Projection Quasicrystals III: Cohomology

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March 31, 2000

Abstract

We compute the cohomology of canonical projection tilings of codimension smaller or equal to 3.

Introduction

This is the third paper of a little series about the cohomology of projection tilings. The first (Paper I [1]) was mainly devoted to the study of various dynamical systems and related groupoids which can be constructed from a tiling. In the second (Paper II [2]) we defined the cohomology of a tiling as the cohomology of one of these groupoids (it turned out that it does not matter which one to take) and interpreted it in various forms, in particular as a standard dynamical invariant of one of the systems. This dynamical system is of the following type: Consider a dense lattice Γ in a euclidian space F as it arises generically if one takes $N > \dim F$ vectors of F and considers the lattice they generate. Let K be a compact set which is the closure of its interior and consider the orbit $S = \partial K + \Gamma \subset F$ of ∂K under Γ. We showed in Paper I how to extend the rather simple dynamical system (F, Γ) (Γ acting by translation) to a dynamical system (F_c, Γ) which coincides with the old one on the dense G_δ-set $F \backslash S$. F_c is locally a Cantor set and obtained from F upon disconnecting it along the points of S. We are interested in calculating the cohomological invariants of this system but for convenience prefer to work with the (isomorphic) homology groups $H_p(\Gamma, C_c(F_c, \mathbb{Z}))$ of the group Γ with coefficients in the compactly supported integer valued continuous functions over F_c.

Already for the main result of Paper II we specialized to the situation in which S is a union of finitely many orbits of hyperplanes. We will restrict our attention to this case here too. In Paper II we studied the case in which the cohomology is infinitely generated. This case is implied by the existence of infinitely many orbits of intersection points, a condition which turned out to be generically satisfied. In this paper we focus on the opposit case: we assume that the number of orbits of intersection points is finite from the beginning and use the very explicit description of the hull obtained in Paper I to carry out the calculation of the cohomology for projection tilings of low codimension with a restriction on the form of the acceptance domain. Although finite number of orbits of intersection points is non-generic this seems to be the case of interest for quasicrystal physics. In particular we will present the Ammann-Kramer tiling as an example. It is a three dimensional analog of the Penrose tilings and often used to model icosahedral quasicrystals.

The material presented here extends [3] to the codimension 3 case. This is important, because all known tilings which are used to model icosahedral quasicrystals are obtained from projection out of a 6-dimensional periodic structure and the codimension is the difference between the rank of the periodicity lattice and the dimension of the tiling (in the case of minimal embedding). We mention that the variant used in [3] to describe the tilings is different from that used in Paper I. It is based on Schlottmann's description [4] and has some benefits if ones restricts attention to the canonical case. In contrast to [3] we use here spectral sequences to prove our formulae.

1 Set up and statement of the results

Given the data $(F, \Gamma, \{H_\alpha\}_\alpha)$, a dense lattice Γ of finite rank in a euclidian space F with a finite family $\{H_\alpha\}_{\alpha=1,\cdots,f}$ of (affine) hyperplanes whose normals span F and form an indecomposable set in the sense of [2], we want to compute the homology $H_*(\Gamma, C_c(F_c, \mathbb{Z}))$ of the group Γ with coefficients in the compactly supported integer valued continuous functions over F_c.We will recall the description of F_c in Section 2.

This set up arises in the context of the projection method for tilings: F may be taken to be the orthocomplement of the physical space (denoted E in Paper I and [3]), Γ is the orthoprojection of the periodicity lattice onto F, and the hyperplanes $\{H_\alpha\}_\alpha$ are the spans of the faces of the acceptance domain K which we assume to be polytopal and such that the orbit of a face under the action of Γ contains the space it spans (this is hypothesis H3 in [3]). In particular, we rule out fractal acceptance domain although this case might be important for quasicrystal physics.

We will consider here the case in which the homology groups $H_p(\Gamma, C_c(F_c, \mathbb{Z}))$ are finitely generated. Since they are torsion free they are entirely determined by their ranks. These depend on the geometry of intersections of singular planes, which to describe we introduce the following notation.

Given a finite family $\{H_\alpha\}_{\alpha=1,\cdots,f}$ of hyperplanes, let J_l be the collection of subsets A of $\{1, \cdots, f\}$ with the properties that $|A| = \dim F - l$ (the number of elements) and

$$H_A := \bigcap_{\alpha \in A} H_\alpha$$

has dimension l. On H_A we have an action of $\Gamma^{|A|}$:

$$\vec{x} \cdot H_A := \bigcap_{\alpha \in A} (H_\alpha - x_\alpha).$$

We call $\vec{x} \cdot H_A$ a singular l-space (l its dimension). Let $\mathcal{J}_l$ be the quotient $\Gamma^{|A|} \times J_l / \sim$ with $(\vec{x}, A) \sim (\vec{x}', A')$ if $\vec{x} \cdot H_A = \vec{x}' \cdot H_{A'}$. It is in one to one correspondence to the set of singular l-spaces. We denote equivalence classes by $[\vec{x}, A]$ and the space $\vec{x} \cdot H_A$ also by $H_{[\vec{x},A]}$. On $\mathcal{J}_l$ we have an action of Γ: $y \cdot [\vec{x}, A] = [y \cdot \vec{x}, A]$ where $(y \cdot \vec{x})_\alpha = y + x_\alpha$. We denote the orbit space $\mathcal{J}_l/\Gamma$ by I_l. Since this action coincides with the geometric action of Γ (by translation) on the singular l-spaces we can use the elements of I_l to label the Γ-orbits of singular l-spaces. Note that the map $\{1, \cdots, f\} \to I_{\dim F - 1}$ which assigns to α the orbit of $[0, \{\alpha\}]$ is surjective but not necessarily injective. Nevertheless we denote the elements of $I_{\dim F-1}$ also by α. The stabilizer of a singular l-space depends only on its label, we denote it by Γ^Θ, $\Theta \in I_l$.

