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Large and small Density Approximations to the thermodynamic Bethe Ansatz

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Abstract

We provide analytical solutions to the thermodynamic Bethe ansatz equations in the large and small density approximations. We extend results previously obtained for leading order behaviour of the scaling function of affine Toda field theories related to simply laced Lie algebras to the non-simply laced case. The comparison with semi-classical methods shows perfect agreement for the simply laced case. We derive the Y-systems for affine Toda field theories with real coupling constant and employ them to improve the large density approximations. We test the quality of our analysis explicitly for the Sinh-Gordon model and the $(G_2^{(1)}, D_4^{(3)})$-affine Toda field theory.

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1 Introduction

In the context of 1+1-dimensional integrable quantum field theories numerous methods have been developed to compute various quantities in an exact manner, that is non-perturbative in the coupling constant. Sometimes it is even possible to perform the related computations analytically. Often the evaluation within one particular approach lacks information, typically a constant, which might be supplied by an entirely different method. Ideally one would like to achieve a situation in which each approach is self-consistent.

An example for the situation just outlined is given for instance in the form-factor program [1], which allows in principle to compute correlation functions. The lowest non-vanishing form-factor is not fixed within this approach and is typically obtained from elsewhere. For instance the vacuum expectation value of the energy-momentum tensor can be extracted from the thermodynamic Bethe ansatz (TBA) [2]. Alternatively, one can compute correlation functions by perturbing around the conformal field theory [3]. Also in this approach one appeals to the thermodynamic Bethe ansatz for the vacuum expectation value of the energy-momentum tensor and to the Bethe ansatz [4] for the relation between the coupling constant and the masses. The latter correspondence is also needed in an approach initiated recently in [5, 6, 7], where it was observed that the ultraviolet asymptotic behaviour for many theories may be well approximated by zero-mode dynamics. The considerations in [5] exploit the knowledge of an exact reflection amplitude, which on one hand results from certain manipulation on the three-point function of the underlying ultraviolet conformal field theory and on the other hand has a semi-classical counterpart in the related quantum mechanical problem. The question why this approach allows to compute scaling functions with a high accuracy is still to be settled [8].

The computation of scaling functions by means of the TBA does conceptually not require any additional input from other methods. However, up-to-now it is only possible to tackle the problem numerically due to the nonlinear nature of the central equation involved. Several attempts have been made to formulate analytical approximations. This is desirable for various reasons, one being that the numerical effort becomes quite considerable for some models with increasing particle species content. A further reason is of course that analytical expressions allow to study further the deeper structures of the theories. For instance for affine Toda field theories (ATFT) [9] related to simply laced Lie algebras some analytical expressions have been

provided [10, 11, 12]. In the approximation method of [10, 11, 12] a constant was left undetermined, which can be fixed in the same spirit as outlined for the other methods in the preceding paragraph, namely by appealing to another approach. Contrary to the claim in [6], we demonstrate in the present manuscript that it is possible to fix the constant in this way without approximating higher order terms. Extending the analysis of [11, 12] also to the non-simply laced case in the present manuscript, we will demonstrate in addition that the constant may also be well approximated from within the TBA-analysis. Furthermore, we give simple analytical expressions for improved approximations in the large and small density regime.

Our manuscript is organized as follows: In section 2 we provide large and small density approximations for the solutions of the TBA-equations. In section 3 we assemble the necessary data for ATFT needed to extend our previous analysis to the non-simply laced case. We derive universal TBA-equations and Y-systems for *all* ATFT and show how they may be utilized to improve on the analytical approximations. We derive the related scaling function. We test the quality of the various approximations for the explicit example of the Sinh-Gordon model and the $(G_2^{(1)}, D_4^{(3)})$-ATFT. We state our conclusions in section 4.

2 Large and small density approximations

2.1 The TBA

The object of investigation of the TBA is a multiparticle system containing n different particle species with masses m_i $(1 \leq i \leq n)$ whose dynamical interaction is described by a factorizable diagonal scattering matrix $S_{ij}(\theta)$, which is a function of the rapidity difference θ. We assume the statistical interaction to be of fermionic type*. Adopting the notation of [12] the thermodynamic Bethe Ansatz equations [2], which characterize the thermodynamic equilibrium of such a system, are the n coupled nonlinear integral equations

$$rm_i \cosh\theta + \ln\left(e^{L_i(\theta)} - 1\right) = \sum_{j=1}^{n} \left(\varphi_{ij} * L_j\right)(\theta)\,. \tag{1}$$

*In order to keep the discussion as simple as possible we do not treat general statistics here as for instance Haldane type [13]. Generalizations of our arguments in this sense are straightforward.

Here the scaling parameter r is given by the inverse temperature times a mass scale. The convolution of two functions is abbreviated as usual by $(f * g)(\theta) := 1/2\pi \int d\theta' f(\theta - \theta')g(\theta')$. The TBA kernel reads

$$\varphi_{ij}(\theta) := -i\frac{d}{d\theta}\ln S_{ij}(\theta). \tag{2}$$

The functions L_i, which are to be determined as solutions of the TBA equations (1), are related to the particle densities ρ_r^i and the densities of available states ρ_h^i as $L_i = \ln(1+\rho_r^i/\rho_h^i)$, such that for physical reasons $L_i \geq 0$. Keeping this definition in mind, we speak of the large density regime when $L_i > \ln 2$ and of the small density regime when $L_i < \ln 2$. It is sometimes useful to express matters in terms the pseudo-energies $\varepsilon_i(\theta) := -\ln[\exp(L_i(\theta)) - 1)]$.

