Dirac operators and conformal invariants of tori in 3-space

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Abstract

We show how to assign to any immersed torus in $\mathbb{R}^3$ or S^3 a Riemann surface such that the immersion is described by functions defined on this surface. We call this surface the spectrum or the spectral curve of the torus. The spectrum contains important information about conformally invariant properties of the torus and, in particular, relates to the Willmore functional. We propose a simple proof that for isothermic tori in $\mathbb{R}^3$ (this class includes constant mean curvature tori and tori of revolution) the spectrum is invariant with respect to conformal transformations of $\mathbb{R}^3$. We show that the spectral curves of minimal tori in S^3 introduced by Hitchin and of constant mean curvature tori in $\mathbb{R}^3$ introduced by Pinkall and Sterling are particular cases of this general spectrum. The construction is based on the Weierstrass representation of closed surfaces in $\mathbb{R}^3$ and the construction of the Floquet–Bloch varieties of periodic differential operators.

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1 Introduction

In this paper we show how to assign to any torus immersed into the three-space $\mathbb{R}^3$ or the unit three-sphere S^3 a complex curve such that the immersion is described by functions defined on this curve (a Riemann surface which is generically of infinite genus). We call this curve the spectrum of a torus (with a fixed conformal parameter). This spectrum has many interesting properties and, in particular, relates to the Willmore functional whose value is encoded in it.

The construction of the spectra for tori in $\mathbb{R}^3$ was briefly explained in [45]. In this text we do that also for immersed tori in S^3.

Our conjecture that the spectrum of a torus in $\mathbb{R}^3$ is invariant under conformal transformations of $\mathbb{R}^3$ was proven modulo some analytic facts by Grinevich and Schmidt [18]. In fact, their proof is rather physical which may be expected because the construction of the spectrum originates in soliton theory.

In this text we give a complete proof of the conformal invariance of the spectra for isothermic tori. This case already covers many interesting surfaces such as constant mean curvature tori and tori of revolution in $\mathbb{R}^3$.

Some spectral curves of finite genus already appeared in studies of harmonic tori in S^3 by Hitchin [22] and constant mean curvature (CMC) tori in $\mathbb{R}^3$ by Pinkall and Sterling [39]. It was shown that such tori are expressed in terms of algebraic functions corresponding to these complex curves [22, 5]. We show that for minimal tori in S^3 and CMC tori in $\mathbb{R}^3$ these spectral curves are particular cases of the general spectrum.

The general construction is based on the global Weierstrass representation of closed surfaces introduced in [43, 45] and a general construction of the Floquet(or Bloch) variety for a periodic differential operator. An existence of this variety is derived from the Keldysh theorem but an effective construction which gives more information about analytic behavior of this complex curve was proposed by Krichever in [28] who used perturbation methods. It is as follows. Take an immersed torus with the induced metric $e^{2\alpha}dzd\bar{z}$ and consider differential operators

$$\mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \quad \text{with } U = \frac{1}{2}He^{\alpha} \text{ for a torus in } \mathbb{R}^3,$$

$$\mathcal{D}^S = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} V & 0 \\ 0 & \bar{V} \end{pmatrix} \quad \text{with } V = \frac{1}{2}(H-i)e^{\alpha} \text{ for a torus in } S^3,$$

where H is the mean curvature.

Let Λ be the period lattice of a torus which means that the torus is an immersion of $\mathbb{C}/\Lambda$ with a conformal parameter $z \in \mathbb{C}$ on it. Take a basis γ_1, γ_2 for Λ which is also considered as a basis for $H_1(T^2) \approx \Lambda$. Now consider all solutions ψ to the equations

$$\mathcal{D}\psi = 0 \qquad \text{or} \qquad \mathcal{D}^S\psi = 0$$

satisfying the following conditions

$$\psi(z + \gamma_j) = \mu_j \psi(z).$$

These are Floquet(–Bloch) functions and the pairs (μ_1, μ_2) form a complex curve in $\mathbb{C}^2$. This is the Floquet zero-level spectrum of $\mathcal{D}$ and, by the definition, this is the spectrum of the immersed torus. The analytic properties of this curve are described by Pretheorem which is a modification of its analogs for two-dimensional scalar Schrödinger and heat operators proven in [28] and it is clear that the proof of Pretheorem may be obtained by slight modifications of the reasonings of [28].

One of the most interesting properties of this construction is its relation to a conformal geometry and the Willmore functional which equals

$$4\int_{\mathbb{C}/\Lambda} U^2 dx \wedge dy \quad \text{or} \quad 4\int_{\mathbb{C}/\Lambda} |V|^2 dx \wedge dy$$

for tori in $\mathbb{R}^3$ or S^3, where $z = x + iy$.

The global Weierstrass representation of closed surfaces represents any closed surface Σ in terms of a solution to the equation $\mathcal{D}\psi = 0$ (a harmonic spinor) where ψ takes values in some bundle over the constant curvature surface Σ_0 which is is conformally equivalent to Σ (see Theorems 1–3 in 2.3) [43, 45]. The Willmore functional $\int_\Sigma (H^2 - K) d\mu = 4\int_{\Sigma_0} U^2 dx \wedge dy - 2\pi\chi(\Sigma)$ measures the L_2-norm of the potential U of the surface. For small values of this functional the equation $\mathcal{D}\psi = 0$ does not admit solutions which describe closed surfaces in $\mathbb{R}^3$ and therefore that explains physical meaning of lower bounds for the Willmore functional proposed by the Willmore conjecture and its generalizations. This gives a hint that the spectral properties of $\mathcal{D}$ have to have a geometric meaning. We shall discuss that in details in 4.4.

In [46] it was established that the dimension of the kernel of $\mathcal{D}$ gives lower estimates for the Willmore functional for spheres. Actually we proved for spheres with one-dimensional potentials U (examples of them are spheres of revolution but not only) and conjectured for all spheres the following inequality

$$\int_{\Sigma_0} U^2 dx \wedge dy \geq 4\pi N^2 \quad \text{with } N = \dim_{\mathbb{C}} \ker \mathcal{D}. \tag{1}$$

The proof of it is based on the inverse scattering problem for the one-dimensional Dirac operator and it also works for general Dirac operators with S^1-symmetry (i.e., with one-dimensional potentials) on special spinor bundles over the 2-sphere. This inequality can not be improved and the equality is achieved on "soliton spheres" [46].

Another treatment of the global representation belongs to Pedit and Pinkall who proposed to consider spinor $\mathbb{C}^2$-bundles introduced in [43] as quaternionic line bundles and consider harmonic spinors as holomorphic quaternionic sections of such bundles. This enables them to apply ideas of algebraic geometry to surface theory and to generalize this representation for surfaces in $\mathbb{R}^4$ [37]. Very recently they managed to relate (1) to the quaternionic analog of the Plücker formula and by that prove our conjecture, i.e., establish the inequality (1) for all spheres together with its generalizations for higher genus surfaces.

This paper is organized as follows.

In section 2 we recall the notion of the Weierstrass representation.

In section 3 we prove that the multipliers (μ_1, μ_2) of Floquet functions form a spectral curve in $\mathbb{C}^2$ and discuss its analytic properties.

In section 4 we show how to assign such a spectrum to an immersed torus in $\mathbb{R}^3$, show the Willmore functional appear in this picture and discuss the modern state of the Willmore conjecture.

In section 5 we show that the spectra of CMC and isothermic tori are particular cases of the spectrum defined in section 4 and also prove our conjecture that the spectra of an isothermic torus and its dual surface coincide [44].

In section 6 we show how to assign a spectral curve to a torus in S^3 and prove that for a minimal torus in S^3 it coincides with a spectral curve defined by Hitchin for harmonic tori in S^3 [22].

In section 7 we prove that the spectrum of an isothermic torus in S^3 is invariant with respect to conformal transformations of $\bar{\mathbb{R}}^3 = \mathbb{R}^3 \cup \{\infty\}$ preserving the torus in $\mathbb{R}^3$.

2 The Weierstrass representation

2.1 Basic equations of surface theory in $\mathbb{R}^3$

In this subsection we recall the main definitions and some well-known important facts from the classical surface theory.

Let $\mathcal{U}$ be a domain in $\mathbb{R}^2$, with coordinates (x^1, x^2), regularly immersed into $\mathbb{R}^3$:

$$F : \mathcal{U} \to \mathbb{R}^3.$$

At every point $p \in \mathcal{U}$ the vectors

$$F_1 = \frac{\partial F}{\partial x^1}, \quad F_2 = \frac{\partial F}{\partial x^2}, \quad N = \frac{[F_1 \times F_2]}{|F_1||F_2|}$$

form a linear basis $\sigma = (F_1, F_2, N)^\top$ for $\mathbb{R}^3$, where F_1 and F_2 are tangent vectors to the surface $\Sigma = F(\mathcal{U})$, and N is a unit normal vector. The variables (x^1, x^2) are local coordinates on Σ and the induced metric on it is

$$\mathbf{I} = g_{kl} dx^k dx^l \quad \text{with } g_{kl} = \langle F_k, F_l \rangle \quad \text{(the \textit{first fundamental form})}$$

where $\langle a, b \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$.

The derivatives of the basic vectors are expanded in σ as

$$\frac{\partial^2 F}{\partial x^k \partial x^l} = \Gamma^j_{kl} \frac{\partial F}{\partial x^j} + b_{kl} N, \qquad \frac{\partial N}{\partial x^k} = -b^j_k \frac{\partial F}{\partial x^j} \tag{2}$$

where Γ^j_{kl} are the *Christoffel symbols*, $\mathbf{II} = b_{kl} dx^k dx^l$ is the *second fundamental form*, and $b^j_k = g^{jl} b_{lk}$. The equations (2) are the *Gauss–Weingarten derivation equations* and have the form

$$\frac{\partial \sigma}{\partial x^1} = \mathbf{U}\sigma, \quad \frac{\partial \sigma}{\partial x^2} = \mathbf{V}\sigma, \tag{3}$$

where U and V are (3×3)-matrices. The compatibility conditions for (3) are the *Codazzi equations*:

$$\frac{\partial^2 \sigma}{\partial x^1 \partial x^2} - \frac{\partial^2 \sigma}{\partial x^2 \partial x^1} = \left(\frac{\partial \mathbf{U}}{\partial x^2} - \frac{\partial \mathbf{V}}{\partial x^1} + [\mathbf{U}, \mathbf{V}] \right) \sigma = 0,$$

which are equivalent to the *zero-curvature equations*

$$\frac{\partial \mathbf{U}}{\partial x^2} - \frac{\partial \mathbf{V}}{\partial x^1} + [\mathbf{U}, \mathbf{V}] = 0 \tag{4}$$

for the connection $(\partial/\partial x^1 - \mathbf{U}, \partial/\partial x^2 - \mathbf{V})$.

At every point p of the surface the fundamental forms are diagonalized as

$$\mathbf{I}(p) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{II}(p) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

and the *principal curvatures* k_1 and k_2 satisfy the equation $\det(b_{kl} - kg_{kl}) = 0$, which divided by $\det g_{jk}$ takes the form $k^2 - 2Hk + K = 0$ where H is the *mean curvature* and K is the *Gaussian curvature*:

$$H = \frac{k_1 + k_2}{2}, \quad K = k_1 k_2.$$

A point p is called an *umbilic* point if the principal curvatures coincide at p: $k_1 = k_2$, which is equivalent to $H^2 - K = 0$.

Let $z = x^1 + ix^2$ be a conformal parameter on the surface, i. e., the first fundamental form is

$$\mathbf{I} = e^{2\alpha(z,\bar z)} dz\, d\bar z,$$

which means that

$$\langle F_z, F_z\rangle = \langle F_{\bar z}, F_{\bar z}\rangle = 0, \quad \langle F_z, F_{\bar z}\rangle = \frac{1}{2}e^{2\alpha}.$$

The family $\widetilde{\sigma} = (F_z, F_{\bar z}, N)^\top$ is a basis for $\mathbb{C}^3$ and the Gauss–Weingarten equations are written as

$$\widetilde{\sigma}_z = \widetilde{\mathbf{U}}\sigma, \quad \widetilde{\sigma}_{\bar z} = \widetilde{\mathbf{V}}\sigma,$$

with

$$\widetilde{\mathbf{U}} = \begin{pmatrix} 2\alpha_z & 0 & A \\ 0 & 0 & B \\ -2e^{-2\alpha}B & -2e^{-2\alpha}A & 0 \end{pmatrix}, \quad \widetilde{\mathbf{V}} = \begin{pmatrix} 0 & 0 & B \\ 0 & 2\alpha_{\bar z} & \bar A \\ -2e^{-2\alpha}\bar A & -2e^{-2\alpha}B & 0 \end{pmatrix},$$

$A = \langle F_{zz}, N\rangle$, and $B = \langle F_{z\bar z}, N\rangle$. The second fundamental form equals

$$\mathbf{II} = (2B + (A+\bar A))(dx^1)^2 + 2i(A - \bar A)dx^1 dx^2 + (2B - (A + \bar A))(dx^2)^2,$$

and we have

$$H = 2Be^{-2\alpha}, \quad K = 4(B^2 - A\bar A)e^{-4\alpha}.$$

Now the Codazzi equations $\widetilde{\mathbf{U}}_{\bar z} - \widetilde{\mathbf{V}}_z + [\widetilde{\mathbf{U}}, \widetilde{\mathbf{V}}] = 0$ take the form

$$\alpha_{z\bar z} + e^{-2\alpha}(B^2 - A\bar A) = 0, \quad A_{\bar z} - B_z + 2\alpha_z B = 0. \tag{5}$$

The first of them is the *Gauss egregium theorem* and another equation

$$A_{\bar z} = \frac{1}{2}H_z\, e^{2\alpha}$$

splits into two real-valued equations.

A quadratic differential $\omega = A\,dz^2$ is called the *Hopf differential* and has important geometrical properties. For instance, ω vanishes at a point if and only if this is an umbilic point.

