$

Let in addition $u_{n+1} - u_n \geq u_* > 0,\ n \in \mathbb{Z}$. Then the functional $E : \ell^1_{\mathbb{R}} \to \mathbb{R}$ has analytic continuation in the domain $\mathcal{J}^1(\rho)$ and the following estimates are fulfilled:

$$|E(\eta)| \leq C_L\|\eta\|_1(1 + C_A^2\|\eta\|_1^2)), \qquad \eta \in \mathcal{J}^1(\rho), \tag{2.12}$$

$$\|h\|_1 \leqslant \|l\|_1(1 + \frac{40}{u_*}\|l\|_1), \quad h \in \ell^1_{\mathbb{R}}, \tag{2.13}$$

and for any $h \in \ell^1_{\mathbb{R}}$ the derivative has the form

$$\partial_n E(h) = \begin{cases} \nu_n \cdot \int\limits_{\sigma(h)} \frac{dt}{(t-z_n)^2}, & \text{if } h_n \neq 0, \\ 2z'(u_n, h), & \text{if } h_n = 0. \end{cases} \tag{2.14}$$

Consider now the analytic properties of the conformal mapping $z(k,\eta)$. Assume that the following condition $u_{n+1} - u_n \geq u_* > 0,\ n \in \mathbb{Z}$ is fulfilled. Note that by the definition, for any fixed $k \in \mathbb{C}$ the function $z(k,h),\ h \in \ell^2_{\mathbb{R}}$ is even with respect to each variable h_n. Then in order to find the derivative $\partial_n z(k,h)$ it is enough to compute the derivative for the case $h_m \geqslant 0, m \in \mathbb{Z}$.

Define the ball $B^p_\omega(\eta, r) = \{f : \|f - \eta\|^p_\omega \leqslant r\} \subset \ell^p_\omega$, and the disk $B(z_0, r) = \{z : |z - z_0| < r\} \subset \mathbb{C}$, where $\rho > 0$. In the case $\ell^p_{\omega,c}$ the ball is denoted by $B^p_{\omega,c}(\eta,\rho)$. For fixed $\varepsilon_n \in [0,1]$ and $h \in \ell^\infty$ introduce two domains

$$\mathcal{K}(h,\varepsilon) = K(h)\setminus \bigcup_{n\in\mathbb{Z}} (\overline{B(\widetilde{u}_n, r_n)} \cup B(\overline{\widetilde{u}_n}, r_n));$$

$$K_n(h) = \{|\operatorname{Re}(r - u_n)| < u_*\} \setminus \{k = u_n + iv, |v| \geqslant |h_n|\},$$

where $\widetilde{u}_n, r_n$ depend on the two cases: let $R_n = h_* u_* \varepsilon/8$,

$$r_n = \frac{\varepsilon u_*}{4}, \quad \widetilde{u}_n = u_n, \quad if \quad |h_n| < R_n, \quad (I\ case);$$

$$r_n = \frac{R_n}{2}, \quad \widetilde{u}_n = u_n + i(|h_n| - \frac{h_* R_n}{4}), \quad if \quad |h_n| \geq R_n, \quad (II\ case).$$

Note that in the second case $\overline{B(\widetilde{u}_n, r_n)} \subset \mathbb{C}_+$, since we have

$$(h_* r_n/2) + r_n \leqslant\ 5r_n/4 < 2r_n \leqslant |h_n|.$$

We formulate our basic results about the analytic properties of the conformal mappings.

Theorem 2.5. *For any fixed $h \in \ell^2_{\mathbb{R}}, \varepsilon = \{\varepsilon_n\}$, where $\varepsilon_n \in [0,1]$, $n \in \mathbb{Z}$ the function $z(k, h + \varepsilon\eta)$ has an analytic continuation from $D = K(h,\varepsilon) \times B^2(\rho)$ into the domain $D_C = K(h,\varepsilon) \times B^2_C(\rho)$. This continuation is such that if the set $K_n(h,\varepsilon)$ is connected for some $n \in \mathbb{Z}$ then the function $z(k, h + \varepsilon\eta)$, $(k,\eta) \in D_C$ has an analytic extension in the domain $\widetilde{D}_n = \widetilde{K}_n \times B^2_C(\rho)$ ("left" if this continuation coincides with $z(k, h+\varepsilon\eta)$ as $(k,\eta) \in \widetilde{D}_n \cap \{\operatorname{Re} k < u_n\}$ and "right" if this continuation coincides with $(k, h+\varepsilon\eta) \in \widetilde{D}_n \cap \{\operatorname{Re} k > u_n\}$). Moreover, for any $h \in \ell^2_{\mathbb{R}}, n \in \mathbb{Z}, k \in K(h)$ the derivative has the following form:*

$$\partial_n z(k,h) = \frac{\nu_n}{z(k,h) - z_n}, \quad k \neq u_n + ih_n, \quad k \in K(h). \tag{2.15}$$

In particular, the formula (2.15) is true for $k \in [u_n, u_n + ih_n)$, if $z(k,h)$ is the left-hand limit (the right-hand limit).

In this Theorem an analytic continuation into the domain $\mathcal{J}^p_\omega(\rho)$ is new, moreover, the identitty (2.15) for the derivatives is also new.

We use the results of the Theorem to get the Löwner equation for our case, when the corresponding conformal mappings have the asymptotics $z(iv,h) = iv - (Q_0 + o(1))/iv$ as $v \to \infty$ (the normalisation at infinity). It is important that in this case the point of the normalisation is the infinity, i.e. this point belongs to boundary of our domain. Remark that in the classical case (see [G]) this point lies inside the domain. Let $h = \{h_n\}_{n\in\mathbb{Z}}$ belong to $\ell^2_{\mathbb{R}}$ and let $h_m > 0$ for some $m \in \mathbb{Z}$. For any $\xi \in [0, h_m]$ define the sequence h_ξ by the following formula:

$$(h_\xi)_n = \begin{cases} h_n, & \text{if} \quad n \neq m, \\ h_m - \xi, & \text{if} \quad n = m, \end{cases}$$

and let $K_\xi = K_+(h_\xi)$, $\;g(z,\xi) = k(z, h_\xi)$. Then $g(\cdot,\xi)$ is the conformal mapping from $\mathbb{C}_+$ onto K_ξ and we have the following asymptotics

$$g(z,\xi) = z - \frac{Q_0(h_\xi)}{z} + \ldots, \quad z \to \infty.$$

Moreover, by Theorem 2.3, the function $\beta(\xi) = Q_0(h_\xi)$ is increasing on the interval $[0,\ h_m]$ and $\beta \in C^\infty([0,\ h_m]$, and

$$\frac{d\beta}{d\xi} = -\nu_m(h_\xi).$$

Introduce the function $f(z,\xi) = g^{-1}(g(z,0),\xi)$ which has the asymptotics

$$f(z,\xi) = z - \frac{(\beta(0) - \beta(\xi))}{z} + \dots, \qquad z \to \infty.$$

Differentiating the function f with respect to ξ and using Theorem 2.5 we obtain

$$\frac{\partial f}{\partial \xi} = -\frac{\nu_m}{f - z_m},$$

where $z_m(\xi) = g^{-1}(u_m + h_m i) \in \mathbb{R}$. Defining the new parameter $t = \beta(0) - \beta(\xi),\ \xi \in [0,\ h_m]$ we have the Löwner equation for the function f on the interval $[0,\ h_m]$:

$$\frac{\partial f}{\partial t} = \frac{1}{z_m - f}, \qquad f(z,0) = z, \quad z \in \mathbb{C}.$$

Using such calculus, the identity $g(f(z,t),t) = g(z,0)$ and in standard way ([G], [A]) we obtain another Löwner equation for the function g:

$$\frac{\partial g}{\partial t} = \left(\frac{1}{z - z_m}\right)\frac{\partial g}{\partial z}, \qquad z \in \mathbb{C}, \qquad t \in [0,\ h_m].$$

Note that $g(\cdot,0)$ is the conformal mapping from $\mathbb{C}_+$ onto $B_0 = K_+(h)$.

Now we estimate the Dirichlet integral $I_D(h)$ (or $Q_0(h)$) for general sequences $\{u_n\}_{n\in\mathbb{Z}}$, using the following geometric construction.

For the vector $h \in l^\infty_{\mathbb{R}}, h_n \to 0$ as $n \to \infty$, we introduce the sequence $\widetilde{h} = \widetilde{h}(h,u)$ which is defined in the following way. We take an integer n_1 such that $|h_{n_1}| = \max_{n\in\mathbb{Z}} |h_n| > 0$ and set $\widetilde{h}_{n_1} = h_{n_1}$; assume that we define the numbers $n_1, n_2, \dots, n_k$, then we take n_{k+1} such that

$$|h_{n_{k+1}}| = \max_{n\in B} |h_n| > 0, \quad B = \{n \in \mathbb{Z} : |u_n - u_{n_l}| > |h_{n_l}|, 1 \le l \le k\}$$

and set $\widetilde{h}_{n_{k+1}} = h_{n_{k+1}}$. Then, for number n, which disagrees with some n_k, we define $\widetilde{h}_n = 0$. Now we formulate the following results

Theorem 2.6. *Let $h \in l^\infty_{\mathbb{R}}, h_n \to 0$ as $|n| \to \infty$; and let $\widetilde{h} = \widetilde{h}(h,u)$. Then the following estimates are fulfilled:*

$$\frac{1}{\pi^2}\|\widetilde{h}\|_2^2 \le Q_0(h) = \frac{1}{2} I_D(h) \le \frac{2\sqrt{2}}{\pi}\|\widetilde{h}\|_2^2. \tag{2.16}$$

Remark that some analog of Theorem 2.6 for $\|\cdot\|_1$ follows from result of Shirokov [Sh], devoted to estimates of the analytic capacity. We have

Theorem (Shirokov) *1). Let $h \in \ell^\infty_{\mathbb{R}};\ h_n \to 0, \quad n \to \infty; \quad \widetilde{h}(h,u)$. Then*

$$\|\widetilde{h}\|_1 \le \|h\|_1 \le C\|\widetilde{h}\|_1$$

where C is the absolute constant.

2) Assume there exist $L > 0, M \in \mathbb{N}$ such that $u_{n+M} - u_n > L$ for any $n \in \mathbb{Z}$. Then for each $h \in l^\infty_{\mathbb{R}}$ the folowing estimates are fulfilled:

$$\frac{1}{2}\|l(h)\| \le \|h\|_1 \le C\|l(h)\|_1(1 + \frac{M}{L}\|l(h)\|_1), \tag{2.17}$$

where C is some the absolute constant.

Application. Consider the electrostatic field in the plane $K = \mathbb{C} \setminus \cup_{n\in\mathbb{Z}}\Gamma_n$, where $\Gamma_n, n \in \mathbb{Z}$ is the system of neutral conductors. In other words, we imbed the system of neutral conductors $\Gamma_n, n \in \mathbb{Z}$, in the external homogeneous electrostatic field $E_0 = (0, -1) \in \mathbb{R}^2$ on the plane. Then on each conductor there exists the induced charge, positive $e_n > 0$ on the lower half of the conductor Γ_n and negative $(-e_n) < 0$ on the upper half of the conductor Γ_n, since their sum equals zero. As a result we have new perturbed electrostatic field $\mathcal{E} \in \mathbb{R}^2$. It is well known that there exists the complex potential $z(k, h), k \in K$ such that

$$\mathcal{E} = \overline{iz'(k)} = -\nabla y(k), \quad k \in K, \quad z = x + iy. \tag{2.18}$$

Recall that $z(k)$ is the conformal mapping from K onto the domain $\mathcal{Z} = \mathbb{C} \setminus \cup\gamma_n$, where $\gamma_n = (z_n^-, z_n^+)$. The function $y(k)$ is called the potential of the electrostatic field in K, and the equation $x(k) = const$ is defined the line of force directed from a positive charge to negative charge. It is well known that the potential $y(k)$ of the electrostatic field on the conductor γ_n is constant. Then the field $\mathcal{E}$ on the conductor has the horizontal direction and the following formula is fulfilled:

$$\mathcal{E} = \overline{iz'(k)} = -(y_u(k), 0) \quad k \in \Gamma_n. \tag{2.19}$$

The density of the charge on the conductor has the form (see [LS])

$$\rho_e(k) = \frac{|z'(k)|}{4\pi}, \quad k \in \Gamma_n, \tag{2.20}$$

and then (2.20) yields

$$\rho_e(k) = \frac{|y_u(k)|}{4\pi}, \ k \in \Gamma_n^+ = \Gamma_n \cap \mathbb{C}_+, \tag{2.21}$$

Using (2.21) we find the induced charge e_n on the upper half of the conductor γ_n^+:

$$e_n = \frac{1}{4\pi}\int_{\Gamma_n^+} x_v(k)dv = \frac{1}{4\pi}|\gamma_n|. \tag{2.22}$$

i.e. we get the nice formula: the value of the charge equals the gap length $|\gamma_n|/4\pi$. Introduce the mapping $e : \ell^p \to e = \{e_n\} \in \ell^p$ by the formula: $e(h) = \{e_n, n \in \mathbb{Z}\}$, where $e_n = |\gamma_n|/4\pi$. Define the charge $E = \sum e_n$, i.e. the sum of all positive induced charges, and the bipolar moment of the conductor with the induced charge density by the formula

$$d_n = \frac{1}{4\pi}\int_{\Gamma_n} vx_v(k)dv.$$

We transform this value into the form

$$d_n = \frac{1}{2\pi} \int_{\gamma_n} v(x)dx = \frac{1}{4} A_n, \tag{2.23}$$

and define the mapping $d : \ell^p \to d = \{d_n\} \in \ell^p$. Then if we know $K(h)$, i.e. we have $h = \{h_n\}$, then we know the electrostatic field $\mathcal{E} \in \mathbb{R}^2$ and all other physical parameters. Moreover, we obtain the value of the positive induced charges $e_n > 0$ and the bipolar moment d_n. An inverse is also true. If we have the value $e = \{e_n\} \in \ell^p, p \in [1, 2]$, then by Theorem 2.1, we know the heightes of all conductors. Moreover, there exist the estimates (see (2.1)) and E is an analytic function of h.

Furthermore, if we have the value $d = \{d_n\} \in \ell^p, p \in [1, 2]$, then by Theorem 2.2, we get the heightes of all conductors. We have the estimates (2.1).

Preliminaries. Below we need some results about conformal mappings. The function $z(\cdot, h)$ has an analytic extension (by the symmetry) from the domain $K_+(h)$ onto the domain

$$K(h) = K(h, U) = \mathbb{C} \setminus \bigcup_{n\in\mathbb{Z}} \Gamma_n, \qquad \Gamma_n = [u_n - i|h_n|, u_n + i|h_n|], \tag{2.24}$$

and $z(\cdot, h)$ maps $K(h)$ conformally onto the domain $Z = \mathbb{C} \setminus \cup_{n\in\mathbb{Z}} \overline{\gamma}_n$. For any nondegenerate gap γ_n the function $k(\cdot, h)$ has an analytic continuation (from above or from below) across the interval γ_n. This sufices to extend the function $-i(k(\cdot, h) - u_n)$ by the symmetry. Similarly the function $z(\cdot, h)$ has analytic continuation (from left or from right) across the vertical slit $(u_n - i|h_n|, u_n + i|h_n|)$ by the symmetry.

Introduce the effective masses $\mu_n^\pm$ for the end $z_n^\pm$ of the gap $|\gamma_n| \neq 0$ by the formula

$$z(k, h) - \mu_n^\pm = \frac{(k - u_n)^2}{2\mu_n^\pm} + O((k - u_n)^3), \quad z \to z_n^\pm. \tag{2.25}$$

Define now the effective masses ν_n in the plane $K(h)$ for the end of the vertical slit $[u_n + i|h_n|, u_n - i|h_n|]$ by the following asymptotics

$$k - (u_n + i|h_n|) = \frac{(z(k, h) - z_n)^2}{2i\nu_n^\pm} + O((z(k, h) - z_n)^3), \qquad k \to u_n + i|h_n|. \tag{2.26}$$

By virtue of symmetry the effective masses $\nu_n^\pm$ coincide, i.e. $\nu_n^\pm \equiv \nu_n$. We have the simple formula

$$\nu_n = \begin{cases} |k''(z_n, h)|^{-1} \operatorname{sign} h_n, , & \text{if} \quad l_n \neq 0, \\ 0, & \text{if} \quad l_n = 0, \end{cases} \tag{2.27}$$

Below we will use the Lindelöf principle (see [J]), which is formulated in the following form, convenient for us.

Lemma 2.7. *Let $h,\ h^1 \in \ell^\infty_{\mathbb{R}};\ \ |h_n^1| \le |h_n|,\ n \in \mathbb{Z}$. Then the following estimates are fulfilled:*

$$y(k, h^1) \ge y(k, h), \quad k \in K_+(h), \tag{2.28}$$

$$|z'(k, h^1)| \ge |z'(k, h)|;\ k \in (u_n, u_n + i|h_n|),\ n \in \mathbb{Z}; \tag{2.29}$$

$$|z'(u,h^1)| \geq |z'(u,h)|;\ u \in \mathbb{R},\ u \neq u_n,\ n \notin \Lambda; \tag{2.30}$$

$$Q_0(h^1) \leq Q_0(h) \ \ \textit{and if}\ Q_0(h^1) = Q_0(h), \ \ \textit{then}\ h^1 = h, \tag{2.31}$$

$$|l_m(h^1)| \geq |l_m(h)|. \tag{2.32}$$

Estimates (2.28-29) are simple and follow from (2.25-27), which are well known and for our case are discussed in the paper [KK1]. We show the possibility of this principle in the following Lemma.

Lemma 2.8. *Let $h \in \ell^\infty_{\mathbb{R}}$. Then for each $n \in \mathbb{Z}$ the following estimates are fulfilled:*

$$\nu_n \leq |h_n|, \tag{2.33}$$

$$\|h\|^2_\infty \leq I_D(h) = 2Q_0(h). \tag{2.34}$$

Proof. Applying estimate (2.28) to h and to the new sequence: $h^1_m = h_n$ if $m = n$ and $h^1_n = 0$ if $m \neq n$. It is clear that $z(k,h^1) = \sqrt{(k-u_n)^2 + h_n^2}$ (the principal value). Then

$$y(k,h) \leq \operatorname{Im}(\sqrt{(k-u_n)^2 + h_n^2}),\ k \in K_+(h),$$

and using the asymptotics (2.26) of the function $z(k,h)$ as $k \to u_n + i|h_n|$, we obtain estimate (2.33).

In order to prove (2.34) we use (2.31) since $Q_0(h^1) = h_m^2/2$. $\square$

Below we need the following estimates (see [KK1])

$$\max\{\frac{l_n^2}{4}, \frac{l_n h_n}{\pi}\} \leqslant A_n = \frac{2}{\pi}\int_{\gamma_n} v(x)dx \leqslant \frac{2 l_n h_n}{\pi}, \tag{2.35}$$

$$v(x) \geqslant v_n(x) = |(x - z_n^-)(z_n^+ - x)|^{1/2}, \quad x \in \gamma_n. \tag{2.36}$$

We need also some results about the Cauchy and the Poisson Integrals. Recall that the function $f(z),\ z \in \mathbb{C}_+$, belongs to the class $\mathcal{R}_0$, if there exists the following formula

$$f(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{t - z}, \quad z \in \mathbb{C}$$

where μ is a Borel measure on $\mathbb{R}$ and $M = \mu(\mathbb{R}) < \infty$. It is well known that $f \in \mathcal{R}_0$ if and only if, one of the two following conditions is fulfilled:

1) $\operatorname{Im} f(z) \geq 0,\ z \in \mathbb{C}_+;\quad M_1 = \sup_{y>0} |f(iy)y| < \infty;$

2) $f(iy) = -\frac{M_2}{iy} + o(\frac{1}{y}),\quad y \to +\infty.$

Note that $M = M_1 = M_2$.

Using this representation and some results from the paper [KK1] it is possible to get the following formulas.

Let $h \in \ell^2_{\mathbb{R}}$. Then the following identities are fulfilled

$$k(z,h) = z + \frac{1}{\pi}\int_{\mathbb{R}} \frac{v(t,h)}{t - z}\,dt, \quad z \in \mathbb{C}_+; \tag{2.37}$$

$$k(z,h) = z - \frac{Q_0(h) + o(1)}{z}, \quad z = iy, \quad y \to +\infty; \tag{2.38}$$

$$u'(x,h) = 1 + \frac{1}{\pi}\int_{\mathbb{R}} \frac{v(t,h)}{(t-x)^2}\,dt, \quad x \in \mathbb{R} \setminus \bigcup_{n\in\mathbb{Z}} \overline{g}_n; \tag{2.39}$$

$$v(x,h) = v_n(x)\Big(1 + \frac{1}{\pi}\int_{\mathbb{R}\setminus\gamma_n} \frac{v(t,h)}{|t-x|v_n(t)}\,dt\Big), \quad x \in \gamma_n. \tag{2.40}$$

Below we need the formulas, where the integrals on the line in the upper half plane. For each $b > \|h\|_\infty$ the following estimates are fulfilled:

$$(k+ib) - z(k+ib,h) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{v(t+ib,h)}{t-k}\,dt, \quad k \in \mathbb{C}_+; \tag{2.41}$$

$$Q_0(h) = \frac{1}{\pi}\int v(t+ib,h)\,dt; \tag{2.42}$$

$$z(k,h) = k + \frac{Q_0(h) + o(1)}{k}, \qquad k = iq, \quad q \to +\infty. \tag{2.43}$$

Note that (2.41-43) are true for more general case. For example, if

$$f \in L^1(\mathbb{R}), \quad V(k) = \operatorname{Im} F(z(k,h)),\ k \in K(h); \quad b > \|h\|_\infty, \qquad S = \frac{1}{\pi}\int_{\mathbb{R}} f(t)\,dt;$$

where the function F has the form

$$F(z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)}{t-z}\,dt, \quad z \in \mathbb{C}_+; \tag{2.44}$$

Then we have

$$F(iv) = iv + \frac{S + o(1)}{iv}, \quad v \to +\infty; \qquad S = \frac{1}{\pi}\int_{\mathbb{R}} V(t+ib,h)\,dt. \tag{2.45}$$

We have also the following inequalities

$$\operatorname{Im}(k(z,h) - z) \geq 0,\ z \in \mathbb{C}_+; \ \operatorname{Im}(k - z(k,h)) \geq 0,\ k \in K_+(h). \tag{2.46}$$

Let us consider the branch points of our conformal mappings $z_n^\pm$, $u_n \pm i|h_n|$, $n \notin \Lambda$. It is well known that $z_n^\pm$ is a branch point of a second order and the following identity is fulfilled:

$$k(z,h) = \sum_{m=0}^{\infty} b_m^\pm\big(\sqrt{\pm(z - z_n^\pm)}\big)^m, \quad z \in \mathbb{C}_+, \tag{2.47}$$

in some disc with the center $z_n^\pm$ (here $\sqrt{\cdot}$ is the principal value in $\mathbb{C}_+ \setminus (-\infty, 0]$, such that $\sqrt{1} = 1$). Note that $b_0^\pm = u_n, \pm b_1^\pm > 0$, and the effective masses $\mu_n^\pm = 2(b_1^\pm)^2$.

The points $u_n \pm i|h_n|$ are the branch points of the second order for the function $z(\cdot, h)$ and the following identity is fulfilled:

$$z(k,h) = \sum_{m=0}^{\infty} c_m(\sqrt{-i(k-(u_n+i|h_n|))})^m, \quad k \in K(h), \tag{2.48}$$

in some disc with the center $u_n \pm i|h_n|$ (here $\sqrt{\cdot}$ is the principal value such that $\sqrt{1} = 1$). It is clear that $c_0 = z_n$, $-ic_1 > 0$ and $\nu_n = -c_1^2$.

We need another notion of analyticity. Let U be an artitrary subset of a Banach space X, and let X_0 be another Banach space. The map $f : U \to X_0$ is weakly analytic on U, if for each $x \in U, h \in X$ and $g \in X^*$, the function $F(z) = (f(x+zh), g)$ is analytic in a disk $\{z : |z| < r\} \subset \mathbb{C}$ for some $r > 0$. in the usual sense of one complex variable. The notion of a weakly analytic map is weaker than that of an analytic map. Remarkably, a weakly analytic map is analytic, if, in addition, it is locally bounded, that is, bounded in some neighborhood of each point of its definition (see [Di]).

Theorem 2.9. (Analyticity) *Let $f : U \to X_0$ be a map from an open subset U of a complex Banach space X into a complex Banach space X_0. Then the following two statements are equivalent.*

i) f is analytic on U,

ii) f is locally bounded and weakly analytic on U.

3 Dirichlet problem with parameter

In this Section we assume that $u_{n+1} - u_n \geq u_* > 0$, $n \in \mathbb{Z}$. For the set $U = \{u : u = u_n, n \in \mathbb{Z}\}$ define the following set

$$\Gamma = \Gamma(U) = \mathbb{R} \cup \{k \in \mathbb{C}_+ : \operatorname{Re} k \in U\},$$

Below we will use the functions $\beta_\pm : \Gamma(U) \to \mathbb{C}$ which satisfies

Condition B. *1) $\beta_+(k) = \beta_-(k)$, $k \in \mathbb{R}$,*

2) $\beta_\pm \in C(\Gamma \setminus U)$, and for each $t > 0$ the following estimate is fulfilled

$$\beta_\infty(t) \equiv \sup_{k \in \Gamma, \operatorname{Im} k < t} |\beta_\pm(k)| < \infty$$

In this Section we study the family of Dirichlet problems depending on a parameter $h \in \ell^\infty$. Namely, for the fixed sequence $h \in \ell^\infty$ we consider the Dirichlet problem in the domain $K_+(h)$, with the boundary function $\beta(k,h)$,

$$\beta(k,h) = \begin{cases} \beta_+(k) = \beta_-(k), & \text{if} \quad k \in \mathbb{R}, ; \\ \beta_a(k), \quad a = \operatorname{sign} h_n, & \text{if} \quad k \in (u_n, u_n + i|h_n|], n \in \mathbb{Z}, \end{cases}$$

It is well known that this problem has a unique solution $\psi(k,h)$ bounded in $K_+(h)$ and $\psi(\cdot,h) \in C(\bar{\mathbb{C}}_+ \setminus U)$ In this case we will say that the function $\beta_\pm$ define the Dirichlet problem with parameter $h \in \ell^\infty$.

