AN ANALOGUE OF THE UP-ITERATION FOR CONSTANT MEAN CURVATURE ONE SURFACES IN HYPERBOLIC 3-SPACE

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1. Introduction

It is well known that a surface of constant mean curvature in a space form has a 1-parameter family of local isometric deformations preserving the principal curvature functions. If such a local deformation can be extended globally, the surface is called *H-deformable*. We denote by $\mathcal{H}^3(-c^2)$ the hyperbolic 3-space of constant curvature $-c^2$. It is well known that surfaces of constant mean curvature c have quite similar properties to minimal surfaces in $\mathbf{R}^3$, in particular, they have an analogue of the Weierstrass representation called the *Bryant representation*. A complete minimal surface of finite total curvature in $\mathbf{R}^3$ is H-deformable if and only if it is rational. However, for CMC-c surfaces in $\mathcal{H}^3(-c^2)$, H-deformability is independent of the rationality, and only a few such examples are known (c.f. [UY3]).

Recently, the first author [M] gave a method, called the UP-iteration, for constructing new rational minimal surfaces of genus zero from a given rational minimal surface. In this paper, we shall give an analogue of it for CMC-c surfaces in hyperbolic 3-space $\mathcal{H}^3(-c^2)$, and give countably many non-trivial families of new complete H-deformable CMC-c surfaces of finite total curvature in $\mathcal{H}^3(-c^2)$. The hyperbolic 3-space $\mathcal{H}^3(-c^2)$ can be realized as the Poincare ball of radius $1/c$. The original UP-iteration can be viewed as the limit of this hyperbolic analogue as $c \to 0$.

2. Preliminaries

Let $\mathrm{Herm}(2)$ be the set of 2-dimensional Hermitian matrices. The hyperbolic 3-space can be expressed as ([Bry])

$$\mathcal{H}^3(-c^2) = \{X \in \mathrm{Herm}(2)\,;\, \det(X) = 1/c^2,\ \mathrm{trace}(X) > 0\}.$$

Let $f : M \to \mathcal{H}^3(-c^2)$ be a conformal CMC-c immersion. We denote by $\tilde{M}$ the universal covering space of M. There exists a null holomorphic immersion

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \tilde{M} \to \mathrm{PSL}(2,\mathbf{C})$$

such that $f = (1/c)FF^*$, where $\mathrm{PSL}(2,\mathbf{C}) = \mathrm{SL}(2,\mathbf{C})/\{\pm 1\}$ and 'null' means that the pull-back by F of the bi-invariant metric on $\mathrm{PSL}(2,\mathbf{C})$ vanishes ([Bry]). We call the null holomorphic immersion $F : M \to \mathrm{PSL}(2,\mathbf{C})$ the *lift* of f. The lift F has an ambiguity of the right multiplication of the constant matrix in $\mathrm{SU}(2)$. Since F is null, it satisfies the identity

$$\det(F^{-1}F') = 0, \tag{1}$$

where $F' = dF/dz$, and z is a local complex coordinate.

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For a CMC-c immersion f, the *hyperbolic Gauss map* G and the *secondary Gauss map* g are defined as follows (See [UY4]):

$$G := \frac{A'}{C'} = \frac{B'}{D'} = \frac{\alpha^{\#}_{11}}{\alpha^{\#}_{21}} = \frac{\alpha^{\#}_{12}}{\alpha^{\#}_{22}}, \tag{2}$$

$$g := -\frac{D'}{C'} = -\frac{B'}{A'} = \frac{\alpha_{11}}{\alpha_{21}} = \frac{\alpha_{12}}{\alpha_{22}}, \tag{3}$$

where $(\alpha_{ij}) = F^{-1}F'$ and $(\alpha^{\#}_{ij}) = F(F^{-1})' = -F'F^{-1}$. The hyperbolic Gauss map G is single valued on M, but the secondary Gauss map g is defined on the universal cover $\tilde{M}$. When we replace the lift F by Fb^{-1} ($b = (b_{ij}) \in \mathrm{SU}(2)$), the hyperbolic Gauss map is invariant, but the secondary Gauss map is changed. The new secondary Gauss map $\tilde{g}$ is given by