Fix $\hat\Theta \in \mathcal{J}_{l+k}$, $l+k < \dim F$ and let $\mathcal{J}_l^{\hat\Theta} := \{\hat\Psi \in \mathcal{J}_l | H_{\hat\Psi} \subset H_{\hat\Theta}\}$. Then Γ^Θ (Θ the orbit class of $\hat\Theta$) acts on $\mathcal{J}_l^{\hat\Theta}$ (diagonally, by the same formula as above) and we let $I_l^{\hat\Theta} = \mathcal{J}_l^{\hat\Theta}/\Gamma^\Theta$, the orbit space. It labels the Γ-orbits of singular l-spaces in the $l+k$-dimensional space $H_{\hat\Theta}$. We can naturally identify $I_l^{\hat\Theta}$ with $I_l^{\hat\Theta'}$ if $\hat\Theta$ and $\hat\Theta'$ belong to the same Γ-orbit and so we write I_l^Θ, for the class $\Theta \in I_{l+k}$. Finally we denote

$$L_l = |I_l|, \quad L_l^\Theta = |I_l^\Theta| \tag{1}$$

where $\Theta \in I_{l+k}$, $l+k < \dim F$. We call the singular 0-spaces also intersection points. The derivation of the following results requires the assumption that L_0, the number of orbits of intersection points, is finite. A density arguement shows that this implies that for all $\Theta \in I_l$ the rank of the stabilizer Γ^Θ is independently of Θ equal to

$$\mathrm{rk}\,\Gamma^\Theta = \nu_l := l\,\frac{\mathrm{rk}\,\Gamma}{\dim F}.$$

Let us write

$$D_p = \mathrm{rk}\, H_p(\Gamma, C_c(F_c, \mathbb{Z}))$$

for the rank. Paper II states that for 1-dimensional F

$$D_p = \binom{\nu_1}{p+1}, \quad p > 0, \tag{2}$$

$$D_0 = (\nu_1 - 1) + L_0. \tag{3}$$

We reprove that theorem below under stronger assumptions as well the following result from [3], both are steps to prove Theorem 2. If $\{M_i | i \in I\}$ is a family of submodules of some bigger module we denote by $\langle M_i | i \in I\rangle$ their span. For a finitely generated lattice G we let ΛG be the exterior module generated by it.

Theorem 1 *Given data $(F, \Gamma, \{H_\alpha\}_\alpha)$ as described in the beginning of this section but with $\dim F = 2$. Then*

$$D_p = \binom{\nu_2}{p+2} + L_1\binom{\nu_1}{p+1} - r_{p+1} - r_p, \quad p > 0, \tag{4}$$

$$D_0 = \binom{\nu_2}{2} - \nu_2 + 1 + L_1(\nu_1 - 1) + e - r_1 \tag{5}$$

where

$$r_p = \mathrm{rk}\,\langle \Lambda_{p+1}\Gamma^\alpha | \alpha \in I_1\rangle,$$

and the Euler number is

$$e := \sum_p (-1)^p D_p = -L_0 + \sum_{\alpha\in I_1} L_0^\alpha.$$

Theorem 2 *Given data* $(F, \Gamma, \{H_\alpha\}_\alpha)$ *with* $\dim F = 3$. *Then, for* $p > 0$,

$$D_p = \binom{\nu_3}{p+3} + L_2 \binom{\nu_2}{p+2} + \tilde{L}_1 \binom{\nu_1}{p+1} - R_p - R_{p+1}, \tag{6}$$

$$\begin{aligned} D_0 &= \sum_{j=0}^{3} (-1)^j \binom{\nu_3}{3-j} + L_2 \sum_{j=0}^{2} (-1)^j \binom{\nu_2}{2-j} \\ &+ \tilde{L}_1 \sum_{j=0}^{1} (-1)^j \binom{\nu_1}{1-j} + e - R_1 \end{aligned} \tag{7}$$

where $\tilde{L}_1 = -L_1 + \sum_{\alpha \in I_2} L_1^\alpha$,

$$R_p = \mathrm{rk}\,\langle \Lambda_{p+2}\Gamma^\alpha | \alpha \in I_2 \rangle - \mathrm{rk}\,\langle \Lambda_{p+1}\Gamma^\Theta | \Theta \in I_1 \rangle + \sum_{\alpha \in I_2} \mathrm{rk}\,\langle \Lambda_{p+1}\Gamma^\Theta | \Theta \in I_1^\alpha \rangle,$$

and the Euler number is

$$e := \sum_p (-1)^p D_p = L_0 - \sum_{\alpha \in I_2} L_0^\alpha + \sum_{\alpha \in I_2} \sum_{\Theta \in I_1^\alpha} L_0^\Theta - \sum_{\Theta \in I_1} L_0^\Theta.$$

In particular, for $\dim F = 3$ and $\mathrm{rk}\,\Gamma = 6$ we obtain as the only nonzero ranks

$$D_3 = 1 \tag{8}$$
$$D_2 = 6 + L_2 - R_2 \tag{9}$$
$$D_1 = 15 + 4L_2 + \tilde{L}_1 - R_1 - R_2 \tag{10}$$
$$D_0 = 10 + 3L_2 + \tilde{L}_1 + e - R_1. \tag{11}$$

2 CW-like complexes defined by $(F, \Gamma, \{H_\alpha\}_\alpha)$

Let $\mathcal{C}'$ be an arbitrary countable collection of affine hyperplanes of F', a linear space, and define $\mathcal{C}'$-topes as in Paper II: compact polytopes which are the closure of their interior and whose boundary faces belong to hyperplanes from $\mathcal{C}'$. For n at most the dimension of F' let $C^n_{\mathcal{C}'}$ be the $\mathbb{Z}$-module generated by the n-dimensional faces of convex $\mathcal{C}'$-topes satisfying the relations

$$[U_1] + [U_2] = [U_1 \cup U_2]$$

for any two faces U_1, U_2, for which $U_1 \cup U_2$ is as well a convex face and $U_1 \cap U_2$ has no interior (i.e. nonzero codimension in U_1). (The above relations then imply $[U_1] + [U_2] = [U_1 \cup U_2] + [U_1 \cap U_2]$ if $U_1 \cap U_2$ has interior.) If we take

$\mathcal{C}' = \mathcal{C} := \{H_\alpha + x | \alpha \in \{1, \cdots, f\}, x \in \Gamma\}$, our collection of singular planes, then $C^n := C^n_{\mathcal{C}}$ carries an obvious Γ-action, namely $x \cdot [U] = [U + x]$. It is therefore an Γ-module. One can now define F_c by identifying $C_c(F_c, \mathbb{Z})$ with $C^{\dim F}$ as Γ-module. This was the way we did it in Paper I but let us provide a more concrete description comparable to that in [3]. For a subset R of Γ let $\mathcal{C}_R := \{H_\alpha + r | \alpha = 1, \cdots, f; r \in R\}$ and $S_R = \{x \in H | H \in \mathcal{C}_R\}$. Let $\mathcal{R}$ be the set of subsets $R \subset \Gamma$ such that all connected components of $F \backslash S_R$ are bounded and have interior. $\mathcal{R}$ is closed under union and hence forms an upper directed system under inclusion. Let F_R be the disjoint union of the closures of the connected components of $F \backslash S_R$. $R \subset R'$ gives rise to a natural surjection $F_{R'} \to F_R$ which is the continuous extension of the inclusion $F \backslash S_{R'} \subset F \backslash S_R$ and F_c is the projective limit of the F_R. An isomorphism between $C^{\dim F}$ and $C_c(F_c, \mathbb{Z})$ is given by assigning to $[U]$ the indicator function on the connected component containing $U \backslash S$ (which is a clopen set). Moreover, C^0 is a free $\mathbb{Z}$-module, its generators are in one to one correspondence to the intersection points. The following proposition is from [3].