Having solved the TBA-equations (1) for the L–functions one is in principle in the position to evaluate the scaling function

$$c(r) = \frac{6r}{\pi^2}\sum_{i=1}^{n} m_i \int_0^\infty d\theta L_i(\theta)\cosh\theta \tag{3}$$

which can be interpreted as off-critical effective central charge belonging to the conformal field theory obtained in the ultraviolet limit, i.e. $r \to 0$. It is our goal in this manuscript to approximate this function in a simple analytical way to high accuracy.

2.2 Approximative analytical Solutions

In general it is possible to solve the TBA-equations numerically, where the convergence of the iterative procedure is guaranteed by means of the Banach fixed point theorem [12]. However, the numerical problem becomes quite complex when one increases the number of particle species. For this reason, and more important because one would like to gain a deeper structural insight into the solutions of (1), it is desirable to obtain analytical solutions to the TBA-equations. Due to the nonlinear nature of (1) only few analytical solutions are known. Nonetheless, one may obtain approximated analytical solutions when r tends to zero. For large L_i, i.e. for large particle densities, it was shown in [10, 11, 12] that the integral equation (1) may be turned into a set of differential equations of infinite order. Under certain natural assumptions, which are however not satisfied universally for all models, one may

approximate these equations by second order differential equations, whose solutions are given by

$$L_i^0(\theta) = \ln\left(\frac{\cos^2(\beta_i\theta)}{2\beta_i^2\eta_i}\right) \qquad \text{for } |\theta| \leq \frac{\arccos(\beta_i\sqrt{2\eta_i})}{\beta_i}. \tag{4}$$

The restriction on the range of the rapidity stems from the physical requirement $L_i \geq 0$. The n constants $\eta_i = \sum_j \eta_{ij}^{(2)}$ are determined by a power series expansion of the TBA kernel

$$\widetilde{\varphi}_{ij}(t) := \int_{-\infty}^{\infty} d\theta\, \varphi_{ij}(\theta) e^{it\theta} = 2\pi \sum_{n=0}^{\infty} (-i)^n \eta_{ij}^{(n)} t^n\,. \tag{5}$$

The dependence on the scaling parameter r enters through the quantity

$$\beta_i = \frac{\pi}{2(\delta_i - \ln(r/2))}\,. \tag{6}$$

Here the β_i, δ_i are constants of integration. There is a very crude lower bound we can put immediately on δ_i. From the fact that $L_i^0(0) \geq \ln 2$, we deduce $\delta_i > 1/\pi/\sqrt{\eta_i} + \ln(r/2)$. For particular models we will provide below a rigorous argument which establishes that in fact they do not depend on the particle type, such that we may replace $\beta_i \to \beta$ and $\delta_i \to \delta$. We will also show that they can be fixed by appealing to the semi-classical approach in [5, 7]. In addition, we provide an argument which determines them approximately from within the TBA analysis by matching the large and small density regimes. Since the constant turns out to be model dependent, we will report on it in detail below when we discuss concrete theories.

The restriction on the range for the rapidities in (4), for which the large density approximation $L_i^0(\theta)$ ceases to be valid, makes it desirable to develop also an approximation for small densities. For extremely small densities we naturally expect that the solution will tend to the one for a free theory. Solving (1) for vanishing kernel yields the well-known solution

$$L_i^f(\theta) = \ln\left(1 + e^{-rm_i \cosh\theta}\right)\,. \tag{7}$$

Ideally we would like to have expressions for both regions which match at some distinct rapidity value, say θ_i^m, to be specified below. Since $L_i^0(\theta)$ and $L_i^f(\theta)$ become relatively poor approximations in the transition region between large and small densities, we seek for improved analytical expressions. This

is easily achieved by expanding (1) around the "zero order" small density approximations. In this case we obtain the integral representation

$$L_i^s(\theta) = \exp\left(-rm_i \cosh\theta + \sum\nolimits_{j=1}^{n} (\varphi_{ij} * L_j^f)(\theta)\right) \ . \tag{8}$$

For vanishing φ_{ij} we may check for consistency and observe that the functions $L^s(\theta)$ become the first term of the expansion in (7). One could try to proceed similarly for the large density regime and develop around L_i^0 instead of L_i^f. However, there is an immediate problem resulting from the restriction on the range of rapidities for the validity of L_i^0, which makes it problematic to compute the convolution. We shall therefore proceed in a different manner for the large density regime and employ Y-systems for this purpose.

In many cases the TBA-equations may be expressed equivalently as a set of functional relations referred to as Y-systems in the literature [14]. Introducing the quantities $Y_i = \exp(-\varepsilon_i)$, the determining equations can always be cast into the general form

$$Y_i(\theta + i\pi\mu)Y_i(\theta - i\pi\mu) = \exp(g_i(\theta)) \tag{9}$$

with μ being some real number and $g_i(\theta)$ being a function whose precise form depends on the particular model. We can formally solve the equation by Fourier transformations

$$Y_i(\theta) = \exp\left[(g_i * \gamma_i)(\theta)\right] \ , \qquad \gamma_i(\theta) = \left[2\mu\cosh(\theta/2/\mu)\right]^{1/2} \tag{10}$$

i.e. substituting (10) into the l.h.s. of (9) yields $\exp(g_i(\theta))$. Of course this identification is not completely compelling and we could have chosen also a different combination of Y's. However, in order to be able to evaluate the $g_i(\theta)$ we require a concrete functional input for the function $Y_i(\theta)$ in form of an approximated function. Choosing here the large density approximation L^0 makes the choice for $g_i(\theta)$ with hindsight somewhat canonical, since other combinations lead generally to non-physical answers.