$$\tilde{g} := \frac{b_{11}g + b_{12}}{b_{21}g + b_{22}}. \tag{4}$$

The geometric meanings of G and g are as follows: Identify the ideal boundary S^2 of $\mathcal{H}^3(-c^2)$ with $\mathbf{C} \cup \{\infty\}$ by stereographic projection. Then the normal geodesic ray emanating from each point $z = P$ of the surface meets the ideal boundary at $G(P)$ ([Bry]). The $(2,0)$-part $\mathcal{Q}$ of the complexified second fundamental form is called the *Hopf differential* of the immersion f. The second fundamental form h of f is expressed

$$h = \mathcal{Q} + \bar{\mathcal{Q}} + c\, ds^2, \tag{5}$$

where ds^2 is the first fundamental form. Moreover, the first fundamental form ds^2 has the expression ([UY2])

$$ds^2 = (1 + |g|^2)^2 \left| \frac{\mathcal{Q}}{dg} \right|^2. \tag{6}$$

The hyperbolic Gauss map G, the secondary Gauss map g and the Hopf differential $\mathcal{Q}$ satisfy the following identity ([UY1], [UY3])

$$S_g dz^2 - S_G dz^2 = 2c\mathcal{Q}, \tag{7}$$

where S_u denotes the Schwarzian derivative of a holomorphic function u defined by

$$S_u := \left(\frac{u''}{u'}\right)' - \frac{1}{2}\left(\frac{u''}{u'}\right)^2, \qquad (u' = \frac{du}{dz},\ u'' = \frac{d^2u}{dz^2}). \tag{8}$$

The pair $(g, \mathcal{Q}/dg)$ is called the Weierstrass data of the CMC-c immersion f.

It is well known that CMC-1 surfaces in $\mathcal{H}^3(-1)$ locally correspond to CMC-c surfaces in $\mathcal{H}^3(-c^2)$ with the same first fundamental form and the same Hopf differential for any non-zero real number c. As a special limiting case, when $c \to 0$, the CMC-c surface locally induces a minimal surface with the same first fundamental form and the same Hopf differential. (The secondary Gauss map of the CMC-c surface is the same as the Gauss map of the corresponding minimal surface.) We call this the *canonical correspondence*. The lift F is related to the Weierstrass data as follows (See [Bry],[UY1] and [UY2].):

$$F^{-1}dF = c \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \frac{\mathcal{Q}}{dg}. \tag{9}$$

This analogue of the Weierstrass representation is called the Bryant representation. Since the Weierstrass data is determined by the Hopf differential and the secondary Gauss map, the corresponding minimal immersion $f_0 : \tilde{M} \to \mathbf{R}^3$ has the expression

$$f_0(z) = \mathrm{Re} \int_{z_0}^{z} (1 - g^2, i(1 + g^2), 2g) \frac{\mathcal{Q}}{dg},$$

with the same Weierstrass data $(g, \mathcal{Q}/dg)$.

Suppose that the first fundamental form ds^2 of the CMC-c immersion f is complete and of finite total curvature. As in the case of minimal surfaces, there exists a compact Riemann surface $\bar{M}$ and a finite number of points $P_1, \cdots, P_N \in \bar{M}$ such that M is bi-holomorphic to $\bar{M} \setminus \{P_1, \cdots, P_N\}$. Each point P_j is called an *end* of the surface. An end of a CMC-c surface is *regular* if the hyperbolic Gauss map extends meromorphically across the end; otherwise it is called *irregular*. A meromorphic function on M is called *rational* if it extends meromorphically on $\bar{M}$. For minimal surfaces, the immersion $f_0 : M \to \mathbf{R}^3$ is called *rational* if the lift $F_0 : M \to \mathbf{C}^3$ (such that $f_0 = F_0 + \bar{F}_0$) extends meromorphically across the ends. Similarly, the CMC-c immersion f is called *rational* if the lift F into $\mathrm{PSL}(2, \mathbf{C})$ of f is single valued on M and can be extended meromorphically on $\bar{M}$. When f is H-deformable (namely, it admits a global non-trivial isometric deformation preserving the principal curvatures), then the lift F is single valued on M and the corresponding minimal surface f_0 is rational. (See [UY3; p.216].) One particular consequence is that the secondary Gauss map g of f is rational. Though the secondary Gauss map g of an H-deformable CMC-c surface is rational, the hyperbolic Gauss map G might have essential singularities. In fact, all of the examples we construct in the next section will have an irregular end. This shows that H-deformable CMC-c surfaces f may not be rational in general. More precisely, the H-deformability and the rationality are mutually independent concepts for CMC-c surfaces in $\mathcal{H}^3(-c^2)$. In fact, there are CMC-1 surfaces which admit a rational lift into $\mathrm{PSL}(2, \mathbf{C})$ but may not be H-deformable. (See Example 3 in [UY3, §3]). The following fact is a key to the construction of complete H-deformable CMC-c surfaces.