We study the properties of the function $\psi(k,h)$ as the function of two variables $k \in K_+(h), h \in \ell^\infty$. Moreover, we will get the analytic continuation with respect to h in some domain. In order to solve and to analyze these Dirichlet problems with parameter we use the Schwartz alternating method which was proposed in the paper [GT1] for a following simple case: $|h_n| \geqslant 0$, if $|n| \leqslant N$, and $|h_n| \equiv 0$, if $|n| > N$ for some $N > 1$, in fact they considered $\mathbb{R}^N$ case. We extend this result for the case $h \in \ell^\infty$. Define the following domains

$$B_+(0,r) = B(0,r) \cap \mathbb{C}_+, \qquad \widetilde{D}(t) = B(0,1)\backslash[-i, it], \qquad D(t) = \mathbb{C}_+ \cap \widetilde{D}(t), \quad t \in [0,1).$$

Let the functions $T(k,\widetilde{k},t)$ and $\widetilde{T}(k,\widetilde{k},t)$ be the Poisson kernel for the domains $D(t)$ and $\widetilde{D}(t)$ respectively at $t \in [0,1/2)$. We need the following results proved in [GT1], which will be used in this Section

Lemma 3.1. *There exist some constants $0 < h_* < 1/2$ and $0 < \delta < 1$ such that for any $|k| = 1, |\widetilde{k}| = 1/2, k, \widetilde{k} \in \mathbb{C}_+$, the functions $T(k,\widetilde{k},t)$ and $\widetilde{T}(k,\widetilde{k},t)$, have continuations from the interval $t \in (0,h^*)$ onto the disk $\{t : |t| < h_*\} \subset \mathbb{C}$ which are analytic on the interior of this disk and for which the following relations are fulfilled:*

$$T \in C(G), \quad G = (\partial B(0,1) \cap \overline{\mathbb{C}_+}) \times (\partial B(0,1/2) \cap \overline{\mathbb{C}_+}) \times \overline{B(0,h_*)},$$

$$T(k,\widetilde{k},-t) = T(k,\widetilde{k},t), \quad t \in B(0,h^*), \quad \text{for any fixed} \quad |k| = 1, |\widetilde{k}| = 1/2; k, \widetilde{k} \in \overline{\mathbb{C}_+},$$

$$\widetilde{T} \in C(\widetilde{G}), \qquad \widetilde{G} = \partial B(0,1) \times \partial B(0,1/2) \times \overline{B(0,h_*)},$$

$$\sup_{|\widetilde{k}|=\frac{1}{2}, \widetilde{k}\in\mathbb{C}_+, |t|\leqslant h_*} \Big(\int_{\partial B_+(0,1)\cap\mathbb{C}_+} |T(k,\widetilde{k},t))||dk|\Big) \leqslant \delta, \qquad \sup_{|\widetilde{k}|=1/2,\ |t|\leqslant h_*} \Big(\int_{|k|=1} |\widetilde{T}(k,\widetilde{k},t))||dk|\Big) \leqslant \delta.$$

Below the constants h^*, δ from Lemma 3.1 will be often used. We need the following simple result, which helps to prove the analyticity of the mapping from $\ell^\infty_{\mathbb{C}}$ into $L^\infty(d\mu)$, where μ is some measure.

Lemma 3.2. *Let $(M, \mathcal{B}, \mu)$ be the measure space and let $\{E_n\}_{n\in\mathbb{Z}}, \ E_n \in \mathcal{B}$ be the disjoint sequence of $\mathcal{B}$- measurable subsets of M. Assume that each mapping $f_n(z) : B(0,r) \to L^\infty(E_n, d\mu), n \in \mathbb{Z}$, is analytic for some $r > 0$ and*

$$\sup_{n\in\mathbb{Z}, |z|<r} \|f_n(z)\|_{L^\infty(E_n,d\mu)}. < +\infty.$$

Then the mapping $F : B^\infty_{\mathbb{C}}(0,r) \to L^\infty(M, d\mu)$ which defined for $\eta \in B^\infty_{\mathbb{C}}(0,r)$ by the formula

$$F(\eta)(k) = \begin{cases} f_n(\eta_n)(k), & if \quad k \in E_n; \\ 0, & if \quad k \in M \setminus \bigcup_{n\in\mathbb{Z}} E_n \end{cases}$$

is analytic.

For fixed $h \in \ell^\infty$ and a sequence $\varepsilon = \{\varepsilon_n\}_{n\in\mathbb{Z}} \in \ell^\infty, \ \varepsilon_n \in [0,1]$ we introduce the following domains

$$K(h,\varepsilon) = K(h)\backslash \bigcup_{n\in\mathbb{Z}} (\overline{B(\widetilde{u}_n, r_n)} \cup \overline{B(\overline{\widetilde{u}_n}, r_n)});$$

$$K_+(h,\varepsilon)=K(h,\varepsilon)\cap\mathbb{C}_+,\qquad \widetilde{K}_+(h,\varepsilon)=K_+(h)\setminus\bigcup_{n\in\mathbb{Z}}\overline{B(\widetilde{u}_n,r_n/2)}.$$

where the values $\widetilde{u}_n, r_n$ depend on $R_n=h_*u_*\varepsilon_n/8$, in the following way:

$$r_n=\frac{\varepsilon_n u_*}{4},\ \widetilde{u}_n=u_n,\quad \text{if}\ \ |h_n|<R_n,\ \ \text{case I},$$

$$r_n=\frac{R_n}{2},\quad \widetilde{u}_n=u_n+i(|h_n|-\frac{h_*R_n}{4}),\quad \text{if}\ |h_n|\geq R_n,\quad \text{case II}.$$

Note that in the case II we have $\overline{B(\widetilde{u}_n,r_n)}\subset\mathbb{C}_+$, since $(h_*r_n/2)+r_n\leqslant\ 5r_n/4<2r_n\leqslant|h_n|$. With the domains $K_+(h,\varepsilon)$ and $\widetilde{K}_+(h,\varepsilon)$ we associate few sets by the following formulas

$$S_n(h_n,\varepsilon)=\partial K_+(h,\varepsilon)\cap\overline{B(\widetilde{u}_n,r_n)};\quad \widetilde{S}_n(h_n,\varepsilon)=\partial\widetilde{K}_+(h,\varepsilon)\cap\overline{B(\widetilde{u}_n,r_n/2)};$$

$$S(h,\varepsilon)=\bigcup_{n\in\mathbb{Z}}S_n(h_n,\varepsilon),\quad \widetilde{S}(h,\varepsilon)=\bigcup_{n\in\mathbb{Z}}\widetilde{S}_n(h_n,\varepsilon).$$

Recall that $\rho=u_*h_*^2/32$. For $\lambda\in[-\rho,\rho]$ we define the following domain $D_n(\lambda)$:

$$D_n(\lambda)=\widetilde{u}_n+r_nD(t),\qquad t=\frac{|h_n+\lambda\varepsilon_n|}{r_n}\leqslant\frac{9}{16}h_*,\qquad \text{case I};$$

$$D_n(\lambda)=\widetilde{u}_n+r_n\widetilde{D}(t),\quad 0\leqslant t=\frac{(h_*\,r_n/2)+\lambda\,\varepsilon_n\,s_n}{r_n}\leqslant h_*,\quad s_n=\operatorname{sign}h_n,\quad \text{case II}.$$

For fixed sequences $\varepsilon=\{\varepsilon_n\}_{n\in\mathbb{Z}}\in\ell^\infty$, $\varepsilon_n\in[0,1]$ and $h\in\ell^\infty$ we consider the spaces $CB(S)$ ($CB(\widetilde{S})$) of all continuous bounded complex-valued function defined on $S=S(h,\varepsilon)$ (on $\widetilde{S}=\widetilde{S}(h,\varepsilon)$).

For $\eta\in B^\infty(0,\rho)$ we define the operator $\mathcal{A}(\eta)=\mathcal{A}(h,\eta,\varepsilon):\ CB(S)\to CB(\widetilde{S})$ in the following way: for a function $f\in CB(S)$ let $\mathcal{A}f\in CB(\widetilde{S})$ be a function , which is equal on the arc $\widetilde{S}_n$ to the restriction on this arc of the solution of the Dirichlet problem for the domain $D_n(\eta_n)$ and the boundary function which is the same as f on the arc S_n and vanishes on the remaining part of the boundary of the set $D_n(\eta_n)$. Next let $\mathcal{P}=\mathcal{P}(h,\varepsilon):CB(\widetilde{S})\to CB(S)$ be an operator which associates with each function $f\in CB(\widetilde{S})$ the restriction $\mathcal{P}f\in CB(S)$ on the set S of the solution of the Dirichlet problem for the domain $\widetilde{K}_+(h,\varepsilon)$ and the boundary function which is the same as f on the set $\widetilde{S}$ and vanishes on the remaining part of the boundary of the set $\widetilde{K}_+(h,\varepsilon)$. Remark that the operator $\mathcal{P}(h,\varepsilon)$ does not depend on η and $\|\mathcal{P}\|\leqslant 1$. Moreover, by Lemma 3.1, $\|\mathcal{A}(\eta)\|\leqslant\delta$ for any $\eta\in B^\infty(0,\rho)$.

Theorem 3.3. *For each sequences $\varepsilon=\{\varepsilon_n\}_{n\in\mathbb{Z}}$, $\varepsilon_n\in[0,1]$, $h\in\ell^\infty$ the operator-valued function $\mathcal{A}(h,\eta,\varepsilon),\eta\in B^\infty(\rho)$, has an analytic extension in the ball $B_c^\infty(\rho)$ where the following estimate is fulfilled*

$$\|\mathcal{A}(h,\eta,\varepsilon)\|\leqslant\delta,\quad \eta\in B_C^\infty(\rho).$$

Proof. Define the function $T_n(k,\widetilde{k},\tau), \tau\in B(0,\rho)$, by the formulas:

$$T_n(k,\widetilde{k},\tau)=T\left(\frac{k-\widetilde{u}_n}{r_n},\frac{\widetilde{k}-\widetilde{u}_n}{r_n},\frac{h_n+\varepsilon_n\tau}{r_n}\right)r_n^{-1},\qquad k\in S_n,\ \widetilde{k}\in\widetilde{S}_n,\ \text{ in case I;}$$

$$T_n(k,\widetilde{k},\tau)=\widetilde{T}\left(\frac{k-\widetilde{u}_n}{r_n},\frac{\widetilde{k}-\widetilde{u}_n}{r_n},\frac{\frac{h_*r_n}{2}+\varepsilon_n\tau(h_n/|h_n|)}{r_n}\right)r_n^{-1},\qquad k\in S_n,\ \widetilde{k}\in\widetilde{S}_n,\ \text{ in case II;}$$

Using the properties of the functions $T,\ \widetilde{T}$ from Lemma 3.1 we deduce that for each $k\in S_n$, $\widetilde{k}\in\widetilde{S}_n$ the function $T_n(k,\widetilde{k},\cdot)$ is analytic in the ball $B(0,\rho)\subset\mathbb{C}$. Moreover, the function $T_n\in C(S_n\times\widetilde{S}_n\times\overline{B(0,\rho)})$, and the function $T_n(k,\widetilde{k},\tau)$ coincides with the Poisson kernel for the domain $D_n(\tau)$ at $\tau\in[-\rho,\rho]$. Then the definition of the operator $\mathcal{A}(\eta)=\mathcal{A}(h,\eta,\varepsilon)$ yields

$$(\mathcal{A}(\eta)f)(k)=\int\limits_{S_n}T_n(k,\widetilde{k},\eta_n)\,f(k)\,|dk|,\qquad \eta\in B^\infty_{\mathbb{R}}(\rho),\quad \widetilde{k}\in\widetilde{S}_n.$$

The operator $\mathcal{A}(\eta)$ maps $CB(S)$ into $CB(\widetilde{C})$. Fix $f\in CB(S)$ and define the new function $F_n(\tau):B(0,\rho)\subset\mathbb{C}\to L^\infty(\widetilde{S}_n)$ by the formula:

$$F_n(\tau)(\widetilde{k})=\int\limits_{S_n}T_n(k,\widetilde{k},\tau)\,f(k)\,|dk|,\quad \tau\in B(0,\rho),\quad \widetilde{k}\in\widetilde{S}_n.$$

Using the theorem about the integral with the parameter the function F_n is analytic in the ball $B(0,\rho)\subset\mathbb{C}$ with the value in $C(\widetilde{S}_n)$. Moreover, the estimates from Lemma 3.1 imply $\|F_n(\tau)\|_\infty\leqslant\delta\|f\|_\infty$, $|\tau|<\rho$. Then for each function $f\in CB(S)$, by Lemma 3.2, the function $\mathcal{A}(\eta)f$ is analytic in the ball $B^\infty_c(\rho)$ as the mapping $B^\infty_c(\rho)\to CB(\widetilde{S})$ and the following estimate is fulfilled $\|\mathcal{A}(\eta)f\|_{\infty,c}\leqslant\delta\|f\|_\infty$. Using the well-known Criterion about the analyticity of operator-valued functions (see [Kato]) and the Theorem about the analyticity of the mappings from the Banach space into another Banach space (see [Di]) we obtain the needed statement. □

For each sequence $\varepsilon=\{\varepsilon_n\}_{n\in\mathbb{Z}}\in\ell^\infty$, $\varepsilon_n\in[0,1]$ we define the multiplication operator $\hat{\varepsilon}$ in the space $\ell^p_{\mathbb{C}}, p\geqslant 1$, by the formula:

$$\hat{\varepsilon}\xi=\{\varepsilon_n\,\xi_n\}_{n\in\mathbb{Z}},\quad \xi\in\ell^p_{\mathbb{C}},$$

We fix sequences $\varepsilon=\{\varepsilon_n\}_{n\in\mathbb{Z}}$, $\varepsilon_n\in[0,1]$ and $h\in\ell^\infty_{\mathbb{R}}$. Introduce the Banach space $\mathcal{H}_+(h,\varepsilon)$ of all complex-valued continuous functions on the set $\mathcal{K}_+(h,\varepsilon)=\overline{K_+(h,\varepsilon)}\backslash U$ and harmonic in $K_+(h,\varepsilon)$ equipped with the sup-norm. Let ψ be the solution of the Dirichlet problem with parameter and with the boundary function β. The function

$$\varphi(\cdot,h,\eta,\varepsilon)\equiv\psi(\cdot,h+\hat{\varepsilon}\eta)|_{\mathcal{K}_+(h,\varepsilon)},\ \eta\in B^\infty(0,\rho)$$

is called the solution of the Dirichlet problem with the scaling parameter.

Below we need the function $\phi(\cdot,h,\eta,\varepsilon)$ which will be used in the method of Schwartz.

Condition F. *The function $\phi(\cdot,h,\eta,\varepsilon)$ is the local analytic of the Dirichlet problem with the scaling parameter if there exists a sequence $\{\phi_n(k,h_n+\varepsilon_n\eta_n)\}_{n\in\mathbb{Z}}$, such that for the fixed sequences $\varepsilon=\{\varepsilon_n\}_{n\in\mathbb{Z}}$, $\varepsilon_n\in[0,1]$ and $h\in\ell^\infty$ the following conditions are fulfilled:*
1. each function $\phi_n(k,h_n+\varepsilon_n\xi)$ is defined and continuous on the set

$$\Sigma_n=\{(k,\xi):\quad k\in D_n(\xi),\ \xi\in[-\rho,\rho]\}\bigcup((S_n\cup\widetilde{S}_n)\times\{\xi\in\mathbb{C}:\ |\xi|\le\rho\}),$$

and $\phi(k,h,\eta,\varepsilon)\equiv\phi_n(k,h_n+\varepsilon_n\eta_n)|_{\Sigma_n}$
2. for each $\tau\in[-\rho,\rho]$ the function $\phi_n(\cdot,h_n+\varepsilon_n\tau)$ is harmonic in $D_n(\tau)$ with the boundary value :

$$\lim_{k\to\zeta,k\in D_n(\tau)}\phi_n(k,h_n+\varepsilon_n\tau)=\beta(\zeta,h);\qquad \zeta\in U_n\equiv\overline{(\partial D_n(\tau)\setminus S_n)}\setminus\{u_n\}$$

3. for each $k\in S_n\cup\widetilde{S}_n$ the function $\phi_n(k,h_n+\varepsilon_n(\cdot))$ is analytic in the disk $B(0,\rho)\subset\mathbb{C}$ and

$$\sup_{n\in\mathbb{Z},|\tau|<\rho}\|\phi_n(\cdot,h_n+\varepsilon_n(\tau)\|_{L^\infty(S_n\cup\widetilde{S}_n)}<+\infty.$$

Below we will study few cases of functions ϕ which satisfy Condition F. In the first case we deal with the function $\beta(k,h)=v,\ \phi_n(k,h,\eta,\varepsilon)=v$.

For $h,\eta\in B^\infty(0,\rho)$ and for some sequence $\{\phi_n\}_{n\in\mathbb{Z}}$, which satisfy Condition F, we introduce the constants

$$c_n(h_n+\varepsilon_n\eta_n)=\sup_{k\in S_n\cup\widetilde{S}_n}|\phi_n(k,h_n+\varepsilon_n\eta_n)|,\qquad b_n(h_n)=\sup_{t\in(0,|h_n|],a=\pm}|\beta_a(u_n+ti)|,$$

$$c_\infty(h,\eta,\varepsilon)=\sup_{n\in\mathbb{Z}}c_n(h_n+\varepsilon_n\eta_n),\qquad \beta_\infty(h)=\max\{\sup_{n\in\mathbb{Z}}b_n(h_n),\ \sup_{x\in\mathbb{R}}|\beta_\pm(x)|\}.$$

In Theorem 3.5 we will describe the method, which gives the analytic continuation of the function φ in the ball $B^\infty_{\mathbb{C}}(0,\rho)$, for some sequence $\{\phi_n\}_{n\in\mathbb{Z}}$, which defines the local analytic continuation of φ.

We need some results about the uniqueness of the analytic extensions of the functions $\varphi(k,h,\eta,\varepsilon^1)$ and $\varphi(k,h,\eta,\varepsilon^2)$ with different sequences ε^1 and ε^2.

Lemma 3.4. *Let a functions $\beta_\pm$ satisfy Condition B and let $h\in\ell^\infty$. Assume that for sequences $\varepsilon^m=\{\varepsilon^m_n\}_{n\in\mathbb{Z}}$, $m=1,2$ such that $0\leqslant\varepsilon^1_n\leqslant\varepsilon^2_n\le1$, $n\in\mathbb{Z}$ each function $\varphi_m(\eta)=\varphi(h,\eta,\varepsilon^m),\eta\in B^\infty(0,\rho)$; $\phi_m:B^\infty(0,\rho)\to\mathcal{H}_+(h,\varepsilon^m)$ have the analytic extension $\phi_m(\eta),\eta\in B^\infty_{\mathbb{C}}(0,\rho)$ in the ball $B^\infty_{\mathbb{C}}(0,\rho)$. Then*

$$\varphi^1(\eta)|_{\mathcal{K}_+(h,\varepsilon^1)}=\varphi^2(\hat\tau(\eta)),\qquad \eta\in B^\infty_c(0,\rho),$$

where

$$\tau=\{\tau_n\}_{n\in\mathbb{Z}},\qquad \tau_n=\begin{cases}\frac{\varepsilon^1_n}{\varepsilon^2_n}, & \text{if } \varepsilon^2_n\neq0,\\ 0, & \text{if } \varepsilon^2_n=0\end{cases}$$

Proof. Let $K_m=\mathcal{K}_+(h,\varepsilon^m),m=1,2$ and it is clear that $K_2\subset K_1$. Introduce the function $f(\eta)=\varphi_1(\eta)|_{K_2}$, $F(\eta)=\varphi_2(\hat\tau(\eta))$, $\eta\in B^\infty_c(0,\rho)$. For $\eta\in B^\infty_c(0,\rho)$ we have

$$f(\eta)=\psi(h+\hat\varepsilon^1(\eta))|_{K_2}=\psi(h+\hat\varepsilon^2(\hat\tau(\eta)))|_{K_2}=r_2(\hat\tau(\eta))=F(\eta).$$

Then two analytic functions $f(\eta)$ and $F(\eta)$ in the ball $B_c^\infty(0,\rho)$ coincide on $B^\infty(0,\rho)$. Hence by uniqueness, they coincide on the ball $B_c^\infty(0,\rho)$. $\square$

In order to estimate the function $\psi(k,h,\eta,\varepsilon)$, $k \in K_+(h,\varepsilon)$ at fixed $\eta \in B_c^\infty(0,\rho)$ we need some estimates of the analytic extensions of the functions $\varphi(\eta,h,\varepsilon)$ for some sequence $\{\phi_n\}_{n\in\mathbb{Z}}$, which defines the local analytic continuation of φ. Moreover, we need the harmonic majorant for the function $\varphi(k,h,\eta,\varepsilon), k \in K_+(h,\varepsilon)$ at fixed $\eta \in B_c^\infty(0,\rho)$. Define the following values

$$M_n(h,\eta,\varepsilon) \equiv \sup_{k\in S_n} |\varphi(k,h,\eta,\varepsilon)|, \qquad W_n(h,\eta,\varepsilon) \equiv \sup_{k\in K_+(h,\varepsilon), |\operatorname{Re} k - u_n| < \frac{u_*}{2}} |\varphi(k,h,\eta,\varepsilon)|,$$

and introduce the Poisson integral $P(\cdot,h,\eta,\varepsilon)$ in $\mathbb{C}_+$ for the function

$$P(u,h,\eta,\varepsilon) = 2\sum_{n\in\mathbb{Z}} (M_n(h,\eta,\varepsilon) + b_n(h_n))\chi_{\Delta_n}(u), \qquad u \in \mathbb{R},$$

where

$$\Delta_n = [u_n - \alpha_n, u_n + \alpha_n], \qquad \alpha_n = |h_n| + \frac{1}{4}(\varepsilon_n u_*).$$

We prove now the basic results of this Section.

Theorem 3.5. *Assume that some sequences $\varepsilon = \{\varepsilon_n\}_{n\in\mathbb{Z}}$, $\varepsilon_n \in [0,1]$ and $h \in \ell^\infty_{\mathbb{R}}$. Let the functions $\beta_\pm$ satisfy Condition B and let some sequence $\{\phi_n\}_{n\in\mathbb{Z}}$ satisfy Condition F. Then the mapping $\varphi(\cdot,h,\cdot,\varepsilon) : B^\infty(0,\rho) \to \mathcal{H}_+(h,\varepsilon), \rho = u_* h_*^2/32$, has the analytic extensions in the ball $B_c^\infty(0,\rho)$ where the following estimate is fulfilled:*

$$M(h,\varepsilon) = \sup_{\|\eta\|_{\infty,c}<\rho} \|\varphi(h,\eta,\varepsilon)\|_\infty \leqslant 2(1-\delta)^{-1}(c_\infty(h+\varepsilon\eta) + \beta_\infty(h)), \tag{3.1}$$

Let, in addition, $\beta(k) = 0$, $k \in \mathbb{R}$, then for $\eta \in B_c^\infty(0,\rho)$ we have

$$M_n(h,\eta,\varepsilon) =\leqslant (1-\delta)^{-1}\left(\frac{8\alpha_n}{u_*} W_n(h,\eta,\varepsilon) + (1+\delta)c_n(h_n,\eta_n,\varepsilon_n) + b_n(h_n)\right), \tag{3.2}$$

$$|\varphi(k,h,\eta,\varepsilon)| \leqslant P(k,h,\eta,\varepsilon), \qquad k \in K_+(h,\varepsilon). \tag{3.3}$$

Proof. For each $\eta \in B^\infty(0,\rho)$ the function $\phi_n(\cdot,\eta_n) - \varphi(\cdot,\eta)$ is harmonic in the domain $D_n(\eta_n)$ and by Definition F, $\phi_n(k,\eta_n) - \varphi(k,\eta) = 0, k \in U_n$. Introduce the following functions

$$F(k,\eta) = \phi_n(k,\eta_n) - \varphi(k,\eta), \quad \Phi(k,\eta) = \phi_n(k,\eta_n), \quad V(k,\eta) = \varphi(k,\eta), \quad k \in S_n,$$

$$\widetilde{F}(k,\eta) = \phi_n(k,\eta_n) - \varphi(k,\eta), \quad \widetilde{\Phi}(k,\eta) = \phi_n(k,\eta_n), \quad \widetilde{V}(k,\eta) = \varphi(k,\eta). \quad k \in \widetilde{S}_n,$$

Then

$$F(\cdot,\eta),\ \Phi(\cdot,\eta),\ V(\cdot,\eta) \in CB(S); \qquad \widetilde{F}(\cdot,\eta), \widetilde{\Phi}(\cdot,\eta), \widetilde{V}(\cdot,\eta) \in CB(\widetilde{S})$$

Let $\widetilde{G}(k)$ be the solution of the Dirichlet problem in the domain $\widetilde{K}_+(h,\varepsilon)$ with the boundary value which is equal to zero for $k \in \widetilde{S}$ and equals $\beta(k,h)$ for $k \in \partial\widetilde{K}(h,\varepsilon)\backslash(\widetilde{S}\cup U)$; $G = \widetilde{G}|_S \in C(S)$. Then using the definitions of the operators $\mathcal{A}$ and $\mathcal{P}$ we obtain:

$$\mathcal{A}F = \widetilde{F}; \quad \mathcal{P}\widetilde{V} = V - G.$$

Hence,

$$\Phi - F - G = V - G = \mathcal{P}\widetilde{V} = \mathcal{P}(\widetilde{\Phi} - \widetilde{F}) = \mathcal{P}(\widetilde{\Phi} - \mathcal{A}F),$$

$$(I - \mathcal{P}\mathcal{A})F = \Phi - \mathcal{P}\widetilde{\Phi} - G.$$

Using Theorem 3.3 we obtain $\|\mathcal{P}\mathcal{A}\| \leqslant \|\mathcal{A}\| \leqslant \delta < 1$ and then there exists the inverse operator

$$\mathcal{R} = (I - \mathcal{P}\mathcal{A})^{-1}, \quad \|\mathcal{R}\| \leqslant (1-\delta)^{-1}.$$

Further

$$V = \Phi - F = \Phi - \mathcal{R}(\Phi - \mathcal{P}\widetilde{\Phi} - G) = (I - \mathcal{R})\Phi + \mathcal{R}(\mathcal{P}\widetilde{\Phi} + G) = \mathcal{R}(G + \mathcal{P}(\widetilde{\Phi} - \mathcal{A}\Phi)).$$

which yields

$$V(\eta) = \mathcal{R}(\eta)(G + \mathcal{P}(\widetilde{\Phi}(\eta) - \mathcal{A}(\eta)\Phi(\eta))). \tag{3.4}$$

It is clear that the mappings $\lambda \to \phi_n(\cdot,\lambda)|_{S_n}$ and $\lambda \to \phi_n(\cdot,\lambda)|_{\widetilde{S}_n}$ are analytic as the functions $B(0,\rho) \subset \mathbb{C} \to C(S_n)$ and $B(0,\rho) \to C(\widetilde{S}_n)$ respectively. Then using the properties of ϕ_n and Lemma 3.2 we deduce that the mappings $\Phi(\eta)$, $\widetilde{\Phi}(\eta)$ have the analytic continuations in the ball $B_c^\infty(0,\rho)$. Hence the last results, the analyticity of $\mathcal{A}$ (see Theorem 3.3), and formula (3.4) give the analytic continuations of V in the ball $B_c^\infty(0,\rho)$.

