The Drinfel'd twisted XYZ model

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Abstract

We construct a factorizing Drinfel'd twist for a face type model equivalent to the XYZ model. Completely symmetric expressions for the operators of the monodromy matrix are obtained.

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1 Introduction

Several integrable quantum spin chain models within the range of the algebraic Bethe ansatz method have a distinguished basis of states, which minimalizes quantum effects. That is, the quasiparticle creation and annihilation operators in this basis have an appearance devoid of polarization clouds. For the XXX and XXZ models with underlying group $sl(2)$ the bases in question were found by Maillet and Sanchez de Santos [?] through the construction of a generalized Drinfeld twist [?].
The ensuing representation of the quantum monodromy matrices coincides, as noted by Terras [?], for the case of the rational XXX model with the representation provided by Sklyanin's functional Bethe ansatz method [?]. An obvious generalization of Sklyanin's method (substituting polynomials in the spectral parameter by polynomials in the exponential of the spectral parameter) leads us to the conclusion that an analogous coincidence holds true for the trigonometric XXZ model.

The purpose of the present communication is to report on the generalization of the above results to the elliptic $sl(2)$-XYZ model. For this sake we will make use of Baxter's map of the XYZ model onto an ice type model [?] (which is akin to a cyclic solid on solid model [?]). This brings us formally near to the XXX and XXZ models and allows us to use the technique developed in [?] for the handling of the analogous problem in the $sl(n)$-XXX model. The plan of the paper is as follows : Section 2 provides a short survey of the XYZ model and its reformulation as an ice-type model. Section 3 deals with the factorizing twists and the computation of the operator valued entries of the monodromy matrix. Section 4 contains the conclusions. The appendix is devoted to proving some relation needed in Sect.3.

2 XYZ model and its relation to ice-type models

In the framework of the Algebraic Bethe Ansatz [?] the XYZ-model is determined by the elliptic solution of the Yang-Baxter equation

$$R_{12}(\lambda_1-\lambda_2)R_{13}(\lambda_1-\lambda_3)R_{23}(\lambda_2-\lambda_3)=R_{23}(\lambda_2-\lambda_3)R_{13}(\lambda_1-\lambda_3)R_{12}(\lambda_1-\lambda_2) \qquad (1)$$

with

$$R^{xyz}(\lambda-\mu)=\begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix} \tag{2}$$

where

$$\begin{aligned} a(\lambda-\mu) &= \frac{\Theta(2\eta)\Theta(\lambda-\mu)}{\Theta(0)\Theta(\lambda-\mu+2\eta)} \\ b(\lambda-\mu) &= \frac{\Theta(2\eta)H(\lambda-\mu)}{\Theta(0)H(\lambda-\mu+2\eta)} \\ c(\lambda-\mu) &= \frac{H(2\eta)\Theta(\lambda-\mu)}{\Theta(0)H(\lambda-\mu+2\eta)} \\ d(\lambda-\mu) &= \frac{H(2\eta)H(\lambda-\mu)}{\Theta(0)\Theta(\lambda-\mu+2\eta)} \end{aligned} \tag{3}$$

with the notation $H(u)=\vartheta_1\left(\frac{u}{2K},q\right)$, $\Theta(u)=\vartheta_4\left(\frac{u}{2K},q\right)$ and $\vartheta_4=\sum_{m\in\mathbb{Z}}(-1)^n q^{n^2}e^{2\pi i n z}$, $\vartheta_1(z,q)=-iq^{\frac{1}{4}}e^{i\pi z}\theta_4(z+\tau/2,q)$ are the standard theta-functions of a single complex variable.[1]
The somewhat different parametrization as compared to [?] is due to the normalization in order to achieve unitarity for the R-matrix.