Lemma 1. *Let g and $\mathcal{Q}$ be a rational function and a meromorphic differential, respectively, on a compact Riemann surface $\bar{M}$ such that the symmetric tensor defined by*

$$ds^2 = (1 + |g|^2)^2 \left| \frac{\mathcal{Q}}{dg} \right|^2 . \tag{10}$$

is complete at the points $\{P_1, \cdots, P_N\}$ $(\subset \bar{M})$ and is positive definite on $M := \bar{M} \setminus \{P_1, \cdots, P_N\}$. Suppose that for some non-zero real number $c = c_0$, there exists a meromorphic function G_{c_0} defined on M such that

$$S_g dz^2 - S_{G_c} dz^2 = 2c\mathcal{Q}. \tag{11}$$

Then there exists a complete CMC-c_0 immersion $f_{c_0} : M \to \mathcal{H}^3(-c_0^2)$ with finite total curvature and whose Weierstrass data is $(g, \mathcal{Q}/dg)$. Furthermore, if meromorphic functions G_c on M satisfying (11) exist for all $c \in \mathbf{R} \setminus \{0\}$, then the original surface f_{c_0} is H-deformable.

Proof. By Theorem 1.6 in [UY3], there exists a unique null meromorphic map (single valued on all of M) $F_{c_0} : M \to \mathrm{PSL}(2, \mathbf{C})$ whose hyperbolic Gauss map and the secondary Gauss map are G_{c_0} and g, respectively. We remark that one can, if necessary, explicitly write down F_{c_0} in terms of 3-jets of G_{c_0} and g without integration (c.f. Small [S]). We set $f_{c_0} = (1/c_0) F_{c_0} F_{c_0}^*$. By (10), f_{c_0} is an CMC-c_0 immersion on M, and since ds^2 is the induced metric, f_{c_0} has a complete metric. Since the total curvature is 4π times the degree of g, the rationality of g implies the finiteness of the total curvature of f_0. This proves the first assertion.

Now we assume a meromorphic function G_c on M satisfying (11) exists for each $c \in \mathbf{R} \setminus \{0\}$. Then the immersions $(f_c)_{c \neq 0}$, constructed as above by replacing c_0 by c, are all single valued on M and have the common Weierstrass data $(g, Q/dg)$. By [UY3, Theorem 3.3], we can conclude that each f_c is H-deformable. □

3. An analogue of the UP-iteration

First, we recall from [M] the definition of the UP-iteration for rational minimal surfaces of genus zero. Let $g : \mathbf{C} \to \mathbf{C}P^1$ be a rational function and $\mathcal{Q}$ a meromorphic 2-differential on $\mathbf{C}P^1 = \mathbf{C} \cup \{\infty\}$, such that the metric given by

$$ds^2 = (1 + |g|^2)^2 \left| \frac{\mathcal{Q}}{dg} \right|^2 . \tag{12}$$

is complete at the points $\{P_1, \cdots, P_N, \infty\}\,(\subset \mathbf{C}P^1)$ and is positive definite on $M := \mathbf{C}P^1 \setminus \{P_1, \cdots, P_N, \infty\}$. Then the map defined by the Weierstrass representation

$$f(z) = \mathrm{Re} \int_{z_0}^{z} (1-g^2, i(1+g^2), 2g)\frac{\mathcal{Q}}{dg}$$

is a conformal minimal immersion on the universal cover $\tilde{M}$ of M. If the map

$$F(z) = \frac{1}{2}\int_{z_0}^{z} (1-g^2, i(1+g^2), 2g)\frac{\mathcal{Q}}{dg}$$

is a rational map on $\mathbf{C}P^1$, f is called a *rational minimal surface* on M. In this procedure, ds^2 is the first fundamental form of the minimal surface, and thus the positivity of the metric ds^2 is crucial. In fact, if ds^2 defined by (12) had degenerate points, such points would be branch points of the surface.