Note that for real $\eta \in B^\infty(0,\rho)$ we have

$$V(\eta)(k) = \beta(k,h), \qquad k \in \Gamma \cap S. \tag{3.5}$$

By the uniqueness theorem for analytic functions, identity (3.5) is fulfilled for $\eta \in B_C^\infty(\rho)$. Define the set

$$X_0 = \{f \in CB(S) : f(k) = \beta(k,h), k \in \Gamma(h) \cap S\}$$

and the mapping J, which takes the function $f \in X_0$ to u where u is the solution of the Dirichlet problem in the domain $K_+(h,\varepsilon)$ with the boundary condition: $u = f$ on S and $u = \beta$ on the rest of the boundary of the domain $K_+(h,\varepsilon)$. It is clear that J is analytic mapping from X_0 into $\mathcal{H}_+(h,\varepsilon)$, since J does not depend on $\eta \in \ell^\infty_{\mathbb{R}}$.

Then the above result yields

$$\varphi(\cdot,\eta) = \psi\Big(\cdot, h + \hat{\varepsilon}\eta\Big)|_{\mathcal{K}_+(h,\varepsilon)} = J(V(\eta)), \quad \eta \in \ell^\infty_{\mathbb{R}}. \tag{3.6}$$

Using (3.4), (3.6) we obtain the needed analytic continuation and the estimate:

$$\|\varphi(\eta)\|_{\infty,c} \leqslant \|V(\eta)\|_{CB(S)} + \beta_\infty(h),$$

$$\|V(\eta)\|_{CB(S)} \leqslant (1-\delta)^{-1}(\|G\|_{CB(S)} + \delta\|\Phi\|_{CB(S)} + \|\widetilde{\Phi}\|_{CB(\widetilde{S})}) \leq (1-\delta)^{-1}((1+\delta)c_\infty + \beta_\infty(h)).$$

which yields (3.1).

Assume $\beta_\pm(k) = 0, k \in \mathbb{R}$, fix n and consider the two following cases. First, let $B(\widetilde{u}_n, r_n) \not\subset B(u_n, u_*/4)$. It is possible for "large" $|h_n|$ and we have $|h_n| \geqslant 2r_n$, $|h_n| + r_n \geqslant u_*/4$. Then $|h_n| \geqslant u_*/6$ and we obtain

$$M_n \leqslant W_n \leqslant \frac{6W_n|h_n|}{u_*}.$$

Second, let $B(\widetilde{u}_n, r_n) \subset B(u_n, u_*/4)$. Put $G_1 = B(u_n, u_*/2) \cap \widetilde{K}_+(h, \varepsilon)$ and note that Eq. (3.4) implies

$$V(\eta) = G + \mathcal{P}(\mathcal{A}(\eta)V(\eta) + \widetilde{\Phi}(\eta) - \mathcal{A}(\eta)\Phi(\eta)).$$

Then by identity (3.6), the function $\varphi(\cdot, \eta)$ is the solution of the Dirichlet problem in the domain $\widetilde{K}_+(h, \varepsilon)$ with the boundary function which equals $\mathcal{A}(\eta)V(\eta) + \widetilde{\Phi}(\eta) - \mathcal{A}(\eta)\Phi(\eta)$ on $\widetilde{S}$ and it equals β on $\partial K(h, \varepsilon)\backslash\widetilde{S}$. Let σ_2 be the part of the boundary of the domain G_1, situated interior to $\mathbb{C}_+ \cap B(u_n, u_*/2)$, $\sigma_1 = \partial B(u_n, u_*/2) \cap \mathbb{C}_+$. We rewrite the function φ in the domain G_1 in the form: $\varphi = \varphi^{(1)} + \varphi^{(2)}$. Here $\varphi^{(2)}$ is the solution of the Dirichlet problem in the domain G_1 with the boundary value φ on σ_2 and one equals zero on the rest of the boundary. First we consider the function $\varphi^{(1)}$. We have the estimate

$$|\varphi^{(1)}(\xi)| \leqslant \vartheta(\xi), \qquad \xi \in S_n, \tag{3.7}$$

where

$$\vartheta(k) = \begin{cases} W_n, & \text{if } \ k \in \sigma_1, \\ 0, & \text{if } \ k \in [u_n - u_*/2, u_n + u_*/2] \end{cases}$$

and ϑ is the solution of the Dirichlet problem in the domain $B(u_n, u_*/2) \cap \mathbb{C}_+$ with the same boundary function. Estimate (3.7) follows from the maximum principle for subharmonic functions. It is clear that the function ϑ has the harmonic continuation in the disk $B(u_n, u_*/2)$ by the symmetry: $\vartheta(\bar{k}) = -\vartheta(k)$. Applying the Caratheodory inequality (see [Ti]) to this function, we obtain

$$|\vartheta(k)| \leqslant \frac{2|k - u_n|}{(u_*/2) - |k - u_n|} W_n, \quad k \in B(u_n, u_*/2),$$

which for $|k - u_n| \leqslant u_*/4$ yields

$$|\vartheta(k)| \leqslant \frac{8|k - u_n|}{u_*} W_n.$$

Moreover, $S_n \subset B(u_n, u_*/4)$ and for $k \in S_n$ we have $|k - u_n| \leqslant \alpha_n = |h_n| + (\varepsilon_n u_*/4)$. Then

$$|\varphi^{(1)}(k)| \leqslant \frac{8\alpha_n}{u_*} W_n, \qquad k \in S_n. \tag{3.8}$$

Second, we consider now the function $\varphi^{(2)}$. By the modulus maximum principle,

$$\|\varphi^{(2)}\|_{L^\infty(S_n)} \leqslant \|\varphi\|_{L^\infty(\widetilde{S}_n)} + b_n.$$

Then

$$\|\varphi\|_{L^\infty(\widetilde{S}_n)} \leqslant \|\mathcal{A}V\|_{L^\infty(\widetilde{S}_n)} + \|\phi_n(\cdot, \eta_n)\|_{L^\infty(\widetilde{S}_n)} + \|\mathcal{A}\Phi\|_{L^\infty(\widetilde{S}_n)}. \tag{3.9}$$

The definition of $\mathcal{A}$ and Theorem 3.1 yield

$$\|\mathcal{A}V\|_{L^\infty(\widetilde{S}_n)} \leqslant \delta M_n; \qquad \|\mathcal{A}\Phi\|_{L^\infty(\widetilde{S}_n)} \leqslant \delta \|\phi_n(\cdot, \eta_n)\|_{L^\infty(S_n)}. \tag{3.10}$$

Using Est. (3.9-10) we have

$$\|\varphi^{(2)}\|_{L^\infty(S_n)} \leqslant \delta M_n + (1+\delta)c_n + b_n. \tag{3.11}$$

and estimates (3.8), (3.11) imply (3.2), indeed

$$(1-\delta)M_n \leqslant \frac{8\alpha_n}{u_*} W_n + (1+\delta)c_n + b_n.$$

We show (3.3). We have $B(\widetilde{u}_n, r_n) \subset B(u_n, \alpha_n)$. The total Euclidean angle under which $[u_n - \alpha_n, u_n + \alpha_n]$ is seen from $\xi \in \partial B(u_n, \alpha_n)$ equals $\frac{\pi}{2}$. Then

$$\frac{1}{\pi} \int_{-\alpha_n}^{\alpha_n} \frac{v\,dt}{|t + u_n - k|^2} \geqslant \frac{1}{2}, \qquad k \in S_n \cup [u_n,\, u_n + i|h_n|].$$

and the maximum principle yields (3.3). □

Let the function β define the Dirichlet problem with parameter and $\beta(u,h) = 0,\ u \in \mathbb{R}$. For a fixed number $n \in \mathbb{Z}$ we define the new function $\beta^{(n)}(k,h)$ which is the same as $\beta(k,h)$ on the ray $\{u_n + it,\ t > 0\}$ and vanish on the remaining part of the Γ. Let $\psi_n(k,h),\ k \in K_+(h),\ h \in \ell^\infty_{\mathbb{R}}$ be corresponding solution of the Dirichlet problem with the boundary function $\beta^{(n)}(k,h)$. Moreover, for the sequences $\varepsilon = \{\varepsilon_m\}_{m\in\mathbb{Z}},\ \varepsilon_m \in [0,1]$ and $\eta, h \in \ell^\infty_{\mathbb{R}}$ we define the solution of the Dirichlet problem with the scaling parameter $\varphi_n(\cdot, h, \cdot, \varepsilon) : K_+(h,\varepsilon) \times B^\infty(0,\rho) \to \mathcal{H}_+(h,\varepsilon)$, by the formula:

$$\varphi_n(\cdot, h, \eta, \varepsilon) = \psi_n(\cdot, h + \hat{\varepsilon}\eta)|_{\mathcal{K}_+(h,\varepsilon)}, \quad \eta \in B^\infty(\rho),$$

Using Theorem 3.5 we get an analytic extension of the function $\varphi_n(\cdot, h, \cdot, \varepsilon) : B^\infty(\rho) \to \mathcal{H}_+(h,\varepsilon)$ in the ball $B^\infty_c(\rho)$. We prove now the estimates which are basic in the next Section.

Corollary 3.6. *Assume that some sequences $\varepsilon = \{\varepsilon_n\}_{n\in\mathbb{Z}},\ \ \varepsilon_n \in [0,1]$ and $h \in \ell^\infty_{\mathbb{R}}$. Let the functions $\beta_\pm$ satisfy Condition B and let some sequence $\{\phi_n\}_{n\in\mathbb{Z}}$ satisfy Condition F. Assume that for some $n \in \mathbb{Z}$ we have*

$$\beta_\pm(k) = 0, \quad k \in (\Gamma \cup \mathbb{R}) \setminus \{k = u_n + it,\ t > 0\}, \quad \phi_m \equiv 0,\ \mu \neq n, m \in \mathbb{Z}.$$

Then for $\eta \in B^\infty_c(0,\rho)$ the following estimates are fulfilled:

$$N_n(h,\eta,\varepsilon) = \sup_{k\in K_+(h,\varepsilon)} |\varphi_n(k,h,\eta,\varepsilon)| \leqslant c_n(h_n + \varepsilon_n\eta_n) + b_n(h_n), \tag{3.12}$$

$$N^0_n(h,\eta,\varepsilon) = \sup_{k\in K_+(h,\varepsilon), |\operatorname{Re} k - u_n| > \frac{u_*}{2}} |\varphi_n(k,h,\eta,\varepsilon)| \leqslant \frac{18\sqrt{6}}{u_*(1-\delta)} \alpha_n (c_n(h_n + \varepsilon_n\eta_n) + b_n(h_n)). \tag{3.13}$$

Proof. To get the estimate (3.12) we can apply the maximum modulus principle to the domain $\widetilde{K}_+(h,\varepsilon) \setminus \overline{D_n(\eta_n)}$ and to the function $\varphi_n(\cdot, h, \eta, \varepsilon)$. If (3.12) is not valid then there exists some point $k_0 \in S_m,\ m \neq n$ such that $|\varphi_n(k_0, h, \eta, \varepsilon)| > \delta N_n(h, \eta, \varepsilon)$. But inequality (3.9) yields

$$|\varphi_n(k_0, h, \eta, \varepsilon)| \leqslant \|\varphi_n(h, \eta, \varepsilon)\|_{L^\infty(\widetilde{S}_m)} \leqslant \|\varphi_n(h, \eta, \varepsilon)\|_{L^\infty(S_m)} \leqslant \delta N_n(h, \eta, \varepsilon).$$

So, we get a contradiction. We prove (3.13). Assume that $|h_n| \geq u_*/6$, then we have

$$N_n^0(h, \eta, \varepsilon) \leqslant N_n(h, \eta, \varepsilon) \leqslant (c_n(h, \eta, \varepsilon) + b_n(h_n)) \leqslant \frac{6}{u_*}(c_n(h, \eta, \varepsilon) + b_n(h_n))|h_n|.$$

Consider the second case $|h_n| < u_*/6$. Let g be a solution of the Dirichlet problem in the domain

$$G_n = \Big\{k \in K_+(h, \varepsilon) : |\operatorname{Re} k - u_n| \leqslant \frac{u_*}{2}\Big\} \cup \Big\{k \in \mathbb{C}_+ : |\operatorname{Re} k - u_n| \geqslant \frac{u_*}{2}\Big\}.$$

with the boundary function which vanishes on $\mathbb{R}$ and equals $\varphi_n(\cdot, \eta)$ on the rest of the boundary. Using (3.3) we have the estimate

$$|g(k)| \leqslant P_n(k) = P_n(k, h, \eta, \varepsilon), \quad k \in G_n,$$

where $P_n(k)$ is the Poisson integral in $\mathbb{C}_+$ for the function $P_n(u) = 2(c_n + b_n)\chi_{\Delta_n}(u),\ \ \Delta_n = [u_n - \alpha_n, u_n + \alpha_n],\ u \in \mathbb{R}$. Note that we have an inequality $\alpha_n \leqslant \frac{5u_*}{12}$. Since the value of the Poisson integral in $\mathbb{C}_+$ for the function $\chi_{\Delta_n}(u),\ u \in \mathbb{R}$ at $k \in \mathbb{C}_+$ is the same as the total Euclidean angle under which Δ_n is seen from k, we have

$$|g(k)| \leqslant \frac{6\sqrt{6}}{u_*}\alpha_n(c_n + b_n),\ k \in G_n.$$

Now we consider the function $f = \varphi_n - g$ in the domain $\widetilde{K}_+(h, \varepsilon) \setminus \overline{D}_n(\eta_n)$. Assume that $D = \sup\limits_{k \in S} |f(k)|$. If we apply the maximum modulus principle to the domain $\widetilde{K}_+(h, \varepsilon) \setminus \overline{D}_n(\eta_n)$ and to the function f we have by the inequality (3.9)

$$D \leqslant \delta D + \frac{12\sqrt{6}\alpha_n(c_n + b_n)}{u_*},$$

$$N_n^0 \leqslant D + \sup_{k \in G} |g(k)| \leqslant (1 - \delta)^{-1} \frac{18\sqrt{6}\alpha_n(c_n + b_n)}{u_*}$$

and we get (3.13). □

4 Analyticity of mappings

Let the functions $\beta_\pm$ define the Dirichlet problem with parameter and $\beta_+(u) = \beta_-(u) = 0,\ u \in \mathbb{R}$. In connection with this problem we consider some nonlinear map $F : \ell^\infty_\mathbb{R} \mapsto \ell^\infty_\mathbb{R}$ wich is defined in the following way. For a number $n \in \mathbb{Z}$ consider the functions $\beta^n_\pm$ wich are the same as $\beta_\pm$ on the ray $\{u_n + it,\ t > 0\}$ and vanish on the remaining part of the Γ. Let $\psi_n(k,h),\ k \in K_+(h),\ h \in \ell^\infty_{\mathbb{R}}$ be corresponding solution of the Dirichlet problem with the boundary function $\beta^n(k,h)$. Then define $F(h) = \{f_n(h)\}_{n\in\mathbb{Z}}$

$$f_n(h) = \int_{\mathbb{R}} \psi_n(u_n + iv(t,h))dt = \int_{\mathbb{R}} \beta^n(u_n + iv(t,h))dt,\ n \in \mathbb{Z},$$

where $v(k,h) = \operatorname{Im} z(k,h)$.

For $h \in \ell^\infty_{\mathbb{R}}$ the quantity $f_n(h)$ determines the asymptotic behavior of the solution of the Dirichlet problem $\psi_n(k,h)$ at infinity. In fact, let $g(z)$ is the Poisson integral in $\mathbb{C}_+$ for the function $\psi_n(u_n + iv(x,h)),\ x \in \mathbb{R}$. Then we have

$$\lim_{y\to\infty} g(iy) = \int_{\mathbb{R}} \psi_n(u_n + iv(t,h))dt.$$

But according to the formula (2.42) and equality $\psi_n(k,h) = g(z(k,h))$ it implies

$$\lim_{y\to\infty} \psi_n(iy,h) = \lim_{y\to\infty} g(z(iy,h)) = f_n(h).$$

In this section we study properties of the map F and consider some important special cases of it. Define the following constants

$$C_A = \frac{48\cdot 6^{1/6}}{(1-\delta)u_* h_*},\quad C_H = \frac{32}{h_*^2}.$$

Theorem 4.1. *Let $h \in \ell^2_{\mathbb{R}}$ be fixed sequence. Let the functions $\beta_\pm,\ \beta_+(u) = \beta_-(u) = 0,\ u \in \mathbb{R}$ define the Dirichlet problem with parameter and let some sequence $\{\phi_n\}_{n\in\mathbb{Z}}$ satisfy Condition F. Define the functions $g_n(\eta) = f_n(h+\eta),\ \eta \in B^2(\rho),\ n \in \mathbb{Z}$. Then all these functions have an analytic extension in the ball $B^2_c(\rho)$ where the following estimates are fulfilled:*

$$|g_n(\eta)| \leqslant C_H(c_n + 2b_n)(|h_n| + |\eta_n|)\Big(1 + C_A^2(\|h\|^2 + \|\eta\|^2)\Big). \tag{4.1}$$

Proof. Fix the number n. Recall that for sequences $\varepsilon = \{\varepsilon_m\}_{m\in\mathbb{Z}},\ \varepsilon_m \in [0,1]$ and $\eta, h \in \ell^\infty_{\mathbb{R}}$ we define the solution of the Dirichlet problem with the scaling parameter $\varphi_n(\cdot,h,\cdot,\varepsilon) : K_+(h,\varepsilon) \times B^\infty(0,\rho) \to \mathcal{H}_+(h,\varepsilon)$, by the formula:

$$\varphi_n(\cdot,h,\eta,\varepsilon) = \psi_n(\cdot,h+\hat{\varepsilon}\eta)|_{\mathcal{K}_+(h,\varepsilon)},\quad \eta \in B^\infty(\rho),$$

Using Theorem 3.5 we get an analytic extension of the function $\varphi_n(\cdot,h,\cdot,\varepsilon) : B^\infty(\rho) \to \mathcal{H}_+(h,\varepsilon)$ in the ball $B^\infty_c(\rho)$.

We consider the mapping $\varphi_n(\cdot,h,\eta,\widetilde{\varepsilon}), \eta \in B^2_c(\rho)$, where the sequence $\widetilde{\varepsilon}_m = 1,\ \ m \in \mathbb{Z}$. Also we introduce the new sequence $\varepsilon, \varepsilon_m = \rho^{-1}|\eta_m|,\ m \in \mathbb{Z}$. Using Lemma 3.4, we have

$$\varphi_n(k,h,\eta,\widehat{\varepsilon}) = \varphi_n(k,h,\widetilde{\eta},\varepsilon),\ k \in K_+(h,\varepsilon),\quad \widetilde{\eta}_m = \begin{cases} \frac{\rho}{|\eta_m|}\eta_m, & \text{if}\quad \eta_m \neq 0,\\ 0, & \text{if}\quad \eta_m = 0.\end{cases}$$

Define the value $M_{n,m} = \|\varphi_n(\cdot, h, \eta, \varepsilon)\|_{L^\infty(S_n)}$, $n, m \in \mathbb{Z}$. Using Theorem 3.5 and Corollary 3.6 for the function $\varphi_n(\cdot, h, \tau, \varepsilon)$ in the domain $K_+(h, \varepsilon)$, we obtain

$$M_{n,m} \leqslant \frac{8\alpha_m}{(1-\delta)u_*}\Big(\frac{18\sqrt{6}}{(1-\delta)u_*}\alpha_n(c_n + b_n)\Big) \leqslant C_1\alpha_m\alpha_n(c_n + b_n), \quad C_1 = \frac{144\sqrt{6}}{(1-\delta)^2 u_*^2}.$$

Due to Theorem 3.5 and Corollary 3.6, for the function $\varphi_n(\cdot, h, \tau, \varepsilon)$ in the domain $K_+(h, \varepsilon)$, there exists the harmonic majorant $P_n = P_n(k, h, \widetilde{\eta}, \varepsilon)$ which is the Poisson integral in $\mathbb{C}_+$ for the function $P_n(u, h, \widetilde{\eta}, \varepsilon)$, $u \in \mathbb{R}$, such that

$$\int_{\mathbb{R}} P_n(u)du = 4(M_{n,n} + b_n)\alpha_n + 4\sum_{m\neq n} M_{n,m}\alpha_m \leqslant 4\alpha_n(c_n + 2b_n)(1 + C_1\sum_{m\neq n}\alpha_m^2),$$

and the simple estimate $\alpha_n = |h_n| + (1/4)\varepsilon_n u_* \leqslant (C_H/4)(|h_n| + |\eta_n|)$, $C_H = 32/h_*^2$, implies

$$\int_{\mathbb{R}} P_n(u)du \leqslant d(c_n + 2b_n)(|h_n| + |\eta_n|)\Big(1 + C^2(\|h\|^2 + \|\eta\|^2)\Big), \quad C_A^2 = \frac{1}{2}C_1 C_H, \qquad (4.2)$$

which yields estimate (4.1). The well known property of the Poisson integral (see [St]) yields

$$\int P_n(u)du = \int P_n(u + iv)du, \qquad \text{for any} \quad v > 0.$$

Fix some $v_0 > \|h\|_\infty + (u_*/4)$. Then for any $\eta \in B_c^2(\rho)$ we get

$$\int |\varphi_n(u + iv_0, h, \eta, \widetilde{\varepsilon})|du \leq \int P_n(u + iv_0)\, du = \int P_n(u)\, du.$$

According Theorem 2.9 we can continue now the functional g_n from $B_R^2(h, \rho)$ on the ball $B_c^2(h, \rho)$ by the formula :

$$g_n(\eta) = f_n(h + \eta) = \int_{\mathbb{R}} \varphi_n(u + iv_0, h, \eta, \widetilde{\varepsilon})du.$$

We show that this extension is an analytic function in the ball $B_c^2(\rho)$. By (4.2), this function is bounded on $B_c^2(\rho)$, then it is enough to check that for any fixed $\eta, \vartheta \in B_c^2(\rho)$ such that $\|vt\|_2 < \rho - \|\eta\|_2$ the function

$$F(t) = f_n(h + \eta + t\vartheta) = g_n(\eta + t\vartheta)$$

is analytic in the ball $B(0, 1)$. (see Theorem 2.9). Define the sequences

$$\varepsilon'_m = \frac{|\eta_m| + |\vartheta_m|}{\rho} \leq 1, \qquad \xi_m(t) = \begin{cases} \frac{\eta_m + t\vartheta_m}{\varepsilon'_m}, & \text{if } \varepsilon'_m \neq 0, \\ 0, & \text{if } \varepsilon'_m = 0 \end{cases}$$

and note that $\|\xi(t)\|_\infty < \rho$ for any $|t| < 1$. Using Lemma 3.4, we get

$$\varphi_n(k, h, \xi(t), \varepsilon') = \varphi_n(k, h + \eta + t\vartheta, \widetilde{\varepsilon}), \quad |t| < 1, \ k \in K_+(h, \varepsilon').$$

Hence

$$F(t) = \int \varphi_n(u + iv_0, h, \xi(t), \varepsilon')dx, \quad |t| < 1.$$

Let now $= P_1(k, h, \xi(t), \varepsilon')$ be the corresponding Poisson integral. Then by Theorem 3.5,

$$|\varphi_n(u + iv_0, h, \eta + t\vartheta, \widetilde{\varepsilon})| = |\varphi_n(u + iv_0, h, \xi(t), \varepsilon')| \leqslant P_1(u + iv_0, h, \xi(t), \varepsilon'), \quad u \in \mathbb{R},$$

$$P_1(u + iv_0, h, \xi(t), \varepsilon') \leqslant Q(u + iv_0), u \in \mathbb{R}$$

where $t \in B(0, 1)$, Q is the Poisson integral in $\mathbb{C}_+$ for the following function

$$Q(u) = 4(\sup_{|\tau|<\rho} \|\varphi_n(\cdot, \tau)\|_{L^\infty(S_n \cup \widetilde{S}_n)} + b_n(h))(\chi_{\Delta_n}(u) + \frac{\alpha_n A}{2} \sum_{m \neq n} \alpha_m \chi_{\Delta_m}(u)),$$

where the interval $\Delta_n = [u_n - \alpha_n, u_n + \alpha_n]$ and $\alpha_m = |h_m| + \frac{1}{4}(\varepsilon'_m u_*)$, $m \in \mathbb{Z}$. We have also

$$\int Q(u + iv_0)\, du = \int Q(u)\, du < +\infty.$$

Hence, the sequence of the functions

$$F_m(t) = \int_{-m}^{m} \varphi(u + iv_0, h, \xi(t), \varepsilon')du, \quad |t| < s,$$

converges to the function $F(t)$, as $n \to \infty$, uniformly on the ball $B(0, 1)$. But each function $F_m(t)$ is analytic in the ball $B(0, 1)$ and then the Weierstrass Theorem yields the needed analyticity.

So we have proved the analyticity of the function f_n in the ball $B_c^2(\rho)$. □

We apply Theorem 4.1 to the case of the function $A_n(h)$.