Proposition 1 *There exist Γ-equivariant module maps δ and Σ such that*

$$0 \to C^{\dim F} \stackrel{\delta}{\to} C^{\dim F - 1} \stackrel{\delta}{\to} \cdots C^0 \stackrel{\Sigma}{\to} \mathbb{Z} \to 0, \tag{12}$$

is an exact sequence of Γ-modules and $\Sigma[U] = 1$ for all vertices U of $\mathcal{C}$-topes.

Proof: For any $R \in \mathcal{R}$, the $\mathcal{C}_R$-topes define a regular polytopal CW-complex

$$0 \to C^{\dim F}_{\mathcal{C}_R} \stackrel{\delta_R}{\to} C^{\dim F - 1}_{\mathcal{C}_R} \stackrel{\delta_R}{\to} \cdots C^0_{\mathcal{C}_R} \to 0, \tag{13}$$

with boundary operators δ_R depending on the choices of orientations for the n-cells ($n > 0$) [5]. Moreover, this complex is acyclic (F is contractible), i.e. upon replacing $C^0_{\mathcal{C}_R} \to 0$ by $C^0_{\mathcal{C}_R} \stackrel{\Sigma_R}{\to} \mathbb{Z} \to 0$ where $\Sigma_R[U] = 1$, (13) becomes an exact sequence. Let us constrain the orientation of the n-cells in the following way: Each that n-cell belongs to a unique singular n-space $H_{\hat\Theta}$, $\hat\Theta \in \mathcal{J}_n$. We choose its orientation such that it depends only on the class of $\hat\Theta$ in I_l (i.e. we choose an orientation for all parallel $H_{\hat\Theta}$ and then the cell inherits it as a subset). By the same principle, all cells of maximal dimensions are supposed to have the same orientation. Then the cochains and boundary operators δ_R share two crucial properties: first, if $R \subset R'$ for $R, R' \in \mathcal{R}$, then we may identify $C^n_{\mathcal{C}_R}$ with a submodule of $C^n_{\mathcal{C}_{R'}}$ and under this identification $\delta_R(x) = \delta_{R'}(x)$ for all $x \in C^n_{\mathcal{C}_R}$, and second, if U and $U + x$ are $\mathcal{C}_R$-topes then $\delta_R[U + x] = \delta_R[U] + x$. The first property implies that the directed system $\mathcal{R}$ gives rise to a directed system of acyclic cochain complexes, and hence its direct limit is an acyclic complex, and the second implies, together with the fact that for all $x \in \Gamma$ and $R \in \mathcal{R}$ also $R+x \in \mathcal{R}$, that this complex becomes a complex of Γ-modules. The statement now follows since $C^n_{\mathcal{C}}$ is the direct limit of the $C^n_{\mathcal{C}_R}$, $r \in \mathcal{R}$. q.e.d.

Let $C^l_{[\vec{x},A]}$ is the restriction of C^l to polytopes which belong to $H_{[\vec{x},A]}$. We can naturally identify $C^l_{[\vec{x},A]}$ with $C^l_{[\vec{x}',A']}$ if $y \cdot [\vec{x},A] = [\vec{x}',A']$ for some $y \in \Gamma$. Moreover, in this case the complexes obtained by restriction of δ,

$$0 \to C^l_{[\vec{x},A]} \stackrel{\delta^{[\vec{x},A]}}{\longrightarrow} C^{l-1}_{[\vec{x},A]} \cdots C^0_{[\vec{x},A]} \to 0,$$

are isomorphic for all labels of the same class in I_l. With this in mind we denote $C^l_{[\vec{x},A]}$ and $\delta^{[\vec{x},A]}$ also by C_Θ and δ^Θ where $\Theta \in I_l$ is the class of $[\vec{x},A]$. Thus every $\Theta \in I_l$ defines a new acyclic complex.

Lemma 1

$$C^l \cong \bigoplus_{\Theta \in I_l} C^l_\Theta \otimes \mathbb{Z}[\Gamma/\Gamma^\Theta]$$

and if $\Theta \in I_{l+k}$, $l+k < \dim F$, *then*

$$C^l_\Theta \cong \bigoplus_{\Psi \in I^\Theta_l} C^l_\Psi \otimes \mathbb{Z}[\Gamma^\Theta/\Gamma^\Psi].$$

Proof: First observe that

$$C^l \cong \bigoplus_{[\vec{x},A] \in \mathcal{J}_l} C^l_{[\vec{x},A]},$$

because an l-face of a $\mathcal{C}$-tope belongs a unique singular l-space. We identify $C^l_{[\vec{x},A]}$ with $C^l_{[\vec{x}',A']}$ if $y \cdot [\vec{x},A] = [\vec{x}',A']$ for some $y \in \Gamma$. Since $y \cdot [\vec{x},A] = [\vec{x},A]$ whenever y leaves H_A invariant the first statement of the lemma follows. The proof for the second is likewise. q.e.d.

3 Group homology

The homology of a discrete group Γ with coefficients in a Γ-module M is defined as the homology of a complex which is obtained from a resolution of $\mathbb{Z}$ by (projective) Γ-modules upon application of the functor $\otimes_\Gamma M$. More specifically we use here a free and finite resolution of $\Gamma = \Gamma \cong \mathbb{Z}^N$ by its exterior module so that the complex looks like

$$0 \to \Lambda_N \Gamma \otimes M \stackrel{\partial}{\to} \cdots \Lambda_0 \Gamma \otimes M \to 0$$

with boundary operator ∂ given by

$$\partial(e_{i_1}\cdots e_{i_k}\otimes m)=\sum_{j=1}^{k}e_{i_1}\cdots\hat{e}_{i_j}\cdots e_{i_k}(e_{i_j}\cdot m-m)$$

$\hat{e}_{i_j}$ denoting that it has to be left out. In particular, $H_k(\Gamma,\mathbb{Z}\Gamma)$ is trivial for all $k>0$ and equal to $\mathbb{Z}$ for $k=0$.