We restrict the Hopf differential $\mathcal{Q}$ to the following form

$$\mathcal{Q} = dz^2,$$

where z is the canonical coordinate on $\mathbf{C}$. By (12), we have

$$ds^2 = |\frac{1}{g'}|^2(1+|g|^2)^2|dz|^2,$$

where $g' = dg/dz$. In this setting, the metric ds^2 never degenerates: Let $\{P_1, \cdots, P_N\}$ be the union of the zeros of g' and the poles of g on $\mathbf{C}$. Then ds^2 is complete at $\{P_1, \cdots, P_N\}$ and at infinity, ∞, and ds^2 is positive definite on $\mathbf{C} \setminus \{P_1, \cdots, P_N\}$. The *Darboux-Bäcklund transformation* of g is given by

$$\hat{g}(z) := \int_{z_0}^{z} \frac{dz}{g'},$$

which is globally defined if and only if the integrand has zero residues. If $\hat{g}$ is defined on M, we call $\hat{g}$ *rational.*

Taking a sequence of Möbius transformations (T_n), we set

$$f_n = \mathrm{Re} \int_{z_0}^{z} (1-g_n^2, i(1+g_n^2), 2g_n)\frac{dz}{g_n'} \qquad (n = 0, 1, 2, 3, ...),$$

where

$$g_0 = g, \qquad g_{n+1} = \widehat{T_n \circ g_n} \quad (n = 0, 1, 2, 3, ...).$$

The surfaces f_n are called the *n-th UP-iterates* of f_0. The following assertion has been shown by the first author.

Theorem 2. *(McCune* [M, Theorem 5.8]*) If $g : \mathbf{C} \to \mathbf{C}P^1$ is rational with double branch points, and if its Schwarzian derivative has zero residues, then each UP-iterate f_n is a rational minimal surface with Hopf differential $\mathcal{Q} = dz^2$ and with rational Gauss map g_n.*

For example, if we take Enneper's surface as the initial surface, the assumptions in the theorem are satisfied and we can construct various rational minimal surfaces with many ends. (See [M].)

As mentioned in the previous section, minimal surfaces locally correspond to CMC-c surfaces with the same Weierstrass data. We denote by $f_{n,c}$ the corresponding CMC-c immersion associated with f_n and call it the *$\mathcal{H}^3(-c^2)$-correspondence of f_n*. By definition, $f_{n,c}$ has the secondary Gauss map g_n and the Hopf differential $\mathcal{Q} = dz^2$. Since the correspondence is local, $f_{n,c}$ may not be single valued on the surface even when f_n is rational. In fact, there is a rational minimal surface whose associated CMC-c immersion is not single-valued (See [UY3, §3]). The main result in this paper is as follows:

Theorem 3. *If $g : \mathbf{C} \to \mathbf{C}P^1$ is rational with double branch points, and if its Schwarzian derivative has zero residues, then the $\mathcal{H}^3(-c^2)$-corresponding $f_{n,c}$ of each UP-iterate f_n is an H-deformable CMC-c surface with Hopf differential $\mathcal{Q} = dz^2$. The surface $f_{n,c}$ has regular ends at the poles and at the branch points of its Gauss map g_n, and it has an irregular end at $z = \infty$.*

Moreover, the original UP-iterate f_n can be viewed as the limit $\lim_{c\to 0} f_{n,c}$ in the Poincare ball of radius $1/c$.

This theorem implies that the UP-iteration has a very strong property: it simultaneously ensures the preservation of the single valued property and the preservation of the H-deformablity for the $\mathcal{H}^3(-c^2)$-correspondence. It should be remarked that f_c and f_{-c} are non-congruent in general. (See Example below.) To prove the theorem, we first prepare some notation and a lemma.

The Schwarzian derivative $S_g(z)$ is single valued on $\mathbf{C}$ and has the following Laurent expansion around a double branch point $z = P$ of g:

$$S_g(z) = -\frac{4}{(z-P)^2} + \frac{\beta_1}{(z-P)} + \beta_2 + \beta_3(z-P) + \cdots .$$

Each coefficient β_j is called the j-th coefficient of $S_g(z)$ at $z = P$.