Corollary 4.2. *Let $\omega = \{\omega_n\}_{n\in\mathbb{Z}}; \omega_n \geq 1, n \in \mathbb{Z}$ be some weight. Then each functional $A_n : \ell^p_\omega \to \mathbb{R}_+, 1 \leq p \leq 2, n \in \mathbb{Z}$ has analytic extension on the domain $\mathcal{J}^p_\omega(\rho)$ where the following estimate is fulfilled:*

$$|A_n(h)| \leqslant \frac{1}{2} C_H^2 |h_n|^2 \Big(1 + C_A^2 \|h\|^2\Big), \qquad h \in \mathcal{J}^p_\omega(\rho). \tag{4.3}$$

The functional $Q_0 : \ell^p_\omega \to \mathbb{R}_+$ has also analytic extension in the layer $\mathcal{J}^p_\omega(\rho)$ by the rule

$$Q_0(h) = \sum_{n\in\mathbb{Z}} A_n(h), \ \ h \in \mathcal{J}^p_\omega(\rho). \tag{4.4}$$

Moreover, for any $m \in \mathbb{Z}$

$$\partial_m Q_0(h) = \sum_{n\in\mathbb{Z}} \partial_m A_n(h), \ \ h \in \mathcal{J}^p_\omega(\rho). \tag{4.5}$$

Proof. Now we apply Theorem 3.5 for the case $\beta_\pm(k,h)=v, k\in\Gamma(U)$ and $\phi_n(k,h+\varepsilon\eta)=v, k=u+iv\in D_n(\eta_n)$. We have the simple estimates for each $n\in\mathbb{Z}$ and any $\eta_n\in B(0,\rho)$

$$c_n(|h_n|,|\eta_n|/\rho)=\|\phi_n(\cdot,h+\frac{|\eta_n|\eta}{\rho})\|_{L^\infty(S_n\cup\widetilde{S}_n)}\le|h_n|+\frac{C_H}{4}|\eta_n|,\qquad\sup_{t\in(0,|h_n|]}|\beta(u_n+it)|=|h_n|,$$

which yields $c_n+2b_n\leqslant(C_H/4)(|h_n|+|\eta_n|)$. Then using (4.1) we obtain the estimate

$$|A_n(h+\eta)|\leqslant\frac{C_H^2}{4}(|h_n|+|\eta_n|)^2\Big(1+C_A^2(\|h\|^2+\|\eta\|^2)\Big).\tag{4.6}$$

We show estimate (4.3). Let $\mu\in\mathcal{J}^p_\omega(\rho),\ h=\operatorname{Re}\mu,\mu=h+\eta$. Then $\eta\in B_c(h,\rho)$ and (4.6) implies (4.3).

At last, note that the analytic extensions of A_n on the different balls $B_c(h,\rho)$, $B_c(h^0,\rho)$ for the different $h,h^0\in\ell^p_\omega$ coincide on $B_c(h,\rho)\cap B_c(h^0,\rho)$, since the analytic extensions coincide on the set $B(h,\rho)\cap B(h^0,\rho)$. At last, according (4.3) the series (4.4) converges uniformly on the balls $B(h,\rho),h\in\ell^p_\omega$. But each function A_n is analytic in the ball $B(h,\rho)$ and then the Weierstrass Theorem yields the needed analyticity and equality (4.5). $\square$

We apply Theorem 4.1 to the case of the function $l_n(h)$.

Corollary 4.3. *Let $\omega=\{\omega_n\}_{n\in\mathbb{Z}};\omega_n\ge1,n\in\mathbb{Z}$ be some weight. Then each functional $l_n:\ell^p_\omega\to\mathbb{R},1\le p\le2,n\in\mathbb{Z}$ has analytic extension on the domain $\mathcal{J}^p_\omega(\rho)$ where the following estimate is fulfilled:*

$$|l_n(h)|\leqslant C_L|h_n|(1+C_A^2\|h\|^2),\quad h\in\mathcal{J}^p_\omega(\rho),\qquad C_L=C_H3\sqrt{2}.\tag{4.7}$$

The functional $E:\ell^1_\omega\to\mathbb{R}$ has also analytic extension in the layer $\mathcal{J}^1_\omega(\rho)$ by the rule

$$E(h)=\sum_{n\in\mathbb{Z}}l_n(h),\ h\in\mathcal{J}^1_\omega(\rho).\tag{4.8}$$

Moreover, for any $m\in\mathbb{Z}$

$$\partial_mE(h)=\sum_{n\in\mathbb{Z}}\partial_ml_n(h),\ h\in\mathcal{J}^1_\omega(\rho).\tag{4.9}$$

Proof. Define a function F by the formula:

$$F(k)=\frac{1}{\pi}\log(\frac{\sqrt{k^2+1}-1}{\sqrt{k^2+1}+1}),\ k\in\mathbb{C}_+\setminus[0,i],$$

where $\sqrt{-t^2}=it,t>0$, and $t\in\mathbb{C}\setminus[0,\infty)$, and we take main value of logarithm on $\mathbb{C}_+$.

It is clear that the function $\operatorname{Im}F$ is the harmonic measure of a double-sided segment $[0,i]$ relative to $\mathbb{C}_+\setminus[0,i]$. As the function F is real on $\mathbb{R}\setminus\{0\}$, it can be continued by Schwartz principle of reflection to the domain $\mathbb{C}\setminus[-i,i]$. Moreover, $F(k)\to0$ as $k\to\infty$; $F(k)=\overline{F(\bar k)}=-F(-k),\ k\in\mathbb{C}\setminus[-i,i]$ and

$$F(k)=-\frac{2}{\pi}\arcsin t,\quad\arcsin t=\int_0^t\frac{dz}{\sqrt{1-z^2}},\quad t=\frac{1}{ik},\quad|k|>1,$$

$$F(k)=\sum_{n=0}^{\infty}\frac{F_n}{k^{2n+1}},\qquad |k|>1,$$

where $F_n\in\mathbb{R},\ n\geq 0$. Hence, using the simple estimate

$$|\arcsin t|\leqslant\sum_{n\geqslant 0}|F_n||k|^{2n+1}\leqslant\frac{|t|}{1-|t|^2},\qquad |t|<1,$$

we obtain

$$|F(k)|\leqslant w(|k|)\equiv\frac{2}{\pi}\frac{|k|}{|k|^2-1},\quad |k|>1.$$

Now we apply Theorem 3.5 for the case

$$\beta_-(k)=-\beta_+(k),\qquad \beta_+(k)=\begin{cases}0, & \text{if } k\in\mathbb{R}\\ 1, & \text{if } k=u_n+iq,\ q>0\end{cases}$$

Introduce also the functions $\phi_n(k,\xi)\equiv(\operatorname{sign}h_n)$ for the case II; in the case I we define the function ϕ_n by the rule:

$$\phi_n(k,\xi)=\operatorname{Im}F\Big(\frac{k-u_n}{h_n+\xi\varepsilon_n}\Big),\qquad k\in\mathcal{D}_n(\xi),\quad -\rho\leqslant\xi\leq\rho$$

$$\phi_n(k,\xi)=-\sum_{n=0}^{\infty}F_n\Big(\frac{h_n+\xi\varepsilon_n}{|k-u_n|}\Big)^{2n+1}\sin n\vartheta,\ k\in S_n\cup\widetilde{S}_n,\ \xi\in B(0,\rho),$$

where $\vartheta\in[0,2\pi)$ is the argument of the complex number $k-u_n$. It is simple to check the hypothesis of a Theorem 4.1 relating to ϕ_n. We have by the maximum modulus principle for harmonic functions

$$\max_{k\in S_n}|\phi_n(k,\xi)|\leq\max_{k\in\widetilde{S}_n}|\phi_n(k,\xi)|\leq w(\frac{8}{9h_*})\leqslant 1,\quad \xi\in B(0,\rho).$$

Then we have the simple estimates for each $n\in\mathbb{Z}$

$$c_n(|h_n|,|\eta_n|/\rho)=\|\phi_n(\cdot,h+\frac{|\eta_n|\eta}{\rho})\|_{L^\infty(S_n\cup\widetilde{S}_n)}\leq 2,\qquad \sup_{t\in(0,|h_n|]}|\beta(u_n+it)|=1,$$

and by (4.1) we obtain the estimate

$$|l_n(h+\eta)|\leqslant C_H 3(|h_n|+|\eta_n|)\Big(1+C_A^2(\|h\|^2+\|\eta\|^2)\Big).\tag{4.10}$$

We show estimate (4.7). Let $\mu\in\mathcal{J}^p_\omega(\rho),\ h=\operatorname{Re}\mu,\mu=h+\eta$. Then $\eta\in B_c(h,\rho)$ and (4.10) imply inequalities (4.7).

At last, note that the analytic extensions of l_n on the different balls $B_c(h,\rho)$, $B_c(h^0,\rho)$ for the different $h,h^0\in\ell^p_\omega$ coincide on $B_c(h,\rho)\cap B_c(h^0,\rho)$, since the analytic extensions coincide on the set $B(h,\rho)\cap B(h^0,\rho)$. At last, according (4.7) the series (4.8) converges uniformly on the balls $B(h,\rho), h \in \ell^1_\omega$. But each function l_n is analytic in the ball $B(h,\rho)$ and then the Weierstrass Theorem yields the needed analyticity and equality (4.9). $\square$

We show that the mappings $l : \ell^p_\omega \to \ell^p_\omega$, and $A : \ell^2_\omega \to \ell^1_\omega$ have the analytic extensions on the domain $\mathcal{J}^p_\omega(\rho)$. Moreover, we consider the mapping $J : \ell^p_\omega \to \ell^p_\omega$.

Theorem 4.4. *Let $\omega = \{\omega_n\}_{n\in\mathbb{Z}}; \omega_n \geq 1, n \in \mathbb{Z}$ be some weight and $1 \leq p \leq 2$. Then*
i) the mappings $l : \ell^p_\omega \to \ell^p_\omega$, and $A : \ell^2_\omega \to \ell^1_\omega$ have the analytic extensions on the domain $\mathcal{J}^p_\omega(\rho)$ and on the domain $\mathcal{J}^2_\omega(\rho)$ respectively, where the following estimate is fulfilled:

$$\|l(h)\|_{p,\omega} \leqslant C_L \|h\|_{p,\omega}(1 + C_A^2\|h\|^2), \quad h \in \mathcal{J}^p_\omega(\rho), \tag{4.11}$$

$$\|A(h)\|_{p,\omega} \leqslant \frac{1}{2} C_H^2 (1 + C_A^2 \|h\|^2)\|h\|_{2,\omega}, \qquad h \in \mathcal{J}^2_\omega(\rho), \tag{4.12}$$

ii) the mapping $J : \ell^p_\omega \to \ell^p_\omega$ has analytic extension on the some domain containing l^p_ω.

Proof. i) The estimate (4.11) is a direct consequence of (4.7). According the estimate (4.7) for each sequence $\eta = \{\eta_n\}_{n\in\mathbb{Z}} \in \ell^q_\omega$ where $2 \leqslant q \leqslant \infty, \frac{1}{p} + \frac{1}{q} = 1$ the series $\sum_{n\in\mathbb{Z}} \omega_n l_n(h)\eta_n$ converges uniformly on bounded subsets of the layer $\mathcal{J}^p_\omega(\rho)$. So, using the Criterion about analyticity of operator-valued functions we obtain the statement about L. The results about J can be proved as above.

ii) Let $h \in \ell^\infty, \|h\|_\infty \leqslant H, |h_n| \geq a > 0$. Then there exists a constant $K(a,H) > 0$ such that $A_n(h) \geq K(a,H)$. To obtain this estimate it is sufficient to note that $A_n(h) \geq lim_{y\to\infty} g(iy) = K(A,H)$ where the function

$$g(k),\ k \in G = (\{k = u + iv \in \mathbb{C}_+ : |u - u_n| < u_*\} \cup \{k = u + iv :\ v > H\}) \setminus [0, ia]$$

is the harmonic measure of a double-sided segment $[0, ia]$ relative to G.

In view of Schwarz‘s lemma we have also by estimate (4.12) a constant $K_1(H)$ such that for each $h \in \ell^2$, $\|h\|_2 \leqslant H$ the following estimate is valid:

$$|A_n(\eta) - A_n(h)| \leqslant K_1(H)\|\eta - h\|_2,\ \eta \in B^2_c(h, \rho/2). \tag{4.13}$$

Fix a sequence $h \in \ell^2$ and some $0 < a < \rho/4$ and choose a number $\rho/2 > \rho_1 > 0$ such that $K_1(\|h\|_2)\rho_1 \leqslant K(a, \|h\|_2)/2$. Then we consider two different cases for the number $n \in \mathbb{Z}$: 1) $|h|_n \geq a$, 2) $|h|_n < a$. In the first case according (4.13) we have the inequality

$$\mathrm{Re}(A_n(h) - A_n(\eta)) > 0,\ \eta \in B^2_c(h, \rho_1)$$

and it implies that J_n has analityc extension on the ball $B_c(h, \rho_1)$.

In the second case we represent the point $\eta \in \ell^2_c$ in the form $\eta = (\eta_n, \widetilde{\eta})$. Consider now the sequence $h_* = (0, \widetilde{h})$ and the set $G = \{(\eta_n, \widetilde{\eta}) : |\eta_n| < \rho/2, \|\widetilde{\eta}\|_2 < \rho/2$. It is clear that

$$\|h - h_*\|_2 < a < \rho/4,\ B^2_c(h, a) \subseteq B^2_c(h_*, 2a) \subseteq G.$$

For any $(\mu_n, \widetilde{\mu}) \in \ell^2$ we have following equalities

$$A_n(\mu_n, \widetilde{\mu}) = A_n(-\mu_n, \widetilde{\mu}),\ A_n(0, \widetilde{\mu}) = 0, \lim_{\mu_n\to 0} \frac{A_n(\mu_n, \widetilde{\mu})}{\mu_n^2} = 1 \tag{4.14}$$

therefore this equalities are valid for any $(\eta_n, \widetilde{\eta}) \in \mathcal{J}^2$.

In view of Schwarz's lemma we have by estimate (4.12) and equalities (4.14)

$$|A_n(\eta_n, \widetilde{\eta})| \leq K_2(\|h\|)|\eta_n|^2, \ (\eta_n, \widetilde{\eta}) \in G, \tag{4.15}$$

where $K_2(\|h\|_2)$ is some constant depending only on $\|h\|_2$.

Now consider the function F:

$$F(\eta_n, \widetilde{\eta}) = \frac{A_n(\eta_n, \widetilde{\eta})}{\eta_n^2}, \ (\eta_n, \widetilde{\eta}) \in G, \ \eta_n \neq 0;$$

$$F(0, \widetilde{\eta}) = 1, \ (0, \widetilde{\eta}) \in G.$$

According to (4.15) we have the estimate

$$F(\eta_n, \widetilde{\eta}) \leq K_2(\|h\|_2), \ (\eta_n, \widetilde{\eta}) \in G. \tag{4.16}$$

Fix some $\mu \in G, \ \nu \in \ell_c^2, \ \nu \neq 0$ and consider the function

$$f(z_1, z) = \frac{A_n(\mu + z_1\nu)}{(\mu_n + z\nu_n)^2}, \quad \mu_n + z\nu_n \neq 0; \quad f(z) = 1, \ \mu_n + z\nu_n = 0.$$

Now we prove that this is an analytic function with respect two variables in the domain $B(0,\varepsilon) \times B(0,\varepsilon) \subset \mathbb{C}^2$ fore some $\varepsilon > 0$. Really if $\mu_n \neq 0$ ore $\mu_n = \nu_n = 0$ it is evident. In opposite case we can apply Hartogs theorem.

It implies the analyticity of the function $F(\mu + z\nu), \ |z| < \varepsilon$ for sufficiently small ε. So by the Criterion about analyticity of functions depending on infinitely many variables we have proved the analyticity of the function F in the domain G.

In view of Schwarz lemma we have also by estimate (4.16)

$$|F(\eta) - F(h_*)| \leqslant K_2(\|h\|_2)\|\eta - h\|_2, \ \eta \in B_c^2(h, \rho/2).$$

Finally we have for a such that $2K_2(\|h\|_2)a < 1/2$

$$\mathrm{Re}(F(h_*) - F(\eta)) = 1 - \mathrm{Re}\, F(\eta) > 0, \ \eta \in B_c^2(h_*, 2a)$$

and required analytical continuation is defined by the following way

$$J_n(\eta) = \eta_n\sqrt{F(\eta)}, \quad \eta \in B_c^2(h_*, 2a).\square$$

Below we need the following sets:

$$K_n = \{k \in \mathbb{C} : 0 < |\,\mathrm{Re}\, k - u_n| < d_n\} \cup (u_n - i|h_n|, u_n + i|h_n|),$$

$$d_n = \min\{u_n - u_{n-1}, \ u_{n+1} - u_n\}, \quad K_n(h, \varepsilon) = K_n \backslash (\overline{B(b_n, r_n)} \cup \overline{B(\overline{b}_n, r_n)}), \quad n \in \mathbb{Z}.$$

Remark. Using Theorem 3.5 for $\eta \in B_C^\infty(\rho)$, we obtain the harmonic extensions of the function $\varphi(\cdot, h)$ across the set $\mathbb{R} \cap K(h, \varepsilon)$ by the symmetry on $K(h, \varepsilon)$. Moreover, if for some $n \in \mathbb{Z}$ the set $K_n(h,\varepsilon)$ is connected, then the function $v - \varphi(k,\eta)$, $k = \underline{u + iv}$, is continued by the symmetry across the interval $(u_n - i|h_n|, u_n + i|h_n|)\backslash(\overline{B(b_n, r_n)} \cup \overline{B(\bar{b}_n, r_n)})$ left or right on the domain $K_n(h,\varepsilon)$.

We consider the analytic extension of the conformal mapping $z(k,\eta)$.

Theorem 4.5. *For fixed $h \in \ell^2_{\mathbb{R}}, \varepsilon = \{\varepsilon_n\}$, where $\varepsilon_n = \xi \in [0,1]$, $n \in \mathbb{Z}$ the function $z(k, h+\varepsilon\eta)$ has the analytic extension from $D = K(h,\varepsilon) \times B^2(\rho)$ on the domain $D_C = K(h,\varepsilon) \times B^2_C(\rho)$. This continuation is such that if the set $K_n(h,\varepsilon)$ is connected for some $n \in \mathbb{Z}$ then the function $z(k, h+\varepsilon\eta)$, $(k,\eta) \in D_C$ has the analytic extension in the domain $\widetilde{D}_n = \widetilde{K}_n \times B^2_C(\rho)$ ("left" if this continuation coincides with $z(k, h+\varepsilon\eta)$ as $(k,\eta) \in \widetilde{D}_n \cap \{\mathrm{Re}\, k < u_n\}$ and "right" if this continuation coincides with $(k, h+\varepsilon\eta) \in \widetilde{D}_n \cap \{\mathrm{Re}\, k > u_n\}$). Moreover, for any $h \in \ell^2_{\mathbb{R}}, n \in \mathbb{Z}, k \in K(h)$ the derivative has the following form:*

$$\partial_n z(k,h) = \nu_n (z(k,h) - \lambda_n)^{-1}, \quad k \neq u_n + ih_n, \quad k \in K(h). \tag{4.17}$$

In particular, the formula (2.13) is true for $k \in [u_n, u_n + ih_n)$, if $z(k,h)$ is the left-hand limit (the right-hand limit).

Proof. Take $v_1 > \|h\|_2 + (u_*/4)$ and then $\{k : \mathrm{Im}\, k > v_1\} \subset K(h,\varepsilon)$. We fix some k_0, $\mathrm{Im}\, k_0 > v_1$. Formula (2.40) yields

$$z(k_0, h+\varepsilon\eta) = k_0 - \frac{1}{\pi}\int_{\mathbb{R}} \frac{\psi(\tau + iv_1, h+\varepsilon\eta)}{\tau - (k_0 - iv_1)} dt, \quad \eta \in B^2(\rho) \tag{4.18}$$

Moreover, for any $k \in K_+(h,\varepsilon)$

$$z'_k(k, h+\varepsilon\eta) = (1 - \psi'_v(k, h+\varepsilon\eta)) + i\psi'_u(k, h+\varepsilon\eta) \tag{4.19}$$

and let $\gamma \subset K(h,\varepsilon)$ be some piecewise smooth path joining k_0 to k. Then

$$z(k,\vartheta) - z(k_0,\vartheta) = \int_\gamma z'_k(\zeta,\vartheta)\, d\zeta \qquad \vartheta = h + \varepsilon\eta \tag{4.20}$$

By Theorem 4.1, the function $\varphi(k,h,\varepsilon,\eta)$ is harmonic with respect to k in the domain $K_+(h,\varepsilon)$ and analytic with respect to $\eta \in B^2_c(\rho)$. Then the function at right-hand side of formula (4.7) is analytic in $K(h,\varepsilon)$. Moreover, due to estimate (4.3) the integral in the right hand side of (4.6) is absolutely integrable. Define $z(k, h+\varepsilon\eta)$ for some $(k,\eta) \in \widetilde{D}$ by (4.6-8). Then for any $\eta \in B^2_C(\rho)$ the function $z(\cdot, h+\varepsilon\eta)$ is analytic in the domain $K(h,\varepsilon)$ and for this function (6.6-8) are fulfilled.

Consider now the function $f(\eta) = z(k_0, h+\varepsilon\eta)$, $\eta \in B^2_C(\rho)$ and show that $f(\eta)$ is analytic in the ball $B^2_C(\rho)$. Due to estimate (4.3) the sequence of the function

$$f_n(\eta) = k_0 - \frac{1}{\pi}\int_{-n}^{n} \frac{\varphi(u + iv_1, h, \varepsilon, \eta)}{u - (k_0 - iv_1)}\, du, \quad \eta \in B^2_C(\rho)$$

converges to f uniformly on $B^2_C(\rho)$. Indeed,

$$|f(\eta) - f_n(\eta)| \leq \frac{1}{\pi}\int_{\mathbb{R}} |\varphi(u + iv_1, h, \varepsilon, \eta)| du \max_{|u|\geq n}(\frac{1}{|u - k_0|})$$

Moreover, by Theorem 4 , each function f_n is analytic and bounded in the ball $B^2_C(\rho)$ and by the Weierstrass Theorem, f is also analytic in this ball.

Let now $k \in K_+(h,\varepsilon)$. Then using (4.7) and the well known formula for harmonic function we obtain

$$z'_k(k,h+\varepsilon\eta) = 1 + \frac{1}{\pi r}\int_{|w|=1}(-w_2+iw_1)\varphi(k+rw,h,\eta,\varepsilon)\,|dw|, \tag{4.21}$$

where $0<r<\operatorname{dist}(\zeta,\partial K_+(h,\varepsilon))$. We fix $k\in K(h,\varepsilon)$. Then formulas (4.8-9) imply

$$z(k,h+\varepsilon\eta) = z(k_0,h+\varepsilon\eta)+(k-k_0)+\frac{1}{\pi r}\int_\gamma dz\int_{|w|=1}(-w_2+iw_1)\varphi(\zeta+rw,h,\eta,\varepsilon)\,|dw| \tag{4.22}$$

where r is less than the distance from γ to $\partial K_+(h,\varepsilon)$. Using Theorem 4.1 and the Theorem about the integral with parameter, we deduce that the function $z(k,h+\varepsilon\cdot)$ is is analytic and bounded on the ball $B^2_C(\rho)$. By formula (4.10), the function $z(k,h+\varepsilon\eta)$ is bounded on any set $S\times B^2_C(\rho)$, where $S\subset K_+(h,\varepsilon)$ is compact.

By the well-known Criterion about analyticity of functions (see Theorem 2.9) enough to show that for any $k\in K_+(h,\varepsilon), k_1\in B(0,1),\ \eta,\eta_1\in B^2_C(\rho)$, the function $F(\zeta)=z(k+k_1\zeta,h+\eta+\eta_1\zeta)$ is analytic in the disk $B(0,s)\subset\mathbb{C}$ for some $s>0$. For small $s>0$ the function F is well define on the disk $B(0,s)$. Define the function

$$G(z_1,z_2)=z(k+z_1k_1,h+\eta+z_2\eta_1),\ |z_1|<s,\ |z_2|<s.$$

By above, the function G is analytic and bounded with respect to any variable at fixed another one. Then by the Hartogs theorem G is analytic with respect two variables, This shows that the function F is analytic in the disk $B(0,s)$. Due to Remark after Theorem 4.1 the function $y(k,\eta)=v-\psi(k,\eta),\ k\in K(h,\varepsilon)$ is harmonic continued by the symmetry across $\mathbb{R}$ and also left or right across any non-empty interval $\tilde K_n\cap\{p=u_n\}$. Now in order to get analytic continuation ("left") in the domain $\tilde D_n$ we repeat the above consideration, we join the fixed point $k_0\in\tilde K_n,\ \operatorname{Re}k_0<u_n$ with any point $k\in F_n$ by some curve lying in the domain $\tilde K_n$. $\square$

5 Estimates

In this section we prove all needed estimates for the real h. Without loss of generality, in this Section, we assume $h_n\geqslant 0, n\in\mathbb{Z}$. First, we need the following Lemma which is some analog of Hardy inequality (see [K4]).

Lemma 5.1. *Let $f\in W^2_1(D)$, where the domain $D=[0,\tau]\times[-h,h]$ for some $h>0,\ \tau>0$. Then*

$$\int_0^h \frac{|f(0,v)-f(0,-v)|^2}{v}\,dv \leqslant \pi\max\left\{1,\frac{h}{\tau}\right\}\iint_D|\nabla f|^2 dv\,du. \tag{5.1}$$

Proof. Let a function f be real and let $h \leqslant \tau$. Then for function $F(v) = f(0,v) - f(0,-v)$ and for any $v \in (0,h)$ we obtain

$$|F(v)| \leqslant v \int_0^{\pi} |\nabla f(ve^{i\varphi})| d\varphi,$$

and hence the Cauchy inequality implies

$$|F(v)|^2 \leqslant \pi v^2 \int\limits_0^{\pi} |\nabla f(ve^{i\varphi})|^2 d\varphi.$$

The integration from 0 to h yields

$$\int\limits_0^h \frac{|F(v)|^2}{v} dv \leqslant \pi \iint_{D_1} |\nabla f|^2 dv\, du \leqslant \pi \iint_D |\nabla f|^2 dv\, du,$$

where $D_1 = B(0,h) \cap \{u \geqslant 0\}$, which gives (5.1) for $h \leqslant \tau$.

Second, let $h > \tau$. Using inequality (5.1) for the function $f_1(u,v) = f(\frac{hu}{\tau}, v)$ for the case $h = \tau$ we deduce that

$$\int\limits_0^h \frac{|F(v)|^2}{v} dv \leqslant \pi \int\limits_0^h\int\limits_0^h \Big(\Big(\frac{\partial f_1}{\partial u}\Big)^2 + \Big(\frac{\partial f_1}{\partial v}\Big)^2\Big)\, du\, dv. \tag{5.2}$$

Moreover, we have $\frac{\partial f_1}{\partial v}(u,v) = \frac{\partial f}{\partial v}(\frac{hu}{\tau}, v)$, $\frac{\partial f_1}{\partial u}(u,v) = \frac{h}{\tau}\frac{\partial f}{\partial u}(\frac{h}{\tau}u, v)$. Then, changing the variables $v' = v$, $u' = \frac{h}{\tau}u$ in the integral in the right side of estimate (5.2), we get

$$\int_0^h |F(v)|^2 \frac{dv}{v} \leqslant \pi \iint_D \Big(\frac{\tau}{h}\Big(\frac{\partial f}{\partial v}\Big)^2 + \frac{h}{\tau}\Big(\frac{\partial f}{\partial u}\Big)^2\Big) dudv \leqslant \frac{\pi h}{\tau} \iint_D |\nabla f(k)|^2 dudv. \quad \square$$

Introduce the function $b(x) = \max\{1, x/u_*\}$ for $x > 0$ and the domains

$$D_n(r) = (u_n, u_n + r) \times (-h_n, h_n), \qquad D_n \equiv (u_n, u_n + u_*) \times (-h_n, h_n), \quad n \in \mathbb{Z}.$$

Using Lemma 5.1 we estimate h_n in terms of the Dirichlet integral.