The monodromy matrix $T(\lambda,\{\lambda_i\})$ (generalized to the inhomogeneous chain [?], [?]) is given as the ordered product of Lax operators $L_i(\lambda-\lambda_i)=R_{0i}(\lambda-\lambda_i)$

$$T(\lambda,\{\lambda_i\})=L_N(\lambda-z_N)\dots L_2(\lambda-z_2)L_1(\lambda-z_1)=\begin{pmatrix} A(\lambda,\{\lambda_i\}) & B(\lambda,\{\lambda_i\}) \\ C(\lambda,\{\lambda_i\}) & D(\lambda,\{\lambda_i\}) \end{pmatrix}. \tag{4}$$

The presence of the Boltzmann weight d in Eq. (2) reflects the non-conservation of spin, which is responsible for the absence of a local vacuum for the Lax operator associated with the above R-matrix.
To circumvent the problems arising from the eight vertex nature, we use the vertex–face map established by Baxter [?] to obtain a XXZ type (six vertex) R-matrix by exploiting the relation

$$R^{xyz}(\lambda-\mu)\phi_{l,l'}\otimes z_{m',l'}=\sum_m w(m,m'|l,l')\phi_{m,m'}\otimes z_{m,l} \tag{5}$$

valid for all integers l,l',m,m' such that $|l-l'|=|m-m'|=1$ and the summation on the r.h.s. is over integers s.t. $|m-m'|=1$, $|m-l|=1$.

[1] For a concise introduction into the theory of theta and elliptic functions we refer the reader to the appendix of [?], whose conventions we shall use throughout our work.

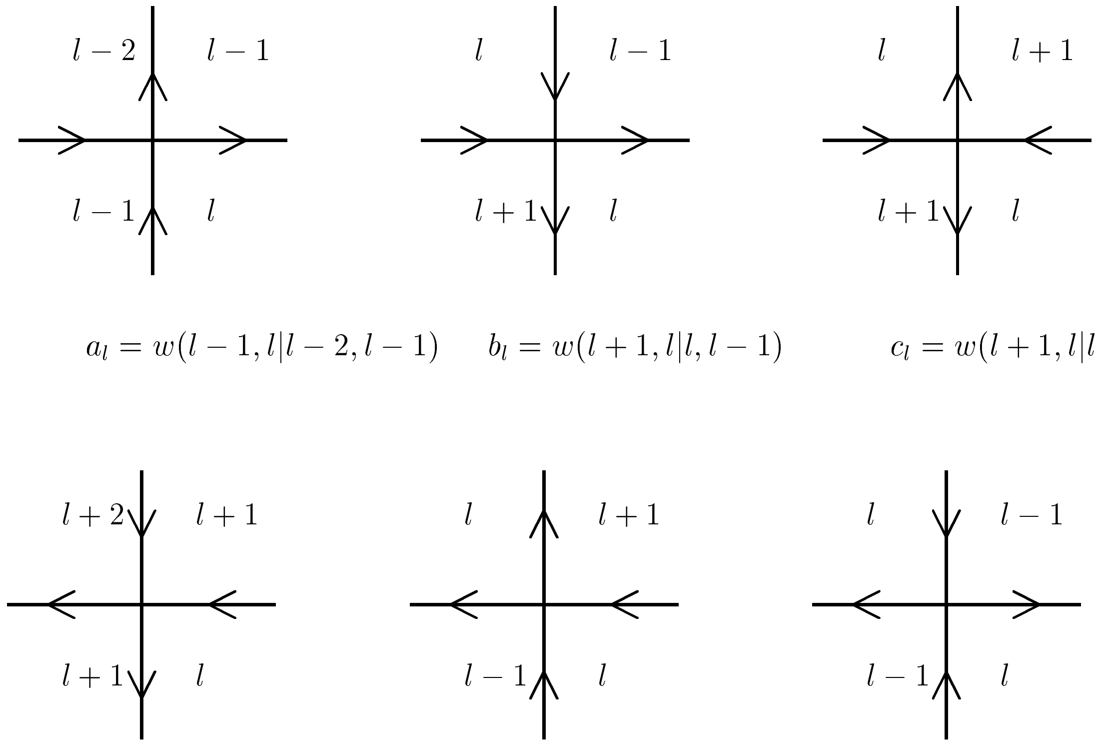
The two dimensional vectors $\phi,\ z$ are given by

$$\begin{aligned} \phi_{l,l+1} &= X(s_l+\mu); \quad z_{l+1,l} = X(s_l+\lambda) \\ \phi_{l+1,l} &= X(t_{l+1}-\mu); \quad z_{l-1,l} = X(t_l-\lambda) \end{aligned} \tag{6}$$

with $X(u) = \binom{H(u)}{\Theta(u)}$ and the abbreviation $s_l = s + 2\eta l$, where s, t are arbitrary complex parameters.