Lemma 4. *If $g : \mathbf{C} \to \mathbf{C}P^1$ is rational with double branch points, and if its Schwarzian derivative has zero residues, then the third coefficient of $S_g(z)$ vanishes at the branch points of g.*

Proof. We first treat the case where g has non-polar double branch points: A *non-polar branch point* of a rational function $g : \mathbf{C} \to \mathbf{C}P^1$ is a branch point p of g such that $g(p) \neq \infty$. Let $z = P$ be such a branch point of g. Then $g'(z)$ has the following expansion at $z = P$

$$g'(z) = b_2(z-P)^2 + b_3(z-P)^3 + b_4(z-P)^4 + \dots . \tag{13}$$

By a direct calculation, we have

$$S_g(z) = -\frac{4}{(z-P)^2} - \frac{2b_3}{b_2(z-P)} + \left(\frac{b_3^2}{2b_2^2} - \frac{2b_4}{b_2}\right) + \left(\frac{b_3^3}{b_2^3} - \frac{2b_3b_4}{b_2^2}\right)(z-P) + \dots .$$

Since S_g has zero residues, we have $b_3 = 0$. The third coefficient of $S_g(z)$ at $z = P$ is given by

$$\frac{b_3^3}{b_2^3} - \frac{2b_3b_4}{b_2^2},$$

which vanishes because $b_3 = 0$.

If some of the double branch points are poles, then we can choose a Möbius transformation μ such that $\mu \circ g$ has non-polar double branch points. Applying the arguments above, we see that the third coefficient of $S_{\mu\circ g}(z)$ is zero. Since $S_{\mu\circ g}(z) = S_g(z)$, this implies that the third coefficient of $S_g(z)$ is zero. □

Proof of Theorem 3. We first assume that g_{n-1} has r double branch points, but no other branch points, and also that its Schwarzian has zero residues. Then for any Möbius transformation ν we obtain a map $g_n = \widehat{\nu \circ g_{n-1}}$. As seen in the proof of [M, Theorem 5.8], there are only r Möbius transformations such that g_n does *not* have double branch points. If ν is such a Möbius transformation, then we may take a sequence of Möbius transformations $(\nu_k)_{k=1,2,3,\dots}$ such that $\lim_{k\to\infty} \nu_k = \nu$ and such that

$$g_{n,k} := \widehat{\nu_k \circ g_{n-1}}$$

has double branch points for each k. Since the maps $g_{n,k}$ converge locally uniformly to the map g_n, and since the Schwarzian of a function is a rational expression in terms of the derivatives of that function, the Schwarzians $S_{g_{n,k}}$ also converge locally uniformly to S_{g_n}. It was shown in [M] that, for each k, $S_{g_{n,k}}$ has zero residues because g_{n-1} has double branch points and because $S_{g_{n-1}}$ has zero residues. Then Lemma 4 implies that the third coefficient of $S_{g_{n,k}}$ vanishes at the branch points of $g_{n,k}$. (The branch points of $g_{n,k}$ will move as k moves and converge to the branch points of g_n.)

Let $P_i(k)$, $i = 1, \dots, N(k)$ be the branch points of $g_{n,k}$. Note that the poles of $S_{g_{n,k}}$ only occur at the branch points of $g_{n,k}$. We set

$$M_k = \mathbf{C} \setminus \{P_i(k)\}_{i=1}^{N(k)},$$

and denote by $\tilde{M}_k$ its universal covering. Then there exists a holomorphic function $G_{c,n,k}$ on $\tilde{M}_k$ such that

$$S_{G_{c,n,k}} = S_{g_{n,k}} - 2c. \tag{14}$$

To see this, consider the ordinary differential equation

$$\psi''(z) + u(z;c,n,k)\psi(z) = 0, \tag{15}$$

where

$$u(z;c,n,k) = \frac{1}{2}(S_{g_{n,k}}(z) - 2c).$$

Since $u(z;c,n,k)$ has pole of order 2 at each branch point of $g_{n,k}$, the ordinary differential equation (15) has regular singularities at these points. Expanding $S_{g_{n,k}}$ at the branch point $P_i(k)$ for a given i shows that the leading coefficient of $S_{g_{n,k}}$ is -4 (because $g_{n,k}$ has double branch points). Therefore the indicial equation is given by (see appendix)

$$\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0.$$

By the appendix, (15) has the two linearly independent solutions $\{X_1, X_2\}$ of the form

$$X_1(z) = (z - P_i)^{-1}\sum_{j=0}^{\infty}\xi_j(z-P_i)^j, \quad X_2(z) = (z-P_i)^2\sum_{j=0}^{\infty}\eta_j(z-P_i)^j + \mu X_1(z)\log(z-P_i),$$