Corollary. 5.2. *For each $n \in \mathbb{Z}$ and any $h \in \ell^{\infty}_{\mathbb{R}}, r > 0$, the following estimates are fulfilled:*

$$2h_n^2 \leqslant \pi \max\{1, \frac{|h_n|}{r}\} \iint_{D_n(r)} |z'(k,h) - 1|^2 dudv, \tag{5.3}$$

Proof. Define the function $f(k) = v - y(k,h)$, $k = u + iv$. Then estimate (5.1) for the function $f(iv + u_n) - f(-iv + u_n) = 2v$, $v \in (-h_n, h_n)$ yields

$$2h_n^2 = \int_0^{h_n} \frac{(2v)^2}{v} dv \leqslant \pi b(|h_n|) \iint_{D_n(r)} |\nabla f|^2 du\, dv,$$

which implies (5.3). $\square$

Corollary 5.2 is important to find the estimates of $\|h\|$ in terms of $\|l\|$ and of the Dirichlet integral. We show now the basic estimates in the case of the space ℓ^2.

Theorem 5.3. *Let $h \in \ell^\infty_{\mathbb{R}}$. Then for l, J, O_0, I_D the following estimates are fulfilled:*

$$\frac{1}{4}\|l\|^2 \leqslant 2Q_0 = I_D \leqslant \frac{2}{\pi}\sum_{n\in\mathbb{Z}} |h_n||l_n|. \tag{5.4}$$

Let in addition, $u_{n+1} - u_n \geqslant u_*$ for any $n \in \mathbb{Z}$, then

$$\|h\|^2 \leqslant \frac{\pi^2}{2} b(\|h\|_\infty) I_D, \qquad I_D = \frac{1}{\pi}\iint_{\mathbb{C}} |z'(k,h) - 1|^2 du dv, \tag{5.5}$$

$$\frac{\pi}{4} I_D \leqslant \|h\|^2 \leqslant \frac{\pi^2}{2}\max\left\{1, \frac{I_D^{1/2}}{u_*}\right\} I_D, \tag{5.6}$$

$$\frac{1}{2}\|l\| \leqslant \|h\| \leqslant \pi b(\|h\|_\infty)\|l\| \leqslant \pi\|l\|\Big(1 + \frac{2}{u_*^2}\|l\|^2\Big), \tag{5.7}$$

$$\frac{\|l\|}{2} \leqslant \|J\| \leqslant \sqrt{2}\|l\|(1 + \frac{\sqrt{2}}{u_*}\|l\|). \tag{5.8}$$

Proof. Estimates (5.4) were proved in the paper [KK1]. Introduce the integral $I_n = \frac{1}{\pi}\iint_{D_n} |z'(k,h) - 1|^2 du dv$. Inequality (5.3) yields

$$\|h\|^2 = \sum_{n\in\mathbb{Z}} h_n^2 \leqslant \frac{\pi^2}{2} b(\|h\|_\infty) \sum_{n\in\mathbb{Z}} I_n \leqslant \frac{\pi^2}{2} b(\|h\|_\infty)\, I_D.$$

Using estimates (5.4) and $|l_n| \leqslant 2|h_n|$ (see (1.1)), we obtain the first inequalities in (5.6-7). The second one in (5.6) follows from (5.5) and (2.33). Moreover, relations (5.4) yields $I_D \leqslant \frac{2}{\pi}\|h\|\|l\|$ and then

$$\|h\|^2 \leqslant \frac{\pi^2}{2} b(\|h\|_\infty)\, I_D \leqslant \pi\, b(\|h\|_\infty)\|h\|\|l\|,$$

which implies $\|h\| \leqslant \pi b(\|h\|_\infty)\|l\|$. Assume that $\|h\|_\infty > u_*$. Then $\|h\| \leqslant (\pi/u_*)\|h\|_\infty\|l\|$, and using (5.4), (2.34) we obtain

$$\|h\|^2 \leqslant \frac{\pi^2}{u_*^2}\|l\|^2 \cdot \frac{2}{\pi}\|h\|\|l\|.$$

Hence $\|h\| \leqslant (2\pi/u_*^2)\|l\|^3$, which yields (5.7).

Identities $\|J\|^2 = 2Q_0 = I_D$ together with (5.4), (5.7) imply (5.8). $\square$

For the case $\ell^p, p \geqslant 1$, we need additional considerations about the comb mappings. Introduce the domain $D_r = \{z \in \mathbb{C} : \ |\operatorname{Re} z| < r\}, r > 0$. We need the following result about the simple mapping.

Lemma 5.4. *The function $f(k)=\sqrt{k^2+h^2},\ k\in\mathbb{C}\setminus[-ih,ih]$, where $h>0$ is the conformal mapping from $\mathbb{C}\setminus[-ih,ih]$ onto $\mathbb{C}\setminus[-h,h]$ and $D_r\setminus[-h,h]\subset f(D_r\setminus[-ih,ih])$ for any $r>0$.*
Proof. Consider the image of the halfline $k=r+iv, v>0$. We have the equations

$$x^2+y^2=\xi\equiv r^2+h^2-v^2,\qquad xy=rv. \tag{5.9}$$

The second identity in (5.9) yields $x>0$ since $y>0$. Then

$$x^4-\xi x^2-r^2v^2=0,$$

and enough to check the following inequality

$$x^2=\frac{\xi+\sqrt{\xi^2+4r^2v^2}}{2}>r^2.$$

The last estimate follows from the simple relations

$$(r^2+h^2-v^2)^2+4r^2v^2>(r^2+v^2-h^2)^2,\qquad 4r^2v^2>4r^2(v^2-h^2).\quad\square$$

We now prove the local estimates for the small slits.
Theorem 5.5. *Let $h\in\ell^\infty_{\mathbb{R}}$. Assume that $(u_n-r,u_n+r)\subset(u_{n-1},u_{n+1})$ and $|h_n|\leqslant r/2$, for some $n\in\mathbb{Z}$ and $r>0$. Then*

$$||h_n|-|\mu_n^\pm||\leqslant\frac{2+\pi}{r}|\mu_n^\pm|\sqrt{I_n}, \tag{5.10}$$

$$0\leqslant|h_n|-\nu_n\leqslant 2\frac{2+\pi}{r}|h_n|\sqrt{I_n}, \tag{5.11}$$

$$0\leqslant|h_n|-\frac{|l_n|}{2}\leqslant\frac{2+\pi}{r}|h_n|\sqrt{I_n}, \tag{5.12}$$

where

$$I_n=\frac{1}{\pi}\iint_{u_n+D_r}|z'(k,h)-1|^2dudv.$$

Proof. Define the functions $f(k)=\sqrt{k^2+h_n^2},\quad k\in D_r\setminus[-ih_n,ih_n], g=f^{-1}$ and $F(w)=z(u_n+g(w),h), w=p+iq$, where the variable $w\in G_1=f((D_r\setminus[-ih_n,ih_n])$. The function F is real for real w then F is analytic in the domain $G=G_1\cup[-h_n,h_n]$ and by Lemma 5.4, $D_r\subset G$. Let now $|w|=r/2$. Then the well known estimate yields

$$\sqrt{\pi}\frac{r}{2}|F'(w)-1|\leqslant(\iint_{B(0,r)}|F'(w)-1|^2dpdq)^{1/2}\leqslant \tag{5.13}$$

$$\leqslant\Big(\iint_{B(0,r)}|(F(w)-g(w))'|^2dpdq\Big)^{1/2}+\Big(\iint_{B(0,r)}|g'(w)-1|^2dpdq\Big)^{1/2}.$$

Using the invariance of the Dirichlet integral with respect to the conformal mapping we obtain

$$\int\int_{B(0,r)}|(F(w)-g(w))'|^2\,dpdq=\int\int_{g(B(0,r))}|z'(k,h)-1|^2dudv\leqslant \tag{5.14}$$

$$\leqslant \int\int\limits_{D_r+u_n} |z'(k,h)-1|^2 dudv = \pi I_n.$$

Moreover, the identity $2Q_0 = I_D$ implies

$$\frac{1}{\pi}\int\int\limits_{B(0,r)} |g'(w)-1|^2 dpdq \leqslant \frac{1}{\pi}\iint_{\mathbb{C}} |g'(w)-1|^2 dpdq == \frac{2}{\pi}\int\limits_{-|h_n|}^{|h_n|} \sqrt{h_n^2-x^2}\, dx = h_n^2. \quad (5.15)$$

Then (5.13-15) for $|w| = r/2$ yields

$$|F'(w)-1| \leqslant \frac{2}{r}(\sqrt{I_n}+h_n),$$

and by Lemma 5.2,

$$h_n^2 \leqslant \frac{\pi^2}{4}\Big(\frac{1}{\pi}\iint_{D_{3r}+u_n} |z'(k,h)-1|^2 dudv\Big) \leqslant \frac{\pi^2}{4} I_n.$$

Then for $|w| = r/2$ we have

$$|F'(w)-1| \leqslant \frac{2}{r}(1+\frac{\pi}{2})\sqrt{I_n} = \frac{2+\pi}{r}\sqrt{I_n}, \quad (5.16)$$

and the maximum principle yields the needed estimates for $|w| \leqslant r/2$.

We prove (5.10) for μ_n^+. The definition of $\mu_n^\pm$ (see Sect. 2) implies

$$F'(|h_n|) = \lim_{x\searrow|h_n|} z'(u_n+g(x),h)\cdot g'(x) = \lim_{x\searrow|h_n|} \frac{g(x)}{\mu_n^+}\cdot\frac{x}{g(x)} = \frac{|h_n|}{\mu_n^+}.$$

and the substitution of the last identity into (5.16) gives (5.10).

We show (5.11). The definition of ν_n (see *Sect.*2) yields

$$(z(k,h)-z_n)^2 = 2i\nu_n(k-u_n-i|h_n|)(1+o(1)), \quad k\to u_n+i|h_n|,$$

$$g(w)-i|h_n| = -\frac{i}{2|h_n|}(w-z_n)^2(1+o(1)), \quad w\to u_n.$$

Then we have $F'(0) = \sqrt{\frac{\nu_n}{|h_n|}}$ and the substitution of the last identity into (5.16) shows $\Big|\sqrt{\frac{\nu_n}{|h_n|}}-1\Big| \leqslant ((2+\pi)/r)\sqrt{I_n}$, and we have (5.11) since by (2.30), $\nu_n \leqslant |h_n|$. Moreover, inequality (5.16) implies (5.12), indeed

$$0 \leqslant 2|h_n|-|l_n| = \int\limits_{-|h_n|}^{|h_n|} (1-F'(x))\, dx \leqslant \frac{2|h_n|}{r}(2+\pi)\sqrt{I_n}, \square$$

We prove the weighted estimates in the norm of the space l^p_ω, $1 \leqslant p \leqslant 2$ and for some weight $\omega_n \geqslant 1, n \in \mathbb{Z}$.

Theorem 5.6. *Let $h \in \ell^\infty$. Then*

$$\pi Q_0 \leqslant \|h\|_p \, \|l\|_q, \quad p \geq 1, \tag{5.17}$$

$$I_D \leqslant (\frac{2}{\pi})^{2/p} \|h\|_p^{2/q} \, \|l\|_p^{2/p}, \quad 1 \leqslant p \leqslant 2, \tag{5.18}$$

$$\pi Q_0 \leqslant \|h\|_\infty \, \|l\|_1 \leqslant \frac{2}{\pi} \|l\|_1^2, \tag{5.19}$$

$$\|h\|_\infty \leqslant \frac{2}{\pi} \|l\|_1, \quad \|l\|_1 \leqslant 2\|h\|_1. \tag{5.20}$$

Proof. Estimate (5.4) and the Hölder inequality yield (5.17).

Using (2.33), $\pi Q_0 \leqslant \sum |h_n||l_n|$ (see (5.4)) and the Hölder inequality we deduce that

$$(\pi/2) I_D \leqslant \|h\|_\infty^{1-\frac{p}{q}} \sum_{n \in \mathbb{Z}} |l_n| \, |h_n|^{p/q} \leqslant (I_D)^{(1-\frac{p}{q})/2} \|l\|_p \|h\|_p^{\frac{p}{q}},$$

$$(\pi/2) I_D^{\frac{(1+\frac{p}{q})}{2}} \leqslant \|l\|_p \|h\|_p^{\frac{p}{q}},$$
$$I_D \leqslant [\frac{2}{\pi}]^{\frac{2}{p}} \, \|l\|_p^{\frac{2}{p}} \|h\|_p^{\frac{2}{q}}.$$

Estimate (5.17) at $q = 1$ implies the first one in (5.19). The last result and (2.33) yield the first inequality in (5.20) and then the second one in (5.19). The second estimate in (5.20) follows from $|l_n| \leqslant 2|h_n|$, $n \in \mathbb{Z}$ (see (1.1)). $\Box$

Consider the case $p \geqslant 1$ more detail and find the double side estimates. Recall that the constant $\alpha_p = (2^{p+2}(2+\pi)/u_*)^p/\pi$

Theorem 5.7. *Let $u_{n+1} - u_n \geqslant u_*$ for any $n \in \mathbb{Z}$. Then the following estimates are fulfilled:*

$$\|h\|_p \leqslant 2\|l\|_p(1 + \alpha_p \, \|l\|_p^p), \quad 1 \leqslant p \leqslant 2, \tag{5.21}$$

$$\|h\|_p \leqslant \frac{2}{\pi} C_p^2 \|l\|_q (1 + [\frac{2C_p\|l\|_q}{\pi u_*}]^{\frac{2}{p-1}}), \quad C_p = (\frac{\pi^2}{2})^{1/p}, \quad p \geqslant 2. \tag{5.22}$$

$$\frac{\|l\|_p}{2} \leqslant \|J\|_p \leqslant \frac{2}{\sqrt{\pi}} \|l\|_p (1 + \alpha_p \|l\|_p^p)^{1/2}, \tag{5.23}$$

$$\frac{\sqrt{\pi}}{2} \|J\|_p \leqslant \|h\|_p \leqslant 4\|J\|_p (1 + \alpha_p 2^p \|J\|_p^p). \tag{5.24}$$

Proof. Let $1 \leqslant p \leqslant 2$. First we show the needed results. Estimate (5.3), at $r = u_*/2$, implies

$$2h_n^2 \leqslant \pi^2 \max\{1, \frac{|h_n|}{r}\} I_n, \quad I_n = \iint_{D_n} |z'(k,h) - 1|^2 du dv,$$

where $D_n = \{k : |\operatorname{Re}(k - u_n)| < u_*/2\}$, and then

$$h_n \leqslant \frac{\pi^2}{u_*} I_n, \quad if \quad h_n > \frac{u_*}{4}, \quad and \quad h_n \leqslant \frac{\pi}{u_*}\sqrt{I_n}, \quad if \quad h_n \leqslant \frac{u_*}{4},$$

Moreover, (5.12) yields

$$h_n \leqslant (l_n/2) + 2h_n \frac{2+\pi}{u_*}\sqrt{I_n}, \qquad if \quad h_n < u_*/4,$$

and then

$$h_n \leqslant 2\pi \frac{2+\pi}{u_*} I_n, \qquad if \quad h_n < u_*/4, \ l_n \leqslant h_n,$$

since $h_n \leqslant (\pi/2)\sqrt{I_n}$, and hence

$$if \quad l_n \leqslant h_n, \quad then \quad h_n \leqslant 2\pi \frac{2+\pi}{u_*} I_n = C_1 I_n,$$

where $C_1 = 2\pi(2+\pi)/u_*$. The last inequality and (2.33) yield

$$\|h\|_p \leqslant \Big(\sum_{h_n < l_n} h_n^p\Big)^{1/p} + \Big(\sum_{l_n \leqslant h_n} h_n h_+^{p-1}\Big)^{1/p} \leqslant \|l\|_p + C_1^{\frac{1}{p}} I_D^{\frac{p+1}{2p}}, \tag{5.25}$$

and if we assume that $C_1^{\frac{1}{p}} I_D^{\frac{p+1}{2p}} \leqslant \|l\|_p$, then we obtain $\|h\|_p \leqslant 2\|l\|_p$.

An inverse, let $\|l\|_p \leqslant C_1^{\frac{1}{p}} I_D^{\frac{p+1}{2p}}$. Then (5.25), (5.18) implies

$$\|h\|_p \leqslant 2C_1^{\frac{1}{p}} \Big[\Big(\frac{2}{\pi}\Big)^{2/p} \|h\|_p^{2/q} \, \|l\|_p^{2/p}\Big]^{\frac{p+1}{2p}},$$

and then

$$\|h\|_p^{1/p^2} \leqslant 2C_1^{\frac{1}{p}} \Big[\Big(\frac{2}{\pi}\Big)\|l\|_p\Big]^{\frac{p+1}{p^2}},$$

and

$$\|h\|_p \leqslant 2^{p^2} C_1^p \Big(\frac{2}{\pi}\Big)^{p+1} \|l\|_p^{1+p}.$$

which yields (5.21).

Let $p \geqslant 2$. Using inequality (5.5), (2.33) we obtain

$$\|h\|_p \leqslant \Big(\sum h_+^{p-2} h_n^2\Big)^{1/p} \leqslant C_p b^{1/p} I_D^{1/2}, \qquad b = b(h_+). \tag{5.26}$$

Consider the case $b \leqslant 1$. Then (5.26), (5.17) imply

$$\|h\|_p^2 \leqslant C_p^2 I_D \leqslant C_p^2 (2/\pi) \|h\|_p \|l\|_q,$$

and then

$$\|h\|_p \leqslant (2C_p^2/\pi)\|l\|_q,$$

Consider the case $b > 1$. Then the substitution of (2.33), (5.17) yield

$$\|h\|_p \leqslant C_p I_D^{\frac{p+1}{2p}} u_*^{-1/p} \leqslant u_*^{-1/p} C_p [(2/\pi)\|h\|_p \|l\|_q]^{\frac{p+1}{2p}},$$

and

$$\|h\|_p^{\frac{p-1}{2p}} \leqslant C_p u_*^{-1/p} [(2/\pi)\|l\|_q]^{\frac{p+1}{2p}},$$

and

$$\|h\|_p \leqslant (C_p u_*^{-1/p})^{\frac{2p}{p-1}} (2/\pi)^{\frac{p+1}{p-1}} \|l\|_q]^{\frac{p+1}{p-1}},$$

and combining these two cases we have (5.22).

Inequality $l_n \leqslant 2J_n$ (see (2.34)) yields the first estimate in (5.23). Relation (2.34) implies

$$\|J\|_p^p = \sum |J_n|^p \leqslant \sum (2/\pi)^{p/2} h_n^{p/2} l_n^{p/2} \leqslant (2/\pi)^{p/2} \|h\|_p^{p/2} \|l\|_p^{p/2},$$

and using (5.21) we obtain the second estimate in (5.23):

$$\|J\|_p \leqslant \sqrt{2/\pi} \|l\|_p^{1/2} [2\|l\|_p (1 + \alpha_p \|l\|_p^p)]^{1/2} = \frac{2}{\sqrt{\pi}} \|l\|_p (1 + \alpha_p \|l\|_p^p)^{1/2}.$$

Inequality $J_n^2 \leqslant 4h_n^2/\pi$ (see (2.34)) yields the first estimate in (5.24). Using (5.21) and $\|l\|_p \leqslant 2\|J\|_p$ (see (5.23)) we deduce that

$$\|h\|_p \leqslant 2\|l\|_p (1 + \alpha_p \|l\|_p^p) \leqslant 4\|J\|_p (1 + \alpha_p 2^p \|J\|_p^p). \quad \square$$

Consider now the examples, which show the exactness our estimates.

Example 1. Define the sequence $h_n = N, |n| \leqslant N$, and $h_n = 0$, if $|n| > N$ and assume that $u_n = n, \ n \in \mathbb{Z}$. Introduce the conformal mapping

$$g : \mathbb{C} \backslash \overline{B(0, R)} \to \mathbb{C} \backslash [-2R, 2R], \quad R^2 = 8N^2,$$

$$g(k) = k + \frac{R^2}{k}, \quad |k| > R$$

and note that $\cup_{|n| \leqslant N} [-i|h_n| + u_n, u_n + i|h_n|] \subset \overline{B(0, R)}$, then for our Theorem ? we have $\|h\|_\infty = N, \ u_* = 1, \ \|h\|_2^2 = 2N^3, \ \|h\|_1 = 2N^2$ and using (5.5) we obtain

$$N^2 = \|h\|_\infty^2 \leqslant 2Q_0 = I_D \leqslant 16N^2.$$

Hence inequality (5.7) is precise. Moreover, the following estimate

$$\frac{1}{\pi} \sum_{n \in \mathbb{Z}} |l_n||h_n| \leqslant 2Q_0 \leqslant \frac{2}{\pi} \sum_{n \in \mathbb{Z}} |l_n||h_n|$$

(see (1.2)) yields

$$\frac{\pi N}{2} \leqslant \frac{\pi Q_0}{N}, \quad \|l\|_1 \leqslant \frac{2\pi Q_0}{N} \leqslant 16\pi N.$$

Then we deduce that (5.21) at $p = 1$, is precise. $\square$

In order to consider the relation (5.8) we need the following simple result.

Lemma 5.8. *Let $h \in l^\infty_{\mathbb{R}}$, $u_{n+1} - u_n = u_n - u_{n-1} = 1$, $\min\{|h_{n-1}|, |h_{n+1}|\} = M$, for some $n \in \mathbb{Z}$. Then*

$$|l_n| \leqslant ((M^2 - |h_n|^2 + 1)^2 + 4|h_n|^2)^{1/4}. \tag{5.27}$$

Proof. Define the sequence

$$\widetilde{h}_m = \begin{cases} 0, & \text{if } |m-n| \geqslant 2, \\ M, & \text{if } |m-n| = 1, \\ h_n, & \text{if } m = n. \end{cases}$$

Inequality (2.29) implies $|\widetilde{l}_n| \geqslant |l_n|$, then enough to show estimate (5.25) only for the sequence $h = \widetilde{h}$ and it is possible to assume that $u_n = 0$. Define the sequence

$$\eta_m = \begin{cases} \widetilde{h}_m, & \text{if } m \neq n, \\ 0, & \text{if } m = n. \end{cases}$$

Then

$$z(k,h) = \sqrt{(z(k,\eta))^2 + |z(i|h_n|,\eta)|^2}.$$

Hence the maximum principle yields

$$|l_n| \leqslant \operatorname{Im} z(i|h_n|,\eta) \leqslant |\sqrt{M^2 + (1 + i|h_n|)^2}| \leqslant$$

$$((M^2 - |h_n|^2 + 1)^2 + 4|h_n|^2)^{1/4}. \quad \square$$

Example 2. Introduce now the sequence

$$h_n = \begin{cases} N - |n|, & \text{if } 0 \leqslant |n| \leqslant N, \\ 0, & \text{if } |n| > N. \end{cases}$$

We estimate $\|l\|_2$. Using Lemma 5.8 we obtain for $|m| \leqslant N - 2$:

$$|l_m| \leqslant ((N - |m|^{-1})^2 - (N - |m|)^2 + 1)^2 + 4((N - |m|)^2)^{1/4} =$$

$$((2 - 2(N - |m|))^2 + 4(N - |m|)^2)^{1/4} \leqslant (8(N - |m|)^2)^{1/4} = 2^{3/4}\sqrt{N}.$$

Moreover, the simple inequality $|l_n| \leqslant |h_n|$ implies $|l_{N-1}| \leqslant 1$, $|l_{1-N}| \leqslant 1$. Consider now $A = \sum_{|n| \leq N} l_n$. By the Theorem Ivanov-Pomerenke, $\frac{A}{4}$ is equal the capacity of the compact set $E = \cup_{|n| \leq N}[u_n - ih_n, u_n + h_n]$. The capacity of the set E is less than the diameter which is equal to $2N$. Then $A \leq 8N$ and we have

$$\|l\|_2^2 = \sum_{m \in \mathbb{Z}} |l_m|^2 \leqslant \sqrt[4]{8}\,\sqrt{N} \sum_{n \in \mathbb{Z}} |l_m| \leqslant 8\sqrt[4]{8}\, N^{3/2} = BN^{3/2}.$$

Assume that the following estimate are fulfilled

$$\|h\|_2 \leqslant C(\|l\|_2 + \|l\|_2^p) \tag{5.28}$$

with some $p > 1$, then for our Example 2 for large N we obtain the inequality $\|h\|_2 \leqslant 2CB^{p/2}N^{3p/4}$. On the other hand we have $\|h\|_2 \geqslant C_1 N^{3/2}$ for some constant $C_1 > 0$. Then $N^{\frac{3}{2}(1-p/2)} \leqslant \frac{2C}{C_1} B^{p/2}$. It is possible only for $p \geqslant 2$. Hence the estimate (5.31) is true only for some $p \geqslant 2$ (we have $p = 3$). $\square$

We have considered the estimates for the weight $\omega_n \leqslant 1$. Now we find the counterexample which shows impossibility of double-sided estimates in the space ℓ^∞_ω with $\omega_n \leqslant 1$.
Counterexample 3. Consider now the uniform comb with $u_n = \pi n, \ \ H = h_n = \|h\|_\infty$, $n \in \mathbb{Z}$. It is clear (see [LS]), that in this case $l = l_n = 2 \arcsin \operatorname{th} H, \ \ n \in \mathbb{Z}$. Then $l \to \pi$ as $H \to \infty$ and in this case for any sequence $\omega = \{\omega_n\}_{n\in\mathbb{Z}}, \ \ \omega_n > 0, \ \ n \in \mathbb{Z}; \ \sum\limits_{n\in\mathbb{Z}} \omega_n < +\infty$ and any $1 \leqslant p < +\infty$ the sequence h belongs to ℓ^p_ω and

$$\|h\|^p_{p,\omega} = \sum_{n\in\mathbb{Z}} h^p_n \omega_n = H^p \sum_{n\in\mathbb{Z}} \omega_n;$$

$$\|l\|^p_{p,\omega} = l^p \sum_{n\in\mathbb{Z}} \omega_n.$$

Hence the estimate $\|h\|_{p,\omega} \leqslant F(\|l\|_{p,\omega})$ is impossible for some function F. Moreover, by the same reason, the following inequality $\|h\|_\infty \leqslant F(\|l\|_\infty)$ is impossible. $\square$

Now we consider any strongly increasing sequence $\{u_n\}_{n\in\mathbb{Z}}, u_n \to \pm\infty$ as $n \to \pm\infty$. Let $h \in l^\infty_\mathbb{R}, h_n \to 0$ as $n \to \infty$, then we introduce the new sequence $\widetilde{h} = \widetilde{h}(h)$ which is defined by the following way. We take such number n_1 that $|h_{n_1}| = \max_{n\in\mathbb{Z}} |h_n| > 0$ and let $\widetilde{h}_{n_1} = h_{n_1}$; assume that the numbers $n_1, n_2, \ldots, n_k$ have been defined, then we take such n_{k+1} that

$$|h_{n_{k+1}}| = \max_{n\in B} |h_n| > 0, B = \{n \in \mathbb{Z} : |u_n - u_{n_l}| > |h_{n_l}|, 1 \le l \le k\}$$

and let $\widetilde{h}_{n_{k+1}} = h_{n_{k+1}}$. Moreover, we let $\widetilde{h}_n = 0$, if the number $n \notin \{n_k, k \in \mathbb{Z}\}$. We prove Theorem 2.6.
Theorem 5.9. *Let $h \in l^\infty_\mathbb{R}, h_n \to 0$ as $n \to \infty$. Then*

$$\frac{1}{\pi^2}\|\widetilde{h}\|^2_2 \le Q_0(h) = \frac{1}{2} I_D(h) \le \frac{2\sqrt{2}}{\pi}\|\widetilde{h}\|^2_2. \tag{5.29}$$

Proof. The Lindelöf principal yields $Q_0(\widetilde{h}) \le Q_0(h)$. On the other hand open squares $P_k = (u_{n_k} - t_k, u_{n_k} + t_k) \times (-t_k, t_k), t_k \equiv |h_{n_k}| = |\widetilde{h}_{n_k}|, k \in Z$, does not overlap. Then applying Lemma 5.2 to the function $(z(k, \widetilde{h}) - k)$ and P_k, we obtain

$$2t^2_k \le \pi \int\int_{P_k} |z'(k, \widetilde{h}) - 1|^2 \, dudq,$$

and

$$Q_0(\widetilde{h}) = \frac{1}{2} I_D(\widetilde{h}) \ge \frac{1}{\pi^2} \sum_{k\ge1} t^2_k = \frac{1}{\pi^2}\|\widetilde{h}\|^2_2$$

which yields the first inequality in (5.29).

