TWISTOR SPINORS AND NORMAL CARTAN CONNECTIONS IN CONFORMAL GEOMETRIES. *

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June 14, 2000

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Abstract

We formulate conformal (spin) geometry with arbitrary signature in the context of almost Hermitian symmetric geometry and construct the canonical normal Cartan connection of conformal geometry. It is shown that twistor spinors on a conformal spin manifold M may be interpreted as parallel sections in a conformal spinor bundle $E \to M$ with respect to a covariant derivative ∇^E, which arises from the canonical normal Cartan connection. We also describe every conformally flat spin manifold M, which admits twistor spinors, with the aid of the holonomy representation of the fundamental group $\pi_1(M)$.

Subj. Class.: Differential Geometry.
1991 MSC: 53A30, 53C27, 53C50.
Keywords: Conformal spin geometry, twistor spinors, normal Cartan connections, development and holonomy.

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*Supported by the SFB 288 of the DFG.

0 Introduction

In the present paper we reformulate the twistor equation as parallelity equation with respect to the normal conformal Cartan connection in a certain conformal spinor bundle and classify conformally flat semi-Riemannian spin manifolds, which admit twistor spinors. On a smooth semi-Riemannian spin manifold (M, g) of dimension n and index k there exists a conformal covariant differential operator acting on spinor fields $\Gamma(S)$ defined by

$$\mathcal{T} : \Gamma(S) \stackrel{\nabla^S}{\to} \Gamma(T^*M \otimes S) \stackrel{p}{\to} \Gamma(ker\mu),$$

where ∇^S denotes the spinor derivative and p denotes the projection onto the kernel of the Clifford multiplication μ. This differential operator is called twistor operator and elements in the kernel of $\mathcal{T}$ are called twistor spinors. Equivalently, a spinor field ϕ is a twistor spinor if and only if it satisfies the conformally invariant twistor equation

$$\nabla^S_X \phi + \frac{1}{n} X \cdot D\phi = 0 \quad \forall X \in \Gamma(TM),$$

where D denotes the Dirac operator. Twistor spinors were first introduced by R. Penrose in Genaral Relativity. Special classes of twistor spinors are Killing spinors and parallel spinors. The theory of parallel spinors is closely related to the classification of holonomy groups of a semi-Riemannian manifold. Killing spinors are related to parallel spinors via the construction of the cone of a semi-Riemannian manifold. The construction of solutions of the twistor equation in the general form is more complicated. In [PM72] the twistor equation on a space-time is set in relation to a 'twistor connection' on the vector bundle of 'local twistors'. It was H. Friedrich, who showed in [Fri76] that the 'twistor connection' on a space-time is induced by the canonical normal Cartan connection of conformal geometry. We will work out in the following the generalization of this fact to arbitrary semi-Riemannian manifolds. From this point of view the solvability of the twistor equation is related to the holonomy of the normal conformal Cartan connection.

In section 1 we will recall the basic definitions of Cartan geometry and we will construct the development of a flat Cartan geometry and the corresponding holonomy representation of the fundametal group of the base manifold. Then we will define almost Hermitian symmetric structures and normal Cartan connections. In section 2 we will discuss in detail conformal geometries of arbitrary signature, which belong to the class of Hermitian symmetric geometries. In particular, we define the standard conformal spin spaces $\hat{C}^n_k$. It is proved in [BFGK91] that on an arbitrary semi-Riemannian manifold a spinor field ϕ is a 'twistor' if and only if

$$\nabla^{S^g \oplus S^g}_X \begin{pmatrix} \phi \\ D\phi \end{pmatrix} := \begin{pmatrix} \nabla^{S^g}_X & \frac{1}{n} X \\ -\frac{n}{2} L(X) & \nabla^{S^g}_X \end{pmatrix} \begin{pmatrix} \phi \\ D\phi \end{pmatrix} = 0,$$

where L denotes the Schouten tensor. In section 3 we will prove that the above covariant derivative $\nabla^{S^g \oplus S^g}$ is induced by the normal Cartan connection of conformal geometry. Once we have understood twistor spinors as parallel sections in $S^g \oplus S^g$, we are able to characterize conformally flat manifolds, which admit twistor spinors, with the help of a development in the standard conformal spin space $\hat{C}^n_k$. This will be done in the last section.

1 Cartan Geometry, Development and Normal Cartan Connections

In the first part of this section we recall the definition of Cartan geometry and define the development map of a flat Cartan geometry. In the second part we give a brief summary of the construction of the canonical normal Cartan connection on an almost Hermitian symmetric structure due to [CSS]. We start with the definition of Klein geometry, which is a generalized concept for classical geometries like Euclidean or affine geometry (comp. [Sha96]).