It is said that the *Gauss map* of a surface $\Sigma = F(\mathcal{U})$

$$G : \Sigma \to S^2, \qquad G(p) = N(p),$$

is *harmonic* if $\Delta G(p) = \lambda(p) N(p)$, where Δ is the *Laplace–Beltrami operator*: $\Delta = 4e^{-2\alpha}\partial\bar{\partial}$. We have $\Delta F = 2HN$. Since $N_{z\bar{z}} = -2e^{-2\alpha}(\bar{A}_z F_z + A_{\bar{z}} F_{\bar{z}} + (A\bar{A} + B^2)N)$, we conclude that

the Gauss map G is harmonic if and only if the Hopf differential ω is holomorphic, which, by (5), is equivalent to $H_z = H_{\bar{z}} = 0$, i.e., $H = \mathrm{const}$ and Σ is a constant mean curvature (CMC) surface [41].

There are two other important classes of surfaces:

1) a surface is called *minimal* if $H = 0$, which is equivalent to $F_{z\bar{z}} = 0$;

2) a surface is called *isothermic* if there is a conformal parameter on it such that $\mathrm{Im}\, A = 0$.

2.2 The local representation of a surface

In this subsection we follow [26, 43].

Denote by $\mathcal{Q}$ a quadric in $\mathbb{C}^3$ defined by the equation

$$Z_1^2 + Z_2^2 + Z_3^2 = 0, \qquad Z = (Z_1, Z_2, Z_3) \in \mathbb{C}^3.$$

For a conformal parameter z on a surface Σ there is a mapping

$$f : \mathcal{U} \xrightarrow{F} \Sigma \to \mathcal{Q} \ \text{ where } \ f(p) = F_z(p), \tag{6}$$

satisfying the conditions

$$\mathrm{Im} \frac{\partial f}{\partial \bar{z}} = 0. \tag{7}$$

It is clear that any mapping $f : \mathcal{U} \to \mathcal{Q}$ satisfying (7) has the form $f = \partial_z \Phi$ for some real-valued function Φ and therefore has the form (6) for some surface.

The set $\mathcal{Q}$ is parameterized by $(\varphi_1, \varphi_2) \in \mathbb{C}^2$ as follows:

$$Z_1 = \varphi_1^2 - \varphi_2^2, \ \ Z_2 = i(\varphi_1^2 + \varphi_2^2), \ \ Z_3 = 2\varphi_1\varphi_2.$$

For simplicity, renormalize φ as follows:

$$\psi_1 = \sqrt{\frac{2}{i}}\varphi_1, \ \ \psi_2 = \sqrt{\frac{2}{i}}\bar{\varphi}_2.$$

In terms of ψ the equations (7) are

$$\mathcal{D}\psi = 0 \tag{8}$$

where

$$\mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar\partial & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \tag{9}$$

is a Dirac operator with a real-valued potential $U(z, \bar z)$.

Now, if $F(p_0) = (x_0^1, x_0^2, x_0^3) \in \mathbb{R}^3$, then the surface is described by the *Weierstrass formulas*

$$\begin{aligned} x^1(p) &= x^1(p_0) + \int_{p_0}^{p} \left(\frac{i}{2}(\bar\psi_2^2 + \psi_1^2)dz - \frac{i}{2}(\psi_2^2 + \bar\psi_1^2)d\bar z \right), \\ x^2(p) &= x^2(p_0) + \int_{p_0}^{p} \left(\frac{1}{2}(\bar\psi_2^2 - \psi_1^2)dz + \frac{1}{2}(\psi_2^2 - \bar\psi_1^2)d\bar z \right), \\ x^3(p) &= x^3(p_0) + \int_{p_0}^{p} \left(\psi_1\bar\psi_2 dz + \bar\psi_1\psi_2 \right), \end{aligned} \tag{10}$$

which are just

$$F(p) = F(p_0) + \int_{p_0}^{p} (f dz + \bar f d\bar z).$$

These local formulas in different forms were known before (see, for instance, [25, 1, 5] and comments in [43, 45]) but this form was introduced by Konopelchenko [26] who working with the formulas of Eisenhart [15] elaborated them into a form most convenient for applications. He considered them for constructing some surfaces via solutions to (8) and defining soliton deformations of "induced surfaces" but as we see this gives a general local construction of surfaces.

By straightforward computations it is derived that

$$U = \frac{He^\alpha}{2}, \quad e^\alpha = |\psi_1|^2 + |\psi_2|^2. \tag{11}$$

and we see that for $H = 0$ the formulas (10) reduce to the classical formulas for minimal surfaces.

Compute $N = e^{-\alpha}\left(-i(\bar\psi_1\bar\psi_2 - \psi_1\psi_2), -(\bar\psi_1\bar\psi_2 + \psi_1\psi_2), |\psi_2|^2 - |\psi_1|^2\right)$ and taking (8) into account derive

$$A = \langle F_{zz}, N\rangle = \psi_{1z}\bar\psi_2 - \bar\psi_{2z}\psi_1, \quad B = \langle F_{z\bar z}, N\rangle = Ue^\alpha. \tag{12}$$

The Gauss–Weingarten equations written in terms of ψ describe the deformations of ψ and the first half of them is just the Dirac equation (8). For obtaining another half of equations differentiate e^α by z:

$$\alpha_z e^\alpha = \bar{\psi}_1 \psi_{1z} + \psi_2 \bar{\psi}_{2z},$$

and, taking (12) into account, obtain

$$\psi_{1z} = \alpha_z \psi_1 + A e^{-\alpha} \psi_2, \quad \psi_{2\bar{z}} = -\bar{A} e^{-\alpha} \psi_1 + \alpha_{\bar{z}} \psi_2.$$

Now the Gauss–Weingarten equations are written as

$$\left[\frac{\partial}{\partial z} - \begin{pmatrix} \alpha_z & A e^{-\alpha} \\ -U & 0 \end{pmatrix}\right] \psi = \left[\frac{\partial}{\partial \bar{z}} - \begin{pmatrix} 0 & U \\ -\bar{A} e^{-\alpha} & \alpha_{\bar{z}} \end{pmatrix}\right] \psi = 0 \qquad (13)$$

and the compatibility conditions for them, the Codazzi equations, are

$$A_{\bar{z}} = (U_z - \alpha_z U) e^\alpha, \quad \alpha_{z\bar{z}} + U^2 - A\bar{A} e^{-2\alpha} = 0. \qquad (14)$$

In fact, the equation (8) is already the compatibility condition for an existence of a surface with the Gauss map given by ψ. The other half of the equations (13) follows from it.

In a paper by Friedrich [17] this representation was explained by classical means of the theory of Dirac operators. We also would like to mention a paper by Matsutani [35] where it was considered from the physical point of view.

2.3 The global Weierstrass representation

Here we follow [43] where this global representation was introduced.

Any closed oriented surface Σ in $\mathbb{R}^3$ is conformally equivalent to a constant curvature surface Σ_0 and a choice of a conformal parameter z on Σ means that a conformal equivalence $\Sigma_0 \to \Sigma$ is fixed.

To define a compact oriented surface globally via the formulas (10) we have to introduce fibre bundles over surfaces and Dirac operators on them.

Consider two cases:

1) *Tori.* Let Σ be a torus immersed into $\mathbb{R}^3$. Then it is conformally equivalent to a flat torus $\Sigma_0 = \mathbb{C}/\Lambda$ and z is a conformal parameter. The vector function ψ is expanded to a section of a $\mathbb{C}^2$-fiber bundle over Σ defined by the monodromy rules

$$\psi(z+\gamma) = \varepsilon(\gamma)\psi(z) \qquad (15)$$

where $\gamma \in \Lambda$ and $\varepsilon : \Lambda \to \{\pm 1\}$ is a character of Λ, i. e., a homomorphism to $\{\pm 1\}$. The Dirac operator $\mathcal{D}$ acts on this bundle and

$$U(z+\gamma) = U(z). \tag{16}$$

Hence, we have

Theorem 1 ([43]) *The formulas (15) and (16) define a $\mathbb{C}^2$ bundle over a flat torus Σ_0. To any section ψ, of this bundle, satisfying the Dirac equation (8) corresponds a surface in $\mathbb{R}^3$ defined by the formulas (10) up to translations in $\mathbb{R}^3$.*

2) *Surfaces of genus $g > 1$.* Let Σ_0 be a hyperbolic surface conformally equivalent to a surface Σ immersed into $\mathbb{R}^3$ and z be a conformal parameter. The surface Σ_0 is isometric to $\mathcal{H}/\Lambda$, where $\mathcal{H}$ is the Lobachevsky upper-half plane and Λ is a discrete subgroup of $PSL(2,\mathbb{R})$.

Any element $\gamma \in \Lambda \subset PSL(2,\mathbb{R})$ is represented by elements

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a,b,c,d \in \mathbb{R}, \quad ad - bc = 1.$$

The action on $\mathcal{H}$ is

$$z \to \gamma(z) = \frac{az+b}{cz+d}.$$

Define over Σ_0 a $\mathbb{C}^2$-bundle by the monodromy rules

$$\psi_1(\gamma(z)) = (cz+d)\psi_1(z), \quad \psi_2(\gamma(z)) = (c\bar{z}+d)\psi_2(z). \tag{17}$$

The Dirac operator acts on this bundle and

$$U(\gamma(z)) = |cz+d|^2 U(z). \tag{18}$$

Theorem 2 ([43]) *The formulas (17) and (18) define a $\mathbb{C}^2$-bundle over a hyperbolic surface Σ_0. To any section ψ, of this bundle, satisfying the Dirac equation (8) corresponds a surface in $\mathbb{R}^3$ defined by the formulas (10) up to translations in $\mathbb{R}^3$.*

Notice that $\psi_1\sqrt{dz}$ and $\bar{\psi}_2\sqrt{d\bar{z}}$ are defined modulo ± 1. In fact, they are spinors and therefore we shall call ψ_1 and ψ_2 also spinors. There are 2^{2g} such nonequivalent spinor bundles over Σ_0 where g is the genus of Σ_0.

This representation of compact oriented surfaces in $\mathbb{R}^3$ via solutions of Dirac equations (i.e., harmonic spinors) in spinor bundles over constant curvature surfaces is called the *Weierstrass representation* of surfaces.

Theorem 3 ([43] *for real-analytic surfaces,* [45] *for C^3-regular surfaces*) *Every smooth closed oriented surface in $\mathbb{R}^3$ has a Weierstrass representation.*

We see from the direct construction in 2.2 that for a surface with a fixed conformal parameter such a representation is unique. This gives rise to the following definition.

Definition 1 *Let (Σ, z) be an immersed surface with a fixed conformal parameter. Then the potential U of its Weierstrass representation is called the potential of a surface.*

To any harmonic spinor on Σ_0 with a potential U there corresponds a surface whose Gauss map descends through Σ_0. The criterion of closedness of such a surface is as follows.

Proposition 1 *A surface represented by a harmonic spinor ψ over a compact surface Σ_0 is closed if and only if*

$$\int_{\Sigma_0} \bar{\psi}_1^2 \, d\bar{z} \wedge \omega = \int_{\Sigma_0} \psi_2^2 \, d\bar{z} \wedge \omega = \int_{\Sigma_0} \bar{\psi}_1 \psi_2 \, d\bar{z} \wedge \omega = 0$$

for any holomorphic differential on Σ_0.

This proposition was proved by M. Schmidt for general tori (in this case $\omega = \mathrm{const} \cdot dz$) and by the author [46] for higher genera surfaces.

One of the most important properties of this representation is the equality

$$4 \int_{\Sigma_0} U^2 dx \wedge dy = \int_\Sigma H^2 d\mu$$

where $d\mu$ is the measure given by the induced metric on Σ [43]. The functional

$$\mathcal{W}(\Sigma) = \int_\Sigma (H^2 - K) d\mu = \int_\Sigma \left(\frac{k_1 - k_2}{2} \right)^2 d\mu$$

is called the *Willmore functional.* By the Gauss–Bonnet theorem, for closed oriented surfaces it equals

$$\mathcal{W}(\Sigma) = \int_\Sigma H^2 d\mu - 2\pi\chi(\Sigma)$$

where $\chi(\Sigma)$ is the Euler characteristic of the surface and, therefore, for tori $\mathcal{W} = \int_\Sigma H^2 d\mu$.

There is a famous Willmore conjecture that for tori the Willmore functional is greater or equal than $2\pi^2$. We shall discuss it in 4.4.

We see the main advantage of the global Weierstrass representation in using the spectral properties of $\mathcal{D}$ for study of conformal geometry of surfaces. The present paper is devoted to developing this idea for tori.

3 The Floquet spectrum of a periodic operator

3.1 Floquet functions and the spectral curve

Let L be a differential operator acting on functions or vector functions on $\mathbb{R}^n$, whose coefficients are periodic with respect to translations by vectors from a lattice Λ, i. e., Λ-periodic. We assume that Λ has the maximal rank, which means that it is isomorphic to $\mathbb{Z}^n$ and $\mathbb{R}^n/\mathbb{Z}^n$ is a torus.

To any vector $\gamma \in \Lambda$ there corresponds a translation operator T_γ:

$$T_\gamma f(x) \to f(x+\gamma),$$

Since L is Λ-periodic, if $Lf = \lambda f$, then $LT_\gamma f = \lambda T_\gamma f$. Moreover

$$[T_\gamma, L] = 0$$

and there are joint eigenfunctions of these commuting operators. Such functions are called *Floquet* (or *Bloch*) functions. The rigorous definition is as follows.

Definition 2 *A function $f : \mathbb{R}^n \to \mathbb{C}$ is called a Floquet function of a Λ-periodic operator L if*

$$Lf = Ef \quad \text{and} \quad f(x+\gamma) = e^{2\pi i \langle k, \gamma\rangle} f(x)$$

for $\gamma \in \Lambda$. The quantities $k_1, \dots, k_n$ are called the quasimomenta of f.

Any Floquet function defines the *multiplier* homomorphism $\mu : \Lambda \to \mathbb{C}$:

$$f(x+\gamma) = \mu(\gamma) \cdot f(x).$$

Consider the case, when $L = \mathcal{D}$, the two-dimensional Dirac operator

$$\mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar\partial & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$$

with a double-periodic continuous potential $U(z)$ where $z = x^1 + ix^2 \in \mathbb{C}$.