Relation (5) provides a change of basis in which the Lax operator has a local vacuum independent of the spectral parameter λ. There exist a whole family of bases labeled by the integer l and the parameters s, t, which achieve this goal.

The six weights $w(m, m'|l, l')$ are represented graphically in Figure 1.



$a_l = w(l-1, l|l-2, l-1)$ $\quad$ $b_l = w(l+1, l|l, l-1)$ $\quad$ $c_l = w(l+1, l|l, l+1)$

$a'_l = w(l+1, l|l+2, l+1)$ $\quad$ $b'_l = w(l-1, l|l, l+1)$ $\quad$ $c'_l = w(l-1, l|l, l-1)$

FIG. 1 The six resulting Boltzmann weights.

The Boltzmann weights are subject to the star-triangle relation

$$\sum_m w(l_2, l_3|l_1, m)w(l_3, l_4|m, l_5)w(m, l_5|l_1, l_6) = \sum_m w(l_3, l_4|l_2, m)w(l_2, m|l_1, l_6)w(m, l_4|l_6, l_5)\ . \tag{7}$$

They are parametrized by $\big(h(u) = \Theta(0)H(u)\Theta(u);\;\; \omega_l = (\frac{s+t}{2} + 2\eta l - K)\big)$:

$$
\begin{aligned}
a_l &= a'_l = 1; \\
b_l &= \frac{h(\lambda)h(\omega_{l-1})}{h(\lambda+2\eta)h(\omega_l)};\;\; b'_l = \frac{h(\lambda)h(\omega_{l+1})}{h(\lambda+2\eta)h(\omega_l)} \\
c_l &= \frac{h(2\eta)h(\omega_l-\lambda)}{h(\lambda+2\eta)h(\omega_l)};\;\; c'_l = \frac{h(2\eta)h(\omega_l+\lambda)}{h(\lambda+2\eta)h(\omega_l)}
\end{aligned} \tag{8}
$$

These weights can be arranged into a matrix

$$
R_{12}(l) = \begin{pmatrix} a_l & 0 & 0 & 0 \\ 0 & b_l & c_l & 0 \\ 0 & c'_l & b'_l & 0 \\ 0 & 0 & 0 & a'_l \end{pmatrix} \tag{9}
$$

which fulfills the modified Yang-Baxter equation [**?**], [**?**]

$$
R_{12}(l-\sigma_3)R_{13}(l)R_{23}(l-\sigma_1) = R_{23}(l)R_{13}(l-\sigma_2)R_{12}(l)\ . \tag{10}
$$

The monodromy matrix related to this modified Yang-baxter equation is [2]

$$
T_{0,1\ldots N}(l) = R_{0N}(l-\sigma_1-\ldots-\sigma_{N-1})\ldots R_{02}(l-\sigma_1)R_{01}(l) \tag{11}
$$

where 0 denotes the horizontal auxiliary space (with the asssociated spectral parameter λ_0), $1,\ldots,N$ label the vertical quantum spaces which span the physical Hilbertspace $\mathcal{H}_N$ (with associated local inhomogeneities $\{\lambda_i\}$), and σ_i equals ± 1 depending on whether the arrow in the i-th space is up or down (right/left for the horizontal space). It also sets our convention to associate the integer in the right lower corner of the graphical representation with the operator (cf. Figure 2).

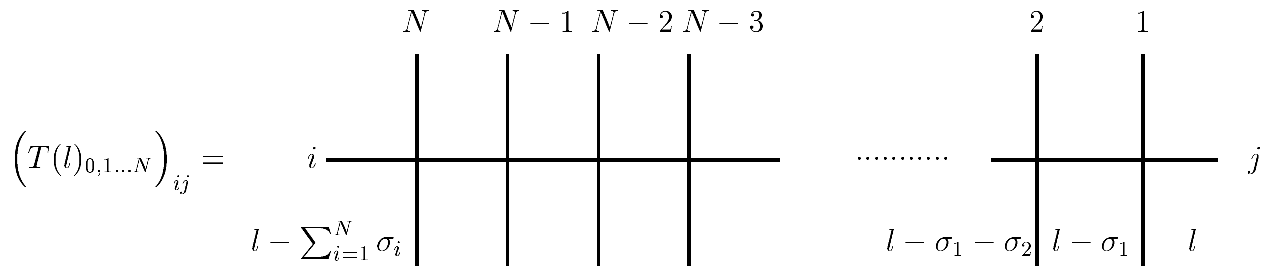


FIG. 2 Elements of the monodromy matrix

[2]We use the convention $X_{0,1\ldots N}$ to denote an operator X represented in space 0 with entries in the tensor-product of spaces $1,\ldots,N$.