We replace now inside the defining relation of $g_i(\theta)$ the Y's by $Y_i\,(\theta) \to \exp(L_i^0(\theta)) - 1$. Analogously to the approximating approach in [10, 11, 12], we can replace the convolution by an infinite series of differentials

$$\varepsilon_i(\theta) = -(g_i * \gamma_i)(\theta) \ = -\sum_{m=0}^{\infty} \nu_i^{(m)} \frac{d^m}{d\theta^m} g_i(\theta) \ , \tag{11}$$

where the ν's are defined by the power series expansion

$$\int_{-\infty}^{\infty} d\theta \gamma_i(\theta)\, e^{it\theta} = 2\pi \sum_{m=0}^{\infty} (-i)^m \nu_i^{(m)} t^m = \pi \sum_{m=0}^{\infty} \frac{E_{2m}}{(2m)!} (\pi\eta_i)^{2m}\, t^{2m} \ . \qquad (12)$$

The E_m denote the Euler numbers, which enter through the expansion $1/\cosh x = \sum_{m=0}^{\infty} x^{2m} E_{2m}/(2m)!$. In accordance with the assumptions of our previous approximations for the solutions of the TBA-equations in the large density approximation, we can neglect all higher order derivatives of the $L_i^0(\theta)$. Thus we only keep the zeroth order in (11). From (12) we read off the coefficient $\nu_i^{(0)} = 1/2$, such that we obtain a simply expression for an improved large density approximation

$$L_i^l(\theta) = \ln[1 + Y_i^l(\theta)] = \ln[1 + \exp(g_i(\theta)/2)] \, . \qquad (13)$$

In principle we could proceed similarly for the small density approximation and replace now $Y_i(\theta) \to \exp(L_i^s(\theta)) - 1$ in the defining relations for the g_i's. However, in this situation we can not neglect the higher order derivatives of the L_i^s such that we have to keep the convolution in (11) and end up with an integral representation instead. We now wish to match L_i^s and L_i^l in the transition region between the small and large density approximations at some distinct value of the rapidity, say θ_i^m. We select this point to be the value when the function $f_i(\theta) = (6/\pi^2) r m_i L_i(\theta) \cosh\theta$, which is proportional to the free energy density for a particular particle species, has its maximum in the small density approximation

$$\left. \frac{d}{d\theta} f_i^s(\theta) \right|_{\theta_i^m} = 0 \ . \qquad (14)$$

In regard to the quantity we wish to compute, the scaling function (3), this is the point in which we would like to have the highest degree of agreement between the exact and approximated solution, since this will optimize the outcome for $c(r)$. Having specified the θ_i^m, the matching condition provides a simple rational to fix the constant δ_i

$$L_i^l(\theta_i^m) = L_i^s(\theta_i^m) \qquad \Rightarrow \quad \delta_i^m \ . \qquad (15)$$

Clearly, in general we can not solve these equations analytically, but it is a trivial numerical problem which is by no means comparable with the one of

solving (1). Needless to say that the outcome of (15) is not to be considered as exact, but as our examples below demonstrate it will lead to rather good approximations. One of the reasons why this procedure is successful is that $L_i^s(\theta_i^m)$ is still very close to the precise solution, despite the fact that is at its worst in comparison with the remaining rapidity range.

Combining the improved large and small density approximation we have the following approximated analytical L-functions for the entire range of the rapidity

$$L_i^a(\theta) = \begin{cases} L_i^l(\theta) & \text{for } |\theta| \leq \theta_i^m \\ L_i^s(\theta) & \text{for } |\theta| > \theta_i^m \end{cases} , \tag{16}$$

such that the scaling function becomes well approximated by

$$c(r) \simeq \sum_{i=1}^{n} \int_0^\infty d\theta f_i^a(\theta) \ . \tag{17}$$

To develop matters further and report on the quality of L^0, L^f, L^s, L^l we have to specify a particular theory at this point.

3 Affine Toda field theory

Affine Toda field theories [9] form a well studied class of relativistic integrable quantum field theories in 1+1 space-time dimensions. To each of these field theories a pair of affine Lie algebras $(X_n^{(1)},\hat{X}^{(\ell)})$ [15] is associated whose structure allows universal statements concerning its properties, like the S-matrix, the mass spectrum, the fusing rules, etc. Here $\hat{X}^{(\ell)}$ denotes a twisted affine Lie algebra w.r.t. a Dynkin diagram automorphism of order ℓ. Both algebras are chosen to be dual to each other, i.e. $\hat{X}^{(\ell)}$ is obtained from the non-twisted algebra $X_n^{(1)}$ of rank n by exchanging roots and co-roots. For $X_n^{(1)}$ simply-laced both algebras coincide, i.e. $X_n^{(1)} \cong \hat{X}^{(\ell)}$, $\ell = 1$, which is reflected in the quantum theory by a strong-weak self-duality in the coupling constant. Moreover, the mass spectrum renormalises by an overall factor and the poles of the S-matrix in the physical sheet do not depend on the coupling constant. For non-simply laced Lie algebras these features cease to be valid. The quantum masses are now coupling dependent and flow between the classical masses associated with $X_n^{(1)}$ and $\hat{X}^{(\ell)}$ in the weak and strong coupling limit, respectively. Consequently, the physical poles of the S-matrix shift depending on the coupling and the strong-weak self-duality is broken.