where the coefficient μ is called *the log-term coefficient*. By (iii) of the corollary in the appendix, we have

$$\mu = \frac{1}{2}\beta_3 - \frac{1}{4}\beta_1(\beta_2 - 2c) + \frac{1}{32}(\beta_1)^3.$$

where β_j $(j = 1,2,3)$ is the j-th coefficient of $S_{g_{n,k}}$. Since $S_{g_{n,k}}$ has no residue, we have $\beta_1 = 0$, and by Lemma 4, β_3 also vanishes. Thus $\mu = 0$. In particular, X_1 and X_2 are both single valued around $z = P_i$. We have (see [L] or [M])

$$S_{X_1/X_2} = S_{g_{n,k}} - 2c.$$

This implies that X_1/X_2 and $G_{c,n,k}$ differ only by a Möbius transformation, and that $G_{c,n,k}$ is meromorphic at $z = P_i(k)$. Hence $G_{c,n,k}$ is an entire function.

Then by Lemma 1, there exists CMC-c surface in $\mathcal{H}^3(-c^2)$ whose hyperbolic Gauss map and the secondary Gauss map are $G_{c,n,k}$ and $g_{n,k}$, respectively. Since the periods are continuous, taking a limit $k \to \infty$, we can conclude that the $\mathcal{H}^3(-c^2)$-corresponding surface $f_{n,c}$ is also single valued on M. We assumed that g_{n-1} had double branch points, but by using multiple sequences of Möbius transformations, it is enough to assume that our initial Gauss map g had double branch points.

Since c is arbitrary, $f_{n,c}$ is H-deformable by Lemma 1. An end of a CMC-c surface in $\mathcal{H}^3(-c^2)$ is regular if and only if the Hopf differential is at most pole of order -2 ([Bry] and [UY1]). Since $f_{n,c}$ has the same first fundamental form and the Hopf differential as f_n, the branch points and poles of g_n are regular ends. In fact, the metric ds^2 is complete and the Hopf differential is holomorphic at those points. The metric ds^2 is also complete at $z = \infty$, but the Hopf differential has pole of order -4. This implies $z = \infty$ is an irregular end of $f_{n,c}$. In [UY2], it is shown that the associate minimal immersion can be obtained as a limit of associated CMC-c immersion in the Poincare ball of radius $1/c$ as $c \to 0$. The final assertion follows from it. $\square$

Example 1. For producing concrete examples, Enneper's surface is the initial data used in the UP-iteration. This minimal surface has Weierstrass data (z, dz^2), and the $\mathcal{H}^3(-c^2)$-corresponding surface f_c has the hyperbolic Gauss map $G(z) = Tanh(cz)$. Under the transformation $z \mapsto iz$, the first fundamental form $ds^2 = (1+|z|)^2|dz|^2$ is unchanged and the Hopf differential changes sign. This implies that f_c and f_{-c} are congruent. (The surfaces f_c and f_{-c} are congruent if and only if the first fundamental form does not change under the transformation $z \mapsto iz$.) In Figure 1, the original minimal Enneper's surface is shown on the left. The corresponding CMC1 surface f_1 in $\mathcal{H}^3(-1)$, called Enneper's cousin, is shown in the center. To the right, the dual surface $f_1^{\#}$ of the Enneper cousin f_1 is shown. An explicit formula for the lift F can be found in [Bry, p.340]. The surface $f_1^{\#}$ is obtained by using the lift F^{-1},

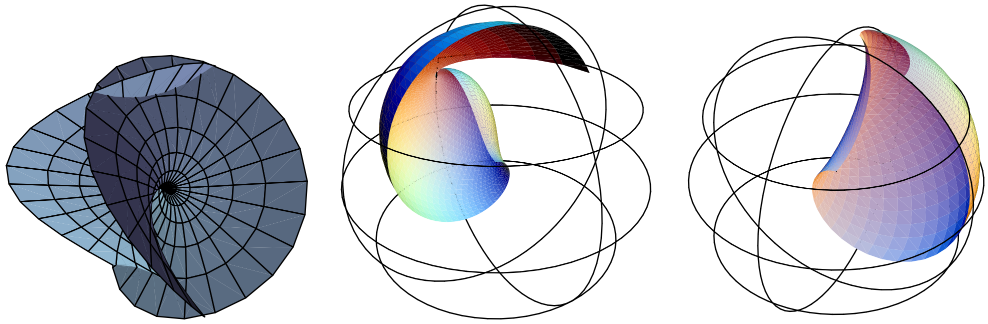

FIGURE 1. Enneper's surface, its cousin, and the cousin's dual. (The last two pictures are courtesy of Wayne Rossman.)

rather than the lift F of f_1, the effect being that the hyperbolic and secondary Gauss maps are interchanged. The dual surface of Enneper's cousin is also a complete surface, but it has infinite total curvature.