From (10) follows the equation for the monodromy matrices

$$R_{00'}(l-\sigma_1-\ldots-\sigma_N)T_0(l)\otimes T_{0'}(l-\sigma_0)=T_{0'}(l)\otimes T_0(l-\sigma_{0'})R_{00'}(l) \tag{12}$$

and can be represented graphically as in Figure 3.

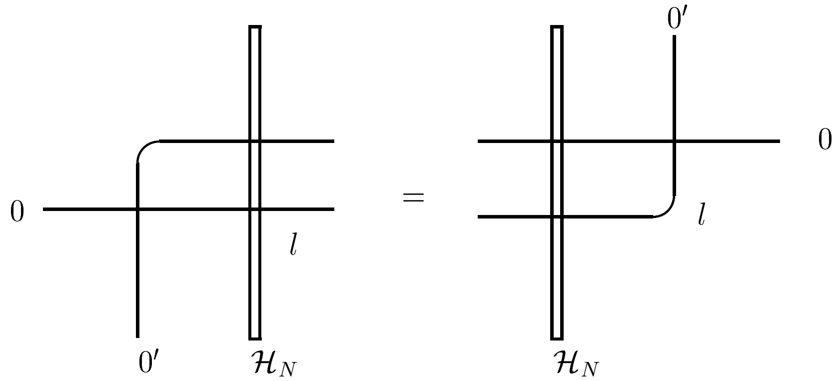


FIG. 3 The Yang–Baxter equation for the monodromy matrix

The unitarity relation $R_{21}R_{12}=\mathbb{1}$ can be checked directly by using the following relation

$$\begin{aligned}&h(u+x)h(u-x)h(v+y)h(v-y)-h(u+y)h(u-y)h(v+x)h(v-x)\\&=h(x-y)h(u-y)h(v+x)h(v+x)\end{aligned} \tag{13}$$

which is a consequence of the equality $H(u,k)\Theta(u,k)=H((1+k)u,2\sqrt{k}/(1+k))$ with k the nome of the elliptic function (Eq. 15.10.20 of [**?**]), and the analogue of (13) for $H(u)$ (Eq. 15.3.10 of [**?**]).

The representation of eigenvalues of the transfer matrix (4) through these of (11) is explained in reference [**?**] (One has however to keep in mind that our model (11) differs from that of [**?**] by an additional change of basis in the quantum space).
We will concentrate in what follows on the computation of a factorizing F-matrix for the monodromy matrix (11) for an arbitrary fixed value of l.

3 The F basis

The factorizing F-matrix for two sites defined by the relation $F_{21}R_{12} = F_{12}$ is

$$F_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & c' & b' & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{[12]} . \tag{14}$$

The proof of the factorization property amounts to checking the same relations as those in the proof of the unitarity of the R-matrix above.

The factorizing F-matrix for N sites (N quantum spaces) turns out to be given by formally the same expression as found in [?] for the XXX model[3]

$$\begin{aligned} F_{1\ldots N}(l) &= \sum_{\alpha\in\mathbb{Z}_2^N} P_\alpha R^{\sigma_\alpha}_{1\ldots N}(l)(z_1,\ldots,z_N) \\ P_\alpha &= \prod_{i=1}^{N} P_i^{\alpha_i} \end{aligned} \tag{15}$$

where $P_i^{\alpha_i}$ projects on the α_i-th component in the i-th space and the permutation σ_α is uniquely determined through the conditions

$$\begin{aligned} \alpha_{\sigma_\alpha(i+1)} &\geq \alpha_{\sigma_\alpha(i)} \quad \text{if} \quad \sigma_\alpha(i+1) > \sigma_\alpha(i) \\ \alpha_{\sigma_\alpha(i+1)} &> \alpha_{\sigma_\alpha(i)} \quad \text{if} \quad \sigma_\alpha(i+1) < \sigma_\alpha(i) . \end{aligned} \tag{16}$$