Suppose that we can split $\Gamma=\Gamma'\oplus\Gamma''$ and let us compute $H_*(\Gamma,\mathbb{Z}\Gamma'')$ where $\mathbb{Z}\Gamma''$ is the free $\mathbb{Z}$-module generated by Γ'' which becomes an Γ-module under the action of Γ given by $(g\oplus h)\cdot h'=h+h'$. Then we can identify

$$\Lambda\Gamma\otimes\mathbb{Z}\Gamma''\cong\bigoplus_{i+j=k}\Lambda_i\Gamma'\otimes\Lambda_j\Gamma''\otimes\mathbb{Z}\Gamma'' \tag{14}$$

and under this identification $\partial\otimes 1$ becomes $(-1)^{deg}\otimes\partial'$ where ∂' is the boundary operator for the homology of Γ''. It follows that

$$H_k(\Gamma,\mathbb{Z}\Gamma'')\cong\bigoplus_{i+j=k}\Lambda_i\Gamma'\otimes H_j(\Gamma'',\mathbb{Z}\Gamma'')=\Lambda_k\Gamma'.$$

As a special case, $H_k(\Gamma,\mathbb{Z})=\Lambda_k\Gamma\cong\mathbb{Z}^{\binom{N}{k}}$. Now let $\Sigma:\mathbb{Z}\Gamma''\to\mathbb{Z}$ be the sum of the coefficients, i.e. $\Sigma[h]=1$ for all $h\in\Gamma''$. We shall later need the following lemma:

Lemma 2 *Under the identifications $H(\Gamma,\mathbb{Z}\Gamma'')\cong\Lambda\Gamma'$ and $H(\Gamma,\mathbb{Z})\cong\Lambda\Gamma$ the induced map $\Sigma_k:H_k(\Gamma,\mathbb{Z}\Gamma'')\to H_k(\Gamma,\mathbb{Z})$ becomes the inclusion $\Lambda_k\Gamma'\hookrightarrow\Lambda_k\Gamma$.*

Proof: Using the decomposition (14) it is easy to see that the induced map $\Sigma_k:\bigoplus_{i+j=k}\Lambda_i\Gamma'\otimes H_j(\Gamma'',\mathbb{Z}\Gamma'')\to\bigoplus_{i+j=k}\Lambda_i\Gamma'\otimes H_j(\Gamma'',\mathbb{Z})$ preserves the bidegree and must be the identity on the first factors of the tensor product. Since $H_k(\Gamma'',\mathbb{Z}\Gamma'')$ is trivial whenever $k\neq 0$ and one dimensional for $k=0$, Σ_k can be determined by evaluating Σ_0 on the generator of $H_0(\Gamma'',\mathbb{Z}\Gamma'')$ and one readily checks that this gives a generator of $H_0(\Gamma'',\mathbb{Z})$ as well. q.e.d.

We note two immediate corollaries of Lemma 1.

Corollary 1

$$H(\Gamma,C^l)\cong\bigoplus_{\Theta\in I_l}H(\Gamma^\Theta,C^l_\Theta).$$

Corollary 2 *If $\Theta\in I_{l+k}$, $l+k<\dim F$, then*

$$H(\Gamma^\Theta,C^l_\Theta)\cong\bigoplus_{\Psi\in I^\Theta_l}H(\Gamma^\Psi,C^l_\Psi).$$

4 The spectral sequences

Taking for M the CW-like complex from above we arrive at a double complex $(\Lambda_p\Gamma \otimes C^q, \delta_{pq}, \partial_{pq})$. Given such a double complex there are two standard spectral sequences attached to it, one for which the first page is the homology of δ and the first differential ∂_*, the induced differential on the homology, and one for which it is that of ∂ and the first differential δ_* [6]. The point is that both sequences converge, since $E^0_{pq} = \Lambda_p\Gamma \otimes C^q$ is non-trivial only for finitely many terms their limits yielding bigraded modules E^∞_{pq} whose total degree submodules are (under suitable conditions which are satisfied in our case) isomorphic (in fact, they are isomorphic to the total homology of E^0_{pq}, but we won't need that here).

It turns out that the spectral sequence which arises when starting with the homology in δ is easy. In fact, if we let $E^1_{pq} = H_\delta(E^0_{pq})$ then

$$E^1_{pq} = \begin{cases} 0 & \text{if } q > 0, \\ \Lambda_p\Gamma & \text{if } q = 0 \end{cases}$$

and therefore already the first differential vanishes. Hence we have $E^\infty = E^1$ yielding

$$\bigoplus_{p+q=k} E^\infty_{pq} \cong \Lambda_k\Gamma. \tag{15}$$

Let us now consider the other case, starting with homology in ∂, i.e. with

$$E^1_{pq} = H_\partial(E^0_{pq}) = H_p(\Gamma, C^q)$$

and $\mathrm{d}^1 = \delta_*$ as first page. We realize that $E^1_{p\dim F}$ is what we want to compute. The strategy to compute it is as follows. First, we know from [7] that $H(\Gamma, C^q)$ has no torsion. Since we can keep track that all modules appearing are finitely generated it suffices to determine their dimension. We may therefore tensor everything with $\mathbb{Q}$ and restrict our attention to the ranks of the modules only. Suppose then that we know $d_{pq} = \dim E^1_{pq}$ for $q < \dim F$. Then we can determine $d_{p\dim F}$ from these data, the dimensions in (15), and the ranks of the differentials.

4.1 Abstract calculations

Let us first consider a more abstract situation. Consider a double complex $(E^0_{pq}, \delta_{pq}, \partial_{pq})$ of finitely generated $\mathbb{Q}$-modules and start the spectral sequence with $E^1_{pq} := H_\partial(E^0_{pq})$ and $\mathrm{d}^1 = \delta_*$. Write $d_{pq} = \dim E^1_{pq}$. Assume furthermore that

- E1 There exist finite M, N such that E^0_{pq} is non-trivial only for $0 \le q \le M$ and $0 \le p \le N$.
- E2 $d_{p0} = 0$ for $p \ge 1$.

Suppose that d_{pq} is known for $q < M$ and also $N_k := \sum_{p+q=k} \dim E^\infty_{pq}$. We want to derive an expression for the d_{pM}. We now carry this out for $M = 1, 2, 3$ using mainly that the higher differentials d^k change the bidegree as follows:

$$\mathrm{d}^k_{pq} : E^k_{pq} \longrightarrow E^k_{p+k-1\,q-k}. \tag{16}$$

4.1.1 $M = 1$

Here (16) implies that $\mathrm{d}^k = 0$ for $k \ge 2$ and hence $E^\infty = E^2$. With the notation

$$a_{pq} := \mathrm{rk}\,\mathrm{d}^1_{p\,q+1} : E^1_{p\,q+1} \to E^1_{pq}$$

we get for $\{\dim E^\infty_{pq} | 0 \le p \le N, 0 \le q \le 1\}$ the table (p increases to the top and q increases to the right).