Theorem 4 *There is an analytic set $Q(\mathcal{D}) \subset \mathbb{C}^3$ of positive codimension such that there exists a Floquet function of $\mathcal{D}$ with the quasimomenta $k = (k_1, k_2)$ and the eigenvalue E if and only if $(k_1, k_2, E) \in Q(\mathcal{D})$.*

Its intersection with the plane $\lambda = 0$ is an analytic set $Q_0(\mathcal{D})$ of complex dimension one.

Proof. Take a constant C such that the operator $\mathcal{D} + C$ is invertible on $L_2(T^2) = L_2(\mathbb{C}/\Lambda)$. Then consider a polynomial operator pencil

$$A_{k,E} = 1 + \begin{pmatrix} U - (C+E) & \pi(k_2 + ik_1) \\ \pi(k_2 - ik_1) & U - (C+E) \end{pmatrix} \begin{pmatrix} C & \partial \\ -\bar{\partial} & C \end{pmatrix}^{-1}.$$

Since for any function g we have

$$e^{2\pi i\langle k,x\rangle}[A_{k,E}(\mathcal{D} + C)]g = [\mathcal{D} - E]e^{2\pi i\langle k,x\rangle}g,$$

there is a Floquet function $f(x)$ with the quasimomenta (k_1, k_2) and the eigenvalue E if and only if there is a Λ-periodic function $g(x)$ satisfying the equation

$$A_{k,E}[\mathcal{D} + C]g = 0.$$

Such a solution exists if and only if there is a solution $\varphi \in L_2(T^2)$ to the equation

$$A_{k,E}\varphi = 0. \tag{19}$$

If such a solution φ exists, then $f = e^{2\pi i\langle k,x\rangle}[\mathcal{D}+C]^{-1}\varphi$ is the desired Floquet function. The operator pencil

$$1 - A_{k,E}$$

is polynomial in k_1, k_2, and E. Since U is bounded, the multiplication operator

$$\times U : L_2(T^2) \to L_2(T^2)$$

is bounded and the pencil $(1 - A_{k,E})$ consists in compact operators on $L_2(T^2)$.

Now we apply the Keldysh theorem (or the polynomial Fredholm alternative) to it. This theorem reads that there is a regularized determinant $\widetilde{\det}\, A_{k,E}$ of this pencil analytic in k_1, k_2, and E such that the equation (19) is solvable if and only if $\widetilde{\det}\, A_{k,E} = 0$ [23, 24]. Now it remains to put

$$Q(\mathcal{D}) = \{\widetilde{\det}\, A_{k,E} = 0\}.$$

In the same manner for the pencil $(1 - A_{k,0})$ we obtain a complex curve

$$Q_0(\mathcal{D}) = \{\widetilde{\det} A_{k,0} = 0\} \subset \mathbb{C}^2.$$

As shown by perturbation methods the codimensions of these sets are positive (see [45]). Also nontriviality of such determinants follows from their construction by Keldysh. This proves the theorem.

We applied this method to the operators $\Delta + u$ and $\partial_t - \Delta$ in 1985. Later it became known to us that Kuchment also proposed the same approach in [30] and therefore our paper was not published and referred in [28, 29] as an unpublished paper. The theory of such determinants is developed in [31] and one can show that in fact they are entire functions of k and E.

We shall discuss another and more effective but technically difficult approach of Krichever in 3.3.

Recall that the dual lattice $\Lambda^* \subset \mathbb{C}^2$ consists of vectors γ^* such that $\langle \gamma, \gamma^* \rangle = 0$ for any $\gamma \in \Lambda$. The following proposition is evident.

Proposition 2 *The sets $Q(\mathcal{D})$ and $Q_0(\mathcal{D})$ are invariant with respect to translations by vectors from Λ^*:*

$$k_1 \to k_1 + \mathrm{Re}\gamma^*, \quad k_2 \to k_2 + \mathrm{Im}\gamma^*. \tag{20}$$

Indeed, this action preserves the multipliers.

In the sequel we shall confine to the set $Q_0(\mathcal{D})$.

Definition 3 *$Q_0(\mathcal{D})$ is called the (zero) Floquet spectral data of $\mathcal{D}$. The genus of the normalization of $Q_0(\mathcal{D})/\Lambda^*$ is called the spectral genus of $\mathcal{D}$.*

Another two properties of the Floquet spectrum are easily derived in the manner usual for soliton theory.

Proposition 3 *If U is real-valued, then $Q_0(\mathcal{D})$ is invariant under an anti-holomorphic involution $k \to -\bar{k}$.*

Proof. If $(\psi_1, \psi_2)^\top$ is a Floquet function with the quasimomenta (k_1, k_2), then $(\bar{\psi}_2, -\bar{\psi}_1)^\top$ is a Floquet function with the quasimomenta $(-\bar{k}_1, -\bar{k}_2)$. This proves the proposition.

Proposition 4 *$Q_0(\mathcal{D})$ is invariant under a holomorphic involution $k \to -k$.*

Proof. Consider the pencil $L_k = A_{k,0}(\mathcal{D}+C)$. We have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \overline{L_k^*} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = L_{-k}. \tag{21}$$

The index of a Fredholm operator A is $\operatorname{ind} A = \dim\ker A - \dim\ker A^*$. Since $\mathcal{D}$ is selfadjoint, its index vanishes. The operators L_k have the same principal terms as $\mathcal{D}$ and, by the index theorem, their indices also vanish. Hence if $k \in Q_0(\mathcal{D})$, then $\dim L_k = \dim L_k^* > 0$ and the identity (21) implies that $\dim L_{-k}^* > 0$. Therefore $(-k) \in Q_0(\mathcal{D})$ and this proves the proposition.

Given a basis (γ_1, γ_2) for Λ, we have a mapping

$$\mathcal{M} : Q_0(\mathcal{D})/\Lambda^* \to \mathbb{C}^2 \quad : \quad \mathcal{M}(k) = (e^{2\pi i \langle k, \gamma_1\rangle}, e^{2\pi i \langle k, \gamma_2 \rangle}),$$

which maps quasimomenta into multipliers.

The submanifold $\mathcal{M}(Q_0(\mathcal{D})/\Lambda^*) \subset \mathbb{C}^2$ is generically singular and its normalization is the normalization of $Q_0(\mathcal{D})/\Lambda^*$.

Definition 4 *A complex curve Γ, which is the normalization of $Q_0(\mathcal{D})/\Lambda^*$, is called the spectral curve of $\mathcal{D}$.*

A normalization sometimes consists in unstucking double points: a pair of points of Γ corresponding to a double point of $Q_0(\mathcal{D})/\Lambda^*$ is called a *resonance pair.*

The definition of $\mathcal{M}$ depends on a choice of a basis for Λ. Given another basis $(\widetilde{\gamma}_1, \widetilde{\gamma}_2)$ for Λ such that

$$\begin{pmatrix} \widetilde{\gamma}_1 \\ \widetilde{\gamma}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}),$$

the multipliers $(\mu_1, \mu_2) = (\mu(\gamma_1), \mu(\gamma_2))$ and $(\widetilde{\mu}_1, \widetilde{\mu}_2) = (\mu(\widetilde{\gamma}_1), \mu(\widetilde{\gamma}_2))$ are related as follows

$$\widetilde{\mu}_1 = \mu_1^a \mu_2^b, \quad \widetilde{\mu}_2 = \mu_1^c \mu_2^d, \tag{22}$$

and the sets of multipliers $\{(\mu_1, \mu_2)\}$ and $\{(\widetilde{\mu}_1, \widetilde{\mu}_2)\}$ with respect to different bases are biholomorphically equivalent.

We call the image of $\mathcal{M}$ the *(Floquet) spectrum* of $\mathcal{D}$ (on the zero energy level, $E = 0$). Given a basis of Λ, this image is uniquely defined. In general, the spectral data are defined modulo the $SL(2,\mathbb{Z})$-action (22) and we say that the spectral data of two operators coincide if the $SL(2,\mathbb{Z})$-orbits of their spectral data coincide.

3.2 Examples of spectra

1) $U = 0$.

Let $\Lambda = \mathbb{Z} + i\mathbb{Z}$. The Floquet functions are parameterized by two planes:

$$\psi^+ = (e^{\lambda_+ z}, 0), \quad \psi^- = (0, e^{\lambda_- \bar{z}}),$$

Γ is a union of these planes compactified by two points at infinities, and ψ has exponential singularities at these points.

The quasimomenta of ψ^+ are

$$k_1 = \frac{\lambda_+}{2\pi i} + n_1, \quad k_2 = \frac{\lambda_+}{2\pi} + n_2, \quad n_j \in \mathbb{Z}, \tag{23}$$

and the quasimomenta of ψ^- are

$$k_1 = \frac{\lambda_-}{2\pi i} + m_1, \quad k_2 = -\frac{\lambda_-}{2\pi} + m_2, \quad m_j \in \mathbb{Z}. \tag{24}$$

Hence

$$Q_0 = \left(\cup_{n_1,n_2 \in \mathbb{Z}} A_{n_1,n_2}\right) \cup \left(\cup_{m_1,m_2 \in \mathbb{Z}} B_{m_1,m_2}\right),$$

where A_{n_1,n_2} and B_{m_1,m_2} are planes described by (23) and (24).

The functions ψ^+ and ψ^- have the same multipliers at the points

$$\lambda_+^{m,n} = \pi(n + im), \quad \lambda_-^{m,n} = \pi(n - im), \quad m, n \in \mathbb{Z}.$$

These are resonance pairs for this potential with $\Lambda = \mathbb{Z} + i\mathbb{Z}$.

Considering the zero potential as Λ-periodic with respect to a general lattice $\gamma_1\mathbb{Z} + \gamma_2\mathbb{Z}$, the Floquet functions are the same but the resonance pairs are

$$\lambda_+^{m,n} = \frac{2\pi i}{\bar{\gamma}_1\gamma_2 - \gamma_1\bar{\gamma}_2}(\bar{\gamma}_1 n - \bar{\gamma}_2 m), \quad \lambda_-^{m,n} = \frac{2\pi i}{\bar{\gamma}_1\gamma_2 - \gamma_1\bar{\gamma}_2}(\gamma_1 n - \gamma_2 m). \tag{25}$$

2) $U = C = \mathrm{const} \neq 0$.

Assume for simplicity that $\Lambda = \mathbb{Z} + i\mathbb{Z}$.

The Floquet functions are

$$\psi(z, \bar{z}, \lambda) = \left(\exp\left(\lambda z - \frac{C^2}{\lambda}\bar{z}\right), -\frac{C}{\lambda}\exp\left(\lambda z - \frac{C^2}{\lambda}\bar{z}\right)\right)$$

where $\lambda \in \Gamma = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Compactify Γ by the points 0 and ∞ and define the Floquet function on Γ globally as

$$\psi(z, \bar{z}, \lambda) = \frac{\lambda}{\lambda - C}\left(\exp\left(\lambda z - \frac{C^2}{\lambda}\bar{z}\right), -\frac{C}{\lambda}\exp\left(\lambda z - \frac{C^2}{\lambda}\bar{z}\right)\right).$$

It has the following asymptotics

$$\psi \approx \begin{pmatrix} \exp(k_+ z) \\ 0 \end{pmatrix} \text{ as } \lambda \to \infty, \ \psi \approx \begin{pmatrix} 0 \\ \exp(k_- \bar{z}) \end{pmatrix} \text{ as } \lambda \to 0 \qquad (26)$$

with $k_+ = \lambda$ and $k_- = -C^2/\lambda$. After the normalization ψ gets a pole at $\lambda = C$.

The resonance pairs (λ, λ') are

$$\lambda = \frac{q\bar{q} \pm \sqrt{(q\bar{q})^2 - 4C^2 q\bar{q}}}{2\bar{q}}, \quad \lambda' = \lambda - q, \quad q = \pi(n + im), \ m, n \in \mathbb{Z}$$

and they are parameterized by $q \in \pi\mathbb{Z}^2 \setminus \{0\}$.

3) $U = U(x)$ *is a function of one variable.*

Let $U(x+T) = U(x)$ where T is the minimal period. Then the equation (8) for Floquet functions $\psi(x,y) = \varphi(x)e^{\lambda y}$ is the *Zakharov–Shabat* system

$$L\varphi = \begin{pmatrix} U & \frac{1}{2}\partial_x \\ -\frac{1}{2}\partial_x & U \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{i}{2}\lambda \\ \frac{i}{2}\lambda & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \qquad (27)$$

and in terms of $\eta_1 = \varphi_1 + i\varphi_2$ and $\eta_2 = \varphi_1 - i\varphi_2$ it is

$$(\partial_x + 2iU)\eta_1 = -i\lambda\eta_2, \quad (\partial_x - 2iU)\eta_2 = -i\lambda\eta_1.$$

We see that $f = \eta_2$ satisfies the equation

$$\left[-\partial_x^2 + (2iU_x - 4U^2)\right] f = \nu f$$

where $\nu = \lambda^2$. The transformation $L \to \left[-\partial_x^2 + (2iU_x - 4U^2)\right]$ is called the *Miura transformation.* The same name is used for the transformation

$$\left[-\partial_x^2 + (2iU_x - 4U^2)\right] \leftrightarrow \left[-\partial_x^2 + (-2iU_x - 4U^2)\right]. \qquad (28)$$

For any $\lambda \in \mathbb{C}$ take a two-dimensional space $\mathcal{V}_\lambda$ of solutions to (27) and consider the monodromy operator

$$\widehat{T} : \mathcal{V}_\lambda \to \mathcal{V}_\lambda \quad : \quad \widehat{T}(\varphi)(x) = \varphi(x+T).$$

For any pair $(\varphi(x,\lambda), \vartheta(x,\lambda))$ of solutions to (27) their Wronskian $W(\vartheta, \varphi) = \vartheta_1(x)\varphi_2(x) - \vartheta_2(x)\varphi_1(x)$ is constant. Take the basis $(c(x,\lambda), s(x,\lambda))$ for $\mathcal{V}_\lambda$ normalized as

$$c(0,\lambda) = s_x(x,\lambda) = 1, \quad c_x(0,\lambda) = s(x,\lambda) = 0.$$

As shown in this basis the entries of the matrix $\widehat{T}$ are entire functions of λ. Since $W(c,s)$ is constant, $\det \widehat{T}(\lambda) = 1$ and the characteristic equation for $\widehat{T}(\lambda)$ takes the form

$$k^2 - \mathrm{Tr}\widehat{T}(\lambda)\, k + 1 = 0. \tag{29}$$

If $\widehat{T}(\lambda)$ is diagonalized, then eigenfunctions of $\widehat{T}$ are the Floquet functions. The operator $\widehat{T}(\lambda)$ is not diagonalized if and only if λ is a simple root of the equation

$$\mathrm{Tr}^2\widehat{T}(\lambda) - 4 = 0, \tag{30}$$

which can have only simple and double roots. In this case the Jordan form for $\widehat{T}(\lambda)$ is a non-diagonal upper triangular matrix, and there is only one (up to multiple) eigenfunction of $\widehat{T}(\lambda)$ with this value of λ.