An algorithm for finding σ_α for a given α is described in the appendix.
The modification of the Yang-Baxter equation (10) enforces a particular rule for the handling of the integer valued parameter l in the formation of the intertwining matrix $R^\sigma(l)$ (related to the permutation σ), which can be read off from the modified composition law

$$\begin{aligned} R^{\sigma\sigma_i}(l) &= R_{\sigma(i),\sigma(i+1)}(\tilde{l}_i)R^\sigma(l) \\ \tilde{l}_i &= l - \sigma_{\sigma(1)} - \ldots - \sigma_{\sigma(i-1)} \end{aligned} \tag{17}$$

where σ_i is the transposition of $i, i+1$, and σ an arbitrary permutation.
$R^\sigma(l)$ has the intertwining property

$$R^\sigma(l)T_{0,1\ldots N}(l) = T_{0,\sigma(1)\ldots\sigma(N)}(l)R^\sigma(l-\sigma_0) .$$

[3]We thank Frank Göhmann for communicating this representation of $F_{1\ldots N}$, which is slightly simpler than that quoted in [?].

The matrix $F_{1\ldots N}(l)$ satisfies the factorizing equation

$$R^{\sigma}_{1\ldots N}(l) = F^{-1}_{\sigma(1\ldots N)}(l) F_{1\ldots N}(l) \ . \tag{18}$$

A proof of the latter equation can be found in [?] – the modification of the composition law going along with the parameter l being rather immaterial. An alternative argument emphasizing the geometric meaning of (18) (loosely speaking a pure gauge field representation of R^{σ} on the symmetric group $\mathcal{S}_N$ as base space) may be sketched as follows:
It is sufficient to prove Eq. (18) for arbitrary elementary transpositions $\sigma = \sigma_i; \sigma_i : \ (i, i+1) \rightarrow (i+1, i)$,

$$R^{\sigma_i}_{1\ldots N} = F^{-1}_{\sigma_i(1\ldots N)} F_{1\ldots N} \ . \tag{19}$$

A special solution $\hat{F}^i$ of (19) is the two site matrix $F_{i\,i+1}$, Eq. (14), embedded into the N-fold tensor product

$$\hat{F}^i = \mathbb{1} \otimes \ldots \otimes F_{i\,i+1} \otimes \ldots \otimes \mathbb{1}$$

and the most general solution of (19) is given by

$$F_{1\ldots N} = X(1, \ldots, \{i, i+1\}, \ldots, N) \hat{F}^i \tag{20}$$

where $X(1, \ldots, \{i, i+1\}, \ldots, N)$ denotes a non-singular matrix in the quantum space $\otimes^N \mathbb{C}^2$ which is symmetric with respect to an exchange of the space labels i and $i+1$ and of the respective local inhomogeneity parameters. So we note that the proof of Eq. (18) is complete if we can show that

$$F_{1\ldots N} \left(\hat{F}^i \right)^{-1} = X^{(i)}_{(1,\ldots N)} \tag{21}$$

is symmetric in the labels i and $i+1$. For this purpose it is convenient to consider the permutation group element σ_α of Eq. (21) as being decomposed into products of elementary transpositions, which also leads to a factorization of R^{σ_α} into a product of elementary (two site) matrices R^{σ_x}, and to subdivide the sum $\sum_\alpha$ in Eq. (15) into terms with the transposition σ_i and into terms without. It is furthermore possible to arrange the order of the transpositions such that R^{σ_i} does occur at most once and then on the very right end of the respective product of matrices R^{σ_x}. Schematically we obtain

$$\sum_\alpha P_\alpha R^{\sigma_\alpha} \ = \ \sum_{\alpha'} P_{\alpha'} R^{\sigma_{\alpha'}} R^{\sigma_i} + \sum_{\alpha''} P_{\alpha''} R^{\sigma_{\alpha''}} \ . \tag{22}$$

Each factor $P_{\alpha'} R^{\sigma_{\alpha'}}$ in the first sum has a counterpart $P_{\alpha''} R^{\sigma_{\alpha''}}$ in the second sum related by an interchange of the labels i and $i+1$.