0	d_{N1}
$\vdots$	$\vdots$
0	d_{11}
$d_{00} - a_{00}$	$d_{01} - a_{00}$

yielding

$$d_{p1} = N_{p+1} + \begin{cases} 0 & \text{for} \quad p > 0, \\ d_{00} - N_0 & \text{for} \quad p = 0. \end{cases} \tag{17}$$

4.1.2 $M = 2$

(16) still implies that $\mathrm{d}^k = 0$ for $k \ge 2$, because of E2. But now $\{\dim E^\infty_{pq} | 0 \le p \le N, 0 \le q \le 2\}$ yields the table

0	$d_{N1} - a_{N1}$	$d_{N2} - a_{N1}$
$\vdots$	$\vdots$	$\vdots$
0	$d_{11} - a_{11}$	$d_{12} - a_{11}$
$d_{00} - a_{00}$	$d_{01} - a_{00} - a_{01}$	$d_{02} - a_{01}$

.

Recall that $\mathrm{d}^1_{p\,q+1} : E^1_{p\,q+1} \to E^1_{pq}$ is the map induced on the homology groups from $\delta_{p\,q+1}$, we denote it by $\delta_{p\,q+1\,*}$. From the exact sequence $0 \to E^0_{p2} \xrightarrow{\delta_{p2}} E^0_{p1} \xrightarrow{\tilde{\delta}_{p1}} Z_\delta(E^0_{p0}) \to 0$ we conclude that

$$a_{p1} = \dim\ker \tilde{\delta}_{p1\,*} = d_{p1} - b_{p0} \tag{18}$$

where

$$b_{pq} := \mathrm{rk}\,\tilde{\delta}_{p\,q+1\,*} : E^1_{p\,q+1} \to H_\partial(Z_\delta(E^0_{pq})).$$

Thus the table can be written as

0	b_{N0}	$d_{N2} - d_{N1} + b_{N0}$
$\vdots$	$\vdots$	$\vdots$
0	b_{10}	$d_{12} - d_{11} + b_{10}$
$d_{00} - a_{00}$	$d_{01} - a_{00} - a_{01}$	$d_{02} - a_{01}$

yielding

$$d_{p2} = N_{p+2} + d_{p1} - b_{p+1\,0} - b_{p0}, \quad p > 0 \tag{19}$$

$$d_{02} = N_2 + d_{01} - b_{10} - N_1 + N_0 - d_{00}. \tag{20}$$

4.1.3 $M = 3$

Now (16) implies that $\mathrm{d}^k = 0$ only for $k > 2$ so that $E^3 = E^\infty$. With the notation

$$\mu_{pq} := \mathrm{rk}\,\mathrm{d}^2_{p-1\,q+2} : E^2_{p-1\,q+2} \to E^2_{pq}$$

we get for $\{\dim E^\infty_{pq} | 0 \le p \le N, 0 \le q \le 3\}$ the table

0	$d_{N1} - a_{N1} - \mu_{N1}$	$d_{N2} - a_{N1} - a_{N2}$	$d_{N3} - a_{N2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	$d_{11} - a_{11} - \mu_{11}$	$d_{12} - a_{11} - a_{12}$	$d_{13} - a_{12} - \mu_{21}$
$d_{00} - a_{00}$	$d_{01} - a_{00} - a_{01}$	$d_{02} - a_{01} - a_{02}$	$d_{03} - a_{02} - \mu_{11}$

.

Let us recall the definition of d^2. Look at the commuting diagram

$$\begin{array}{ccccccc} 0 \to E^0_{p3} & \xrightarrow{\delta_{p3}} & E^0_{p2} & \xrightarrow{\delta_{p2}} & E^0_{p1} & \xrightarrow{\delta_{p1}} & \cdots \\ & & \downarrow\tilde{\delta}_{p2} & \nearrow^{\imath_{p1}} & \downarrow\tilde{\delta}_{p1} & \nearrow^{\imath_{p0}} & \\ & & Z_\delta(E^0_{p1}) & & Z_\delta(E^0_{p0}) & & \end{array} .$$

Let $\theta : H_\partial(Z_\delta(E^0_{p1})) \to H_\partial(E^0_{p-1\,3}) = E^1_{p-1\,3}$ be the connecting map of the exact sequence $0 \to E^0_{p3} \overset{\delta_{p3}}{\to} E^0_{p2} \overset{\tilde{\delta}_{p2}}{\to} Z_\delta(E^0_{p1}) \to 0$. Then $\mathrm{d}^2_{p-1\,3} : E^2_{p-1\,3} \to E^2_{p1}$ is the map induced by $\imath_{p1\,*} \circ \theta^{-1} : E^1_{p-1\,3} \to E^1_{p1}$. Since θ^{-1} identifies $E^2_{p-1\,3} = Z_{d^1}(E^1_{p-1\,3}) \overset{\theta^{-1}}{\cong} H_\partial(Z_\delta(E^0_{p1}))/\mathrm{im}\,\tilde{\delta}_{p2\,*}$ we get

$$\mu_{p1} = \mathrm{rk}\,\imath_{p1\,*} - \mathrm{rk}\,\delta_{p2\,*} = \dim\ker\tilde{\delta}_{p1\,*} - \mathrm{rk}\,\delta_{p2\,*} = (d_{p1} - b_{p0}) - a_{p1}.$$

Furthermore,

$$a_{p2} = \dim\ker\tilde{\delta}_{p2\,*} = d_{p2} - b_{p1}$$

so that we can write the table as

0	b_{N0}	$b_{N1}-a_{N1}$	$d_{N3}-d_{N2}+b_{N1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	b_{p0}	$b_{p1}-a_{p1}$	$d_{p3}-d_{p2}-d_{p+1\,1}+b_{p1}+b_{p+1\,0}+a_{p+1\,1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$d_{00}-a_{00}$	$d_{01}-a_{00}-a_{01}$	$d_{02}-a_{01}-a_{02}$	$d_{03}-d_{11}-a_{02}+a_{11}+b_{10}$

leading to the solution

$$\begin{aligned} d_{p3} &= N_{p+3}+d_{p2}+d_{p+1\,1}-b_{p1}-b_{p+1\,1}-b_{p+1\,0}-b_{p+2\,0}, \quad p>0, && (21)\\ d_{03} &= N_3-N_2+N_1-N_0+d_{02}+d_{11}-d_{01}+d_{00}-b_{11}-b_{20}. && (22)\end{aligned}$$

4.2 Proofs of the theorems

Let us now specify to double complexes of the form $E^0_{pq}=\Lambda_p\Gamma\otimes C^q$, $\partial_{pq}=\partial_p\otimes 1$ and $\delta_{pq}=(-1)^p\otimes\delta_q$ as they arrise in our application however tacitly changing the ring to be $\mathbb{Q}$. Clearly, E^0_{pq} is non-trivial only for $0\le p\le \operatorname{rk}\Gamma$ and $0\le q\le\dim F$. Furthermore, $E^1_{p0}=H_\delta(\Lambda_p\Gamma\otimes C^0)=H_p(\Gamma,C^0)$. Since C^0 is always a free Γ-module one has $H_p(\Gamma,C^0)=0$ for $p>0$ and the dimension of $H_0(\Gamma,C^0)$ is simply $|I_0|$, the number of orbits of intersection points:

$$d_{00}=L_0. \tag{23}$$

We called this number L earlier on and assume it to be finite here. In particular E1 and E2 are satisfied.