Let $A_k = \{n \in \mathbb{Z} : u_n \in [u_{n_k} - t_k, u_{n_k} + t_k]\}$. By the Lindelöf principal the gap length $|l_{n_k}|$ such that $[u_n, u_n + i|h_n|], n \in A_k$ increases if we take off all another slits. By Theorem of Ivanov-Pomerenke (see Section 1), the sum of new gap lengths equals to $4 \times$ *capacity* of the set $E = \cup_{n\in A_k}[u_n - |h_n|, u_n + |h_n|]$, which is less than the diameter of the set E,(see [ ]). Then

$$\sum_{n\in A_k} |l_n(h)| \le 2\sqrt{2}t_k, \tag{5.30}$$

and using (5.28) we obtain

$$\pi Q_0(h) \le \sum_{n\in\mathbb{Z}} |h_n||l_n| \le \sum_{k\ge1}\sum_{n\in A_k} |h_n||l_n| \le$$

$$\sum_{k\ge1} t_k \sum_{n\in A_k} |l_n| \le 2\sqrt{2}\sum_{k\ge1} t_k^2 = 2\sqrt{2}\|\widetilde{h}\|_2^2,$$

since $|h_n| \le t_k, n \in A_k$ and the diameter of the set E is less than or equals $2\sqrt{2}t_k$. □

Note that the proved Theorem shows that estimates (5.7-8) are fulfilled for the more weak conditions on the sequence U.

Corollary 5.10. *Assume that there exist $L > 0, M \in \mathbb{N}$ such that each interval with length L has $m \leqslant M$ the points from the sequence U. Then for any $h \in l^\infty_{\mathbb{R}}$ the following estimates are fulfilled:*

$$\frac{\pi}{4}I_D(h) \le \|h\|_2^2 \le \frac{\pi^2}{2}MI_D(h)(\sqrt{2}\pi\sqrt{I_D(h)} + 1), \tag{5.31}$$

$$\frac{1}{2}\|l\|_2 \le \|h_2\|_2 \le \pi M\|l\|_2 + 8\pi^3(\frac{M}{L})^2\|l\|_2^3,$$

Proof. The left inequality follows from ( ). We prove the right one. Let $|A_k|$ be the number of elements of the set A_k. Then

$$|A_k| \le (\frac{2t_k}{L} + 1)M,$$

and

$$\sum_{n\in\mathbb{Z}} |h_n|^2 \le \sum_{k\ge1}\sum_{n\in A_k} |h_n|^2 \le \frac{2M}{L}\|\widetilde{h}\|_3^3 + M\|\widetilde{h}\|_2^2$$

Using (5.27), we obtain the needed result. Moreover, the left estimate in the second inequality in (5.31) is simple and follows from (5.4), and the right one follows from (5.27) and (5.4). Indeed, we have

$$\|h\|_2^2 \le \pi^2 MI_D(h) = 2\pi^2 MQ_0(h) \le 2\pi M\|h\|_2\|l\|_2;$$

$$\|h\|_2 \le 2\pi M\|l\|_2,$$

or

$$\|h\|_2^2 \le \frac{\pi^3 M\sqrt{2}}{L}I_D(h)^{\frac{3}{2}} = \frac{4\pi^3 M}{L}Q_0(h)^{\frac{3}{2}} \le \frac{4\pi^{\frac{3}{2}}M}{L}\|h\|_2^{\frac{3}{2}}\|l\|_2^{\frac{3}{2}},$$

$$\|h\|_2 \le 16\pi^3 (\frac{M}{L})^2 \|l\|^3. \qquad \square$$

In order to prove the estimates in the space ℓ^p_ω we need some results. Recall that for each $h \in \ell^\infty_{\mathbb{R}}$ the following identity is fulfilled:

$$v(x) = v_n(x)(1 + V_n(x)), \quad V_n(x) = \frac{1}{\pi} \int\limits_{\mathbb{R}\setminus g_n} \frac{v(t)dt}{|t-x|\, v_n(t)}, \quad x \in g_n = (z_n^-, z_n^+), \tag{5.32}$$

where

$$k(z) = k(z,h) = u(z) + iv(z), \quad z = x + iy, \quad v_n(x) = |(x - z_n^+)(x - z_n^-)|^{1/2}.$$

For any $n \in \mathbb{Z}$ there exists the unique point $z_n \in g_n$, such that $|h_n| = v(z_n)$. Let $s = s(h) = \inf |\sigma_n(h)|$ and $h_+ = \|h\|_\infty$. We have

Lemma 5.11. *Let $u_{n+1} - u_n \geqslant u_* > 0$ for any $n \in \mathbb{Z}$. Then for each $h \in \ell^\infty_{\mathbb{R}}$ the following estimates are fulfilled:*

$$s \leqslant u_* \leqslant \frac{\pi s}{2} \exp \max\{2, \frac{5\pi}{2u_*}\|h\|_\infty\}, \tag{5.33}$$

$$1 + \frac{2\|h\|_\infty}{s\pi} \leqslant \exp \frac{9\|h\|_\infty}{u_*}, \tag{5.34}$$

$$\max_{n \in g_n} V_n(x) \leqslant \frac{2\|h\|_\infty}{\pi s}, \quad n \in \mathbb{Z}, \tag{5.35}$$

$$|h_n| \leqslant \frac{1}{2}|l_n|(1 + \frac{2\|h\|_\infty}{\pi s}) \leqslant \frac{1}{2}|l_n| \exp \frac{9\|h\|_\infty}{u_*}, \quad n \in \mathbb{Z}. \tag{5.36}$$

Proof. Let $h_+ = \|h\|_\infty$. Introduce the domain

$$G = \{z \in \mathbb{C} : h_+ \geqslant \text{Im } z > 0, \ \text{Re } z \in (-\frac{u_*}{2}, \frac{u_*}{2})\} \cup \{\text{Im } z > h_+\}.$$

Let g be the conformal mapping from G onto $\mathbb{C}_+$, such that $g(iy) \sim iy$ as $y \nearrow +\infty$ and let α, β be images of the points $u_*/2, (u_*/2) + ih_+$ respectively. Define the function $f = \text{Im}\, g$. Fix any $n \in \mathbb{Z}$. Then the maximum principle yields

$$y(k) = \text{Im}\, z(k,h) \geqslant f(k - p_n), \quad k \in G + p_n, \quad p_n = \frac{1}{2}(u_{n-1} + u_n).$$

Due to the fact that these positive functions equal zero on the interval $(p_n - u_*/2, p_n + u_*/2)$, we obtain

$$\frac{\partial y}{\partial v}(x) = \frac{\partial x}{\partial u}(x) \geqslant \frac{\partial f}{\partial v}(x - p_n), \quad x \in (p_n - \frac{u_*}{2}, p_n + \frac{u_*}{2}).$$

Then

$$z(u_n) - z(u_{n-1}) \geqslant \int\limits_{-u_*/2}^{u_*/2} \frac{\partial f}{\partial v}(x)\, dx = 2\alpha > 0,$$

and the inequality $s \leqslant u_*$ (see [KK1]) implies $2\alpha \leqslant s \leqslant u_*$.

Let w be the inverse function for g. The function $w : \mathbb{C}_+ \to G$ is defined uniquely and the formula of Christoffel-Schwartz yields

$$w(z) = \int_0^z \sqrt{\frac{t^2 - \beta^2}{t^2 - \alpha^2}} dt, \qquad 0 < \alpha < \beta,$$

and then we have

$$\frac{u_*}{2} = \int_0^\alpha \sqrt{\frac{\beta^2 - t^2}{\alpha^2 - t^2}} dt, \quad h_+ = \int_\alpha^\beta \sqrt{\frac{\beta^2 - t^2}{t^2 - \alpha^2}} dt. \tag{5.37}$$

The first integral in (5.37) has the simple double-sided estimate

$$\alpha = \int_0^\alpha dt \leqslant \frac{u_*}{2} \leqslant \int_0^\alpha \frac{\beta dt}{\sqrt{\alpha^2 - t^2}} = \frac{\beta\pi}{2},$$

that is

$$2\alpha \leqslant s \leqslant u_* \leqslant \pi\beta. \tag{5.38}$$

Consider the second integral in (5.37). Let $\varepsilon = \beta/\alpha \geqslant 5$ and using the new variable $t = \alpha \cosh r, \cosh\delta = \varepsilon$, we obtain

$$h_+ = \alpha \int_0^\delta \sqrt{\varepsilon^2 - \cosh^2 r} dr \geqslant \alpha\varepsilon \int_0^{\delta/2} \sqrt{1 - \frac{\cosh^2 r}{\varepsilon^2}} dr \geqslant \beta\delta\frac{2}{5},$$

since for $r \leqslant \delta/2$ we have the simple inequality

$$\frac{\cosh^2 r}{\cosh^2 \delta} \leqslant e^{-\delta}(1 + e^{-\delta})^2 \leqslant \varepsilon^{-1}(1 + \varepsilon^{-1})^2$$

and due to $\varepsilon \leqslant e^\delta$ we get $\varepsilon \leqslant \exp(5h_+/2\beta)$ and estimate (5.38) implies

$$\frac{1}{s} \leqslant \frac{\pi}{2u_*} \exp(\frac{5\pi}{2u_*} h_+), \quad if \quad \varepsilon \geqslant 5. \tag{5.39}$$

If $\varepsilon \leqslant 5$, then using (5.38) again we obtain

$$\frac{1}{s} \leqslant \frac{\varepsilon}{2\beta} \leqslant \frac{\pi\varepsilon}{2u_*} \leqslant \frac{\pi}{2u_*} 5, \qquad if \quad \varepsilon \leqslant 5.$$

and the last estimate together with (5.39) yield (5.33-34).

Identity (5.32) for $x \in g_n = (z_n^-, z_n^+)$ implies

$$V_n(x) = \frac{1}{\pi} \int_{z_n^- - t < s} \frac{v(t)dt}{|t - x| v_n(t)} + \frac{1}{\pi} \int_{t - z_n^+ > s} \frac{v(t)dt}{|t - x| v_n(t)} \leqslant$$

$$\frac{1}{\pi} \int_{z_n^- - t < s} \frac{h_+ dt}{|t - z_n^-|^2} + \frac{1}{\pi} \int_{t - z_n^+ > s} \frac{h_+ dt}{|t - z_n^+|^2} \leqslant \frac{2h_+}{\pi s}.$$

Using (5.32), (5.34) and simple inequality $v_n(z_n) \leqslant |l_n|/2$ we have (5.36). $\square$

We prove the double-sided estimates of $h_n, l_n, \mu_n^{\pm}, J_n$ in the weight spaces.

Theorem 5.12. *Let the distance $u_{n+1} - u_n \geqslant u_*$ and the weight $\omega_n \geqslant 1$ for any $n \in \mathbb{Z}$. Then for each $h \in \ell^p_\omega$ and any $p \in [1, 2]$ the following estimates are fulfilled:*

$$\|h\|_\infty \leqslant \max\{2\pi\|\mu^{\pm}\|_\infty,\ \|J\|_{p,\omega},\ 2\pi^{-1/p}\|l\|_{p,\omega}(1 + \alpha_n\|l\|^p_{p,\omega})^{1/q}\}, \tag{5.40}$$

$$\|l\|_{p,\omega} \leqslant 2\|h\|_{p,\omega} \leqslant \|l\|_{p,\omega} \exp(9\|h\|_\infty/u_*), \tag{5.41}$$

$$\|l\|_{p,\omega} \leqslant 2\|J\|_{p,\omega} \leqslant 2\|l\|_{p,\omega} \exp(5\|h\|_\infty/u_*), \tag{5.42}$$

$$\frac{\sqrt{\pi}}{2}\|J\|_{p,\omega} \leqslant \|h\|_{p,\omega} \leqslant \sqrt{\frac{\pi}{2}}\|J\|_{p,\omega} \exp(5\|h\|_\infty/u_*), \tag{5.43}$$

$$\|l\|_{p,\omega} \leqslant 2\|\mu^{\pm}\|_{p,\omega} \leqslant \|l\|_{p,\omega} \exp(18\|h\|_\infty/u_*). \tag{5.44}$$

Proof. The first estimate in (5.40) follows from $|h_n| \leqslant 2\pi|\mu_n^{\pm}|$ (see [KK1]). The second one in (5.40) follows from $\|h|_-\infty \leqslant \sqrt{I_D} = \|J\| \leqslant \|J\|_p \leqslant \|J\|_{p,\omega}$ since we have the inequality $\|f\|_p \leqslant \|f\|_{p,\omega}$ for any f. Moreover, substituting (5.23) into $\|h|_-\infty \leqslant \sqrt{I_D} = \|J\| \leqslant \|J\|_p$ and using $\|f\|_p \leqslant \|f\|_{p,\omega}$ we obtain the last estimate in (5.40).

Introduce the function $G_+ = \exp(9\|h\|_\infty/u_*)$. The first estimate in (5.41) follows from (1.1). Due to (5.36) we get $2h_n \leqslant G_+ l_n$ and then we obtain the second inequality in (5.41).

The first estimate in (5.42) follows from (2.34). Using (2.34), (5.36) we have $J_n^2 \leqslant 2l_n h_n/\pi \leqslant (G_+/\pi)l_n^2$, and then we get the second inequality in (5.42).

The first estimate in (5.43) follows from (2.34), (1.1). Using (2.32), (5.36) we obtain $h_n^2 \leqslant G_+ l_n h_n/2 \leqslant (\pi G_+/2)J_n^2$, which yields the second inequality in (5.43).

Identity $2\mu_n^{\pm} = \pm|l_n|[1 + V_n(z_n^{\pm})]^2$ (see [KK1]) implies $2|\mu_n^{\pm}| \geqslant |l_n|$ which yields the first inequality in (5.44). Moreover, using (5.36) we obtain the estimate $2|\mu_n^{\pm}| \leqslant G_+^2|l_n|$ which yields the second inequality in (5.44). $\square$

6 Asymptotics

In this section we consider the case when the function $\beta(k, h) = v,\ k = u + iv \in \gamma$. Then (see Sect. 2) for any $h \in \ell^\infty_{\mathbb{R}}$ the function $y(k, h) = v - \psi(k, h),\ k = u + iv \in K_+(h)$, is the unique imaginary part of the conformal mapping $z(k, h) : K_+(h) \to \mathbb{C}_+$. Recall that

$$K_n(h) = \{u + iv \in \mathbb{C} : u \in (u_n - d_n, u_n + d_n)\} \backslash \{u + iv \in \mathbb{C} :\ u = u_n, |v| \geq |h_n|\},$$

where $d_n = \min\{u_n - u_{n-1},\ u_{n+1} - u_n\}$. First we prove results about the convergence of the imaginary part of the conformal mappings and the gap lengths.

Theorem 6.1. *Let $h, h^{(m)} \in \ell^\infty_{\mathbb{R}},\ \ m \geqslant 1,$ such that $\lim\limits_{m\to\infty} h_n^{(m)} = h_n$ for any $n \in \mathbb{Z}$ and $H = \sup\limits_{m\geqslant 1}\|h^{(m)}\| < +\infty$. Then*

$$\psi(k, h^{(m)}) \to \psi(k, h), \quad as \ \ m \to \infty, \tag{6.1}$$

uniformly on bounded subsets in $\overline{\mathbb{C}}_+$. Moreover, for each $n \in \mathbb{Z}$

$$l_n(h^{(m)}) \to l_n(h), \quad \text{as} \quad m \to \infty. \tag{6.2}$$

Remark. *The uniform convergence of the harmonic functions on the bounded subsets yields the uniform convergence of their gradients on the bounded subsets. Hence, if all Conditions of Theorem 6.1 are fulfilled then*

$$grad\ \psi(k, h^m) \to grad\ \psi(k, h), \quad m \to \infty, \tag{6.3}$$

uniformly on bounded subsets in $K(h)$. Moreover, for each $n \in \mathbb{Z}$ we have the convergence (6.3) on the bounded subsets in K_n for harmonic continuations (left or right) across the interval $(u_n - i|h_n|, u_n + i|h_n|)$.

Proof. The functions $y_m(k)) = y(k, h^{(m)})$ and $y(k) = y(k, h)$ have the harmonic continuation across $\mathbb{R}$ in the plane $\mathbb{C}$ by the symmetry principle since $y_m|_{\mathbb{C}} = y|_{\mathbb{R}} = 0$. Moreover, each function y_m has the harmonic continuation across the interval $(u_n - i|h_n^{(m)}|, u_n + i|h_n^{(m)}|)$ in the plane $\mathbb{C}$ by the symmetry principle, since the function equals zero on this interval and is continuous in $\mathbb{C}$. The same is true for the functions $y(\cdot, h), \psi(\cdot, h), \psi(\cdot, h^{(m)})$.

We show that there exists some subsequence $\{\psi_{m_s}\}_{s=1}^{\infty}$ from $\{\psi_m\}$ such that:
1) this subsequence converges uniformly on bounded subsets in $K(h)$;
2) for each $n \in \mathbb{Z}$ a harmonic extensions of ψ_{m_s} (by the symmetry left or right) across the interval $(u_n - i|h_n^{(m)}|, u_n + i|h_n^{(m)}|)$, converges uniformly on bounded subsets in $K_n(h)$.

Let $\psi_m(k) = \psi(k, h^{(m)})$, $\psi(k) = \psi(k, h)$. Then ψ_m is the bounded solution of the Dirichlet problem in $K(h)$ and the following estimates are fulfilled:

$$\sup_{k=u+iv \in K(h^{(m)})} |y_m(k) - v| \leqslant H,$$

and we have uniform boundedness. Hence, $\psi_{m_l} \to \widetilde{\psi}$ as $l \to \infty$ uniformly on compact subsets in $G = \mathbb{C} \setminus \cup_{n \in \mathbb{Z}} \{u_n - i|h_n|, u_n + i|h_n|\}$. Hence, the principle of accumulation for harmonic function (see [G]) yields the existence such sequence since the functions $\{y_m\}$ are uniform bounded on the compact sets in $\mathbb{C}$. Then, the bounded function $\widetilde{\psi}$ is the solution of the Dirichlet problem in $K(h)$ with the boundary value $\beta = v$. Then $\widetilde{\psi} = \psi$ since the solution of the Dirichlet problem is unique. Note that our proof is fulfilled for any subsequence of $\{\psi_m\}$.

We prove the convergence near the tops of the vertical slits. Fix $n \in \mathbb{Z}$ for the case $h_n \neq 0$ and consider the sequence of the following functions

$$f_m(k) = \psi_m(k + (u_n + i|h_n^{(m)}|)), \quad f(k) = \psi(k + (u_n + i|h_n|)), \quad |k| \leqslant \frac{|h_n|}{2}.$$

These functions $f_m, m \geqslant N$, are harmonic in the domain $2^{-1}|h_n|\widetilde{D}(0)$ for some N and above we have showed that $f_m(k) \to f(k)$ uniformly on the circle $|k| = |h_n|/2$. Moreover,

$$\max_{k \in [-i|h_n|/2, 0]} |f_m(k) - f(k)| \leqslant ||h_n^m| - |h_n|| \to 0, \quad m \to \infty.$$

The maximum principle yields $f_m \to f$ uniformly on the disk $|k| \leqslant \frac{|h_n|}{2}$.

Consider the case $h_n = 0$. Define $a = \frac{d_n}{2}$. Then for large m we obtain $|h_n^{(m)}| < a$. The functions ψ and ψ_m are harmonic in the domain $G_1 = u_n + a\, D(\frac{|h_n^{(m)}|}{a})$ and the maximum principle yields

$$\max_{k\in\bar{G}_1} |\psi_m(k) - \psi(k)| \leqslant \max_{k\in\bar{G}_1,\, |k-n_n|=a} |\psi_m(k) - \psi(k)| + |h_n^{(m)}|,$$

since $0 \leqslant \psi = v - y \leqslant v$.

We show (6.2). By the definition,

$$l_n(\phi) = (z(u_n + 0, \phi) - z(u_n - 0, \phi))\, \mathrm{sign}\ \phi_n, \quad \phi \in \ell^\infty_{\mathbb{R}},$$

and note that

$$z'(k,\phi) = \left(1 - \frac{\partial\psi}{\partial v}(k,\phi)\right) + i\frac{\partial\psi}{\partial u}(k,\phi), \quad \phi \in \ell^\infty_{\mathbb{R}}. \tag{6.4}$$

Then the Remark for this Theorem implies

$$z(u_n + 0, h^m) - z(u_n - 0, h^m) = \int_\gamma z'(k, h^m)\, dk \to$$

$$\int_\gamma z'(k,h)\, dk = z(u_n + 0, h) - z(u_n - 0, h), \quad as \quad m \to \infty$$

where the piecewise smooth curve γ combines the point $u_n - 0$ on the left side of the slit $u = u_n$ with the point $u_n + 0$ on the right side of this slit and lies in the domain $K(h)$. Moreover, in the case $h_n \neq 0$ we have $\mathrm{sign}\, h_n^{(m)} \to \mathrm{sign}\, h_n$ as $m \to \infty$.□

We consider now the convergence of the comb mappings on the bounded subsets.

Corollary 6.2. *Let $h^{(m)} \in \ell^p_\omega, m \geqslant 1$, for some $1 \leqslant p \leqslant 2$ and the weight $w_n \geqslant 1, n \in \mathbb{Z}$. Assume that $h_n^{(m)} \to h_n$ as $m \to \infty$ for any $n \in \mathbb{Z}$ and $H = \sup_{m\geqslant 1} \|h^{(m)}\|_{p,\omega} < +\infty$. Then*

$$z(k, h^{(m)}) \to z(k,h), \quad as \quad m \to \infty, \tag{6.5}$$

uniformly on bounded subsets in $K(h)$. Moreover, for each $n \in \mathbb{Z}$ convergence (6.5) is fulfilled uniformly on bounded subsets in $K_n(h)$ for analytic extensions (left or right) across the interval $(u_n - i|h_n|, u_n + i|h_n|)$ and the following estimates are fulfilled:

$$A_n(h^{(m)}) \to A_n(h), \qquad as \qquad n \to \infty, \tag{6.6}$$

$$J_n(h^{(m)}) \to J_n(h). \qquad as \qquad n \to \infty. \tag{6.7}$$

Proof. It is clear that enough to prove the Theorem for the case $p = 2, \omega_n = 1, n \in \mathbb{Z}$. Due to identity (6.4) and the Remark for the Theorem 6.1 enough to prove convergence $z(k_0, h^m)$ to $z(k_0, h)$ for some point $k_0 \in K(h)$. The weak convergence implies

$$\|h^{(m)}\|_\infty \leqslant \|h^{(m)}\| \leqslant H < +\infty, \quad m \geqslant 1.$$

Take $H_0 > H$ and let $k_0 = (H_0+1)i$, $\phi \in \ell^p_\omega$. Then we have the function $f(k) = (k+iH_0) - z(k+iH_0, \phi, k \in \overline{\mathbb{C}}_+$. This function has the following properties

$$\operatorname{Im} f(k) \geqslant 0, \quad k \in \bar{\mathbb{C}}_+; \qquad f(iy) \sim \frac{Q_0}{iy}, \quad y \nearrow +\infty.$$

Then formula (2.4) yields

$$(k+iH_0) - z(k+iH_0, \phi) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\psi(x+iH_0, \phi)}{x-k} dx, \quad k \in \mathbb{C}_+; \quad Q_0 = \frac{1}{\pi} \int_{\mathbb{R}} v(x+iH_0, \phi) dx. \tag{6.8}$$

Applying formula (6.8) to the functions $z(k, h^{(m)}$, $z(k,h)$ at $k = k_0$ we obtain for any $B > 0$

$$\pi |z(k_0, h) - z(k_0, h^m)| \leqslant$$

$$2B \max_{x \in [-B,B]} |\psi(x+iA, h) - \psi(x+iH_0, h^m)| + \frac{1}{B} \int_{\mathbb{R}} \left(\psi(x+iA, h) + \psi(x+iH_0, h^m)\right) dx \leqslant$$

$$2B \max_{x \in [-B,B]} |\psi(x+iA, h) - \psi(x+iH_0, h^m)| + \frac{2}{B} H^2.$$

where we use the following estimates (see (5.6))

$$\int_{\mathbb{R}} \psi_m(x+iH_0)\, dx \leqslant \|h^m\|_2^2, \quad \int_{\mathbb{R}} \psi(x+iH_0)\, dx \leqslant \|h\|_2^2.$$

Hence since $\psi_m \to \psi$ converges uniformly on $[-B+iH_0, B+iH_0]$ and B is large enough we obtain (6.5).