We conclude that the Floquet function is globally defined on the two-sheeted covering of $\mathbb{C}$ branched at points $\lambda_1, \dots$ which are simple roots of (30) and this complex curve is exactly Γ.

Resonance pairs of the spectrum are pairs of points which project into double roots of (30).

If there are finitely many simple roots of (30) L is called *finite gap*. For finite gap operators Γ is compactified by two infinities and the Floquet functions are pasted in a meromorphic function on Γ with the asymptotics (26) at the infinities.

These analytic properties of the Floquet function for a one-dimensional Schrödinger operator are explained in [13] and for the Dirac operator such results are obtained by using the Miura transformation or derived by the same reasonings straightforwardly.

3.3 Spectra via perturbation theory

Generically Floquet functions and the Floquet spectrum can not be found by solving ordinary differential equations as in examples in 3.2. For proving existence of the Floquet spectrum we use in 3.1 the Keldysh theorem. Another approach for finding this spectrum and describing it in rather efficient manner was proposed by Krichever who realized it for a two-dimensional Schrödinger operator and for the operator $\partial_y - \partial_x^2 + U(x,y)$ [28, 29]. It is based on perturbation theory.

The examples discussed above demonstrate how the spectrum deforms under a deformation of U. The main picture is as follows: deforming potential we deform double points on $Q_0(\mathcal{D})/\Lambda^*$ into handles removing singularities. The norm of the deformation measures the "size" of such handles.

For the two-dimensional Dirac operator (9) this is not done until recently but, since it is clear that the method of [28] works for this operator after a slight modification, we explain what is the expected picture:

Pretheorem 1 *For a smooth potential U the spectral curve Γ consists of two parts: M_0 and M_∞ where*

1) M_0 is a complex curve of finite genus whose boundary consists in a pair of circles;

2) M_∞ is diffeomorphic to a union of the domains $|\lambda_\pm| \geq R$ of the $\lambda_\pm$-planes for some R with some resonance pairs $(\lambda_+^{m,n}, \lambda_-^{m,n})$ (25) "joined by handles" with decreasing sizes as $m^2+n^2 \to \infty$;

3) M_0 and M_∞ are pasted along their boundaries.

This "joining by a handle" means that some small disks $|\lambda_+ - \lambda_+^{m,n}| < \varepsilon_+^{m,n}$ and $|\lambda_- - \lambda_-^{m,n}| < \varepsilon_-^{m,n}$ are excluded and replaced by a cylinder pasted to their boundaries and $\varepsilon_\pm^{m,n} \to 0$ as $m^2+n^2 \to \infty$.

To any point of the subsets $M_+, M_- \subset M_\infty$, where

$$M_+ = \mathbb{C} \setminus \{\ \{|\lambda_+| > R\} \cup \left(\cup_{m,n}\{|\lambda_+ - \lambda_+^{m,n}| \leq \varepsilon_+^{m,n}\}\right)\ \},$$

$$M_- = \mathbb{C} \setminus \{\ \{|\lambda_-| > R\} \cup \left(\cup_{m,n}\{|\lambda_- - \lambda_-^{m,n}| \leq \varepsilon_-^{m,n}\}\right)\ \},$$

corresponds a unique (up to multiple) Floquet function. These functions and their multipliers $\mu(\gamma_j, \lambda_\pm)$ asymptotically behave as in the case $U = 0$, and, in particular,

$$\mu(\gamma_j, \lambda_+) = e^{\lambda_+ \gamma_j}\left(1 + O\left(\frac{1}{\lambda_+}\right)\right), \quad \mu(\gamma_j, \lambda_-) = e^{\lambda_- \bar{\gamma}_j}\left(1 + O\left(\frac{1}{\lambda_-}\right)\right)$$

as $\lambda_\pm \to \infty$.

If U does not vanish identically, then Γ is irreducible.

A potential U is finite gap if there is such a representation with no handles joining resonance points in M_∞. In this case Γ is compactified by a pair of infinities $\infty_\pm$ to a Riemann surface of finite genus.

In fact this is a general description of the Floquet spectra of operators with periodic coefficients. This physical picture from [28] was chosen by Feldman, Knörrer, and Trubowitz as a most convenient and reasonable definition of general (non-hyperelliptic) Riemann surfaces of infinite genus and they developed a nice theory of such surfaces for which analogs of many classical theorems on algebraic curves take place [16] (see also their preprints published by ETH).

4 The spectrum of the Weierstrass representation

4.1 The spectral curve of an immersed torus

Let $F : \mathbb{C} \to \mathbb{R}^3$ be a conformal immersion of a plane whose Gauss map descends through a torus $\mathbb{C}/\Lambda$, i. e., double periodic. We shall consider tori as a particular case of such planes, when the immersion is also double periodic.

For immersed plane with a periodic Gauss map we have the Weierstrass representation constructed in 2.2. The double periodic potential $U(z)$ of this representation is the potential of the surface (with fixed conformal parameter).

It is said that two surfaces (with fixed conformal parameters) $F_1 : \mathbb{C} \to \mathbb{R}^3$ and $F_2 : \mathbb{C} \to \mathbb{R}^3$ are *isopotential* if their potentials coincide.

Definition 5 *The spectral curve Γ of the operator $\mathcal{D}$ with the potential U is called the spectral curve of a surface and the spectral genus of $\mathcal{D}$ is called the spectral genus of a surface. Given a basis γ_1, γ_2 for Λ, the image of the multiplier map*

$$\mathcal{M} : Q_0(\mathcal{D})/\Lambda^* \to \mathbb{C}^2 \quad : \quad \mathcal{M}(k) = (e^{2\pi i \langle k, \gamma_1 \rangle}, e^{2\pi i \langle k, \gamma_2 \rangle})$$

is called the spectrum of a surface.

Let us look how these spectral notions depend on a choice of a conformal parameter.

Proposition 5 *The spectral curve and the spectral genus of a surface do not depend on a choice of a conformal parameter.*

The spectrum of a surface depends on a choice of a basis for $H_1(T^2) = H_1(\mathbb{C}/\Lambda) = \Lambda$ and for different bases $(\widetilde{\gamma}_1, \widetilde{\gamma}_2)$ and (γ_1, γ_2) related by a $SL(2, \mathbb{Z})$-transformation

$$\begin{pmatrix} \widetilde{\gamma}_1 \\ \widetilde{\gamma}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix},$$

the spectra $\{(\widetilde{\mu}_1, \widetilde{\mu}_2)\}$ and $\{(\mu_1, \mu_2)\}$ are related as

$$\widetilde{\mu}_1 = \mu_1^a \mu_2^b, \quad \widetilde{\mu}_2 = \mu_1^c \mu_2^d.$$

Proof. It remains to check that all these data are preserved by a passage from z to $w = t^2 z$ with $t \in \mathbb{C} \setminus \{0\}$. For these conformal parameters the surface is defined by functions $\psi(z) = (\psi_1(z), \psi_2(z))^\top$ and $\widetilde{\psi}(w) = (\widetilde{\psi}_1(w), \widetilde{\psi}_2(w))^\top$, which are solutions to the equations

$$\mathcal{D}\psi = \begin{pmatrix} U & \partial_z \\ -\partial_{\bar z} & U \end{pmatrix} \psi = 0, \quad \widetilde{\mathcal{D}}\widetilde{\psi} = \begin{pmatrix} \widetilde{U} & \partial_w \\ -\partial_{\bar w} & \widetilde{U} \end{pmatrix} \widetilde{\psi} = 0.$$

We have $dw = t^2 dz, \partial_z = \frac{1}{t^2}\partial_w$, and, since $e^{2\widetilde{\alpha}(w)} dw d\bar w = e^{2\alpha(z)} dz d\bar z$ and $\widetilde{H}(w) = H(z)$, the formulas (11) imply that $\widetilde{U}(w) = t\bar t U(z)$ for $w = t^2 z$. Therefore

$$\begin{pmatrix} U & \partial_z \\ -\partial_{\bar z} & U \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 0 \quad \text{if and only if} \quad \begin{pmatrix} \widetilde{U} & \partial_w \\ -\partial_{\bar w} & \widetilde{U} \end{pmatrix} \begin{pmatrix} t\,\varphi_1 \\ \bar t\,\varphi_2 \end{pmatrix} = 0$$

and, since $\varphi = (\varphi_1, \varphi_2)^\top$ and $\widetilde{\varphi} = (t\varphi_1, \bar t \varphi_2)^\top$ have the same multipliers.

The transformation of the spectra was already discussed in 3.1 and we just repeat here these formulas because now they appear in another situation.

The proposition is proved.

We say that two planes, which may convert to tori by immersions, are *isospectral* if

1) there are conformal parameters one them such that both of them are represented by mappings $F_1 : \mathbb{C} \to \mathbb{R}^3$ and $F_2 : \mathbb{C} \to \mathbb{R}^3$;

2) the corresponding potentials of Dirac operators are Λ-periodic with the same lattice Λ, and the spectra of these operators with respect to a fixed basis for Λ coincide.

4.2 On the Willmore functional

Given a Weierstrass representation of a torus Σ, we have

$$\mathcal{W}(\Sigma) = 4\int_\Pi U^2 dx \wedge dy = 2i \int_\Pi U^2 dz \wedge d\bar z, \tag{31}$$

(see [43]) and that shows that the Willmore functional measures the deviation of $\mathcal{D}$ from the Dirac operator with the zero potential and geometrically that means that it measures the deviation of a connection in a spinor bundle defined by $\mathcal{D}$ from the trivial connection.

A relation of this spectrum to the Willmore functional

$$\mathcal{W}(\Sigma) = \int_\Sigma H^2\, d\mu,$$

was first established in [44] where it was discussed for surfaces of revolution. In this case there is a direct construction of the Floquet spectrum which a hyperelliptic complex curve (see Example 3 in 3.2). When the curve is of finite genus there is a compactification of it by two infinities and the Floquet function $\psi(z,P)$ is defined on the compactification and is meromorphic outside these infinities. We shall discuss a general case using a construction of the spectrum by perturbation.

Let U be a Λ-periodic potential with $\Lambda = \gamma_1\mathbb{Z} + \gamma_2\mathbb{Z}$ and let Γ be of finite genus and be compactified by two infinities $\infty_\pm$. In fact, as shown in Pretheorem, these infinities are inherited during the perturbation of U from the compactification of the spectrum of the zero potential (see Example 1 in 3.2).

Take the Floquet function $\psi(z,P)$ meromorphic outside the infinities and with the asymptotics

$$\psi \approx \left(\begin{array}{c} \exp(\lambda_+ z) \\ 0 \end{array}\right) \text{ as } P \to \infty_\pm, \ \psi \approx \left(\begin{array}{c} 0 \\ \exp(\lambda_- \bar{z}) \end{array}\right) \text{ as } P \to \infty_- \quad (32)$$

where $\lambda_\pm^{-1}$ are local parameters near $\infty_\pm$.

The theory of finite gap integration gives a recipe for reconstructing U from such asymptotic expansions. Let

$$\psi(z,\lambda_+) = \exp(\lambda_+ z) \cdot \left(\left(\begin{array}{c} 1 \\ 0 \end{array}\right) + \left(\begin{array}{c} \zeta_1 \\ \zeta_2 \end{array}\right)\frac{1}{\lambda_+} + O\left(\frac{1}{\lambda_+^2}\right)\right) \text{ as } \lambda_+ \to \infty.$$

Substitute this expansion into the Dirac equation and expand $\mathcal{D}\psi = 0$ into the powers of λ_+. Every coefficient in this expansion equals zero. Take the first two of them:

$$U + \zeta_2 = 0, \quad U\zeta_2 - \bar{\partial}\zeta_1 = 0. \quad (33)$$

This gives a reconstruction formula for U and also the identity

$$-U^2 = \bar{\partial}\zeta_1.$$

Now let us show how the Willmore functional appears in this picture. By the perturbation theory, we expect that the spectrum asymptotically behaves as the spectrum of the zero potential and this leads to the following conclusion: there is a function $W(\lambda_+)$ defined near ∞_+ such that

1) $W(\lambda_+) = C_1\lambda_+^{-1} + O\left(\lambda_+^{-2}\right)$;

2) $\psi(z,\lambda_+) = e^{\lambda_+ z + W(\lambda_+)\bar{z}} \cdot \varphi(z,\lambda_+)$ where $\varphi(z,\lambda_+)$ is Λ-periodic.

The function W measures the deviation of the Floquet spectrum from the spectrum of the zero potential. Indeed, the multipliers of ψ are

$$(\mu(\gamma_1),\mu(\gamma_2)) = (e^{\lambda_+\gamma_1 + W(\lambda_+)\bar{\gamma}_1}, e^{\lambda_+\gamma_2 + W(\lambda_+)\bar{\gamma}_2})$$

and the multipliers of the Floquet function of the zero potential (which is considered as Λ-periodic) are

$$(\mu_0(\gamma_1), \mu_0(\gamma_2)) = (e^{\lambda+\gamma_1}, e^{\lambda+\gamma_2}).$$

It is easy to notice that

$$\zeta_1 = C_1 \bar{z} \ + \ (\text{ a } \Lambda\text{-periodic function})$$

and hence

$$\int_\Pi U^2 \, dz \wedge d\bar{z} = -C_1 \cdot \mathrm{Vol}\,\Pi$$

where Π is a the parallelogram spanned by γ_1 and γ_2. Now using (31) we conclude that

$$\mathcal{W}(\Sigma) = 4C_1 \cdot (\gamma_1\bar{\gamma}_2 - \bar{\gamma}_1\gamma_2) = -4C_1 \cdot \mathrm{Vol}\,\Pi \tag{34}$$

where Π is the area of the fundamental domain of Λ. This derivation of the formula (34) was exposed in [18]. The analogous derivation for the area of minimal tori is given in [22].