The case $\dim F=1$ (2,3) can now be immediately obtained using (17) and (15).

A proof of Theorem 1 can also be found in [3] but we repeat it here using the framework of spectral sequences. So let $\dim F=2$. We apply our earlier observations with $M=2$, notably (19,20). For that we need to determine d_{p1} and b_{p0}.

To determine d_{p1} we use Corollary 1. $H(\Gamma^\alpha,C^1_\alpha)$ can be computed using the double complex $(\Lambda_p\Gamma^\alpha\otimes C^q_\alpha,\partial^\alpha_{pq},\delta^\alpha_{pq})$. So let us consider the spectral sequence which arrises when starting with homology in ∂^α. As for the case $\dim F=1$ one concludes that E^1_{p0} is non-trivial only for $p=0$ and the dimension of E^1_{p0} equal to L^α_0, the number of orbits of intersection points of H_α. Since $\operatorname{rk}\Gamma^\alpha=\nu_1$ we get

$$\begin{aligned} d_{p1} &= L_1\binom{\nu_1}{p+1}, \quad p>0, && (24)\\ d_{01} &= L_1(\nu_1-1)+\sum_{\alpha\in I_1}L^\alpha_0. && (25)\end{aligned}$$

It remains to compute

$$b_{p0} = \mathrm{rk}\, \tilde{\delta}_{p1\,*} : H_p(\Gamma, C^1) \to H_p(\Gamma, Z_\delta(C^0)).$$

For that look at the following commuting diagram of exact sequences

$$\begin{array}{ccccccccc}
0 \to & C^1_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha] & \stackrel{\delta_1^\alpha \otimes 1}{\to} & C^0_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha] & \to & \mathbb{Z}[\Gamma/\Gamma^\alpha] & \to 0 \\
 & \downarrow \tilde{\delta}_1^\alpha \otimes 1 & & \downarrow & & \downarrow \Sigma^\alpha & \\
0 \to & Z_\delta(C^0) & \hookrightarrow & C^0 & \stackrel{\epsilon}{\to} & \mathbb{Z} & \to 0
\end{array}$$

where the middle verticle arrow is the inclusion, the right vertical arrow the sum of the coefficients, $\Sigma^\alpha[\gamma] = 1$, and the direct sum over all $\alpha \in I_1$ of the left vertical arrows is $\tilde{\delta}_{p1}$. The above commutative diagram gives rise to two long exact sequences of homology groups together with vertical maps, all commuting, $(\tilde{\delta}^\alpha_{p1} \otimes 1)_* : H_p(\Gamma, C^1_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha]) \to H_p(\Gamma, Z_\delta(C^0))$ being one of them. Now we use that for $p > 0$, $H_p(\Gamma, C^0_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha]) = 0$ and $H_p(\Gamma, C^0) = 0$ which implies $H_p(\Gamma, C^1_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha]) \cong H_{p+1}(\Gamma, \mathbb{Z}[\Gamma/\Gamma^\alpha]) \cong H_{p+1}(\Gamma^\alpha, \mathbb{Z})$ and $H_p(\Gamma, Z_\delta(C^0)) \cong H_{p+1}(\Gamma, \mathbb{Z})$ and

$$(\tilde{\delta}^\alpha_{p1} \otimes 1)_* = \Sigma^\alpha_{p+1}.$$

By Lemma 2 the map Σ^α_p can be identified with the inclusion $\Lambda_p \Gamma^\alpha \hookrightarrow \Lambda_p \Gamma$. For $p > 0$ therefore,

$$b_{p0} = \mathrm{rk}\, \langle \mathrm{im}\, (\tilde{\delta}^\alpha_{p1} \otimes 1)_* | \alpha \in I_1 \rangle = \mathrm{rk}\, \langle \Lambda_{p+1} \Gamma^\alpha | \alpha \in I_1 \rangle. \tag{26}$$

Direct application of (19,20) yields now the formulas for the dimensions stated in Theorem 1. We have performed our calculations, however, for $\mathbb{Q}$-modules and it remains to check that they remain valid for $\mathbb{Z}$-modules. Since we know from [7] that the result for the homology groups will be torsion free this follows if all dimensions appearing in the calculations are finite. This follows from the assumption that L_0 is finite.

4.2.1 Proof of Theorem 2

Let $\dim F = 3$. To apply (21,22) we need to determine $d_{p1}, d_{p2}, b_{p0}, b_{p1}$. To calculate d_{p1} we use Corollary 1 and (17) to obtain, for $\Theta \in I_1$,

$$\dim H_p(\Gamma^\Theta, C^1_\Theta) = \binom{\nu_1}{p+1} + \begin{cases} 0 & \text{for} \quad p > 0, \\ L_0^\Theta - 1 & \text{for} \quad p = 0. \end{cases}.$$

Hence

$$d_{p1} \quad = \quad L_1 \binom{\nu_1}{p+1} + \begin{cases} 0 & \text{for} \quad p > 0, \\ -L_1 + \sum_{\Theta \in I_1} L_0^\Theta & \text{for} \quad p = 0 \end{cases}.$$

To calculate d_{p2} we consider the complexes, $\alpha \in I_2$,

$$0 \to C^2_\alpha \xrightarrow{\delta^\alpha} C^1_\alpha \xrightarrow{\delta^\alpha} C^0_\alpha \to 0.$$

The appropriate double complex to consider is therefore $(\Lambda_p\Gamma^\alpha \otimes C^q_\alpha, \partial^\alpha_{pq}, \delta^\alpha_{pq})$. Repeating the arguments of the proof of Theorem 1 we get for the dimensions of the modules of bidegree $(p, 2)$ of the first page

$$\begin{aligned} d^\alpha_{p2} &= \binom{\nu_2}{p+2} + L^\alpha_1 \binom{\nu_1}{p+1} - r^\alpha_p - r^\alpha_{p+1} \quad p > 0, \\ d^\alpha_{02} &= \binom{\nu_2}{2} - \nu_2 + 1 + L^\alpha_1(\nu_1 - 1) - r^\alpha_1 - L^\alpha_0 + \sum_{\Theta \in I^\alpha_1} L^\Theta_0. \end{aligned}$$

where

$$r^\alpha_p = \mathrm{rk}\,\langle \Lambda_{p+1}\Gamma^\Theta | \Theta \in I^\alpha_1 \rangle.$$