Example 2. In applying the UP-iteration to Enneper's surface, there is one parameter of freedom arising from the choice of Möbius transformation. A Möbius transformation ν is composed with the Gauss map $g_0(z) = z$, and then the Darboux-Bäcklund transformation is performed to yield the new Gauss map g_1. If we choose

$$\nu = \begin{pmatrix} 0 & -1 \\ 1 & k_1 \end{pmatrix},$$

then the UP-iterate is

$$g_1 = \int \frac{1}{(\nu \circ g_0)'(z)} = \int \frac{1}{(z+k_1)^2} = k_1^2 z + k_1 z^2 + \frac{z^3}{3},$$

up to an additive constant of integration.

Choosing the constant $k_1 = 1$ yields the Gauss map $g(z) = z + z^2 + z^3/3$, and the first fundamental form is $ds^2 = (1 + |z + z^2 + z^3/3|^2)^2/(|1 + 2z + z^2|^2)$. Since ds^2 is not invariant under the transformation $z \mapsto iz$, this Gauss map yields non-congruent $\mathcal{H}^3(-1)$-corresponding surfaces, f_1 and f_{-1}. The two hyperbolic Gauss maps are

$$G_1(z) = \frac{-(1+z)Cosh(1+z) + Sinh(1+z)}{-Cosh(1+z) + (1+z)Sinh(1+z)}$$

and

$$G_{-1}(z) = \frac{(1+z)Cos(1+z) - Sin(1+z)}{Cos(1+z) + (1+z)Sin(1+z)},$$

with Schwarzians

$$S_{G_1}(z) = -2\frac{3 + 2z + z^2}{(1+z)^2}$$

and

$$S_{G_{-1}}(z) = 2\frac{-1 + 2z + z^2}{(1+z)^2},$$

respectively. The original minimal surface, along with the $\mathcal{H}^3(-1)$ corresponding surface for $c = 1$, are shown in Figure 2. The $\mathcal{H}^3(-1)$ corresponding surface for $c = -1$ is shown from both front and back in Figure 3.

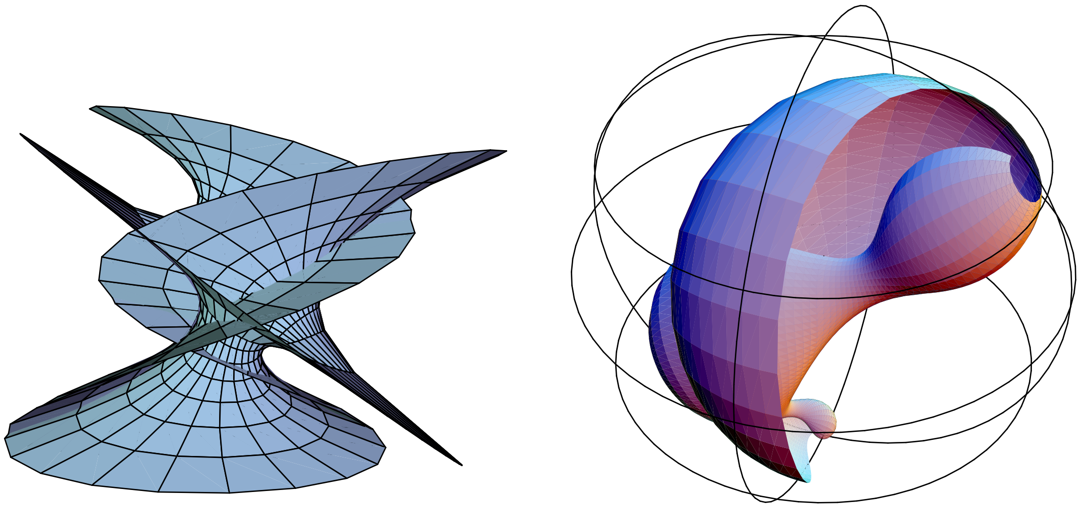

FIGURE 2. A first level UP-iterate and one of its $\mathcal{H}^3(-1)$-corresponding surfaces.