In order to show (6.6) we take the large contour c_n. Then substituting (6.5) into (1.2) we have

$$A_n(h^{(m)}) = \frac{1}{\pi i} \int_{c_n} z(k, h^{(m)}) dk \to \frac{1}{\pi i} \int_{c_n} z(k,h) dk = A_n(h).$$

Asymptotics (6.6) yield (6.7). $\square$

We prove asymptotics of our parameters as $n \to \pm\infty$ and in order to do it we use some results from [K9]. For each $h \in \ell^\infty_{\mathbb{R}}$ there exists the following identity

$$v(x) = v_n(x)(1 + V_n(x)), \quad v_n(x) = |(x-z_n^+)(x-z_n^-)|^{1/2}, \quad x \in g_n, \tag{6.9}$$

(see [KK1]), where $k(z) = k(z,h) = u(z) + iv(z)$, $z = x+iy$, and

$$V_n(x) = \frac{1}{\pi} \int\limits_{\mathbb{R} \setminus g_n} \frac{v(t)}{|t-x|\, v_n(t)}\, dt, \quad x \in g_n = (z_n^-, z_n^+). \tag{6.10}$$

For each $n \in \mathbb{Z}$ there exists an unique point $z_n \in g_n$ such that $|h_n| = v(z_n)$. We have the following identities

$$V_n'(x) = \frac{1}{\pi} \int\limits_{\mathbb{R} \setminus g_n} \frac{v(t)\, dt}{(t-x)|t-x| v_n(t)}, \quad x \in g_n, \tag{6.11}$$

$$V_n''(x) = \frac{2}{\pi} \int\limits_{\mathbb{R}\setminus g_n} \frac{v(t)\,dt}{|t-x|^3\, v_n(t)} > 0, \qquad x \in g_n. \tag{6.12}$$

Then

$$|V_n'(x)| \leqslant \frac{V_n(x)}{s}, \quad 0 < V_n''(x) \leqslant \frac{2V_n(u)}{s^2}, \quad x \in g_n. \tag{6.13}$$

where $s = \min$ Recall that $\Lambda = \{n \in \mathbb{Z} : \ h_n = 0\}$ and note that if $n \in \Lambda$, then

$$k'(z_n) = 1 + \frac{1}{\pi} \int \frac{v(t)dt}{(t-z_n)^2} = 1 + V_n(z_n) \tag{6.14}$$

Introduce the space

$$C_0 = \{\xi \in \ell^\infty_{\mathbb{R}} : \ \xi_n \to 0, |n| \to \infty\}.$$

and the sequence $H = \{H_n, n \in \mathbb{Z}\}$, where

$$H_n = \frac{1}{\pi} \sum_{m \neq n} \frac{|h_m l_m|}{|m-n|^2}.$$

It is well known that if $\{l_n h_n\} \in \ell^p, p \geqslant 1$, then $H \in \ell^p$ and if $\{l_n h_n\} \in C_0$, then $H \in C_0$. First we find the asymptotics of V_n as $n \to \infty$.
Lemma 6.3._Let $h \in C_0$. Then_

$$\max_{x \in \gamma_n} V_n(x) \leqslant H_n s^{-2}, \qquad n \in \mathbb{Z}, \tag{6.15}$$

$$1 \leqslant k'(z_n) = 1 + V_n(z_n) \leqslant 1 + H_n s^{-2}, \qquad n \in \Lambda. \tag{6.16}$$

Let $h, h^0 \in \ell^2$ have the same components $h_n = h_n^0$ for $n \neq 1$ and $h_1 = t \to 0 = h_1^0$, where $t \in (0, r)$ for some small $r > 0$. Then

$$V_1(z_1(h), h) \to k'(z_1(h^0), h^0) - 1, \quad t \to 0. \tag{6.17}$$

Proof. Using Lemma 5.11 and formula (5.33) we obtain

$$V_n(x) = \frac{1}{\pi} \int_{\mathbb{R}\setminus g_n} \frac{v(t)}{|t-x|\, v_n(t)}\, dt \leqslant \frac{1}{\pi s^2} \sum_{m \neq n} \frac{|h_m||l_m|}{|m-n|^2} = \frac{H_n}{s^2}, \qquad x \in g_n,$$

and $H \in C_0$ since $\{l_n h_n\} \in C_0$. It means that $\max\limits_{x \in g_n} V_n(x) \to 0$ as $n \to \infty$. The following identity

$$k'(u_n) = 1 + \frac{1}{\pi} \int_{\mathbb{R}} \frac{v(t)dt}{(t-u_n)^2}, \quad n \in \Lambda,$$

yields

$$0 \leqslant k'(u_n) - 1 \leqslant \frac{1}{\pi s^2} \sum_{m \neq n} \frac{|l_m h_m|}{|m-n|^2} = \frac{H_n}{s^2}.$$

We prove (6.17). Due to Lemma 6.2 we deduce that $z_1^{\pm}(h) \to z_1(h^0) = z_1^0$ as $t \to 0$. Define the function

$$f(h) = \frac{1}{\pi} \int_{\mathbb{R} \setminus g_1} \frac{v(x)dx}{(x - z_1)^2}.$$

By Lemma 6.2, $V_1(z_1(h), h) - f(h) \to 0$ as $t \to 0$. Then enough to show that $f(h) - f(h^0) \to 0$ as $t \to 0$, where $f(h^0) = k'(z_1(h^0), h^0) - 1$.

Rewrite the set γ in the form $\gamma = \gamma^N \cup G_N, G_N = \cup_{|n|>N} \gamma_n$ and then

$$f(h) = f_N + F_N, \quad F_N = \frac{1}{\pi} \int_{G_N} \frac{v(x)dx}{(x - z_1)^2}.$$

Using Lemma 6.3, we obtain $(x - z_1(h))^2 > (Ns)^2$ for any $t \in (0, r)$. Then we have

$$F_N \leqslant \frac{\|h\|_\infty}{\pi} \int_{G_N} \frac{dx}{(x - z_1)^2} \leqslant \frac{\|h\|_\infty}{\pi N s}.$$

Then the value $F_N \to 0$ as $N \to \infty$ uniformly for $t \in (0, r)$ We take some large fixed N and F_N is small. In order to consider f_N as $t \to 0$ we rewrite the function f_N in the form

$$f_N(h) = \sum_{n \neq 1, |n| \leqslant N} \frac{1}{\pi} \int_{\gamma_n} \frac{v(x)dx}{(x - z_1)^2} = \frac{1}{2\pi i} \sum_{n \neq 1, |n| \leqslant N} \int_{c_n} \frac{dk}{(z(k, h) - z_1)}.$$

Using Lemma 6.2 we have $f_N(h) \to f_N(h^0)$ as $t \to 0$, and since N is large and fixed then (6.17) is proved. $\square$

Introduce the function $\delta_n(x) = x - a_n$ where $a_n = (z_n^- + z_n^+)/2$. We find the estimates and the asymptotics, here we use some results from [K9].

Theorem 6.4. *Let $h \in C_0$ (or $h \in \ell^p$, $1 \leqslant p < +\infty$); $u_{n+1} - u_n \geqslant u_* > 0$, $n \in \mathbb{Z}$. Then*

$$\delta_n(z_n) = \frac{v_n^2(z_n) V_n'(z_n)}{1 + V_n}. \tag{6.18}$$

$$|\frac{1}{2} l_n - v_n(z_n)| \leqslant v_n(z_n)^3 \frac{V_n'(z_n)^2}{(1 + V_n(z_n))^2}, \tag{6.19}$$

$$|h_n - \frac{1}{2} l_n(1 + V_n(z_n))| \leqslant |l_n/2|^3 \frac{V_n'(z_n)^2}{(1 + V_n(z_n))}, \tag{6.20}$$

$$|v_n(z_n) - \nu_n(1 + V_n(z_n))| \leqslant |\nu_n| v_n(z_n)^2 [3V_n'(z_n)^2 + V_n''(z_n)], \tag{6.21}$$

$$\frac{l_n \nu_n}{v_n^2(z_n)} = 2 + o(1), \quad as \quad n \to \infty, \ n \notin \Lambda, \tag{6.22}$$

Proof. We have $v'(z_n) = 0$ and $v'(x) = v_n'(x)(1 + V_n(x)) + v_n(x) V_n'(x)$. The derivative $(v_n^2)'(x) = -2\delta_n(x) = -2(x - a_n)$ yields (6.18). Using the identity $(l_n/2)^2 = v_n^2(x) + \delta_n^2(x)$ we obtain

$$\left(\frac{l_n}{2}\right)^2 = v_n^2(z_n) \left(1 + \frac{v_n^2(z_n)(V_n'(z_n))^2}{(1 + V_n(z_n))^2}\right), \tag{6.23}$$

which implies (6.19).

Inequality (6.19) together with the identity $h_n = v_n(z_n)(1+V_n(z_n))$ yield

$$|h_n - \frac{1}{2}l_n(1+V_n(z_n))| = (1+V_n(z_n))|(l_n/2) - v_n(z_n)| \leqslant v_n(z_n)^3 \frac{V_n'(z_n)^2}{(1+V_n(z_n))},$$

and we have (6.20).

Differentiating the function $v'(x)v_n(x)$ at z_n we obtain

$$(v_nv')' = \left(\left(\frac{v_n^2}{2}\right)'(1+V_n) + v_n^2V_n'\right)' = -(1+V_n) + (\frac{v_n^2}{2})\prime V_n' + +(v_n^2)'V_n' + v_n^2V_n''.$$

and the relation $(v_nv')' = v_nv''$ at z_n implies

$$v_n(z_n)v''(z_n) = -(1+V_n(z_n)) - 3V_n'(z_n)\delta_n(z_n) + v_n^2(z_n)V_n''(z_n),$$

and using the definition of the value ν_n we deduce that

$$v_n(z_n) = \nu_n[1+V_n(z_n) + 3V_n'(z_n)\delta_n(z_n) - v_n(z_n)^2V_n''(z_n)]. \tag{6.24}$$

Then identity (6.18), estimate $v_n(z_n) \leqslant l_n/2$ yield

$$|v_n(z_n) - \nu_n(1+V_n(z_n))| \leqslant |\nu_n|v_n(z_n)^2[3V_n'(z_n)^2 + V_n''(z_n)].$$

Moreover, inequalities (6.19-21), (6.13), (6.15) and the definition of the value ν_n imply (6.22). □

We find the asymptotics of the actions A_n.

Lemma 6.5. *For each $h \in l^\infty_{\mathbb{R}}$ and $n \in \mathbb{Z}$ the following estimates are fulfilled:*

$$|A_n - (l_n^2/4)(1+V_n(a_n))| \leqslant 2(l_n/4)^4 \max_{x\in\gamma_n}|V_n''(x)| \leqslant 2(l_n/4)^4\frac{H_n}{s^2}, \tag{6.25}$$

$$|A_n - \frac{1}{2}l_nh_n| \leqslant 2(l_n/2)^4[\max_{x\in\gamma_n}|V_n''(x)| + \max_{x\in\gamma_n}V_n'(x)^2], \tag{6.26}$$

$$|\partial_nA_n - 2\nu_n| \leqslant \frac{|\nu_n|}{2(\pi s)^2}H_n. \tag{6.27}$$

Proof. We use the Taylor's formula with integral remainder for the function $V_n(x)$:

$$V_n(x) = V_n(a_n) + V_n'(a_n)(x-a_n) + \int_{a_n}^x V_n'(s)(x-s)ds,$$

and substituting this identity into the next integral, we obtain

$$\pi A_n = \int_{\gamma_n} v(x)dx = \int_{\gamma_n} v_n(x)(1+V_n(x))dx =$$

$$\int_{\gamma_n} v_n(x)(1+V_n(a_n)+V_n'(a_n)(x-a_n))dx+\int_{\gamma_n} v_n(x)dx\int_{a_n}^{x} V_n''(s)(x-s)ds,$$

The first integral is simple

$$\int_{\gamma_n} v_n(x)(1+V(a_n)+V_n'(a_n)(x-a_n))dx=\pi(l_n/4)^2(1+V_n(a_n)),$$

Consider the second one. The function $V_n(x)'', x\in\gamma_n$, is positive then for some $x_n\in\gamma_n$ we get

$$\int_{\gamma_n} v_n(x)dx\int_{a_n}^{x} V_n'(s)(x-s)ds\leqslant V_n''(x_n)\int_{\gamma_n} v_n(x)dx\int_{a_n}^{x}(x-s)ds=\pi V_n''(x_n)(l_n/4)^4$$

which yields (6.25).

We have the following identity for some $x_1\in\gamma_n$:

$$h_n=(l_n/2)(1+V_n(a_n))+[h_n-(l_n/2)(1+V_n(z_n))]+(l_n/2)\delta(z_n)V_n'(x_1),$$

and substituting this relation into $|A_n-(h_nl_n/2)|$ and using (6.25), (6.20), (6.18) we obtain

$$|A_n-(h_nl_n/2)|\leqslant$$

$$|A_n-(l_n^2/4)(1+V_n(a_n))|+|l_n/2|[h_n-(l_n/2)(1+V_n(z_n))]+(l_n^2/4)|\delta(z_n)V_n'(x_1)|\leqslant$$
$$2(l_n/4)^4\max_{x\in\gamma_n}|V_n''(x)|+(l_n/2)^4V_n'(z_n)^2+(l_n/2)^4\max_{x\in\gamma_n}V_n'(x)^2$$

which yields (6.26).

Identities (7.13), (6.9) imply

$$|\partial_nA_n-2\nu_n|=\frac{|\nu_n|}{\pi}\int_{\gamma\setminus\gamma_n}\frac{v(t)dt}{(t-z_n)^2}\leqslant\frac{|\nu_n|}{\pi}\sum_{m\neq n}\frac{h_ml_m}{s^2(n-m)^2}=\frac{\nu_n}{\pi s^2}H_n.\quad\square$$

Consider now J_n.

Corollary 6.6. *For each $h\in l^\infty_{\mathbb{R}}$ and $n\in\mathbb{Z}$ the following estimates are fulfilled:*

$$|J_n-(l_n/2)\sqrt{1+V_n(a_n)}|\leqslant(l_n/4)^3\max_{x\in\gamma_n}|V_n''(x)|\leqslant\sqrt{2}(l_n/4)^3\frac{H_n}{s^2},\tag{6.28}$$

$$|J_n-\sqrt{l_nh_n/2}\,\mathrm{sign}\,h_n|\leqslant(l_n/2)^3[\max_{x\in\gamma_n}|V_n''(x)|+\max_{x\in\gamma_n}V_n'(x)^2],\tag{6.29}$$

$$|\partial_nJ_n-(\nu_n/J_n)|\leqslant\frac{\nu_n}{J_n}\frac{H_n}{(2\pi s)^2}.\tag{6.30}$$

Proof. Using inequality $A_n=J_n^2\geqslant l_n^2/4$ (see(2.35)) and estimates (6.25-26) we obtain (6.28-29).

Identity $\partial_nA_n=2J_n\partial_nJ_n$ and estimate (6.27) imply (6.30). $\square$

7 Derivatives and mappings

In this Section we assume $u_{n+1} - u_n \geq u_* > 0$, $n \in \mathbb{Z}$. Note, that for fixed $k \in \mathbb{C}$ the function $z(k,h)$, $h \in \ell^2_{\mathbb{R}}$ is even with respect to each variable h_n. Then in order to find the derivative $\partial_n z(k,h)$ we consider only the following case $h_m \geqslant 0, m \in \mathbb{Z}$.

Theorem 7.1. *i) Let $h \in \ell^2_{\mathbb{R}}$. Then for each $n \in \mathbb{Z}$ the derivatives have the following form*

$$\partial_n z(k,h) = 0, \quad h_n = 0, \qquad k \neq u_n, \ k \in \overline{\mathbb{C}}_+, \tag{7.1}$$

$$\partial_n z(k,h) = \frac{\nu_n}{(z(k,h) - z_n)}, \quad h_n \neq 0, \qquad k \neq u_n + ih_n, \ k \in \overline{\mathbb{C}}_+. \tag{7.2}$$

Moreover, formula (7.2) is true for $k \in [u_m, u_m + ih_m)$, if $z(k,h)$ is defined as the limit value from left (right).
ii) Let $h \in \ell^2$. Then

$$\partial_n Q_0(h) = \nu_n. \tag{7.3}$$

Proof. Recall that the function $z(k,h)$, $h \in \ell^2_{\mathbb{R}}$ is even with respect to each variable h_n for fixed $k \in K$. Then in order to find the derivatives $\partial_m z$ we consider only the case $h_m \geqslant 0$, $m \in \mathbb{Z}$.
i) First we consider $h_n > 0$. For $t > 0$ and $t \to 0$ we define the sequence

$$h^t = \{h^t_m\}_{m \in \mathbb{Z}}, \qquad h^t_m = \begin{cases} h_m, & \text{if} \quad m \neq n, \\ h_n - t & \text{if} \quad m = n, \end{cases}$$

and the functions

$$z^t(k) = z(k, h^t), \ z(k) = z(k,h), \qquad w^t(\xi) = z^t(k(\xi, h)), \ \xi \in \overline{\mathbb{C}}_+.$$

It is clear that the function w^t is the conformal mapping from $\mathbb{C}_+$ onto $\mathbb{C}_+$ with the cut S_t, which is the image of the segment $[u_n + i(h_n - t), \ u_n + ih_n]$ by the mapping z^t. It is easy to show that $w^t(iy) = iy + o(1)$, $y \to +\infty$ (see (2.36)), then the Herglotz yields the following formula

$$w^t(\xi) = \xi + \frac{1}{\pi} \int_{I_t} \frac{a^t(x)}{x - \xi} dx, \qquad \xi \in \overline{\mathbb{C}}_+ \setminus I_t,$$

where $a^t = \operatorname{Im}(w^t)$, $w^t(I_t) = S_l$. Hence, if $\xi = z(k)$, then we obtain

$$z^t(k) - z(k) = \frac{1}{\pi} \int_{I_t} \frac{a^t(x)}{x - z(k)} \, dx. \tag{7.4}$$

We fix some δ such that $0 < \delta^2 < \min\{u_*, h_n\}$ and consider the following functions

$$f(\xi, t) = z^t(- i\xi^2 + (u_n + i(h_n - t))), \quad f(\xi) = z(- i\xi^2 + (u_n + ih_n)), \quad \xi \in B_+(\delta),$$

where $B_+(\delta) = B(\delta) \cap \mathbb{C}_+$. Then the functions f and $f(t)$ for small t are analytic inside the semidisk $B_+(\delta)$ and continuous on the closed semidisk $\bar{B}_+(\delta)$. Note that $\operatorname{Im} f(x,t) =$

$\operatorname{Im} f(x) = 0$, $x \in [-\delta, \delta]$, and then these functions have the analytic extension in the disk with the radius δ and with zero as the centrum. Using Corollary 6.2, we obtain

$$f(\xi, t) \to f(\xi), \quad t \to 0, \qquad |\xi| = \delta,$$

and the maximum principle yields the uniform convergence in the disk with the radius δ. Note that by the Theorem Weierstrass, for fixed $r > 0$ we have the convergence:

$$f_\xi(\xi, t) \to f'(\xi), \quad f_{\xi\xi}(\xi, t) \to f''(\xi), \quad t \to 0, \qquad |\xi| \leqslant r < \delta.$$

Hence

$$|f(\xi, t) - f(0, t) - f_\xi(0, t)\xi| \leqslant C|\xi|^2, \quad |\xi| \leqslant r < \sqrt{\delta},$$

where the constant C does not depend from t. Then using the identity $f'(0) = \sqrt{\nu_n}$, we obtain at $\xi = i\sqrt{s}$, $s \in (0, t)$:

$$\operatorname{Im} z^t(p_n + is) = \sqrt{2\nu_n}\sqrt{s} + o(s), \quad p_n = u_n + i(h_n - t) \quad 0 \leqslant s \leqslant t, \quad t \to 0.$$

Then due to this fact we have $|I_t| \to 0$ as $t \to 0$. Define the functions $z'_\pm(\xi) = z'_\pm(\xi_\pm)$, $\xi \in [u_n,\, u_n + ih_n)$. Hence we obtain

$$\begin{aligned}
\int_{I_t} a^t(x)dx &= \int_0^t \operatorname{Im} z^t(p_n + is)\left(|z'_+(p_n + is)| + |z'_-(p_n + is)|\right) ds \\
&= \int_0^t \left(\sqrt{2\nu_n}\sqrt{s} + o(s)\right) \times \left(\sqrt{\frac{2\nu_n}{(t-s)}} + o(1)\right) ds \\
&= 2\nu_n \int_0^t \sqrt{s/(t-s)}ds + o(t),
\end{aligned}$$

and using

$$\int_0^t \sqrt{\frac{s}{(t-s)}}ds = \frac{\pi}{2}t,$$

we have

$$\frac{1}{\pi}\int_{I_l} a^t(x)dx = t\nu_n + o(t). \tag{7.5}$$

Then using the formula (7.4) and $|I_t| \to 0$, we have

$$\frac{z^t(k) - z(k)}{-t} = \frac{\nu_n}{z(k, h) - z_n} + o(1),$$

which yields (7.2). The function $z(k, \cdot)$ is even with respect to the variable h_n and then we have (7.1).

ii) Asymptotics (7.5) and the definition of the value Q_0 imply

$$Q_0(h^t) - Q_0(h) = (-t)\nu_n + o(t),$$

which yields (7.3) at $h_n \neq 0$. Formula (7.3) at $h_n = 0$ follows from the real analyticity and evenness of the function $Q_0(h)$ with respect to any variable h_n. $\square$

We prove the following main result.

Proof of Theorem 2.3. By Corollary 4.2, the functional $Q_0 : \ell^2_{\mathbb{R}} \to \mathbb{R}_+$ has the analytic continuation in the domain $\mathcal{J}^p_\omega(\rho)$. Using (4.3-4) we have (2.8), and (5.6) yields (2.9). $\square$

We prove the next main result.

Proof of Theorem 2.4. By Corollary 4.3, we get the needed the analytic continuation of $E(h)$ and estimate (2.12). Formula (7.9) implies (2.14), and (5.19) gives (2.11). and (5.21) yields (2.13). $\square$

We prove again the following main result.

Proof of Theorem 2.5. Using Theorem 4.5 we obtain the needed the analytic continuation of $z(k,h)$. Identity (7.3) implies (2.15). $\square$

Below we need simple result from the operator theory.

Lemma 7.2. *Let $h \in \ell^p_\omega$ for some $1 \leq p \leq 2$ and some weight $\omega = \{\omega_n, n \in \mathbb{Z}\}$, where $\omega_n \geqslant 1$. Then the operator $B : \ell^p_\omega \to \ell^p_\omega$ which is defined by the following formula:*

$$(Bf)_m = \sum_{n \neq m} h_n h_m (n-m)^{-2} f_n, \quad f \in \ell^p_\omega,$$

is compact. Moreover, this operator B is nucleus if $p = 2$.

Recall that $v_n(x) = |(z_n^+ - x)(x - z_n^-)|^{1/2}, x \in g_n$ Below we need the following result about uniqueness.

Theorem 7.3. *Let $h \in \ell^\infty$ and $h_n \to 0$ as $|n| \to \infty$. Assume operator $B : \ell^\infty \to \ell^\infty$ is defined by the formula $(Bf)_m = \sum B_{m,n} f_n$, where real coefficients $B_{m,n}$ satisfy the following conditions:*

i) for each $n \in \mathbb{Z}$ the number $B_{n,n} > 0$ and $\sup B_{n,n} < \infty$,

ii) if $|g_n| \neq 0$, then $B_{n,n} \geqslant \alpha_n |l_n| / v_n(z_n)^2$, where the number α_n such that: $\sup \alpha_n < \infty$, and the number $\alpha_n = 0$ if $|\gamma_n| = 0$ and $\alpha_n > 0$ if $|\gamma_n| > 0$,

iii) if $m \neq n$, then $|B_{m,n}| \leqslant \alpha_m |l_n| / v_n(z_m)^2$,

Let $f \in \ell^\infty$ is the solution of the equation $Bf = 0$. Then $f = 0$.

Proof. Let $n \in \Lambda$, then Condition iii) yields $B_{n,n} f_n = 0$ and hence $f_n = 0$.

Assume that there exists only one nondegenerate gap then Conditions i), iii) ipmly $f = 0$.

Let the number of the nondegenerate gaps is greater than one. First we show that $f_n \to 0$ as $|n| \to \infty$. Using the identity $B_{m,m} f_m = \sum_{n \neq m} B_{m,n} f_n$ and Conditions ii)-iii) at $m \notin \Lambda$ we obtain the estimate

$$\frac{\alpha_m |l_m|}{v_m^2} |f_m| \leqslant \alpha_m |\sum_{n \neq m} B_{m,n} f_n| \leqslant$$

$$\alpha_m \|f\|_\infty \sum_{n \neq m} \int_{g_n} \frac{dt}{(t - \lambda_m)^2} < \alpha_m \|f\|_\infty \int_{|t| \geqslant s} \frac{dt}{t^2} = \alpha_m \|f\|_\infty (2/s),$$

which yields $|f_n| \leqslant |l_n| \|f\|_\infty (2/s)$ and then $f_n \to 0$ as $|n| \to \infty$.

Second, without loss of generality, we assume that $f_1 = \sup |f_n| = 1$. Then Condition iii) implies

$$B_{1,1} = -\sum_{n\neq 1} B_{1,n} f_n \leqslant \sum_{n\neq 1} \alpha_1 \int_{\gamma_n} \frac{dt}{(t-\lambda_1)^2} <$$

$$\alpha_1 [\int_{-\infty}^{a_1^-} \frac{dt}{(t-\lambda_1)^2} + \int_{a_1^+}^{-\infty} \frac{dt}{(t-\lambda_1)^2}] = \alpha_1 [\frac{1}{(a_1^+ - \lambda_1)} + \frac{1}{(\lambda_1 - a_1^-)}] = \frac{\alpha_1 |l_1|}{v_1(z_1)^2},$$

which contradicts Condition ii) and hence $f = 0$. $\square$

We find the Frechet derivatives for the mapping $l : \ell^p_\omega \to \ell^p_\omega$. This Theorem will give the formulas of the derivatives of needed parameters. Recall $\Lambda(h) = \{n \in \mathbb{Z} : \quad h_n = 0\}$ and let I - the identity operator in ℓ^p_ω.