Writing $\mathcal{F} = \sum_{\alpha \in I_2} L^\alpha_1$ we thus get

$$\begin{aligned} d_{p2} &= L_2 \binom{\nu_2}{p+2} + \mathcal{F} \binom{\nu_1}{p+1} - \sum_{\alpha \in I_2} (r^\alpha_p + r^\alpha_{p+1}), \quad p > 0, \\ d_{02} &= L_2 \left(\binom{\nu_2}{p+2} - \nu_2 + 1 \right) + \mathcal{F}\,(\nu_1 - 1) + \sum_{\alpha \in I_2} \left(\sum_{\Theta \in I^\alpha_1} L^\Theta_0 - L^\alpha_0 - r^\alpha_1 \right). \end{aligned}$$

We now compute b_{p0} and b_{p1}. For the first we can again repeat the arguments which yield (26) to obtain

$$b_{p0} = \mathrm{rk}\,\langle \Lambda_{p+1}\Gamma^\Theta | \Theta \in I_1 \rangle.$$

b_{p1} is the rank of $\tilde{\delta}_{p2\,*}$, i.e. the rank of the map induced from $\tilde{\delta}_2 : C^2 \to Z_\delta(C^1)$. To determine it we consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & C^2_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha] & \stackrel{\delta^\alpha_2 \otimes 1}{\to} & C^1_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha] & \stackrel{\tilde{\delta}^\alpha_1 \otimes 1}{\to} & Z_\delta(C^0_\alpha) \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha] & \to 0 \\ & \downarrow \tilde{\delta}^\alpha_2 \otimes 1 & & \downarrow & & \downarrow & \\ 0 \to & Z_\delta(C^1) & \hookrightarrow & C^1 & \stackrel{\delta_1}{\to} & Z_\delta(C^0) & \to 0. \end{array}$$

The middle and right vertical maps are the obvious inclusions. By Corollary 1 the direct sum of the left vertical arrows of the diagrams which arrise if $\alpha \in I_2$ is $\tilde{\delta}_2$. The diagram gives rise to a commutative diagram of which the following is one degree

$$\begin{array}{ccccccccc} \to & H_p(\Gamma, C^2_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha]) & \to & H_p(\Gamma, C^1_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha]) & \stackrel{(\tilde{\delta}^\alpha_{p1} \otimes 1)_*}{\to} & H_p(\Gamma, Z_\delta(C^0_\alpha) \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha]) & \to \\ & \downarrow (\tilde{\delta}^\alpha_{p2} \otimes 1)_* & & \downarrow & & \downarrow & \\ \to & H_p(\Gamma, Z_\delta(C^1)) & \to & H_p(\Gamma, C^1) & \stackrel{\delta_{p1\,*}}{\to} & H_p(\Gamma, Z_\delta(C^0)) & \to \end{array} \quad (27)$$

also with exact sequences. In particular, $b_{p1} = \mathrm{rk}\,\langle \mathrm{im}\,(\tilde\delta^\alpha_{p2}\otimes 1)_*|\alpha\in I_2\rangle$. Now we can follow the same analysis as in the proof of Theorem 1 to see that, for $p>0$, (27) is isomorphic to

$$\begin{array}{cccccccc} \to & H_p(\Gamma, C^2_\alpha\otimes\mathbb{Z}[\Gamma/\Gamma^\alpha]) & \to & \bigoplus_{\Theta\in I_1^\alpha}\Lambda_{p+1}\Gamma^\Theta & \stackrel{\beta_p^\alpha}{\to} & \Lambda_{p+1}\Gamma^\alpha & \to \\ & \downarrow (\tilde\delta^\alpha_{p2}\otimes 1)_* & & \downarrow j_p^\alpha & & \downarrow \imath_p^\alpha & \\ \to & H_p(\Gamma, Z_\delta(C^1)) & \to & \bigoplus_{\Theta\in I_1}\Lambda_{p+1}\Gamma^\Theta & \stackrel{\gamma_p}{\to} & \Lambda_{p+1}\Gamma & \to \end{array} \tag{28}$$

where β_p^α is the direct sum over $\Theta\in I_1^\alpha$ of the inclusions $\Lambda_{p+1}\Gamma^\Theta\hookrightarrow\Lambda_{p+1}\Gamma^\alpha$, γ_p is the direct sum over $\Theta\in I_1$ of the inclusions $\Lambda_{p+1}\Gamma^\Theta\hookrightarrow\Lambda_{p+1}\Gamma$, and j_p^α and $\imath_p^\alpha$ are the obvious inclusions. As we are working with finitely generated $\mathbb{Q}$-modules we obtain from (28)

$$\mathrm{im}\,(\tilde\delta^\alpha_{p2}\otimes 1)_* \cong \mathrm{im}\,\imath^\alpha_{p+1}/(\mathrm{im}\,\imath^\alpha_{p+1}\cap\mathrm{im}\,\gamma_{p+1})\oplus j_p^\alpha(\ker\beta_p^\alpha).$$

Since $I_1 = \bigcup_{\alpha\in I_2} I_1^\alpha$ the direct sum over $\alpha\in I_1^\alpha$ of j_p^α is surjective. Hence $\mathrm{im}\,\gamma_p\subset\langle\mathrm{im}\,\imath_p^\alpha|\alpha\in I_2\rangle$ and therefore

$$\langle\mathrm{im}\,\imath_p^\alpha/(\mathrm{im}\,\imath_p^\alpha\cap\mathrm{im}\,\gamma_p)|\alpha\in I_2\rangle = \langle\mathrm{im}\,\imath_p^\alpha|\alpha\in I_2\rangle/\mathrm{im}\,\gamma_p.$$

Furthermore, using also that $\imath_p^\alpha$ is injective, we get $j_p^\alpha(\ker\beta_p^\alpha) = \mathrm{im}\,j_p^\alpha\cap\ker\gamma_p$ and

$$\langle\mathrm{im}\,j_p^\alpha\cap\ker\gamma_p|\alpha\in I_2\rangle = \ker\gamma_p.$$

Thus we have

$$\langle\mathrm{im}\,(\tilde\delta^\alpha_{p2}\otimes 1)_*|\alpha\in I_2\rangle \cong \langle\mathrm{im}\,\imath^\alpha_{p+1}|\alpha\in I_2\rangle/\mathrm{im}\,\gamma_{p+1}\oplus\ker\gamma_p$$

which, since $\mathrm{rk}\,\gamma_p = b_{p0}$, implies

$$b_{p1} = \mathrm{rk}\,\langle\Lambda_{p+2}\Gamma^\alpha|\alpha\in I_2\rangle - b_{p+1\,0} + L_1\begin{pmatrix}\nu_1\\ p+1\end{pmatrix} - b_{p0}.$$

With these results we obtain from (21,22) the ranks stated in Theorem 2. The argument given at the end of the proof for Theorem 1 applies also here so that these are the ranks of the $\mathbb{Z}$-modules. q.e.d.