3.1 The universal S-matrix

Remarkably, despite these structural differences the S-matrix of ATFT can be cast into a universal form covering the simply-laced as well as the non-simply laced case [16, 17]. For our purposes the formulation in form of an integral representation is most useful

$$S_{ij}(\theta) = \exp \int_0^\infty \frac{dt}{t}\, \phi_{ij}(t)\, \sinh \frac{t\theta}{i\pi}\,, \tag{18}$$

$$\phi_{ij}(t) = 8 \sinh(t\vartheta_h) \sinh(t_j \vartheta_H t)\ \left([K]_{q(t)\bar{q}(t)}\right)^{-1}_{ij} \quad . \tag{19}$$

Denoting a q-deformed integer n as common by $[n]_q = (q^n - q^{-n})/(q^1 - q^{-1})$, we introduced here a "doubly q-deformed" version of the Cartan matrix K [16, 18, 17] of the non-twisted Lie algebra

$$[K_{ij}]_{q\bar{q}} = (q\bar{q}^{\,t_i} + q^{-1}\bar{q}^{\,-t_i})\delta_{ij} - [I_{ij}]_{\bar{q}} \tag{20}$$

for the generic deformation parameters $q, \bar{q}$. The incidence matrix $I_{ij} = 2\delta_{ij} - K_{ij}$ of the $X_n^{(1)}$ related Dynkin diagram is symmetrized by the integers t_i, i.e. $I_{ij}t_j = I_{ji}t_i$. With α_i being a simple root we fix the length of the long roots to be 2 and choose the convention $t_i = \ell\alpha_i^2/2$. Inside the integral representation (18) we take

$$q(t) = e^{t\vartheta_h}, \quad \bar{q}(t) = e^{t\vartheta_H}, \qquad \text{with} \quad \vartheta_h := \frac{2-B}{2h}, \quad \vartheta_H := \frac{B}{2H} \tag{21}$$

for the deformation parameters, where $0 \le B \le 2$ is the effective coupling constant. We further need the Coxeter numbers h, $\hat{h}$ and the dual Coxeter numbers $h^\vee$, $\hat{h}^\vee$ of $X_n^{(1)}$ and $\hat{X}^{(\ell)}$, respectively, as well as the ℓ-th Coxeter number $H = \ell\hat{h}$ of $\hat{X}^{(\ell)}$. Complete tables of these quantities for individual algebras may be found in [15].
The incidence matrix satisfies the relation [16, 17]

$$\sum_{j=1}^{n} [\, I_{ij}]_{\bar{q}(i\pi)}\, m_j = 2\cosh(\theta_h + t_i\theta_H)\, m_i \ , \tag{22}$$

which will turn out to be crucial for the arguments below. We introduced here the imaginary angles $\theta_h = i\pi\vartheta_h$ and $\theta_H = i\pi\vartheta_H$.

3.2 The TBA-kernel

From the universal integral representation (18), we can now immediately derive the Fourier transformed TBA-kernel (5) for ATFT. However, when taking the logarithmic derivative one has to be careful about interchanging the derivative with the integral, since these two operations do not commute. Comparison with the block representation of the S-matrix [17] yields

$$\varphi_{ij}(\theta) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, \phi_{ij}(t)\, \exp\frac{t\theta}{i\pi}, \tag{23}$$

such that the Fourier transformed universal TBA-kernel (5) acquires the form

$$\widetilde{\varphi}_{ij}(t) = -\pi\phi_{ij}(\pi t) = -8\pi \sinh t\pi\vartheta_h \sinh t_j\pi\vartheta_H \ \left([K]_{q(\pi t)\bar{q}(\pi t)}\right)^{-1}_{ij} \ . \tag{24}$$

To be able to carry out the discussion of the previous section we require the second order coefficient $\eta^{(2)}_{ij}$ in the power series expansion (5). From (24) we read off directly

$$\eta^{(2)}_{ij} = \frac{\pi^2}{h\,H} B(2-B) K^{-1}_{ij}\, t_j = \frac{\pi^2}{h\,h^\vee} B(2-B)\,(\lambda_i\cdot\lambda_j) \ . \tag{25}$$

In the latter equality we used the fact that the inverse of the Cartan matrix is related to the fundamental weights as $\lambda_i = \sum_j K^{-1}_{ij}\ \alpha_j$, $t_i = \ell\alpha_i^2/2$ and $H = \ell\,\hat{h} = \ell\,h^\vee$. This implies on the other hand that

$$\eta_i = \frac{\pi^2}{h\,h^\vee} B(2-B)\,(\lambda_i\cdot\rho) \tag{26}$$

with $\rho = \sum_i \lambda_i$ being the Weyl vector. Therefore

$$\eta = \sum_{i=1}^{n} \eta_i = B(2-B)\frac{\pi^2\rho^2}{h\,h^\vee} = nB(2-B)\frac{\pi^2(h+1)}{12h} \ . \tag{27}$$

We used here the Freudenthal-de Vries strange formula $\rho^2 = h^\vee/12\ \dim X^{(1)}_n$ (see e.g. [19]) and the fact that $\dim X^{(1)}_n = n(h+1)$. Thus we have generalized the result of [12] to the non-simply laced case. Notice that in terms of quantities belonging to the non-twisted Lie algebra $X^{(1)}_n$ the formula (27) is identical for the simply laced and the non-simply laced case.