Definition 1.1 *A Klein geometry is a pair (G, H), where G is a Lie group and H is a closed subgroup of G such that the homogenous space G/H, which is called the Klein model, is connected.*

Example: Let $Aff(n)$ denote the group of affine transformations on $\mathbb{R}^n$. Affine geometry is the pair $(Aff(n), Gl(n))$ and its standard model is the affine n-space $\mathbb{A}^n = Aff(n)/Gl(n)$.

A Cartan geometry on a C^∞-manifold M is modeled on a Klein geometry (G, H). Let $(\mathfrak{g}, \mathfrak{h})$ denote the pair of Lie algebras to (G, H).

Definition 1.2 *Let M be a C^∞-manifold, (G, H) a Klein geometry and $P \to M$ an H-principal fiber bundle. A $\mathfrak{g}$-valued 1-form $\omega : TP \to \mathfrak{g}$, which satisfies the conditions*

(1) $\omega(A^*) = A$, where A^* is the fundamental vector field on P corresponding to $A \in \mathfrak{h}$

(2) $R_a^*\omega = Ad(a^{-1})\omega$

(3) $\omega_p : T_pM \to \mathfrak{g}$ is an isomorphism for each $p \in P$,

is called a Cartan connection of type (G, H). The pair (P, ω) is called a Cartan geometry on M.

The third condition of Definition 1.2 implies for a Cartan geometry (P, ω) of type (G, H) on M that $\dim P = \dim G$ holds.

Example: Let $P := Aff(n) \xrightarrow{\pi} \mathbb{A}^n$ be the $Gl(n)$-principal fiber bundle of linear frames on the affine n-space $\mathbb{A}^n$ and denote by

$$\eta : TP \to \mathfrak{gl}(n)$$

the standard connection on P with vanishing torsion and curvature. The canonical form θ on P is defined by

$$\begin{aligned} \theta \ &: \ TP \to \mathbb{R}^n, \\ &\quad X_p \mapsto p^{-1}\pi_*(X_p), \qquad p \in P \end{aligned}$$

where a linear frame p is viewed as a linear isomorphism $p : \mathbb{R}^n \to T_{\pi(p)}\mathbb{A}^n$ as usual. The 1-form

$$\omega = \theta + \eta : TP \to \mathbb{R}^n \oplus \mathfrak{gl}(n) \cong \mathfrak{aff}(n)$$

is a Cartan connection on $P \to \mathbb{A}^n$ of affine type $(Aff(n), Gl(n))$ and (P, ω) is a Cartan geometry on $\mathbb{A}^n$.

Cartan connections induce connections in the usual sense on an enlarged bundle. Let (P, ω) be a Cartan geometry on M of type (G, H). The enlarged bundle $\bar{P} := P \times_H G \to M$ is a G-principal fiber bundle and the mapping

$$p \in P \mapsto [p, e] \in \bar{P}$$

is a natural embedding of principal fiber bundles. The image of the Cartan connection ω on P in $\bar{P}$ can be uniquely extended to a $\mathfrak{g}$-valued 1-form $\bar{\omega}$, which is a principal fiber bundle connection on $\bar{P}$. The $\mathfrak{g}$-valued 2-form

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]$$

is the Cartan curvature of ω on P. If the curvature form Ω of (P, ω) vanishes, then the Cartan geometry (P, ω) is called flat. The curvature form $\bar{\Omega} = d\bar{\omega} + \frac{1}{2}[\bar{\omega}, \bar{\omega}]$ of the corresponding connection $\bar{\omega}$ on $\bar{P}$ also vanishes in this case.

Example: Let (G, H) be a Klein geometry. Then $G \to G/H$ is an H-principal bundle and the Maurer-Cartan form $\omega_G : TG \to \mathfrak{g}$ defined by

$$X_a \in T_a G \mapsto \omega_G(X_a) = L_{a^{-1}}(X_a) \in T_a G \cong \mathfrak{g}$$

is a Cartan connection on G. Since by the Maurer-Cartan equation

$$\Omega_G = d\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0,$$

the Cartan geometry (G, ω_G) on G/H is flat. There exists a canonical parallel section with respect to $\bar{\omega}_G$ in the enlarged principal fiber bundle $\bar{G} := G \times_H G$:

$$\begin{aligned} \bar{\alpha} \ : \ & G/H \to \bar{G}. \\ & aH \mapsto [a, a^{-1}] \end{aligned}$$