Keeping this in mind one verifies straightforwardly by inspection that the multiplication of Eq. (22) by

$$\hat{F}^{-1}_{\sigma_i} = \left(\sum_{2 \geq \alpha_i > \alpha_{i+1}} \mathbb{1}_{i,i+1} P_i^{\alpha_i} P_{i+1}^{\alpha_{i+1}} + \sum_{2 \geq \alpha_{i+1} \geq \alpha_i} R^{\sigma_i} P_i^{\alpha_i} P_{i+1}^{\alpha_{i+1}} \right) \times diag\{1, b^{-1}(\lambda_j, \lambda_i), b^{-1}(\lambda_i, \lambda_j), 1\}$$

from the right side produces an expression which is symmetric in i and $i+1$.

The operators of the monodromy matrix (11) in the F basis are obtained by using a recursion relation, which enables one to express the monodromy matrix for N sites in terms of that for $N-1$ sites. Starting from the definition $\tilde{T}_{0,1\ldots N}(l) = F_{1\ldots N}(l) T_{0,1\ldots N}(l) F^{-1}_{1\ldots N}(l-\sigma_0)$ we use the identity $F_{1\ldots N}(l) = F_{2\ldots N}(l - \frac{1+\sigma_1}{2}) F_{1,2\ldots N}(l)$ with $F_{1,2\ldots N}(l) = (P_1^1 + P_2^1 T_{1,23\ldots N})$ (for the proof of this identity we refer the reader to the appendix)

$$\begin{aligned} & \tilde{T}_{0,1\ldots N}(l) = F_{1\ldots N}(l) T_{0,1\ldots N}(l) F^{-1}_{1\ldots N}(l-\sigma_0) \\ = \;& F_{2\ldots N}(l - \frac{1+\sigma_1}{2}) F_{1,2\ldots N}(l) T_{0,1\ldots N}(l) F^{-1}_{1,2\ldots N}(l-\sigma_0) F^{-1}_{2\ldots N}(l - \sigma_0 - \frac{1+\sigma_1}{2}) \;. \end{aligned}$$

Splitting $T_{0,12\ldots N}(l)$ into $T_{0,2\ldots N}(l-\sigma_1) R_{01}(l)$ we arrive at

$$\begin{aligned} & \tilde{T}_{0,1\ldots N}(l) \\ = \;& F_{2\ldots N}(l - \frac{1+\sigma_1}{2}) F_{1,2\ldots N}(l)\, F^{-1}_{2\ldots N}(l-\sigma_1) F_{2\ldots N}(l-\sigma_1)\, T_{0,2\ldots N}(l-\sigma_1) R_{01}(l) \;\times \\ & \times\, F^{-1}_{2\ldots N}(l-\sigma_0-\sigma_1) F_{2\ldots N}(l-\sigma_0-\sigma_1)\, F^{-1}_{1,2\ldots N}(l-\sigma_0) F^{-1}_{2\ldots N}(l-\sigma_0 - \frac{1+\sigma_1}{2}) \\ = \;& \tilde{F}_{1,2\ldots N}(l) \tilde{T}_{0,2\ldots N}(l-\sigma_1) R_{01}(l) \tilde{F}^{-1}_{1,2\ldots N}(l-\sigma_0) \end{aligned}$$

Inserting the explicit form of

$$\tilde{F}_{1,2\ldots N}(l) = F_{2\ldots N}\left(l - \frac{1+\sigma_1}{2}\right) F_{1,2\ldots N}(l) F^{-1}_{2\ldots N}(l-\sigma_1) = \begin{pmatrix} \mathbb{1} & 0 \\ \tilde{C}^1_{2\ldots N}(l) & \tilde{D}^1_{2\ldots N}(l) \end{pmatrix}_{[1]}$$

we obtain the final expression

$$\tilde{T}_{0,1\ldots N}(l) = \begin{pmatrix} \mathbb{1} & 0 \\ \tilde{C}^1_{2\ldots N}(l) & \tilde{D}^1_{2\ldots N}(l) \end{pmatrix}_{[1]} \tilde{T}_{0,2\ldots N}(l-\sigma_1) R_{01}(l) \begin{pmatrix} \mathbb{1} & 0 \\ \tilde{C}^1_{2\ldots N}(l-\sigma_0) & \tilde{D}^1_{2\ldots N}(l-\sigma_0) \end{pmatrix}^{-1}_{[1]} . \tag{23}$$