3.3 Universal TBA equations and Y-systems

In analogy to the discussion for simply-laced Lie algebras [12], the universal expression for the kernel (24) can be exploited in order to derive universal TBA-equations for *all* ATFT, which may be expressed equivalently as a set of functional relations referred to as Y-systems. Fourier transforming (1) in a suitable manner and invoking the convolution theorem we can manipulate the TBA equations by using the expression (24). After Fourier transforming back we obtain

$$\varepsilon_i + \sum_{j=1}^{n} \Delta_{ij} * L_j = \sum_{j=1}^{n} \Gamma_{ij} * (\varepsilon_j + L_j)\,. \tag{28}$$

The universal TBA kernels Δ and Γ are then given by

$$\gamma_i(\theta) = \left(2(\vartheta_h + t_i\vartheta_H)\cosh\tfrac{\theta}{2(\vartheta_h+t_i\vartheta_H)}\right)^{-1}, \tag{29}$$

$$\Gamma_{ij}(\theta) = \sum_{k=1}^{I_{ij}} \gamma_i(\theta + i(2k-1-I_{ij})\theta_H), \tag{30}$$

$$\Delta_{ij}(\theta) = \left[\gamma_i(\theta + (\theta_h - t_i\theta_H)) + \gamma_i(\theta - (\theta_h - t_i\theta_H))\right]\delta_{ij}\,. \tag{31}$$

The key point here is that the entire mass dependence, which enters through the on-shell energies $m_i\cosh\theta$, has dropped out completely from the equations due to the identity (22). Noting further that

$$[\,I_{ij}]_{\bar{q}(i\pi)}\, m_j\cosh\theta = \sum_{k=1}^{I_{ij}} m_j\,\cosh\left[\theta + (2k-1-I_{ij})\theta_H\right]\,, \tag{32}$$

we have assembled all ingredients to derive functional relations for the quantities $Y_i = \exp(-\varepsilon_i)$. For this purpose we may either shift the TBA equations appropriately in the complex rapidity plane or use again Fourier transformations, see [12]

$$Y_i\,(\theta+\theta_h+\,t_i\theta_H)\,Y_i\,(\theta-\theta_h-\,t_i\theta_H) = \frac{[1+Y_i(\theta+\theta_h-t_i\theta_H)][1+Y_i(\theta-\theta_h+t_i\theta_H)]}{\prod_{j=1}^{n}\prod_{k=1}^{I_{ij}}\left[1+Y_j{}^{-1}(\theta+(2k-1-I_{ij})\theta_H)\right]}\,. \tag{33}$$

These equations are of the general form (9) and specify concretely the quantities μ and $g_i(\theta)$. We recover various particular cases from (33). In case the associated Lie algebra is simply-laced, we have $\theta_h + t_i\theta_H \to i\pi/h$, $\quad\theta_h - t_i\theta_H \to$

$i\pi/h(1-B)$ and $I_{ij} \to 0, 1$, such that we recover the relations derived in [12]. As stated therein we obtain the system for minimal ATFT [14] by taking the limit $B \to i\infty$.

The concrete formula for the approximated solution of the Y-systems in the large density regime, as defined in (13), reads

$$Y_i^l(\theta) = \frac{\cos(2\theta\beta_i)+\cos(2(\theta_h - t_i\theta_H)\beta_i)}{4\eta_i\beta_i^2} \prod_{j=1}^{n}\prod_{m=1}^{I_{ij}} \left(1 - \frac{2\eta_j\beta_j^2}{\cos^2(\beta_j(\theta+(2m-1-I_{ij})\theta_H)}\right)^{\frac{1}{2}} \quad . \quad (34)$$

Exploiting possible periodicities of the functional equations (33) they may be utilized in the process of obtaining approximated analytical solutions [20]. As we demonstrated they can also be employed to improve on approximated analytical solution in the large density regime. In the following subsection we supply a further application and use them to put constraints on the constant of integration δ_i in (6).

3.4 The constants of integration β and δ

There are various constraints we can put on the constants β_i and δ_i on general grounds, e.g. the lower bound already mentioned. Having the numerical data at hand we can use them to approximate the constant. In [12] this was done by matching L^0 with the numerical data at $\theta = 0$ and a simple analytical approximation was provided $\delta^{\text{num}} = \ln[B(2-B)2^{1+B(2-B)}]$. Of course the idea is to become entirely independent of the numerical analysis. For this reason the argument which led to (15) was given.

When we consider a concrete theory like ATFT, we can exploit its particular structure and put additional constraints on the constants from general properties. For instance, when we restrict ourselves to the simply laced case it is obvious to demand that the constants respect also the strong-weak duality, i.e. $\beta_i(B) = \beta_i(2-B)$ and $\delta_i(B) = \delta_i(2-B)$.