Let (P, ω) be a flat Cartan geometry of type (G, H) on a connected and simply connected C^∞-manifold M. The enlarged G-principal fiber bundle $\bar{P} = P \times_H G$ is equipped with the flat connection $\bar{\omega}$. We fix a point $\bar{p} \in \bar{P}$. Since $\bar{\omega}$ is flat and M is connected and simply connected, the parallel displacement on $\bar{P}$ defines in a unique way a G-equivariant C^∞-map

$$\bar{\delta}_{\bar{p}} : \bar{P} \to G$$

with $\bar{\delta}_{\bar{p}}(\bar{p}) = e$ and $\bar{\delta}_{\bar{p}}^* \omega_G = \bar{\omega}$. Moreover, there exists for every C^∞-map $\bar{\delta} : \bar{P} \to G$ with the property that $\bar{\delta}^* \omega_G = \bar{\omega}$ an element $a \in G$ such that

$$\bar{\delta} = L_a \circ \bar{\delta}_{\bar{p}},$$

where L_a denotes the left translation by a. The natural embedding of P in $\bar{P}$ gives rise to a C^∞-map defined by

$$\begin{aligned} \delta_{\bar{p}} \ : \ P &\to \ G. \\ p &\mapsto \bar{\delta}_{\bar{p}}([p,e]) \end{aligned}$$

This map $\delta_{\bar{p}}$ is H-equivariant, and since ω is the pullback of $\bar{\omega}$ on $\bar{P}$, it holds $\omega_G \circ d\delta_{\bar{p}} = \omega$. We can conclude that $\delta_{\bar{p}} : (P,\omega) \to (G,\omega_G)$ is a local isomorphism of Cartan geometries and every local isomorphism $\delta : (P,\omega) \to (G,\omega_G)$ is of the form $L_a \circ \delta_{\bar{p}}$ for some $a \in G$. The induced map

$$\delta^M_{\bar{p}} : M \to G/H$$

of $\delta_{\bar{p}}$ on the base spaces is called a development map of M in the Klein model G/H.

Let M be a connected C^∞-manifold with flat Cartan geometry (P,ω). The universal covering space $\tilde{M} \xrightarrow{\pi} M$ is equipped with the flat Cartan geometry $(\tilde{P},\tilde{\omega}) := (\pi^*P,\pi^*\omega)$, which is the pullback of (P,ω) by π. The fundamental group $\pi_1(M)$ of M acts on $\tilde{M}$ by deck transformations and these deck transformations preserve the Cartan geometry $(\tilde{P},\tilde{\omega})$. This induces a group homomorphism

$$w : \pi_1(M) \to Aut(\tilde{P},\tilde{\omega})$$

into the automorphism group of the Cartan geometry $(\tilde{P},\tilde{\omega})$ on $\tilde{M}$. Let $\delta : \tilde{M} \to G/H$ be any development defined as before. For each $\gamma \in \pi_1(M)$ a unique $a_\gamma \in G$ exists such that

$$\delta \circ w(\gamma) = L_{a_\gamma} \circ \delta.$$

This unique relation gives rise to the holonomy representation of the fundamental group:

$$\begin{aligned} \rho \ \ : \pi_1(M) \ &\to G \ . \\ \gamma \ \ &\mapsto a_\gamma \end{aligned}$$

The holonomy representations corresponding to different developments are conjugated in G.

We come now to the definition of almost Hermitian symmetric structures on a C^∞-manifold. One particularity of these structures is that they admit a uniquely determined normal Cartan connection. We will explain this in a short manner. The approach that we present here is due to the detailed work [CSS] on almost Hermitian symmetric structures and normal Cartan connections. The important point for us is that conformal structures belong to the class of almost Hermitian symmetric structures and, in particular, it exists a canonical normal conformal Cartan connection. We will discuss the conformal cases explicitly in the next sections.

First remember that a Lie algebra $\mathfrak{g}$ is called $|1|$-graded if

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

is a direct sum of subspaces such that

$$[\mathfrak{g}_i,\mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \quad \forall i,j \in \{-1,0,1\}.$$

Then it is known that the map $\mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_{-1})$ is the inclusion of a subalgebra and the Killing form of $\mathfrak{g}$ identifies $\mathfrak{g}_1$ as a $\mathfrak{g}_0$-module with the dual of $\mathfrak{g}_{-1}$ (see [Och70]).