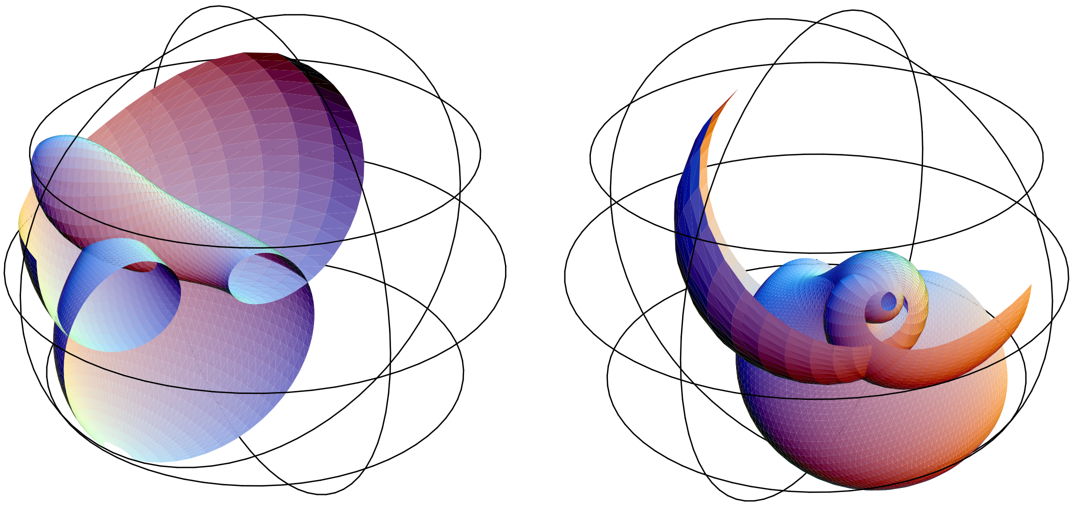

FIGURE 3. Two views of another $\mathcal{H}^3(-1)$-corresponding surface.

APPENDIX: A COMPUTATION OF THE LOG TERM COEFFICIENTS

We shall discuss on the solution of the ODE with a regular singularity at $z = P$ ($P \in \mathbf{C}$)

$$(z-P)^2 y''(z) - q(z)y(z) = 0, \tag{16}$$

where $q(z) = \sum_{j=0}^{\infty} q_j (z-P)^j$ ($q_j \in \mathbf{C}$). It is well known that (16) has the two linearly independent solutions $\{X_1, X_2\}$ of the form

$$X_1(z) = z^{\lambda_1} \sum_{j=0}^{\infty} \xi_j (z-P)^j, \qquad X_2(z) = z^{\lambda_2} \sum_{j=0}^{\infty} \eta_j (z-P)^j + \mu X_1 \log(z-P),$$

where λ_1 and λ_2 are the solutions of the indicial equation of (16),

$$\lambda^2 - \lambda - q_0 = 0;$$

explicitly,

$$\lambda_1 = \frac{1}{2}\{1+m\}, \quad \lambda_2 = \frac{1}{2}\{1-m\}, \quad m = \sqrt{1+4q_0}. \tag{17}$$

The coefficient μ is called *the log-term coefficient* at the regular singular point $z = P$, which might be non-zero only when

$$m := \lambda_1 - \lambda_2 \in \mathbf{Z}.$$

The following assertion holds. (c.f. [CL])

Proposition. *Suppose the difference of the solutions of indicial equation $m := \lambda_1 - \lambda_2$ is a positive integer. Then the log-term coefficient μ is given by*

$$\mu = \frac{1}{m}\sum_{k=0}^{m-1} q_{m-k}a_k \tag{18}$$

where

$$\begin{aligned} a_0 &= m \\ a_j &= \frac{1}{j(j-m)}\sum_{k=0}^{j-1} q_{j-k}a_k \end{aligned}$$

By a direct calculation, we obtain the following

Corollary. *The solutions of $(z-P)^2y''(z) - q(z)y(z) = 0$ have no log-term at $z = P$ if and only if*

(i) $q_1 = 0$ *for* $m = 1$,
(ii) $q_2 - (q_1)^2 = 0$ *for* $m = 2$,
(iii) $q_3 - q_1q_2 + \frac{1}{4}(q_1)^3 = 0$ *for* $m = 3$.

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