Now consider the whole series

$$W(\lambda) = C_1\lambda^{-1} + C_2\lambda^{-2} + \dots.$$

Since an involution $k \to -k$ inverts λ and preserves the spectrum, we have $C_{2k} = 0$ for $k = 1, 2, \dots$. The quantities C_{2k-1} for $k \geq 2$ depend on choices of a conformal parameter z and a parameter λ on the spectral curve. Given a parameter λ, the holomorphic differentials

$$\mathcal{W}_k = -4(C_{2k-1}\,\mathrm{Vol}\,\Pi)\, dz^{2k-2}$$

are geometric invariants. The first of them is the Willmore functional $\mathcal{W} = \mathcal{W}_1$.

If the conformal parameter is fixed, then C_{2k-1} are first integrals of the modified Novikov–Veselov equation and the question of what are their geometrical meanings was posed in [43].

On a surface of revolution there is a distinguished conformal parameter $z = x + iy$ where x is a parameter on the rotating curve and y is the angle of rotation, $0 \leq y \leq 2\pi$, and there is a distinguished parameter λ which is the eigenvalue of the Miura transformation of the Dirac operator (see 3.2). For these parameters C_{2k-1} are the *Kruskal–Miura integrals* of the modified Korteweg–de Vries equations. They are geometric invariants of surfaces of revolution [44].

Critical points of the Willmore functional are *Willmore surfaces*. For higher functionals critical points were not studied but using trace formulas we showed in [44] that for spheres of revolution higher invariants are not bounded both from above and below.

4.3 Surfaces in terms of theta functions

Assume that the spectrum Γ of a torus Σ has finite genus which equals g and take two different points $\infty_\pm$ on Γ with local parameters $\lambda_\pm^{-1}$ near them such that $\lambda_\pm^{-1}(\infty_\pm) = 0$. Then the theory of Baker–Akhieser functions [27] (see also [12]) reads that for a generic effective divisor D, i. e. , a formal sum of points on Γ, of degree $g+1$ ($D = P_1 + \ldots + P_{g+1}$) there is a unique function $\psi(z, \bar{z}, P)$ such that

1) ψ is meromorphic in $P \in \Gamma \setminus \{\infty_\pm\}$ and has the asymptotics (32) as $P \to \infty_\pm$;

2) ψ has poles only in D.

Then this function is constructed in terms of theta functions of a Riemann surface Γ. From this function one can reconstruct a Dirac operator $\mathcal{D}$ by (33). Therefore to each point $P \in \Gamma \setminus \{\infty_\pm, P_1, \ldots, P_{g+1}\}$ there corresponds a surface in $\mathbb{R}^3$ constructed from $\psi(P)$ via (10).

In this event, the Riemann surface Γ parameterizes isopotential surfaces and each of these surfaces is described in terms of theta functions of Γ. The detailed formulas are given in [45, 47]. For CMC tori in $\mathbb{R}^3$ such formulas were derived in [5].

For tori of infinite spectral genera one can apply the theory of theta functions on such surfaces developed by Feldman, Knörrer, and Trubowitz [16].

4.4 On Willmore surfaces and the Willmore conjecture

First the Willmore functional $\mathcal{W}$ appeared in the 20s in the papers by Blaschke [4] and Thomsen [48]. In this event it was called the *conformal area* and its extrema were called *conformally minimal surfaces*. Blaschke and Thomsen also established the main properties of this functional which are:

1) the Willmore functional is invariant with respect to conformal transformations of the ambient space: let $\Sigma \subset \mathbb{R}^3$ be a compact immersed oriented surface, $z = x + iy$ be a conformal parameter on it, and $G : \bar{\mathbb{R}}^3 \to \bar{\mathbb{R}}^3$ be a conformal transformation which maps Σ into $\mathbb{R}^3$, then $\mathcal{W}(\Sigma) = \mathcal{W}(G(\Sigma))$,

and this follows from the conformal invariance of the quantity [1]

$$(k_1 - k_2)^2 d\mu = 4(H^2 - K)d\mu = 16|A|^2 e^{-2\alpha} dx \wedge dy$$

where $e^{2\alpha}dzd\bar{z}$ is the first fundamental form and Adz^2 is the Hopf differential of the surface;

2) if Σ is a minimal surface in S^3 and $\pi : S^3 \to \bar{\mathbb{R}}^3$ is the stereographic projection which maps Σ into $\mathbb{R}^3$, then $\pi(\Sigma)$ is a conformally minimal (Willmore) surface.

Hence in difference with minimal surfaces there are compact immersed Willmore surfaces in $\mathbb{R}^3$.

All Willmore spheres were described by Bryant [8]. The classification of Willmore tori is not complete until recently and we only mention the papers [2, 3] where the finite gap integration was applied to this problem. In [20] the Dorfmeister–Pedit–Wu (DPW) method [11], was applied for constructing general Willmore surfaces.

The simplest example of a Willmore torus is the stereographic projection of the Clifford torus $\{(x^1)^2 + (x^2)^2 = 1/2, (x^3)^2 + (x^4)^2 = 1/2\} \subset S^3 \subset \mathbb{R}^4$ into $\mathbb{R}^3$. In another way it may be obtained as a circle torus of revolution such that the ratio of the distance from the center of the circle to the axis of revolution and the radius of the circle equals $\sqrt{2}$. This torus in $\mathbb{R}^3$ is also called the Clifford torus.

Willmore conjectured that

the Willmore functional achieves its minimum for tori on the Clifford torus and its conformal transformations and this minimum equals $2\pi^2$ (the Willmore conjecture)

and checked this conjecture for circle tori of revolution [50].

This conjecture implies the Hsiang–Lawson conjecture that the area of a minimal torus in S^3 is greater or equal than $2\pi^2$, the area of the Clifford torus in S^3, but not equivalent to it since there are Willmore tori in $\mathbb{R}^3$ which are not images of minimal tori under the stereographic projection [38].

The Willmore conjecture is still open and there are some particular cases for which it was proved:

1) Langer and Singer proved it for tori of revolution [33] and Hertrich-Jeromin and Pinkall generalized their result for channel tori which are the enveloping tori for one-parameter families of spheres [21];

2) Li and Yau proved it for tori conformally equivalent to flat tori $\mathbb{C}/\{\mathbb{Z} + \tau\mathbb{Z}\}$ where $\tau = a + ib, 0 \leq a \leq 1/2, b > 0$ and $\sqrt{1-a^2} \leq b \leq 1$ [34] and later

[1] Pinkall indicated that he and Pedit recently proved that the quantity $Ae^{-\alpha}$ is already conformally invariant.

Montiel and Ros improved the latter inequality to $(a-1/2)^2+(b-1)^2 \leq 1/4$ [36].

Simon proved that the minimum of the Willmore functional is achieved on a real-analytic torus [42].

For higher genera surfaces a generalization of the Willmore conjecture was proposed by Kusner [32]. He conjectured that for such surfaces the Willmore functional attain its minima of the stereographic projections of some minimal surfaces in S^3 constructed by Lawson.

In [45] (see also [47]) we conjectured that

for a fixed conformal classes of tori the minimum of $\mathcal{W}$ is attained on tori with minimal spectral genus.

If this conjecture is valid then we may reduce the Willmore conjecture to estimating $\mathcal{W}$ for Willmore tori of small spectral genera and this can be done by using soliton theory.

The global Weierstrass representation gives a physical explanation for lower bounds for $\mathcal{W}$: for small perturbations of the zero-potential $U=0$ the surfaces constructed from solutions to (8) via (10) do not convert into tori and the lower bound for $\mathcal{W}$, the squared L_2-norm of the perturbation, shows how large a perturbation has to be to convert planes into tori.

5 The spectra of integrable tori

5.1 Constant mean curvature tori

Let Σ be a CMC torus in $\mathbb{R}^3$, i.e., $H = \text{const}$, and let it be conformally equivalent to $\mathbb{C}/\Lambda$ with $\Lambda = \gamma_1\mathbb{Z} + \gamma_2\mathbb{Z}$.

As shown in 2.3 for CMC surfaces the Hopf differential $\omega = A\,dz^2$ is holomorphic and, since the space of quadratic holomorphic differentials on a torus is one-dimensional, we have $A\,dz^2 = \text{const}\,dz^2$. This differential does not vanish because otherwise all points are umbilics which is impossible for tori in $\mathbb{R}^3$. Hence assume that

$$\omega = \frac{1}{2}\,dz^2, \quad H = 1$$

and this is achieved by rescaling z and by a homothety in $\mathbb{R}^3$. Now the Codazzi equations in terms of $u = 2\alpha$ read

$$u_{z\bar{z}} + \sinh u = 0 \tag{35}$$

which is the *sinh-Gordon equation.* The Codazzi equations (14) give a commutation representation for (35):

$$\left[\frac{\partial}{\partial z} - \left(\begin{array}{cc} \alpha_z & -\frac{\lambda^2}{2}e^{-\alpha} \\ -\frac{1}{2}e^{\alpha} & 0 \end{array} \right)\right]\psi = 0, \quad \left[\frac{\partial}{\partial \bar{z}} - \left(\begin{array}{cc} 0 & \frac{1}{2}e^{\alpha} \\ \frac{\lambda^{-2}}{2}e^{-\alpha} & \alpha_{\bar{z}} \end{array} \right)\right]\psi = 0 \tag{36}$$

where $\lambda^2 = -1$ in (14).

Notice that

1) for any $\lambda \neq 0$ the compatibility condition for (36) is (35);

2) if $|\lambda| = 1$ then (36) are the Codazzi equations (14) for the surface defined by $\psi(\lambda, z, \bar{z})$ via (10).

There is another representation of (35) which gives rise to the *spectral curve of a CMC torus* [39, 5].

Consider the equation (35) as the compatibility condition for the system

$$\left[\frac{\partial}{\partial z} - \frac{1}{2}\left(\begin{array}{cc} -u_z & -\lambda \\ -\lambda & u_z \end{array} \right)\right]\varphi = 0, \quad \left[\frac{\partial}{\partial \bar{z}} - \frac{1}{2\lambda}\left(\begin{array}{cc} 0 & e^{-u} \\ e^{u} & 0 \end{array} \right)\right]\varphi = 0 \tag{37}$$

which contains the linear problem

$$L\varphi = \partial_z\varphi - \frac{1}{2}\left(\begin{array}{cc} -u_z & 0 \\ 0 & u_z \end{array} \right)\varphi = \frac{1}{2}\left(\begin{array}{cc} 0 & -\lambda \\ -\lambda & 0 \end{array} \right)\varphi$$

for a general Λ-periodic potential u. Since L is a first order 2×2-matrix operator, for every $\lambda \in \mathbb{C}$ the system (37) has a two-dimensional space $\mathcal{V}_\lambda$ of solutions and these spaces are invariant under the translation operators

$$\widehat{T}_j\varphi(z) = \varphi(z + \gamma_j), \quad j = 1, 2.$$

Since $\widehat{T}_1, \widehat{T}_2$, and L commute, they have common eigenvectors and these vectors are glued into a meromorphic function $\Psi(z, \bar{z}, P)$ on a two-sheeted covering

$$\Gamma(L) \to \mathbb{C} : P \in \Gamma(L) \to \lambda \in \mathbb{C},$$

which ramifies at points where $\widehat{T}_j$ and L are not diagonalized simultaneously.

To each point $P \in \Gamma(L)$ corresponds a unique (up to a constant multiple) Floquet function φ with multipliers $\mu(\gamma_1, P)$ and $\mu(\gamma_2, P)$.

By the same reasonings as in the example 3 in 3.4 it is shown that there are the Floquet functions defined on a $\Gamma(L)$ such that $\Gamma(L)$ is compactified by four infinities $\infty^1_\pm, \infty^2_\pm$ such that $\infty^1_\pm$ are mapped into $\lambda = \infty$ and $\infty^2_\pm$ are mapped into $\lambda = 0$ and there are asymptotics

$$\psi(z, P) \approx \exp\left(\mp\frac{\lambda z}{2}\right)\left(\begin{array}{c} 1 \\ \pm 1 \end{array} \right) \quad \text{as } P \to \infty^1_\pm,$$

$$\psi(z,P) \approx \exp\left(\mp\frac{\bar{z}}{2\lambda}\right)\left(\begin{array}{c} 1 \\ \pm 1 \end{array}\right) \quad \text{as } P \to \infty^2_{\pm},$$

and therefore their multipliers tend to ∞ as $\lambda \to 0, \infty$ [5].

If $\varphi = (\varphi_1, \varphi_2)^\top$ satisfies (37) for $\lambda = \mu$, then

$$\sigma(\varphi) = (\varphi_1, -\varphi_2) \tag{38}$$

satisfies (37) for $\lambda = -\mu$ and this generates an involution $\sigma : \Gamma(L) \to \Gamma(L)$, descending to an involution of $\mathbb{C} : \lambda \to -\lambda$. We also have $\sigma(\infty^1_{\pm}) = \infty^1_{\mp}$ and $\sigma(\infty^2_{\pm}) = \infty^2_{\mp}$. By (38), the immersion

$$\mathcal{M} : \Gamma(L) \to \mathbb{C}^2 \quad : P \to (\mu(\gamma_1, P), \mu(\gamma_2, P))$$

descends through the quotient of σ, i.e., $\mathcal{M} : \Gamma(L) \to \Gamma(L)/\sigma \to \mathbb{C}^2$.