Let G be a connected and semisimple Lie group with $|1|$-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. We denote by B the closed (parabolic) subgroup of G corresponding to the Lie subalgebra $\mathfrak{b} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ in $\mathfrak{g}$. Then there are uniquely defined closed subgroups B_0 and B_1 in B with Lie algebras $\mathfrak{g}_0$ resp. $\mathfrak{g}_1$. The exponential map to $\mathfrak{g}_1$ is a diffeomorphism onto B_1 and B_1 is a normal vector subgroup in B. Moreover, the group B is the semidirect product of its subgroups B_0 and B_1 (see [Och70] and [CSS]):

$$B = B_0 \rtimes B_1, \qquad B_0 \cong B/B_1.$$

Consider now a Lie group G', which isn't necessarily connected and is a covering group of G with covering map $\lambda' : G' \to G$. There exists a uniquely defined closed subgroup B' of G' such that the restriction $\lambda'|_{B'} : B' \to B \subset G$ is a group covering and $G'/B' \cong G/B$. The Lie group B' is isomorphic to the semidirect product $B'_0 \rtimes B_1$, where B'_0 is a closed subgroup of B', which covers B_0, and $B_1 = \exp \mathfrak{g}_1$ is the simply connected normal vector subgroup in B' generated by the Lie algebra $\mathfrak{g}_1$. Such a pair (G', B') is called a Hermitian symmetric geometry and the corresponding Klein model G'/B' is a Hermitian symmetric space.

Definition 1.3 *Let (G, B) be a Hermitian symmetric geometry, let M be a C^∞-manifold of dimension $n = \dim \mathfrak{g}_{-1}$ and let $B(M) \to M$ be a B-principal fiber bundle on M equipped with a 1-form*

$$\theta = \theta_{-1} \oplus \theta_0 \in \Omega^1(B(M), \mathfrak{g}_{-1} \oplus \mathfrak{g}_0)$$

such that

(1) $\theta_{-1}(X) = 0$ iff $X \in TB(M)$ is vertical

(2) $\theta_0(A^* + B^*) = A$ for all $A \in \mathfrak{g}_0$, $B \in \mathfrak{g}_1$

(3) $R_b^*\theta = Ad(b^{-1})\theta$ for all $b \in B$, where Ad means the restriction of the adjoint action to the vector space $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$.

The form θ is called the soldering form and $(B(M), \theta)$ is called an almost Hermitian symmetric structure on M of type (G, B).

We use the following convention. Let P_1 and P_2 be principal fiber bundles over a base space M with structure groups G_1 resp. G_2. If $\bar{\lambda} : P_1 \to P_2$ is a principal fiber bundle morphism over the idenity of M associated to a homomorphism $\lambda : G_1 \to G_2$, which is a covering of a subgroup in G_2, we call the bundle P_1 a G_1-reduction of P_2.

A (G, B)-structure $(B(M), \theta)$ defined as above may be understood as a 'second order structure' on M (comp. [CSS] II.1.8.), whereas the induced B_0-principal fiber bundle $B_0(M) := B(M)/B_1$ with induced soldering form θ_{-1} is a B_0-reduction of the first order frame bundle $Gl^{(1)}(M)$ on M. There is also a way of constructing a (G, B)-structure from a B_0-reduction of $Gl^{(1)}(M)$ (comp. [CSS] II.1.). But it isn't true in general that an almost Hermitian symmetric structrue $(B(M), \theta)$

is uniquely determined by its first order reduction $B_0(M)$, since $B_0(M)$ need not contain any geometric information. This happens in the case of projective geometry, where the first order reduction $B_0(M)$ is isomorphic to the linear frame bundle $Gl^{(1)}(M)$ itself (comp. [Kob72], p. 143).

Let $(B(M), \theta)$ be an almost Hermitian symmetric structure and $B_0(M)$ its first order reduction. There exist B_0-equivariant sections $\sigma \in \Gamma(B_0(M); B(M))$. For such a section σ the induced 1-form

$$\sigma^*\theta = \sigma^*\theta_{-1} + \sigma^*\theta_0$$

is a Cartan connection and the $\mathfrak{g}_0$-valued component $\sigma^*\theta_0$ is a connection on $B_0(M)$. Moreover, there exists a uniquely defined Cartan connection $\omega^\sigma = \theta_{-1} + \theta_0 + \omega_1^\sigma$ on $B(M)$ satisfying $\omega_1^\sigma|_{T\sigma(B_0(M))} = 0$ (see [CSS] I.3. and II.1.7.).

The curvature function $\kappa \in C^\infty(B(M), \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g})$ of an arbitrary Cartan connection ω on $B(M)$ is defined by

$$\kappa(u)(X,Y) := \Omega(\omega^{-1}(X), \omega^{-1}(Y))(u), \quad X, Y \in \mathfrak{g}_{-1},\ u \in B(M).$$

The map $ad : \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_{-1})$ is injective and therefore the $\mathfrak{g}_0$-valued component κ_0 of κ can be viewed as a function on $B(M)$ with values in $\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_1^*$.