Theorem 7.4. *i) Let $h \in \ell^p_\omega$ for some $1 \le p \le 2$ and the weight $\omega_n \geqslant 1$. Then the operator $l' = l'(h)$ has the following matrix in the canonical basis:*

$$l'_{m,n} = \partial_n l_m(h) = -\frac{l_m \nu_n}{v_m(z_n)^2}, \quad m \neq n, \tag{7.6}$$

$$l'_{n,n} = \partial_n l_n(h) = \begin{cases} 2z'(u_n, h), & \text{if } \ h_n = 0, \\ l_n \nu_n / v_n(z_n)^2, & \text{if } h_n \neq 0, \end{cases} \tag{7.7}$$

$$l'_{n,n} = 2 + o(1), \quad n \to \pm\infty. \tag{7.8}$$

Moreover, the operator $l'(h) - 2I$ is compact and the operator $l'(h)$ has an inverse in ℓ^p_ω.
ii) Let $h \in \ell^1$ for some weight $\omega_n \geqslant 1, n \in \mathbb{Z}$. Then

$$\partial_n E(h) = \begin{cases} \nu_n \cdot \int_\sigma \frac{dt}{(t-z_n)^2}, & \text{if } \quad h_n \neq 0, \quad \sigma = \sigma(h, U), \\ 2z'(u_n, h), & \text{if } \quad \text{if } h_n = 0. \end{cases} \tag{7.9}$$

Proof. i) If $h_n > 0$, then $l_m = z(u_m + 0) - z(u_m - 0)$ and formula (7.2) yields

$$\partial_n l_m(h) = \nu_n \left(\frac{1}{z(u_m + 0) - z_n} - \frac{1}{z(u_m - 0) - z_n} \right)$$

$$= \nu_n \left(\frac{1}{a_m^+ - z_n} - \frac{1}{a_m^- - z_n} \right) = -\frac{(a_m^+ - a_m^-)\nu_n}{(a_m^+ - z_n)(a_m^- - z_n)} = -\frac{\nu_n l_m}{v_n^2(z_n)}.$$

If $h_n < 0$, then using oddness of the function $l_m(h)$ with respect to the variable h_n and evenness with respect to each variable h_m, $m \neq n$, we get (7.6).

Consider the case $h_n = 0$. If $m \neq n$, then in order to prove (7.6) we use formula (7.6). Hence we have to consider the case $m = n, h_n = 0$.

If $h_n = 0$, then asymptotics (6.20), (6.17) and identity (6.16) yield the derivative $\partial_n l_n(h) = 2z(u_n, h)$.

Lemma 6.3 implies $|z_m^\pm - z_n| \ge s|m - n|$, $m \neq n$, for some $s > 0$. Then using (7.6), (1.1), (2.32) we deduce that

$$|(l'(h))_{m,n}| \leqslant \frac{|h_m h_n|}{2s^2(n-m)^2}, \quad m \neq n,$$

and (6.22) implies asymptotics (7.8). Hence all Conditions of Lemma 7.2 are fulfilled and the operator $l'(h) - 2I$ is compact.

We prove the invertibility of the operator $l'(h)$ in ℓ^p_ω, for any $h \in \ell^p_\omega$. The operator $l'(h) - 2I$ is compact, then by the Fredholm Theorem, enough to show that the equation

$$(l'(h))^* f = 0, \quad f \in (\ell^p_\omega)^* = \ell^q_\omega, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{7.10}$$

has only zero solution. We use the proof by contradiction. Assume that Eq. (7.10) has the solution $f \neq 0$. Then using the form of the operator $l'(h)$ we obtain the equation

$$(\partial_m l_m) f_m + \sum_{n \neq m} \nu_m l_n v_n (z_m)^{-2} f_n = 0, \quad f \in \ell^q_\omega.$$

Above we show the estimate $|B_{m,n}| \leqslant |h_m h_n| (2s^2(n-m)^2)^{-1}$, $m \neq n$, and the asymptotics $B_{n,n} = 2 + o(1), n \to \infty$, then $f \in \ell^\infty$. For the matrix $B_{m,n} = \nu_m l_n v_n (z_m)^{-2}, m \neq n$, and the coefficients $B_{n,n}$ (see (7.6-8)) all Conditions of Lemma 7.3 at $\alpha_n = \nu_n$ are fulfilled and hence $f = 0$.

ii) We have $E = \sum l_n$ and by Corollary 4.3, the series (which defines E) can been differentiated term by term. Then if $h_n = 0$, then the first point i) of this Theorem yields

$$\partial_n E(h) = \sum_{m \in \mathbb{Z}} \partial_n l_m(h) = 2z'(u_n, h).$$

if $h_n \neq 0$, then

$$\partial_n E(h) = \nu_n \sum_{m \in \mathbb{Z}} \left(\frac{1}{z_m^+ - z_n} - \frac{1}{z_m^- - z_n} \right) = \nu_n \sum_{m \in \mathbb{Z}} \int_{z_m^-}^{z_m^+} \frac{1}{(x - z_n)^2} dx.$$

which implies (7.9). □

Introduce the contour

$$c_n = \{k : \mathrm{dist}(k, \Gamma_n) = u_*/2\} \subset K(h).$$

Consider now the function $A_n(h)$.

Lemma 7.6. *For each $1 \leq p \leq 2$ and the weight $\omega_n \geqslant 1$ the functional $A_n : \ell^p_\omega \to \mathbb{R}$ has analytic continuation in the domain $\mathcal{J}^p_\omega(\rho)$ for some $\rho > 0$ by the formula*

$$A_n(h) = \frac{1}{\pi i} \int_{c_n} z(k, h) dk, \ \ h \in \mathcal{J}^p_\omega(\rho). \tag{7.11}$$

Moreover, for any $m \in \mathbb{Z}$ the derivative has the following form:

$$\partial_m A_n(h) = \frac{\nu_m}{\pi i} \int_{c_n} \frac{dk}{z(k, h) - z_m}, \ \ h \in \mathcal{J}^p_\omega(\rho), \tag{7.12}$$

$$\partial_n A_n(h) = 2\nu_n + \frac{\nu_n}{\pi}\int_{\gamma\setminus\gamma_n}\frac{v(t)dt}{(t-z_n)^2},\ \ h \in \ell^p_\omega, \tag{7.13}$$

$$\partial_m A_n(h) = \frac{2\nu_m}{\pi}\int_{z_n^-}^{z_n^+}\frac{v'(x,h)dx}{(z_m - x)}, \qquad h \in \ell^p_\omega, \tag{7.14}$$

$$\partial_m A_n(h) = \frac{2\nu_m}{\pi}\int_{z_n^-}^{z_n^+}\frac{v(x,h)dx}{(x-z_m)^2}, \quad m \neq n,\ \ h \in \ell^p_\omega, \tag{7.15}$$

Proof. Let j_n be the contour (counterclockwise) around the slit g_n. The integration by parts (for real h) implies

$$A_n(h) = -\frac{1}{\pi i}\int_{j_n} k(z)dz = \frac{1}{\pi i}\int_{c_n} z(k,h)dk,$$

and the analyticity of the integrand function $z(k,h)$ with respect to $h \in \mathcal{J}^p_\omega(\rho)$ (se Theorem 4.5) yields the analyticity of the function A_n in $\mathcal{J}^p_\omega(\rho)$.

Using (7.11) and (7.2) we obtain (7.12).

Substituting (2.38) into (7.12) we have

$$\partial_n A_n(h) = \frac{\nu_n}{\pi i}\int_{j_n}[1 + \frac{1}{2\pi}\int_g \frac{v(t)dt}{(t-z)^2}]\frac{dz}{z-z_n},$$

then computing the residue and changing the order of the integration we obtain

$$\partial_n A_n(h) - \nu_n = \frac{\nu_n}{2\pi^2 i}\int_{j_n}\frac{dz}{z-z_n}\int_g \frac{v(t)dt}{(t-z)^2} =$$

$$\frac{\nu_n}{2\pi^2 i}\int_g v(t)dt\int_{j_n}\frac{dz}{(z-z_n)(t-z)^2},$$

Decomposing the integral into two ones we have

$$\partial_n A_n(h) - 2\nu_n = \frac{\nu_n}{2\pi^2 i}\int_{g_n} v(t)dt\int_{j_n}\frac{dz}{(z-z_n)(t-z)^2} + \frac{\nu_n}{2\pi^2 i}\int_{g\setminus g_n} v(t)dt\int_{j_n}\frac{dz}{(z-z_n)(t-z)^2} =$$

$$\frac{\nu_n}{2\pi^2 i}\int_{g\setminus g_n} v(t)dt\int_{j_n}\frac{dz}{(z-z_n)(t-z)^2} = \frac{\nu_n}{\pi}\int_{g\setminus g_n}\frac{v(t)dt}{(t-z_n)^2}$$

since the first equals zero.

Identity (7.12) yields

$$\partial_m A_n(h) = \frac{\nu_m}{\pi i}\int_{d_n}\frac{k'(z)dz}{z-z_m} = -2\nu_n\int_{z_n^-}^{z_n^+}\frac{v'(x,h)dx}{(x-z_m)},$$

which implies (7.14). The integration by parts in (7.13) gives (7.15). □

Consider now the mapping J.

Lemma 7.7. *For each $1 \le p \le 2$ and the weight $\omega_n \geqslant 1$ the function $J : \ell^p_\omega \to \mathbb{R}$ is real analytic For any $h \in \ell^p_\omega$ the operator $J'(h) = J'$ has the following matrix in the canonical basis:*

$$J'_{m,n} = \partial_n J_m = \frac{1}{2J_m}\partial_n A_m(h), \qquad h_m \neq 0, \tag{7.16}$$

$$J'_{m,n} = \partial_n J_m = 0, \quad if \quad h_m h_n = 0, \quad m \neq n, \tag{7.17}$$

$$J'_{n,n} = \sqrt{z'(u_n, h)}, \quad h_n = 0, \tag{7.18}$$

$$J'_{n,n} = 1 + o(1), \quad |n| \to \infty. \tag{7.19}$$

Moreover, the operator $J'(h) - I$ is compact and the operator $J'(h)$ has an inverse in ℓ^p_ω.
Proof. By Thoerem 4.4, the mapping $J : \ell^p_\omega \to \mathbb{R}$ is real analytic.

Identity $J_m^2 = A_m$ and the analyticity of A_m, J_m imply (7.16).

Evenness of the function $J_m(h)$ with respect to $h_n, m \neq n$, yields (7.17).

Using (6.20), (6.28) we obtain the asymptotics $J_n(h)$ as $h_n \to 0$:

$$J_n(h) - \frac{h_n}{\sqrt{(1 + V_n(z_n))}} = O(h_n^3), \quad h \to 0.$$

which together with (6.17) gives (7.17).

Using (6.16) we have the asymptotics (7.18) as $n \in \Lambda$.

Using (6.19), (6.21), (6.28) we obtain $\nu_n / J_n = 1 + o(1)$, as $|n| \to \infty, h_n \neq 0$. Then estimate (6.30) gives (7.18) as $n \notin \Lambda$.

In order to prove compactness we need the following inequality

$$|J'_{m,n}| \leqslant \frac{|h_n h_m|}{s^2(n-m)^2}, \qquad m \neq n. \tag{7.20}$$

If $h_m = 0$ or $h_n = 0$, then due to identity (7.16) we have $J'_{m,n} = 0$ and inequality (7.19) is true.

Let $h_m \neq 0$ and $h_n \neq 0$. Using Lemma 6.3 we get $|a_m^\pm - z_n| \geq s|m-n|$, $m \neq n$, for some $s > 0$. Hence (7.14), (2.32) yield

$$|J'_{m,n}| \leqslant |\partial_n J_m(h)| = \frac{|\nu_n|}{2\pi |J_m|} \int_{a_m^-}^{a_m^+} \frac{v(x,h)dx}{(x - z_n)^2}$$

$$\frac{|\nu_n|}{|J_m| s^2 (n-m)^2} \int_{a_m^-}^{a_m^+} v(x,h)\frac{dx}{\pi} = \frac{|\nu_n J_m|}{2s^2(n-m)^2} \leqslant \frac{|h_n h_m|}{s^2(n-m)^2}.$$

and (7.19) is proved. Then all Conditions of Lemma 7.2 are fulfilled and the operator $J'(h) - I$ is compact.

We prove the invertibility of the operator $J'(h)$ ℓ^p_ω for any $h \in \ell^p_\omega$. The operator $J'(h) - I$ is compact, then by the Fredholm Theorem, enough to show that the equation

$$(J'(h))^* f = 0, \quad f \in (\ell^p_\omega)^* = \ell^q_\omega, \quad (1/p) + (1/q) = 1, \tag{7.21}$$

has only zero solution. We use the proof by contradiction. Assume that Eq. (7.20) has the solution $f \neq 0$. Then using the form of the operator $J'(h)$ we obtain the equation

$$(\partial_m J_m) f_m + \sum_{n \neq m} \partial_m J_n f_n = 0, \quad f \in \ell^q_\omega,$$

then the inequality $J'_{n,n} \neq 0$ and estimate (7.18) imply $f_n = 0, n \in \Lambda$. Then without loss of generality we assume $\Lambda = \emptyset$. In this case the operator $[J'(h)]^*$ has the following matrix

$$J'_{m,n} = \partial_m J_n = \frac{1}{2J_n} \partial_m A_n = \frac{\nu_m}{J_n \pi} \int_{\gamma_n} \frac{v(x)dx}{(x - z_m)^2}.$$

Using (6.11) and (2.32) we obtain the inequality for $h_n \neq 0$:

$$h_n^2 \leqslant (h_n l_n/2)[1 + \|h\|_\infty/(\pi s)] \leqslant \pi J_n^2 [1 + \|h\|_\infty/(\pi s)].$$

Then we deduce that $\xi = \{\xi_n, n \in \mathbb{Z}\}, \xi_n = f_n h_n/(\pi J_n)$, belongs to ℓ^q_ω and for ξ the following equation is fulfilled:

$$B_{m,m}\xi_m + \sum_{n \neq m} B_{m,n}\xi_n = 0, \quad \xi \in \ell^q_\omega, \quad B_{m,n} = \frac{\pi}{h_n} \partial_m A_n. \tag{7.22}$$

We check all Conditions in Lemma 7.3 for the matrix B.

i) Using (7.17), (7.15), (7.13) we have $B_{n,n} > 0$ and the asymptotics (7.18), (6.28) imply

$$B_{n,n} = \frac{\pi J_n}{h_n} J'_{n,n} = \pi + o(1), \quad |n| \to \infty.$$

ii) Identity (7.13) yields

$$B_{n,n} = \frac{\nu_n}{h_n} \int_{\gamma_n} \frac{v(x)'dx}{(z_m - x)} =$$

$$\frac{\nu_n}{h_n} [\int_{z_n^-}^{z_n} \frac{v(x)'dx}{(z_m - x)} + \int_{z_n}^{z_n^+} \frac{v(x)'dx}{(z_m - x)}] \geqslant \frac{\nu_n l_n}{v_n(z_n)^2}.$$

iii) Formula (7.14) implies

$$|B_{m,n}| = |\frac{\nu_m}{h_n}| \int_{\gamma_n} \frac{v(x)dx}{(z_m - x)^2} \leqslant |\nu_m| \int_{\gamma_n} \frac{dx}{(z_m - x)^2} = \frac{\nu_m l_n}{v_n(z_n)^2}.$$

Then All Condition of Lemma 7.3 are fulfilled , $\xi = 0$ and hence $f = 0$.□

8 Mappings

In order to prove the basic results about real analytic isomorphism we need the following fact from nonlinear functional analysis.

Theorem 8.1. *Let $X,\ X_1$ be real Banach spaces equipped with norms $\|\cdot\|, \|\cdot\|_1$. Assume that there exist the Schauder basis $\{u_n\}_{n=1}^{\infty},\ \{e_n\}_{n=1}^{\infty}$ in the spaces $X,\ X_1$ such that*

$$x = \sum_{-\infty}^{\infty} \widetilde{u}_m(x) \cdot u_m, \quad x \in X; \qquad y = \sum^{\infty} \widetilde{e}_m(y) \cdot e_m, \quad y \in X_1.$$

where $\{\widetilde{u}_n\}_{-\infty}^{\infty},\ \widetilde{u}_n \in X^*,\ \ \{\widetilde{e}_n\}_{-\infty}^{\infty},\ \widetilde{e}_n \in X_1^*$ are the sequences of the functionals defined by these basis.

Suppose that the mapping $f : X \to X_1$ satisfies the following conditions:
1) f is real analytic (or of class $C^p, p \geqslant 1$),
2) for each $x \in X$ the operator $f'(x)$ has an inverse,
3) there is a nondecreasing function $g : [0, \infty) \to [0, \infty),\ \ g(0) = 0$ such that $\|x\| \leqslant g(\|f(x)\|_1)$, for all $x \in X$,
4) for each bounded sequence $\{x_n\}_{n=1}^{\infty} \subset X$ there exist subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ and x from X such that for any $m \in \mathbb{Z}$

$$\lim_{k\to\infty} \widetilde{e}_m(f(x_{n_k})) = \widetilde{e}_m(f(x));$$

5) if $\widetilde{e}_m(f(x)) = 0$, for some $x \in X,\ m\nu\mathbb{Z}$ then $\widetilde{u}_m(x) = 0$.
Then f is a real analytic (respectively, C^p-) isomorphism between X, X_1.

Proof. Using Conditions 1), 2) and the inverse function theorem, we see that the set $f(X)$ is open. We prove that it is also closed .

Let $f_m(\cdot) = \widetilde{e}_m(f(\cdot)) : X \to \mathbb{R}$ be the component of the mapping f with number m. Suppose that $y_n = f(x_n) \to y$ strongly as $n \to \infty$. Then Condition 3) yields $\|x_n\| \leqslant g(\|y_n\|_1) \leqslant g(\sup_{n\geqslant 1} \|y_n\|_1)$. Hence by Condition 4), there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ and some $x \in X$ such that for any $m \in \mathbb{Z}$ we have $f_m(x_{n_k}) \to f_m(x)$ as $k \to \infty$. The continuity of the functional $\widetilde{e}_m$ yields $\widetilde{e}_m(f(x_{n_k})) = f_m(x_{n_k}) \to \widetilde{e}_m(y)$, as $k \to \infty$. Then $\widetilde{e}_m(y) = \widetilde{e}_m(f(x))$ for any $m \in \mathbb{Z}$, and then $y = f(x)$. Hence $f(X) = X_1$ since the set $f(X)$ is open and closed simultaneously but X_1 is connected.

We show that f is an injection. We introduce the sets

$$K_m = \{h \in X_1 :\ \widetilde{e}_n(h) = 0,\ |n| > m\} \subset X_1, \quad M_m = f^{-1}(K_m) \subset X_1,$$

$$L_m = \{h \in X :\ \widetilde{u}_n(h) = 0,\ |n| > m\} \subset X,$$

The map f is a smooth local isomorphism so that M_m is a real smooth submanifold of X of dimension m. Note that L_m is the subspace in X of dimension m and by Condition 5), $M_m \subset L_m$. Moreover, for each bounded sequence $\{x_n\}_{n=1}^{\infty} \subset M_m$ there exist subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ which converge to some $x \in M_m$. Denoting by E_m the set of points in K_m that have more than one preimage, we see that E_m is open because f is a local isomorphism. But E_m is also closed. Indeed, suppose there are distinct points $x_n \neq t_n$ in M_m such that $f(x_n) = f(t_n) \to z$ as $n \to \infty$. Then, by Condition 3), the sequences $\{x_n\}_{n=1}^{\infty},\ \{t_n\}_{n=1}^{\infty}$ are bounded. Then there exist subsequences such that

$$\lim_{k\to\infty} x_{n_k} = x, \quad \lim_{k\to\infty} t_{n_k} = t,$$

for some x, t. If $x = y$, then $x_{n_k} = t_{n_k}$ for large k, since the map f is a local homeomorphism. Hence $x \neq t, f(x) = f(t) = z$ since f is continuous. Then $z \in E_m$ and E_m is closed. But $0 \notin E_m$, whence $E_m = \emptyset$. Thus, $f : M_m \to K_m$ is an isomorphism.

Suppose that $f : X \to X_1$ is not an injection. Then some point $z \in X_1$ has at least two pre-images. Since f is a local homeomorphism, the same is true of every point in some neighbourhood of z. In this neighborhood there exists

$$z_N = \sum_{m=-N}^{N} \widetilde{e}_m(z) e_m \in K_N$$

for some large $N \in \mathbb{N}$. But this contradicts the fact that $f : M_N \to K_N$ is an isomorphism. □

Remark 1. Instead of Condition 2 in Theorem 8.1 it is possible to write the following

Condition 2') For each $x \in X$ the operator $f'(x)$ is Fredholm and $\operatorname{Ker} f'(x) = 0$ *(or* $\operatorname{Ker}(f'(x))^* = 0$*).*

Indeed Condition 2 follows from Condition 2') since we have the Fredholm Theorem.

Remark 2. Condition 4) in Theorem 8.1 follows from the following assumption.

Condition 4') The space X is reflexive and for each $n \geqslant 1$ the coordinate function $f_n(x) = \widetilde{e}_n(f(x))$ is compact, i.e. if $x_m \to x$ weakly then $f_n(x_m) \to f(x)$ as $m \to \infty$.

Remark 3. This Theorem for the case of the Hilbert space was proved in the paper [KK1], the proof of Theorem 8.1 repeats really the proof in [KK1]. In order to give the detail picture only, we refresh this result.

We prove now our basic results. First we consider the mapping $l : \ \ell^p_\omega \to \ell^p_\omega$ and we prove Theorem 2.1 with few addisional results.

Theorem 8.2. *Let $u_{n+1} - u_n \geq u_* > 0$ for any $n \in \mathbb{Z}$ and let $\omega = \{\omega_n\}_{n\in\mathbb{Z}}, \omega_n \geq 1, n \in \mathbb{Z}$ be some weight. Then for each $1 \leq p \leq 2$ we have*

i) the mapping $l : \ \ell^p_\omega \to \ell^p_\omega$ is real analytic and has analytic continuation in the domain $\mathcal{J}^p_\omega(\rho)$, where estimates (2.1-2) are fulfilled,

ii) for each $h \in \ell^p_\omega$ the operator $l'(h)$ has an inverse,

iii) estimates (2.2-3) are fulfilled,

iv) for each bounded sequence $h^{(n)}, n \geqslant 1$, from ℓ^p_ω there exist subsequence $h^{(n_k)}$ and $h \in \ell^p_\omega$ such that $l_m(h^{(n_k)}) \to l_m(h)$ as $k \to \infty$ for any $m \in \mathbb{Z}$.

Moreover, the mapping $l : \ \ell^p_\omega \to \ell^p_\omega$ is a real analytic isomorphism.

Proof. Apply Theorem 8.1 for the case $X = X_1 = \ell^p_\omega$ and the mapping $f = l$ and we take the canonical basis ℓ^p_ω in the capacity of the Schauder basis. In order to get Condition 1)-4) Theorem 8.1 we check all statements i)- iv) of Theorem 8.2 (see also Remark 1 after Theorem 8.1). Last Condition 5) in Theorem 8.1 is very simple since if $l_n(h) = 0$, then by the definition, the gap $\gamma_n(h)$ is empty and $h_n = 0$, hence Condition 5) is true.

i) Using Theorem 4.4 and Corollary 4.3, we obtain (2.1-2) and that the mapping $l : \ \ell^p_\omega \to \ell^p_\omega$ is real analyticand has analytic continuation in the domain $\mathcal{J}^p_\omega(\rho)$.

ii) By Theorem 7.4, the operator $l'(h)$ has an inverse for any $h \in \ell^p_\omega$. Recall that here we use Remark 1 after Theorem 8.1.

iii) Theorem 5.12 implies estimate (2.3) and (5.21), (1.1) yield (2.4).

iv) Let $h^{(n)}, n \geqslant 1$, be some bounded sequence from ℓ^p_ω. Then using the "diagonal process", we obtain some subsequence $h^{(n_k)}, k \geqslant 1$, such that $\lim h^{(n_k)}_m = h_m$, for any $m \in \mathbb{Z}$ as $k \to \infty$ and for some $h \in \ell^p_\omega$. Using Theorem 6.1 we obtain statement iv). $\square$

Consider now the mapping J and we prove Theorem 2.2 with few addisional results.

Theorem 8.3. *Let $u_{n+1} - u_n \geq u_* > 0,\ n \in \mathbb{Z}$ and let $\omega = \{\omega_n\}_{n\in\mathbb{Z}}, \omega_n \geq 1, n \in \mathbb{Z}$ be some weight. Then for some each $1 \leq p \leq 2$ we have*
i) the mapping $J: \ \ell^p_\omega \to \ell^p_\omega$ is real analytic,
ii) for each $h \in \ell^p_\omega$ the operator $J'(h)$ has an inverse,
3) estimates (2.5-7) are fulfilled,
4) for each bounded sequence $h^{(n)}, n \geqslant 1$, from ℓ^p_ω there exist subsequence $h^{(n_k)}, k \geqslant 1$, and $h \in \ell^p_\omega$ such that $J_m(h^{(n_k)}) \to J_m(h)$ as $k \to \infty$ for any $m \in \mathbb{Z}$.

Moreover, the mapping $J: \ \ell^p_\omega \to \ell^p_\omega$ is a real analytic isomorphism.

Proof. Apply Theorem 8.1 for the case $X = X_1 = \ell^p_\omega$ and the mapping $f = J$ and we take the canonical basis ℓ^p_ω in the capacity of the Schauder basis. In order to get Condition 1)-4) Theorem 8.1 we check all statements i)- iv) of Theorem 8.2 (see also Remark 1 after Theorem 8.1). Last Condition 5) in Theorem 8.1 is very simple since if $J_n(h) = 0$, then by the definition, the gap $\gamma_n(h)$ is empty and $h_n = 0$, hence Condition 5) is true.
i) Theorem 4.4 yields real analyticity of the mapping $J: \ell^p_\omega \to \ell^p_\omega$.
ii) By Theorem 7.7, the operator $J'(h)$ has an inverse for any $h \in \ell^p_\omega$. Recall that here we use Remark 1 after Theorem 8.1.
iii) (5.43), (5.23-24) imply estimate (2.5-7).
iv) Let $h^{(n)}$ be some bounded sequence from ℓ^p_ω. Then using the "diagonal process", we obtain some subsequence $h^{(n_k)}$ such that $\lim h^{(n_k)}_m = h_m$, for any $m \in \mathbb{Z}$ as $k \to \infty$ and for some $h \in \ell^p_\omega$. Using Corollary 6.2 we obtain iv). $\square$

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