Definition 6 *The complex curve $\Gamma(L)/\sigma$ is called the spectral curve of a CMC torus Σ. It is said that $\mathcal{M}(\Gamma(L)/\sigma)$ is the spectrum of this torus.*

By straightforward computations, we obtain

Proposition 6 *$\varphi = (\varphi_1, \varphi_2)^\top$ satisfies (37) if and only if $\psi = (\lambda\varphi_2, e^{\alpha}\varphi_1)^\top$ satisfies (36).*

This proposition implies that the Floquet functions of L and $\mathcal{D}$ have the same multipliers:

Theorem 5 *Given a CMC torus Σ, the spectrum of the CMC torus form a component, of the spectrum of the surface as defined in 4.1, containing both asymptotic ends where $\mu(\gamma_j) \approx e^{\lambda_+ \gamma_j}$ and $\mu(\gamma_j) \approx e^{\lambda_- \bar{\gamma}_j}$ as $\lambda_{\pm} \to \infty$.*

Therefore, the spectral curve of the CMC torus Σ is an irreducible component of the spectral curve of this surface as defined in 4.1.

Assuming that Pretheorem is valid, we have more:

the spectrum and the spectral curve of a CMC torus coincide with the spectrum and the spectral curve of this surface as defined in 4.1.

In fact, for this conclusion we need only one part of Pretheorem, which states that the spectral curve is irreducible for $U \neq 0$.

Notice that the genus of the spectral curve of a CMC torus is finite [39].

5.2 Isothermic tori

A surface is called *isothermic* if near every point there is a conformal parameter $z = x + iy$ such that the fundamental forms are

$$\mathbf{I} = e^{2\alpha}(dx^2 + dy^2), \quad \mathbf{II} = e^{2\alpha}(k_1 dx^2 + k_2 dy^2).$$

To each isothermic surface $F : \mathcal{U} \to \mathbb{R}^3$ there corresponds the *dual isothermic surface* $F^* : \mathcal{U} \to \mathbb{R}^3$ defined up to translations by the formulas

$$F^*_z = e^{-2\alpha} F_{\bar{z}}, \quad F^*_{\bar{z}} = e^{-2\alpha} F_z.$$

The fundamental forms of the dual surface are

$$\mathbf{I} = e^{-2\alpha}(dx^2 + dy^2), \quad \mathbf{II} = -k_1 dx^2 + k_2 dy^2,$$

the Gauss maps of F and F^* are antipodal: $N = -N^*$, and $F^{**} = F$ (modulo translations).

It is obtained by straightforward computations that

Proposition 7 *If an isothermic surface F is represented via (10) by a vector function $\psi = (\psi_1, \psi_2)^\top$, then the dual surface F^* is represented via (10) by the function $\psi^* = (ie^{-\alpha}\psi_2, ie^{-\alpha}\psi_1)$.*

The potentials of these surfaces are

$$U = \frac{k_1 + k_2}{4} e^{\alpha}; \quad U^* = \frac{k_2 - k_1}{4} e^{\alpha}$$

and the Hopf differentials $A\,dz^2$ and $A^*\,dz^2$ are

$$A = \frac{k_1 - k_2}{4} e^{2\alpha}, \quad A^* = -\frac{k_1 + k_2}{4}.$$

Corollary 1 *Given an isothermic surface Σ and its Hopf differential Adz^2 and the metric $e^{2\alpha}dzd\bar{z}$, there is an equality*

$$Ae^{-\alpha} = \frac{k_1 - k_2}{4} e^{\alpha} = -U^*,$$

where U^* is the potential of the dual surface.

The simplest examples of isothermic surfaces are surfaces of revolution and constant mean curvature surfaces.

Let Σ be an isothermic plane, which may convert into an immersed torus, whose Gauss map descends through $\mathbb{C}/\Lambda$ with $\Lambda = \gamma_1\mathbb{Z} + \gamma_2\mathbb{Z}$.

By Proposition 7, the Gauss–Weingarten equations (13) are written as

$$\psi_z = \mathbf{U}\psi, \quad \psi_{\bar z} = \mathbf{V}\psi,$$

with

$$\mathbf{U} = \begin{pmatrix} \alpha_z & \frac{k_1-k_2}{4}e^{\alpha} \\ -\frac{k_1+k_2}{4}e^{\alpha} & 0 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} 0 & \frac{k_1+k_2}{4}e^{\alpha} \\ \frac{k_2-k_1}{4}e^{\alpha} & \alpha_{\bar z} \end{pmatrix},$$

and their compatibility conditions are

$$\alpha_{xx} + \alpha_{yy} + k_1 k_2 e^{2\alpha} = 0, \quad k_{2x} - (k_1 - k_2)\alpha_x = k_{1y} + (k_1 - k_2)\alpha_y = 0. \quad (39)$$

The equations (39) are also the compatibility conditions for linear problems with a spectral parameter:

$$\widehat{\varphi}_z = \widehat{\mathbf{U}}(\lambda)\widehat{\varphi}, \quad \widehat{\varphi}_{\bar z} = \widehat{\mathbf{V}}(\lambda)\widehat{\varphi}, \quad (40)$$

where

$$\widehat{\mathbf{U}}(\lambda) = \begin{pmatrix} \mathbf{U} & \lambda \mathbf{J}^- \\ \lambda \mathbf{J}^+ & \mathbf{U} + \alpha_z \mathbf{E} \end{pmatrix}, \quad \widehat{\mathbf{V}}(\lambda) = \begin{pmatrix} \mathbf{V} & \lambda \mathbf{J}^+ \\ \lambda \mathbf{J}^- & \mathbf{V} + \alpha_{\bar z} \mathbf{E} \end{pmatrix},$$

$$\mathbf{J}^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{J}^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

First such representation with a spectral parameter had been found in [10] in terms of 5×5-matrices and, by using the 4-dimensional spinor representation of $SO(5,\mathbb{C})$, had been written in terms of 4×4-matrices in [7]. Here we use another representation which is gauge equivalent to the latter one.

Now the reasonings for describing the spectrum of an isothermic torus are the same as for a one-dimensional Dirac operator (see Example 3 in 3.2) and CMC tori and we will only sketch them.

For every $\lambda \in \mathbb{C}$ the system (40) has a four-dimensional space $\mathcal{V}_\lambda$ of solutions. On each such a space the translation operators act

$$\widehat{T}_j \widehat{\varphi}(z) = \widehat{\varphi}(z + \gamma_j), \quad j = 1, 2.$$

Since $\widehat{T}_1, \widehat{T}_2$, and L commute, they have common eigenvectors and these vectors are glued into a meromorphic function $\Phi(z, \bar z, P)$ on a four-sheeted covering

$$\Gamma(\mathbf{U}) \to \mathbb{C} : P \in \Gamma(\mathbf{U}) \to \lambda \in \mathbb{C},$$

which ramifies at points where $\widehat{T}_j$ and L are not diagonalized simultaneously.

To each point $P \in \Gamma(\mathbf{U})$ there corresponds a unique (up to a constant multiple) Floquet function $\widehat{\varphi}$ with multipliers $\mu(\gamma_1, P)$ and $\mu(\gamma_2, P)$. If $\Gamma(\mathbf{U})$ is of finite genus, it is compactified by four "infinities" and $\widehat{\varphi}$ is normalized to a meromorphic function on $\Gamma(\mathbf{U})$ with exponential singularities at these "infinities".

Notice that if $\widehat{\varphi} = (\widehat{\varphi}_1, \widehat{\varphi}_2, \widehat{\varphi}_3, \widehat{\varphi}_4)^\top$ satisfies (40) for $\lambda = \mu$ then

$$\sigma(\widehat{\varphi}) = (\widehat{\varphi}_1, \widehat{\varphi}_2, -\widehat{\varphi}_3, -\widehat{\varphi}_4)^\top \tag{41}$$

satisfies (40) for $\lambda = -\mu$ and this generates an involution $\sigma : \Gamma(\mathbf{U}) \to \Gamma(\mathbf{U})$, which descends to an involution of $\mathbb{C}$: $\lambda \to -\lambda$.

By (41), the immersion

$$\mathcal{M} : \Gamma(\mathbf{U}) \to \mathbb{C}^2 \quad : \quad P \to (\mu(\gamma_1, P), \mu(\gamma_2, P))$$

descends through the quotient of σ, i. e., $\mathcal{M} : \Gamma(\mathbf{U}) \to \Gamma(\mathbf{U})/\sigma \to \mathbb{C}^2$.

Definition 7 *The complex curve $\Gamma(\mathbf{U})/\sigma$ is called the spectral curve of an isothermic surface Σ. It is said that $\mathcal{M}(\Gamma(\mathbf{U})/\sigma))$ is the spectrum of this surface.*

It is again checked by straightforward computations that

Proposition 8 *If $\widehat{\varphi}$ satisfies (40), then*

1) $\psi = (e^{-\alpha}\varphi_3, e^{-\alpha}\varphi_4)$ satisfies the Dirac equation $\mathcal{D}\psi = 0$ with the potential

$$U = \frac{k_1 + k_2}{4} e^{\alpha};$$

2) $\psi^* = (e^{-\alpha}\varphi_2, e^{-\alpha}\varphi_1)$ satisfies the Dirac equation $\mathcal{D}\psi^* = 0$ with the potential

$$U^* = \frac{k_2 - k_1}{4} e^{\alpha}.$$

We see that the Floquet functions of $(\partial_z - \mathbf{U})$ and the Dirac operators $\mathcal{D}$ with potentials U and U^* have the same multipliers and conclude

Theorem 6 *Given an isothermic surface Σ, the spectral curve and the spectrum of this isothermic isothermic surface coincide with*

1) a component, of the spectrum of Σ as defined in 4.1, containing both asymptotic ends where $\mu(\gamma_j) \approx e^{\lambda_+ \gamma_j}$ and $\mu(\gamma_j) \approx e^{\lambda_- \bar{\gamma}_j}$ as $\lambda_\pm \to \infty$;

2) a component, of the spectrum of the dual surface Σ^* as defined in 4.1, containing both asymptotic ends where $\mu(\gamma_j) \approx e^{\lambda_+ \gamma_j}$ and $\mu(\gamma_j) \approx e^{\lambda_- \bar{\gamma}_j}$ as $\lambda_\pm \to \infty$.

The spectral curve of the isothermic surface is an irreducible component of the spectral curves of this surface and its dual as defined in 4.1.

Of course, speaking about irreducibility we exclude the case $U \equiv 0$.

Now Pretheorem or, more precisely, its statement about irreducibility of the spectral curve implies that

the spectrum and the spectral curve of an isothermic surface coincide with the spectrum and the spectral curve of this surface and its dual as defined in 4.1.

We consider here a general case when surface may be an immersed plane but not only torus because usually the dual surface to an isothermic torus is not closed.

In [44] we introduce a particular case of the conjecture on the isospectrality of an isothermic surface and its dual. We conjectured that for surfaces of revolution, for which the isospectrality is equivalent to coincidence of all Kruskal–Miura integrals. Theorem 6 proves the general conjecture modulo Pretheorem and, since for surfaces of revolution Pretheorem holds (see Example 3 in 3.2 and [44]), implies the following

Theorem 7 *A torus of revolution Σ and its dual surface Σ^* have the same values of the Kruskal–Miura invariants $\mathcal{W}_k$.*

Tori of revolution also explain this passage from $\Gamma(\mathbf{U})$ to $\Gamma(\mathbf{U})/\sigma$. The spectra of one-dimensional Schrödinger operators related by the Miura transformation (28) coincide because they are both double covered by the spectrum of L [14, 47]. Moreover the Floquet function ψ of L consists in two components η_1 and η_2 which are the Floquet functions of the Schrödinger operators. In the same manner the spectra of Dirac operators corresponding to an isothermic surface and its dual surface are double covered in by the spectrum of $\widehat{L} = [\partial_z - \widehat{\mathbf{U}}(\lambda)]$ for $\lambda = 0$ and the Floquet function of $\widehat{L}$ consists in the Floquet functions ψ and ψ^* of the Dirac operators (Proposition 7).

Recall the formulas for the Kruskal–Miura invariants for surfaces of revolution. Let Σ be parameterized by x, a parameter on the rotating curve, and y, which is the angle of rotation and $0 \leq y \leq 2\pi$, and let $z = x + iy$ be a conformal parameter on Σ. Let U be the potential of its Weierstrass representation, which is periodic in x for tori. It also may be defined for spheres of revolution and in this case it is fast decaying and defined on the whole real line [44].

The densities of the Kruskal–Miura integrals for the KdV equation are

$$R_1 = -q, \quad R_{n+1} = -R_{nx} - \sum_{k=1}^{n-1} R_k R_{n-k}.$$

Since R_{2n} are the derivatives of fast decaying functions, only the integrals

$$H_n(q) = \int_0^T R_{2n-1} dx$$

do not vanish identically. Here the integration is taken over $[0, T]$, where T is the minimal period, for tori and over $\mathbb{R}$ for spheres. The simplest integrals are

$$H_1(q) = \int q dx, \quad H_2(q) = \int q^2 dx, \quad H_3(q) = \int (2q^3 - q_x^2) dx.$$

Put $q = 2iU_x - U^2$ and define the *Kruskal–Miura* invariants as

$$\mathcal{K}_l(U) = 2\pi H_l(q).$$

Notice that $\mathcal{K}_1$ is the Willmore functional and others are multiples of $\mathcal{W}_l$.

6 Surfaces in the three-sphere

6.1 The Dirac equation for surfaces in the three-sphere

Let G be the Lie group $SU(2)$ and $\mathcal{G}$ be its Lie algebra $su(2)$ identified with the tangent space T_eG to G at the unity e. This Lie algebra is spanned by

$$e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{42}$$

which satisfy the commutation relations $[e_j, e_k] = 2\,\varepsilon_{jkl}\, e_l$. Take a biinvariant metric on G:

$$\langle \xi, \eta \rangle = -\frac{1}{2}\mathrm{Tr}(\xi\eta), \quad \xi, \eta \in \mathcal{G} = T_eG,$$

in which the basis $\{e_1, e_2, e_3\}$ is orthonormal and G is isometric to the unit 3-sphere in $\mathbb{R}^4$.

Let Σ be a surface immersed into G, let $z = x + iy$ be a conformal parameter on Σ, let $f : \Sigma \to G$ be the immersion, and let $\mathbf{I} = e^{2\alpha} dz d\bar{z}$ be the induced metric.