5 A 3-dimensional example: Ammann-Kramer tilings

In Paper II we showed that generically, canonical projection tilings have infinitely generated cohomology. Nevertheless, all tilings known to us which are used to describe quasicrystals have finitely generated cohomology. In [8] a list of results for the known 2-dimensional tilings used by quasicrystallographers was presented. Here we present the first result for a 3-dimensional projection tiling with finitely generated cohomology. It is the Ammann-Kramer tiling. The tiling was invented before the discovery of quasicrystals [9] and rediscovered in [10, 11]. It has been used to describe icosahedral quasicrystals in [12, 13]. It is sometimes also called 3-dimensional Penrose tiling because it generalizes in a way Penrose's 2-dimensional tilings. A short description of further 3-dimensional tilings related icosahedral symmetry is given in [14]. The tiling is obtained by the canonical projection method from the data $(\mathbb{Z}^6, E)$, $\mathbb{Z}^6$ being the lattice generated by an orthonormal basis of $\mathbb{R}^6$, and E being obtained from symmetry considerations involving the representation theory of the icosahedral group. Projecting $\mathbb{Z}^6$ orthogonally onto the orthocomplement F of E one obtaines the lattice Γ generated by the six vectors

$$\begin{pmatrix} \tau \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\tau \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \tau \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \tau \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -\tau \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \tau \end{pmatrix}$$

with respect to an orthonormal basis of F. Here $\tau = \frac{\sqrt{5}-1}{2}$ and we have the usual relation $\tau^2 + \tau - 1 = 0$. The acceptance domain is the orthogonal projection of the 6-dimensional unit cube into F and forms the triacontrahedron whose vertices are the linear combinations with coefficients $0, 1$ of the above vectors. It has 30 faces which are all triangles and the affine hyperplanes $\{H_\alpha\}_\alpha$ are the planes spanned by these triangles. Although we have an action of the icosahedral group (which even permutes the faces) the geometry of their intersections is quite complex, but can be summarized as follows.
Singular subspaces in F: The two triangles which are in opposite position of the triacontrahedron belong to the same orbit (under the action of Γ) and representatives of the 15 orbits of singular planes are given by the spans of all possible pairs of vectors from our list of 6 vectors above. Thus I_2 has $L_2 = 15$ elements. The 15 planes can be gathered into five orthogonal triples. Pairs of these planes intersect in singular lines and triples intersect in singular points. Singular lines occur in three classes: Intersections of 5 planes simultaneously at a line (class I - 6 possible orientations): Intersections of 3 planes only (class II - 15 possible orientations): Intersections of 2 planes only (class III - 10 possible orientations). Each of the class I and III orientations is a single orbit, but each of the class II orientations splits into two orbits. Thus if we write $I_1 = I_1^{\mathrm{I}} \cup I_1^{\mathrm{II}} \cup I_1^{\mathrm{III}}$, a disjoint union according to the class, we have $|I_1^{\mathrm{I}}| = 6$, $|I_1^{\mathrm{II}}| = 2 \times 15$, and $|I_1^{\mathrm{III}}| = 10$. In particular, we have $L_1 = 46$ orbits of singular lines, arranged in 31 possible orientations. A similar calculation

finds $L_0 = 32$ orbits of singular points formed at the intersection of three or more singular planes: Briefly, every singular point lies at the intersection of some orthogonal triple of planes. Given a particular othogonal triple, those points which are at the intersection of three planes in this orientation split into 8 orbits (because intersections of orthogonal planes give class II lines). However, two of those orbits are found at intersections of planes parallel to every other orientation (e.g. the orbit of the origin), and the remaining 6 are unique to the particular orthogonal triple chosen. This adds up to $6 \times 5 + 2 = 32$ orbits therefore.
Singular subspaces in H_α, $\alpha \in I_2$: Since all H_α lie in one orbit of the icosahedral group it doesn't matter which α to take. Within a singular plane H_α there are 2 directions of each class of lines, but each of the class II lines is in two orbits. Thus if we write $I_1^\alpha = I_1^{\alpha\,\mathrm{I}} \cup I_1^{\alpha\,\mathrm{II}} \cup I_1^{\alpha\,\mathrm{III}}$ as disjoint union according to the class we have $|I_1^{\alpha\,\mathrm{I}}| = 2$, $|I_1^{\alpha\,\mathrm{II}}| = 2 \times 2$, and $|I_1^{\alpha\,\mathrm{III}}| = 2$, giving a total of $L_1^\alpha = 8$ orbits of singular lines in 6 possible directions. In particular $\tilde{L}_1 = 15 \times 8 - 46 = 74$. A careful calculation finds the singular points arranged into $L_0^\alpha = 8$ orbits.
Singular subspaces in H_Θ, $\Theta \in I_1$: In each class I and class III line we have exactly two orbits of singular points. In each class II line we have exactly 4 orbits. Hence $L_0^\Theta = 2$ if Θ is of class I or III whereas otherwise $L_0^\Theta = 4$. This gives $\sum_{\Theta \in I_1} L_0^\Theta = 2\times 6 + 4\times 30 + 2\times 10 = 152$ and $\sum_{\Theta \in I_1^\alpha} L_0^\Theta = 2 \times 2 + 4 \times 4 + 2 \times 2 = 24$. Altogether $e = 120$.

To find the ranks R_1, R_2, a more detailed analysis of the generators of the stabilizers and their inner products must be made. Since this involves finding the rank of matrices of up to 15×15 in size, it is best checked by computer: $R_1 = 69$ and $R_2 = 9$. This yields the following numbers for the ranks of the homology groups:

D_0	D_1	D_2	D_3
180	71	12	1

.

We are grateful to Franz Gähler for verifying these results on the computer.

Acknowledgements. We wish to thank F. Gähler for helpful discussions. The collaboration of the first two authors was initiated by the William Gordon Seggie Brown Fellowship at The University of Edinburgh, Scotland, and recieved continuing support from a Collaborative Travel Grant from the British Council and the Research Council of Norway with the generous assistance of The University of Leicester, England, and the EU Network "Noncommutative Geometry" at NTNU Trondheim, Norway. The collaboration of the first and third authors was supported by the Sonderforschungsbereich

288, “Differentialgeometrie und Quantenphysik” at TU Berlin, Germany, and by the EU Network and NTNU Trondheim. The first author is supported while at NTNU Trondheim, as a post-doctoral fellow of the EU Network and the third author is supported by the Sfb288 at TU Berlin. All three authors are most grateful for the financial help received from these various sources.

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