Finally we present a brief argument which establishes that the constants β_i are in fact independent of the particle type i. We replace for this purpose in the functional relations (33) the Y-functions by $Y_i^h(\theta))$ and consider the equation at $\theta = 0$, such that

$$\frac{\cosh^2[\pi\beta_i(\vartheta_h + t_i\vartheta_H)] - 2\beta_i^2\eta_i}{\cosh^2[\pi\beta_i(\vartheta_h + t_i\vartheta_H)]} = \prod_{j=1}^{n}\prod_{m=1}^{I_{ij}} \frac{(\cosh^2[\pi\beta_i(2m-1-I_{ij})\vartheta_H] - 2\beta_i^2\eta_i)^{\frac{1}{2}}}{\cosh[\pi\beta_i(2m-1-I_{ij})\vartheta_H]} \quad . \quad (35)$$

Keeping in mind that β_i is a very small quantity in the ultraviolet regime, we expand (35) up to second order in β_i, which yields after cancellation

$$4t_i\vartheta_h\vartheta_H = \frac{\alpha_i^2}{2}\frac{B(2-B)}{hh^\vee} = \sum_{j=1} K_{ij}\frac{\beta_j^2}{\beta_i^2}\frac{\eta_j}{\pi} = \frac{B(2-B)}{hh^\vee}\sum_{j=1} K_{ij}\frac{\beta_j^2}{\beta_i^2}(\lambda_j\cdot\rho)\,. \tag{36}$$

We substituted here the expression (26) for the constants η_j in the last equality. Using once more the relation $\lambda_i = \sum_j K_{ij}^{-1}\alpha_j$, we can evaluate the inner product such that (36) reduces to

$$\alpha_i^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} K_{ij}\frac{\beta_j^2}{\beta_i^2}K_{jk}^{-1}\alpha_k^2\ . \tag{37}$$

Clearly this equation is satisfied if all the β_i are identical. From the uniqueness of the solution of the TBA-equations follows then immediately that we can always take $\beta_i \to \beta$. Since the uniqueness is only rigorously established [12] for some of the cases we are treating here, it is reassuring that we can obtain the same result also directly from (37). From the fact that the β_i are real numbers and all entries of the inverse Cartan matrix are positive follows that $\beta_i^2 = \beta_j^2$ for all i and j. The ambiguity in the sign is irrelevant for the use in $L(\theta)$.

3.5 The Scaling Functions

In [12] it was proven that the leading order behaviour of the scaling function is given by

$$c(r) \simeq n - \frac{3\eta}{(\delta-\ln(r/2))^2} = n\left(1 - \frac{\pi^2 B(2-B)(h+1)}{4h(\delta-\ln(r/2))^2}\right)\ . \tag{38}$$

From our arguments in section 3.2, which led to the general expression for the constant η in form of (27), follows that in fact this expression holds for *all* affine Toda field theories related to a dual pair of simple affine Lie algebras $(X_n^{(1)},\hat{X}^{(\ell)})$. However, strong-weak duality is only guaranteed for $\ell = 1$.

Restricting ourselves to the simply laced case, we can view the results of [5, 6, 7] obtained by means of a semi-classical treatment for the scaling function as complementary to the one obtained from the TBA-analysis and compare directly with the expression (38). Translating the quantities in

[5, 7] to our conventions, i.e. $R \to r$, $B \to B/2$, we observe that $c(r)$ becomes a power series expansion in β. We also observe that the second order coefficients precisely coincide in their general form. Comparing the expressions, we may read off directly

$$\delta^{\text{semi}} = \ln\left(\frac{4\pi\Gamma\left(\frac{1}{h}\right)\left(\frac{2}{B}-1\right)^{\frac{B}{2}-1}}{k\Gamma\left(\frac{1}{h}-\frac{B}{2h}\right)\Gamma\left(1+\frac{B}{2h}\right)}\right) - \gamma_E \tag{39}$$

for all ATFT related to simply laced Lie algebras[†]. Here γ_E denotes Euler's constant and $k = (\prod_{i=1}^{l} n_i^{n_i})^{\frac{1}{2h}}$ is a constant which can be computed from the Kac labels n_i of the related Lie algebra. Contrary to the statement made in [6], this identification can be carried out effortlessly without the need of higher order terms. Recalling the simple analytical expression δ^{num} of [12] we may now compare. Figure 1 demonstrates impressively that this working hypothesis shows exactly the same qualitative behaviour as δ^{semi} and also quantitatively the difference is remarkably small.

To illustrate the quality of our approximate solutions to the TBA-equations, we shall now work out some explicit examples.

3.6 Explicit Examples

To exhibit whether there are any qualitative differences between the simply laced and non-simply laced case we consider the first examples of these series.

3.6.1 The Sinh-Gordon Model

The Sinh-Gordon model is the easiest example in the simply laced series and therefore ideally suited as testing ground. The Coxeter number is $h = 2$ in this case. An efficient way to approximate the L-functions to a very high accuracy is

$$L^a(\theta) = \begin{cases} \ln\left[1 + \frac{\cos(2\beta\theta)+\cosh(\pi\beta(1-B))}{4\eta\beta^2}\right] & \text{for } |\theta| \leq \theta^m \\ \exp\left[-rm\cosh\theta + (\varphi * L^f)(\theta)\right] & \text{for } |\theta| > \theta^m \end{cases} \tag{40}$$

[†]The expressions in [5] and [7] only coincide if in the former case $m = 1$ and in the latter $m = 1/2$. In addition, we note a missing bracket in equation (6.20) of [5], which is needed for the identification. Replace $C \to -4QC$ therein.

with

$$\varphi(\theta) = \frac{4\sin(\pi B/2)\cosh\theta}{\cosh 2\theta - \cos\pi B}, \qquad \eta = \frac{\pi^2 B(2-B)}{8}. \tag{41}$$