Take the pullback of TG to a $\mathcal{G}$-bundle over Σ: $\mathcal{G} \to E = f^{-1}(TG) \xrightarrow{\pi} \Sigma$ and the differential

$$d_{\mathcal{A}} : \Omega^1(\Sigma; E) \to \Omega^2(\Sigma; E),$$

acting on E-valued 1-forms on Σ as follows. Let

$$\omega = f \cdot v dz + f \cdot v^* d\bar{z}$$

where $f \cdot : T_e G \to T_{f(p)} G$ is the left translation by $f(p)$. Then

$$d_{\mathcal{A}} \omega = d'_{\mathcal{A}} \omega + d''_{\mathcal{A}} \omega$$

where

$$d'_{\mathcal{A}} \omega = d'_{\mathcal{A}} (f \cdot v dz) = f \cdot \left(-\bar{\partial} v - \frac{1}{2} [f^{-1} \cdot \bar{\partial} f, v] \right) dz \wedge d\bar{z},$$

$$d''_{\mathcal{A}} \omega = d''_{\mathcal{A}} (f \cdot v^* dz) = f \cdot \left(\partial v^* + \frac{1}{2} [f^{-1} \cdot \partial f, v^*] \right) dz \wedge d\bar{z}.$$

By straightforward computations derive that

$$d_{\mathcal{A}}(df) = 0. \tag{43}$$

Since $*dz = -idz$ and $*d\bar{z} = id\bar{z}$, we have

$$d_{\mathcal{A}}(*df) = f \cdot \left(i\bar{\partial}(f^{-1} \cdot \partial f) + i\partial(f^{-1} \cdot \bar{\partial} f) \right) dz \wedge d\bar{z}.$$

By the definition of the mean curvature H, we have

$$d_{\mathcal{A}}(*df) = f \cdot (e^{2\alpha} \tau(f)) dx \wedge dy = \frac{i}{2} f \cdot (e^{2\alpha} \tau(f)) dz \wedge d\bar{z}$$

where $\tau(f)$ is the tension vector and $f \cdot \tau(f) = 2HN$ with N the normal vector: $f^{-1} \cdot N = -ie^{-2\alpha}[f^{-1} \cdot \partial f, f^{-1} \cdot \bar{\partial} f]$. Finally we derive

$$d_{\mathcal{A}}(*df) = f \cdot \left(H[f^{-1} \cdot \partial f, f^{-1} \cdot \bar{\partial} f] \right) dz \wedge d\bar{z}. \tag{44}$$

The case $H = 0$ is described by the harmonicity equation

$$d_{\mathcal{A}}(*df) = 0.$$

Put

$$df = f \cdot (\Psi dz + \Psi^* d\bar{z}) \tag{45}$$

and rewrite (43) and (44) as

$$\bar{\partial} \Psi - \partial \Psi^* + [\Psi^*, \Psi] = 0, \tag{46}$$

$$\bar{\partial} \Psi + \partial \Psi^* = iH[\Psi^*, \Psi]. \tag{47}$$

Since $f^{-1} \cdot f_x = a^j e_j$ and $f^{-1} \cdot f_y = b^k e_k$ with $a^j, b^k \in \mathbb{R}$, we have

$$\Psi = \sum_{j=1}^{3} Z_j e_j, \quad \Psi^* = \sum_{j=1}^{3} \bar{Z}_j e_j$$

with $Z_j = (a^j - ib^j)/2$ and $\Psi, \Psi^* \in su(2) \otimes \mathbb{C}$. The induced metric is

$$e^{2\alpha} dz d\bar{z} = -\frac{1}{2} \mathrm{Tr}[(\Psi dz + \Psi^* d\bar{z})^2] =$$

$$(Z_1^2 + Z_2^2 + Z_3^2)(dz)^2 + 2(|Z_1|^2 + |Z_2|^2 + |Z_3|^2) dz d\bar{z} + (\bar{Z}_1^2 + \bar{Z}_2^2 + \bar{Z}_3^2)(d\bar{z})^2$$

and we conclude that

$$|Z_1|^2 + |Z_2|^2 + |Z_3|^2 = \frac{1}{2} e^{2\alpha}, \quad Z_1^2 + Z_2^2 + Z_3^2 = 0.$$

Representing solutions to the latter equation as in 2.2

$$Z_1 = \frac{i}{2}(\bar{\psi}_2^2 + \psi_1^2), \ Z_2 = \frac{1}{2}(\bar{\psi}_2^2 - \psi_1^2), \ Z_3 = \psi_1 \bar{\psi}_2, \tag{48}$$

we derive that

$$e^{\alpha} = |\psi_1|^2 + |\psi_2|^2.$$

Now rewriting (46) in terms of ψ_j and expanding it in the basis $\{e_j\}$, show that (46) is equivalent to the system

$$\bar{\partial}(\psi_1 \bar{\psi}_2) - \partial(\bar{\psi}_1 \psi_2) = i(|\psi_2|^4 - |\psi_1|^4), \quad \bar{\partial}(\psi_1^2) + \partial(\psi_2^2) = 2i\psi_1\psi_2 e^{\alpha}.$$

Analogously show that (47) is equivalent to the system

$$\bar{\partial}(\psi_1 \bar{\psi}_2) + \partial(\bar{\psi}_1 \psi_2) = H(|\psi_2|^4 - |\psi_1|^4), \quad \bar{\partial}(\psi_1^2) - \partial(\psi_2^2) = 2H\psi_1\psi_2 e^{\alpha}.$$

Introduce $V_1 = -\partial\psi_2/\psi_1$ and $V_2 = \bar{\partial}\psi_1/\psi_2$. It follows from (46) that $\mathrm{Re}V_1 = \mathrm{Re}V_2, \mathrm{Im}V_1 = -e^{\alpha}/2$, and $\mathrm{Im}V_2 = e^{\alpha}/2$, and (47) implies that $\mathrm{Re}V_1 = \mathrm{Re}V_2 = He^{\alpha}/2$.

Finally we derive that

Theorem 8 *For any immersed surface Σ is S^3 the spinor field ψ defined by (45) and (48) satisfies the Dirac equation*

$$\mathcal{D}^S \psi = 0$$

with

$$\mathcal{D}^S = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} V & 0 \\ 0 & \bar{V} \end{pmatrix}, \quad V = \frac{1}{2}(H - i)(|\psi_1|^2 + |\psi_2|^2). \tag{49}$$

This spinor field is unique by its construction and we say that ψ is the generating spinor for a surface.

Notice that the converse is not always true as in the case of Theorems 1 and 2. Indeed, not to any solution of $\mathcal{D}^S\psi = 0$ corresponds a surface: a solution related to a surface has to satisfy an additional condition

$$|\psi_1|^2 + |\psi_2|^2 = -2\mathrm{Im}V.$$

It is easy to check that if $\mathcal{D}^S\psi = 0$, then $\mathcal{D}^S\varphi = 0$ with $\varphi = (\bar{\psi}_2, -\bar{\psi}_1)^\top$.

Let us write the complete system of the Gauss–Weingarten equations. Recall that the Hopf differential equals $Adz^2 = \langle f_{zz}, N\rangle\, dz^2$ and, since the metric is left-invariant, we have $A = \langle f^{-1}f_{zz}, f^{-1}N\rangle$. Now $\Psi = f^{-1}f_z$ and

$$f^{-1}f_{zz} = \Psi_z + \Psi^2,$$

where

$$\Psi = \begin{pmatrix} iZ_1 & Z_2 + iZ_3 \\ -Z_2 + iZ_3 & -iZ_1 \end{pmatrix}, \quad \Psi^* = \begin{pmatrix} i\bar{Z}_1 & \bar{Z}_2 + i\bar{Z}_3 \\ -\bar{Z}_2 + i\bar{Z}_3 & -i\bar{Z}_1 \end{pmatrix}. \tag{50}$$

We have $\Psi^2 = (Z_1^2 + Z_2^2 + Z_3^2)e_1$ and, since z is a conformal parameter, $\Psi^2 = 0$. Therefore as in 2.2 the Hopf differential takes the same form

$$A\, dz^2 = (\psi_{1z}\bar{\psi}_2 - \bar{\psi}_{2z}\psi_1)dz^2.$$

We also have

$$\alpha_z e^\alpha = \bar{\psi}_1\psi_{1z} + \psi_2\bar{\psi}_{2z}$$

and finally write down the Gauss–Weingarten equations for an immersed surface in S^3 as

$$\left[\frac{\partial}{\partial z} - \begin{pmatrix} \alpha_z & Ae^{-\alpha} \\ -V & 0 \end{pmatrix}\right]\psi = \left[\frac{\partial}{\partial \bar{z}} - \begin{pmatrix} 0 & \bar{V} \\ -\bar{A}e^{-\alpha} & \alpha_{\bar{z}} \end{pmatrix}\right]\psi = 0. \tag{51}$$

The compatibility conditions are the Codazzi equations

$$\alpha_{z\bar{z}} + |V|^2 - |A|^2e^{-2\alpha} = 0, \quad A_{\bar{z}} = (\bar{V}_z - \alpha_z\bar{V})e^\alpha. \tag{52}$$

Examples.

1) The Clifford torus. This torus in $\mathbb{R}^4$ is defined by the equations

$$(x^1)^2 + (x^2)^2 = (x^3)^2 + (x^4)^2 = \frac{1}{2}$$

where $(x^1, \dots, x^4) \in \mathbb{R}^4$. It is immersed into $SU(2)$ by the formula

$$f(x,y) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{ix} & e^{iy} \\ -e^{-iy} & e^{-ix} \end{pmatrix}$$

and has a conformal type of the square torus, $(x,y) \in \mathbb{Z}^2/2\pi\mathbb{Z}^2$. Compute that

$$\Psi = \frac{1}{4}((1+i)e_1 + (1-i)\sin(x-y)e_2 + (1-i)\cos(x-y)e_3),$$

$$\psi_1 = \sqrt{\frac{1-i}{2}} \sin\left(\frac{x-y}{2} - \frac{\pi}{4}\right), \quad \psi_2 = \sqrt{\frac{1+i}{2}} \cos\left(\frac{x-y}{2} - \frac{\pi}{4}\right),$$

$$e^\alpha = \frac{1}{\sqrt{2}}, \quad V = -\frac{i}{2\sqrt{2}}, \quad A = \frac{1}{4}.$$

2) Minimal tori in S^3. In this case we have $V = -ie^\alpha/2$ and derive from (52) that $A_{\bar z} = 0$. This means that the Hopf differential is holomorphic and as in the case of CMC tori in $\mathbb{R}^3$ we conclude that it is constant and by rescaling conformal parameter achieve $A = 1/2$. The case $A = 0$ is also excluded for tori: it is realized by the equatorial S^2-spheres in S^3 (complete CMC surfaces in $\mathbb{R}^3$ with $A = 0$ are the round spheres). The first equation from (52) is

$$u_{z\bar z} + \sinh u = 0, \qquad u = 2\alpha.$$

For CMC tori in S^3 the Hopf differential is also holomorphic and they are described in the same manner.

Now it is clear that the analogs of Theorems 1, 2, and 3 hold for surfaces in S^3. We again have the same spinor bundles over constant curvature surfaces.

Definition 8 *Given a torus, represented in S^3 via ψ satisfying (51), the spectral curve Γ^S of the operator $\mathcal{D}^S$ with the potential (49) is called the spectral curve of the torus. Given in addition a basis γ_1, γ_2 for Λ, the period lattice for U, the image of the multiplier map*

$$\mathcal{M} : Q_0(\mathcal{D}^S)/\Lambda^* \to \mathbb{C}^2 \quad : \quad \mathcal{M}(k) = (e^{2\pi i\langle k, \gamma_1\rangle}, e^{2\pi i \langle k, \gamma_2\rangle})$$

is called the spectrum of the torus in S^3.

Example. Let Σ be the Clifford torus. Then $V = -i/2\sqrt{2}$ and the Floquet eigenfunctions $\psi = (\psi_1, \psi_2)^\top$ satisfy the equation

$$\left(\partial\bar\partial + \frac{1}{8}\right)\psi_j = 0, \quad j = 1, 2.$$

We derive that the general Floquet function is

$$\psi(z,\bar z,\lambda) = \left(e^{\lambda z - \frac{1}{8\lambda}\bar z}, \frac{i}{2\sqrt{2}\lambda} e^{\lambda z - \frac{1}{8\lambda}\bar z}\right)^{\top}$$

and find the spectrum as the image of the multiplier map

$$\lambda \in \mathbb{C}\setminus\{0\} \to \left(e^{2\pi(\lambda - \frac{1}{8\lambda})}, e^{2\pi i(\lambda + \frac{1}{8\lambda})}\right).$$

The spectral curve is the two-sphere, i.e., the punctured plane $\mathbb{C}\setminus\{0\}$ compactified by the points $\lambda = 0, \infty$. This implies that the spectral genus of the Clifford torus equals zero.

We shall not discuss the spectra of tori in S^3 in detail but only mention that Pretheorem also has to hold for them.

6.2 The Hitchin system

Let us compare the previous computations with the Hitchin theory of harmonic tori in the 3-sphere [22]. For Riemannian manifolds N and M a mapping $f : N \to M$ is called *harmonic* if it satisfies the equations

$$d_{\mathcal{A}}(df) = d_{\mathcal{A}}(*df) = 0$$

where $\mathcal{A}$ is the pullback of the Levi-Civita connection on TM and the Hodge operator $*$ is taken with respect to the metric on N. If f is an immersion and the metric on N is the induced metric, then $f(N)$ is a minimal submanifold.

Let N be an immersed surface Σ with the induced metric and $M = G$ be Lie group with a biinvariant metric. We adopt the notation from 6.1.

The harmonicity equations take the form

$$d_{\mathcal{A}}(\Psi dz + \Psi^* d\bar z) = 0,$$

$$d_{\mathcal{A}}(*(\Psi dz + \Psi^* d\bar z)) = -i d_{\mathcal{A}}(\Psi dz - \Psi^* d\bar z) = 0.$$

They describe minimal surfaces in S^3 and are rewritten as

$$\bar\partial\Psi - \partial\Psi^* + [\Psi^*, \Psi] = \bar\partial\Psi + \partial\Psi^* = 0. \tag{53}$$

Following [22] put

$$\Phi = \frac{1}{2}\Psi, \quad \Phi^* = -\frac{1}{2}\Psi^*$$

and rewrite (53) as the *Hitchin system*

$$d''_{\mathcal{A}}\Phi = 0, \quad F_{\mathcal{A}} = d^2_{\mathcal{A}} = [\Phi, \Phi^*] = 0, \tag{54}$$

where F_A is the curvature of the connection

$$d_{\mathcal{A}} : \Omega^p(\Sigma; f^{-1}TG) \to \Omega^{p+1}(\Sigma; f^{-1}TG)$$

and the formula means that $d^2_{\mathcal{A}}$ coincides with the multiplication by $F_{\mathcal{A}}$. The system (54) describes general harmonic mappings of surfaces in S^3 (when the metric on the surface is not necessarily induced) in terms of a connection $\mathcal{A}$ associated to the harmonic map and the *Higgs field* Φ.