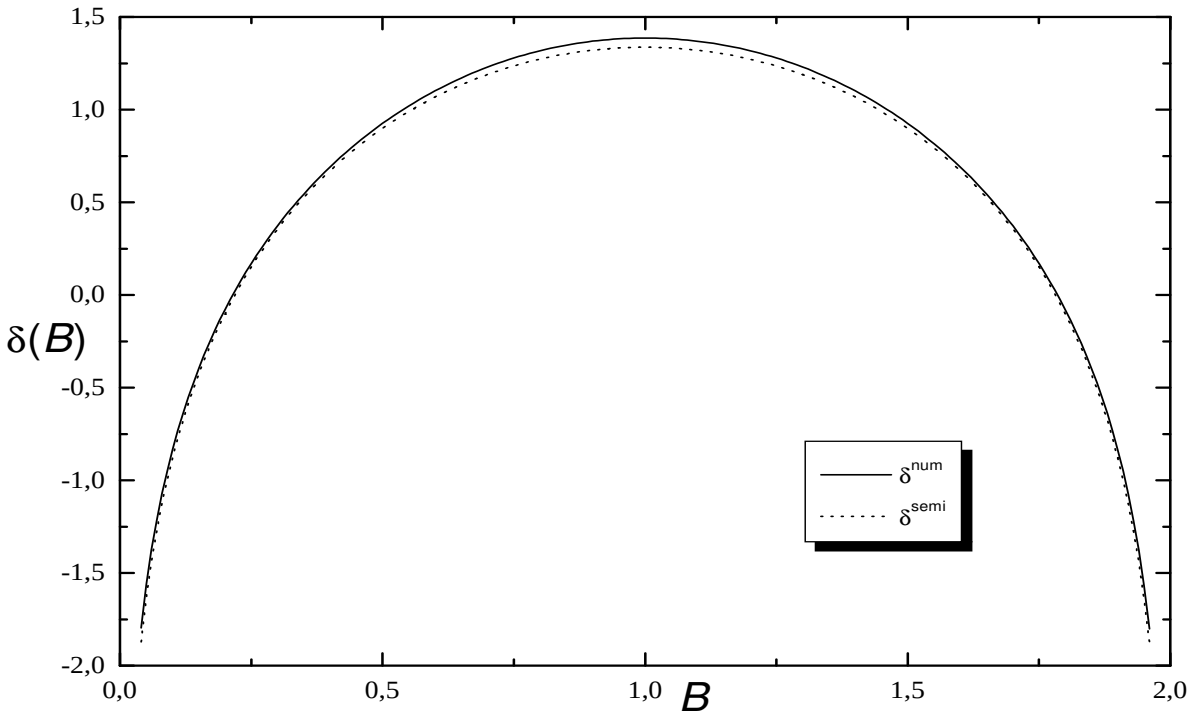

Figure 1: Numerically fitted constant $\delta^{\rm num}$ versus the constant from the semi-classical approach for the Sinh-Gordon model $\delta^{\rm semi}$.

The determining equation for the matching point reads

$$\sinh\theta^m - rm/2\sinh(2\theta^m) + \cosh(\theta^m)(\varphi' * L^f)(\theta^m) = 0 \ . \tag{42}$$

For instance for $B = 0.4$ this equation yields $\theta^m = 11.9999$ such that the matching condition (15) gives $\delta^m = 0.4913$. Figure 2(a) shows that the large and small density approximation L^0 and L^f may be improved in a fairly easy way. In view of the simplicity of the expression L^a the agreement with the numerical solution is quite remarkable. Figure 2(a) also illustrates that when using the constant $\delta^{\rm semi}$ instead of δ^m the agreement with the numerical solutions appears slightly better for small rapidities. When we employ $\delta^{\rm num}$ instead of $\delta^{\rm semi}$ the difference between the two approximated

solutions is beyond resolution. However, as may be deduced from Figure 2(b), with regard to the computation of the scaling function the difference between using δ^m instead of $\delta^{\rm semi}$ is almost negligible. Whereas in the former case the resulting value for the scaling function is slightly below the correct value, it is slightly above by almost the same amount in the latter case. More on the approximation of the scaling function in form of (38) may be found in [12].

3.6.2 $(G_2^{(1)}, D_4^{(3)})$-ATFT

In this case we have $h = 6$ and $H = 12$ for the related Coxeter numbers. The two masses are $m_1 = m\sin(\pi(1/6 - B/24))$ and $m_2 = m\sin(\pi(1/3 - B/12))$. The L-functions are well approximated by

$$L_1^a(\theta) = \begin{cases} \ln[1 + \frac{\cos(2\beta\theta)+\cosh(\pi\beta(\frac{1}{3}-\frac{B}{4}))}{4\eta_1\beta^2}\sqrt{1 - \frac{2\eta_2\beta^2}{\cos^2(\beta\theta)}}\ ] & \text{for } |\theta| \le \theta_1^m \\ \exp[-rm_1\cosh\theta + (\varphi_{11} * L_1^f + \varphi_{12} * L_2^f)(\theta)] & \text{for } |\theta| > \theta_1^m \end{cases}$$

$$L_2^a(\theta) = \begin{cases} \ln[1 + \frac{\cos(2\beta\theta)+\cosh(\pi\beta(\frac{1}{3}-\frac{5B}{24}))}{4\eta_2\beta^2}\prod_{k=-1}^{1}\sqrt{1 - \frac{2\eta_1\beta^2}{\cos^2(\beta(\theta+\frac{kB}{12}))}}\,] & \text{for } |\theta| \le \theta_2^m \\ \exp[-rm_2\cosh\theta + (\varphi_{21} * L_1^f + \varphi_{22} * L_2^f)(\theta)] & \text{for } |\theta| > \theta_2^m \end{cases},$$

with φ given by (23) and

$$\eta_1 = \frac{5\pi^2 B(2-B)}{72}, \quad \eta_2 = \frac{\pi^2 B(2-B)}{8}, \quad \eta = \frac{7\pi^2 B(2-B)}{36}\ . \tag{43}$$