The equation $d_{\mathcal{A}} df = 0$ is equivalent to

$$\bar{\partial}\Psi - \partial\Psi^* + [\Psi^*, \Psi] = 0$$

and means that the connection $\mathcal{A} = (\partial + \Psi, \bar{\partial} + \Psi^*)$ on $f^{-1}TG$ is flat, which is evident from its construction. However the second of the equations (53) implies that this connection is extended to an analytic family of flat connections

$$\mathcal{A}_\lambda = \left(\partial + \frac{1+\lambda^{-1}}{2}\Psi, \bar{\partial} + \frac{1+\lambda}{2}\Psi^*\right)$$

where $\mathcal{A} = \mathcal{A}_1$ and $\lambda \in \mathbb{C} \setminus \{0\}$. This commutation representation with a spectral parameter was found by Pohlmeyer [40] for harmonic maps into $SU(2)$ and later developed by Mikhailov and Zakharov for the case, when the target space is not a group but a symmetric space S^2 [51]. Finally these two papers gave rise to the "integrability" part of the modern theory of harmonic maps [49, 22, 9, 19].

For harmonic tori Hitchin introduced spectral curves and showed that they are of finite genus [22]. Their construction is as follows.

Let Σ be a harmonic torus in S^3. For any $\lambda \in \mathbb{C} \setminus \{0\}$ we have a flat $Sl(2,\mathbb{C})$ connection. Fix a basis $\{\gamma_1, \gamma_2\}$ for $H_1(\Sigma)$. For γ_1 and γ_2 define matrices $H(\lambda), \widetilde{H}(\lambda) \in SL(2,\mathbb{C})$ which describe the monodromies of $\mathcal{A}_\lambda$ along closed loops realizing γ_1 and γ_2. These matrices commute and have joint eigenfunctions $\varphi(\lambda, \mu)$ where μ is a root of the characteristic equation for $H(\lambda)$

$$\mu^2 - \mathrm{Tr} H(\lambda) + 1 = 0$$

and therefore there is a Riemann surface on which the eigenvalues

$$\mu_{1,2} = \frac{1}{2}\left(\mathrm{Tr} H(\lambda) \pm \sqrt{\mathrm{Tr}^2 H(\lambda) - 4}\right)$$

are defined. The complex curve Γ, which is a two-sheeted covering of $\mathbb{C}P^1$, ramifying at the odd zeros of the function $(\mathrm{Tr}^2 H(\lambda) - 4)$ and at 0 and ∞, is called the *spectral curve of a harmonic torus* in S^3.

On Γ the eigenvalues of $H(\lambda)$ paste into a single-valued function μ with singularities at 0 and ∞. Moreover the joint eigenfunctions of $H(\lambda)$ and $\widetilde{H}(\lambda)$ paste into a vector function φ meromorphic on $\Gamma \setminus \{0, \infty\}$.

We shall show that in the case, when the harmonic tori is an immersed tori in S^3 with the induced metric, i. e., in the situation of 6.1, the spectral curve of Hitchin is the same as the spectrum of the torus as defined in 6.1.

Let $f : \Sigma \to S^3$ be an immersion of a minimal torus and $\Psi = f^{-1} f_z, \Psi^* = f^{-1} f_{\bar{z}}$. Let the surface be defined by a spinor ψ.

The Hitchin eigenfunction $\varphi(\lambda, \mu)$ satisfies the equations

$$\left[\partial + \frac{1+\lambda}{2}\Psi\right]\varphi = \left[\bar{\partial} + \frac{1+\lambda^{-1}}{2}\Psi^*\right]\varphi = 0.$$

Take the matrix

$$L = \begin{pmatrix} \bar{a} & -\bar{b} \\ b & a \end{pmatrix} \tag{55}$$

with $a = (-i\bar{\psi}_1 + \psi_2)/\sqrt{2}, b = (-i\psi_1 + \bar{\psi}_2)/\sqrt{2}$. By (48) and (50), compute that

$$L^{-1}\Psi L = e^{\alpha}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad L^{-1}\Psi^* L = e^{\alpha}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

We also have

$$L^{-1}L_z = \begin{pmatrix} \alpha_z & -iV \\ -iAe^{-\alpha} & 0 \end{pmatrix}, \qquad L^{-1}L_{\bar{z}} = \begin{pmatrix} 0 & -i\bar{A}e^{-\alpha} \\ -i\bar{V} & \alpha_{\bar{z}} \end{pmatrix}.$$

The vector function $L^{-1}\varphi$ satisfies the equations

$$\left[\partial + \begin{pmatrix} \alpha_z & -iV \\ -iAe^{-\alpha} & 0 \end{pmatrix} + \frac{1+\lambda}{2}e^{\alpha}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] L^{-1}\varphi = 0,$$

$$\left[\bar{\partial} + \begin{pmatrix} 0 & -i\bar{A}e^{-\alpha} \\ -i\bar{V} & \alpha_{\bar{z}} \end{pmatrix} + \frac{1+\lambda^{-1}}{2}e^{\alpha}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\right] L^{-1}\varphi = 0.$$

These two equations are compatible only for minimal tori, which are described by the condition

$$V = -\frac{ie^{\alpha}}{2}.$$

For $\widetilde{\varphi} = e^{\alpha}L^{-1}\varphi$ we derive that

$$\partial\widetilde{\varphi}_1 + \frac{\lambda}{2}e^{\alpha}\widetilde{\varphi}_2 = 0, \qquad \bar{\partial}\widetilde{\varphi}_2 - \frac{1}{2\lambda}e^{\alpha}\widetilde{\varphi}_1 = 0.$$

Put $\widetilde{\psi}_1 = i\lambda\widetilde{\varphi}_2, \widetilde{\psi}_2 = \widetilde{\varphi}_2$ and notice that $\widetilde{\psi}$ satisfies (49):

$$\left[\left(\begin{array}{cc} 0 & \partial \\ -\bar{\partial} & 0 \end{array}\right) + \left(\begin{array}{cc} V & 0 \\ 0 & \bar{V} \end{array}\right)\right]\widetilde{\psi} = 0 \quad \text{with } V = -ie^{\alpha}/2.$$

As in the proofs of Theorems 5 and 6 we conclude

Theorem 9 *Given a minimal torus $f : \Sigma \to S^3$, the Hitchin eigenfunction $\varphi(\lambda, \mu)$ by the transformation*

$$\left(\begin{array}{c} \varphi_1 \\ \varphi_2 \end{array}\right) \to \left(\begin{array}{c} \widetilde{\psi}_1 \\ \widetilde{\psi}_2 \end{array}\right) = e^{\alpha}\left(\begin{array}{cc} 0 & i\lambda \\ 1 & 0 \end{array}\right) \cdot L^{-1} \cdot \left(\begin{array}{c} \varphi_1 \\ \varphi_2 \end{array}\right)$$

is mapped to a Floquet function $\widetilde{\psi}$ of $\mathcal{D}^S$. There is the mapping of the eigenvalues μ and $\widetilde{\mu}$ of φ with respect to the monodromy operators $H(\lambda)$ and $\widetilde{H}(\lambda)$ to the multipliers of $\widetilde{\psi}$:

$$(\mu, \widetilde{\mu}) \to ((-1)^{\varepsilon(\gamma_1)}\mu, (-1)^{\varepsilon(\gamma_2)}\widetilde{\mu}) \tag{56}$$

where $(-1)^{\varepsilon(\gamma_1)}, (-1)^{\varepsilon(\gamma_2)}$ are the multipliers of the spinor ψ generating a minimal torus.

This mapping (56) establishes a biholomorphic equivalence between the Hitchin spectral curve of a minimal torus and the connected component of the spectrum of this torus as defined in 6.1. This connected component contains both asymptotic ends near which $\widetilde{\psi} \approx (e^{\lambda_+ z}, 0)^{\top}$ or $\widetilde{\psi} \approx (0, e^{\lambda_- \bar{z}})^{\top}$.

If Pretheorem holds for $\mathcal{D}^S$, then the spectrum is irreducible and therefore (56) establishes a biholomorphic equivalence of both spectra.

7 Conformal invariance of the spectra of tori

7.1 The Möbius group

We consider $\mathbb{R}^4$ as the set of matrices

$$a = \left(\begin{array}{cc} x^4 + ix^1 & x^2 + ix^3 \\ -x^2 + ix^3 & x^4 - ix^1 \end{array}\right), \qquad x^1, x^2, x^3, x^4 \in \mathbb{R}, \tag{57}$$

and consider $\mathbb{R}^3$ as a subset described by $x^4 = 0$. The unit sphere $S^3 = SU(2)$ is defined by the equation

$$|x| = 1.$$

Take the north pole $P = (0,0,0,1)$ and denote by π the stereographic projection of S^3 to $\mathbb{R}^3 = \{x^4 = 0\}$ from P:

$$\pi : a \to \frac{1}{1-x^4} \begin{pmatrix} ix^1 & x^2 + ix^3 \\ -x^2 + ix^3 & -ix^1 \end{pmatrix} = (1+a)(1-a)^{-1}.$$

The inverse mapping is

$$\pi^{-1} : b \to (b-1)(b+1)^{-1}.$$

This mapping π establishes a conformal equivalence between S^3 and $\mathbb{R}^3$ compactified by a point at infinity, i.e., by $\pi(P) = \infty$.

The group of conformal transformations of $\mathbb{R}^3 \cup \infty$ is isomorphic to $O^+(4,1)$, the subgroup of $O(4,1)$ formed by isochronic transformations. The geometric picture is as follows. Let $\mathbb{R}^{1,4}$ be a 5-dimensional pseudo-Euclidean space with the metric

$$\langle x, y \rangle_{1,4} = x^0 y^0 - \sum_{j=1}^{4} x^j y^j.$$

The 4-dimensional hyperbolic space $\mathcal{H}^4$ is embedded into $\mathbb{R}^{1,4}$ as the upper half of a hyperboloid: $\langle x, x \rangle_{1,4} = 1, x^0 > 0$, with the metric on tangent vectors $\langle \xi, \xi \rangle = -\langle \xi, \xi \rangle_{1,4}$. The group of isometries of $\mathcal{H}^4$ is $O^+(4,1)$ and it acts on S^3, the absolute of $\mathcal{H}^4$, by conformal transformations.

By the Liouville theorem, the group $O^+(4,1)$ of conformal transformations is generated by

1) isometries of $\mathbb{R}^3$;

2) inversions with centers in $x_0 \in \mathbb{R}^3$: $x \to \frac{x - x_0}{|x - x_0|^2}$;

3) homotheties: $x \to \lambda x$, $\lambda \in \mathbb{R} \setminus \{0\}$.

Any conformal transformation of $\bar{\mathbb{R}}^3 = \mathbb{R}^3 \cup \{\infty\}$ which preserves ∞ is a composition of isometries and homotheties. Notice that in terms of (57) the inversion of $\mathbb{R}^3$ centered at x_0 looks simply as $x \to (x_0 - x)^{-1}$.

We think that

the spectrum of a torus in S^3 which is stereographically projected into a torus in $\mathbb{R}^3$ and the spectrum of this projection coincide.

We can not prove it now but would like to notice that this statement easily implies conformal invariance of both spectra: it is clear that the spectrum of a torus in $\mathbb{R}^3$ is invariant under translations of the torus and the spectrum of a torus in S^3 is invariant under rotations in S^3. However the stereographic projection converts rotations in S^3 into conformal transformations of $\mathbb{R}^3$ which together with translations and homotheties generate the conformal group $O^+(4,1)$. The same holds with translations of $\mathbb{R}^3$ whose compositions with the projection generate together with rotations the group of conformal transformations of S^3.

7.2 Conformal invariance of the spectra for isothermic tori in $\mathbb{R}^3$

Theorem 10 *Let Σ be an isothermic torus in $\mathbb{R}^3$ and let $F : \bar{\mathbb{R}}^3 \to \bar{\mathbb{R}}^3$ be a conformal transformation which maps Σ into a torus $F(\Sigma)$ lying in $\mathbb{R}^3$. Then the spectrum of an isothermic torus Σ coincides with the spectrum of $F(\Sigma)$.*

Notice that the spectrum of an isothermic torus is defined as a component of the general spectrum which contains the asymptotic ends where $\mu(\gamma_j) \approx e^{\lambda_+ \gamma_j}$ and $\mu(\gamma_j) \approx e^{\lambda_j \bar{\gamma}_j}$ as $\lambda_\pm \to \infty$. Pretheorem states that the general spectrum is irreducible and therefore coincides with this component.

Proof. By Theorem 6, the spectrum of an isothermic torus and its dual isothermic surface coincide. The potential of the dual surface equals

$$U^* = \frac{k_2 - k_1}{4} e^{\alpha}$$

and, by the Blaschke theorem, the density of the Willmore functional

$$\left(\frac{k_2 - k_1}{2}\right)^2 d\mu = 4\,(U^*)^2\, dx \wedge dy$$

is invariant under conformal transformations of $\bar{\mathbb{R}}$. As known, conformal transformations maps isothermic surfaces into isothermic ones.

Let z be a conformal parameter on Σ which is mapped into a conformal parameter on $F(\Sigma)$ and V be the potential of $F(\Sigma)$ with respect to this parameter. We see that, by the Blaschke theorem, $V^2 = (U^*)^2$ and therefore $V = \pm U^*$.

It is clear that the spectra of the Dirac operators whose potentials differs by sign coincide. Now we derive that the spectra of the isothermic tori Σ and $F(\Sigma)$ coincide with the spectrum of the isothermic surface with the potential U^*. This proves the theorem.

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