Using now the numerical data $L_1(0) = 4.2524$ and $L_2(0) = 3.67144$ as benchmarks, we compute by matching them with $L_1^a(0)$ and $L_2^a(0)$ the constant to $\delta = 1.1397$ in both cases. This confirms our general result of section 3.4. Evaluating the equations (15) and (14) we obtain for $B = 0.5$ the matching values for the rapidities $\theta_1^m = 12.744$ and $\theta_2^m = 12.278$ such that $\delta_1^m = 1.9539$ and $\delta_2^m = 1.5572$. Figure 2(c) and 2(d) show a good agreement with the numerical outcome.

The approximated analytical expression for the scaling function reads

$$c(r) \simeq 2 - \frac{7\,\pi^2 B(2-B)}{12(\delta - \ln(r/2))^2}\ . \tag{44}$$

This expressions differs from the one quoted in [12], since in there the sign of some scattering matrices at zero rapidity was chosen differently.

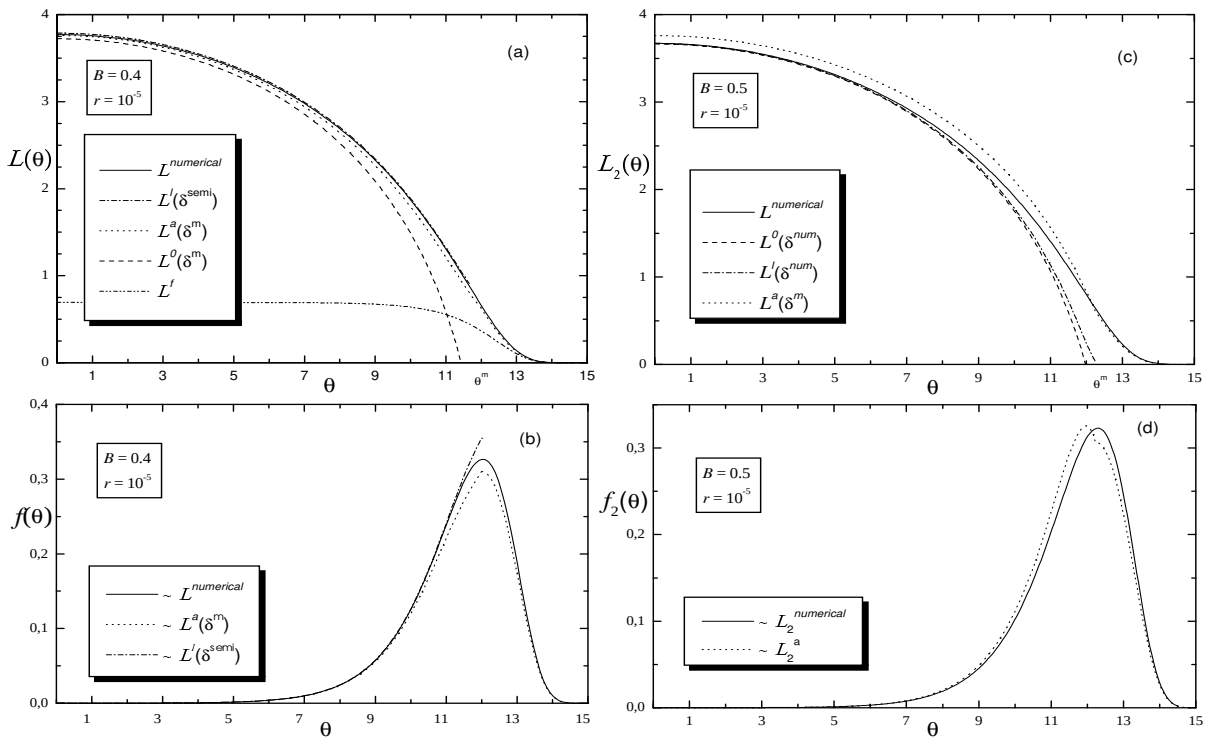

Figure 2: Various L-functions and free energy densities for the Sinh-Gordon model (a), (b) at B=0.4 and r=10 $^{-5}$ and $(G_2^{(1)}, D_4^{(3)})$-ATFT at B=0.5 and r=10 $^{-5}$ (c), (d).

4 Conclusions

We have demonstrated that it is possible to find simple analytical solutions to the TBA-equation in the large and small density regime, which approximate the exact solution to high accuracy. By matching the two solutions at the point in which the particle density and the density of available states coincide, it is possible to fix the constant of integration, which originated in the approximation scheme of [10, 11, 12] and was left undetermined therein. Alternatively the constant may be fixed by a direct comparison with a semi-classical treatment of the problem. It is not necessary for this to proceed to higher order differential equations as was claimed in [6]. Of course one may proceed further to higher orders, but since the solutions to the higher order

differential equations may only be obtained approximately one does not gain any further structural insight and moreover one has lost the virtue of the first order approximation, its simplicity.

We derived the Y-systems for all ATFT and besides demonstrating how they can be utilized to improve on the large density approximations we also showed how they can be used to put constraints on the constant of integration.

We have proven that the expression (38) for the scaling function is of a general nature, i.e. valid for *all* ATFT. It is desirable to extend the semi-classical analysis [7] also to the non-simply laced case. This would allow to read off the constant δ also in that case.

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