Definition 1.4 *(1) An admissable connection*

$$\omega = \omega_{-1} + \omega_0 + \omega_1 \in \Omega^1(B(M), \mathfrak{g})$$

on an almost Hermitian symmetric structure $(B(M), \theta)$ is a Cartan connection with $\omega_{-1} = \theta_{-1}$ and $\omega_0 = \theta_0$.

(2) A normal Cartan connection $\omega \in \Omega^1(B(M), \mathfrak{g})$ is an admissable connection, which satisfies

$$tr\kappa_0(X,Y) := \sum_{i=1}^{\dim \mathfrak{g}_{-1}} \kappa_0(e_i, X)(Y)(e^i) = 0,$$

where e_i is a basis of $\mathfrak{g}_{-1}$ and e^i is its dual basis of $\mathfrak{g}_1$.

Two admissable Cartan connections $\omega, \bar\omega$ on $(B(M), \theta)$ differ only in the $\mathfrak{g}_1$-component and it exists a deformation tensor $\Gamma \in C^\infty(B(M), \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_1)$ such that

$$\bar\omega - \omega = \Gamma \circ \theta_{-1}.$$

The tensor Γ on $B(M)$ is always the pullback of a tensor on the base space M (see [CSS] I.3.10.).

It is proved that on every almost Hermitian symmetric structure $(B(M), \theta) \to M$, there exists a canonical normal Cartan connection ω_N ([CSS] II., [Kob72]). Moreover, for every B_0-equivariant section $\sigma \in \Gamma(B_0(M); B(M))$ the difference between the admissable connection ω^σ and the canonical Cartan connection ω_N is described by the deformation tensor Γ^σ and the tensor Γ^σ can be expressed by a universal formula in terms of the curvature tensor of the connection $\sigma^*\theta_0$ on $B_0(M)$. If the canonical Cartan connection ω_N on $B(M)$ is flat then the Cartan geometry $(B(M), \omega_N) \to M$ is locally isomorphic to $(G, \omega_G) \to G/B$.

2 The Conformal Spin Spaces

After we have defined Klein geometries in general in the last section, we present in this section the standard models of conformal and conformal spin geometry. We introduce these models first in a geometric manner and give then the group-theoretical description.

Let $\mathbb{R}^n_k := (\mathbb{R}^n, \langle\ ,\ \rangle^n_k), n \geq 3$ and $k \in \{0, \ldots, n\}$, denote the n-dimensional Euclidean space of index k with scalar product

$$\langle x, y\rangle^n_k := -\sum_{i=1}^{k} x_i y_i + \sum_{i=k+1}^{n} x_i y_i$$

and let Cl^n_k be the Clifford algebra of $\mathbb{R}^n_k$. The group of (pseudo)-orthogonal transformations on $\mathbb{R}^n_k$ is denoted by $O(n,k)$ and the subgroup

$$SO(n,k) := \{A \in O(n,k) \mid \det A = 1\}$$

denotes the special orthogonal group. The spin group $Spin(n,k)$ is defined in the Clifford algebra Cl^n_k by

$$Spin(n,k) := \{x_1 \cdot \ldots \cdot x_{2l} \in Cl^n_k \mid x_i \in \mathbb{R}^n_k, \langle x_i, x_i\rangle^n_k = \pm 1, i = 1, \ldots, 2l\}$$

and

$$\begin{array}{rccc} \lambda : & Spin(n,k) & \rightarrow & SO(n,k) \\ & s & \mapsto & (x \in \mathbb{R}^n_k \mapsto sxs^{-1} \in \mathbb{R}^n_k) \end{array}$$

is a smooth two-fold group-covering. With $\mathbb{R}^{n+2}_{k+1} := (\mathbb{R}^{n+2}, (\ ,\ )^{n+2}_{k+1})$ we denote the pseudo-Euclidean space of dimension $n+2$ and index $k+1$, but this time equipped with the scalar product

$$(x, y)^{n+2}_{k+1} := x_0 y_{n+1} + x_{n+1} y_0 - \sum_{i=1}^{k} x_i y_i + \sum_{i=k+1}^{n} x_i y_i.$$

Furthermore, we define the space

$$Q^n_k := \{[x] \in \mathbb{R}P^{n+2} \mid x \in \mathbb{R}^{n+2}_{k+1} \setminus \{0\},\ (x,x)^{n+2}_{k+1} = 0\},$$

which is a regular quadric in the projective space $\mathbb{R}P^{n+2}$. The regular quadric Q^n_k is two-fold covered by the set $\hat{Q}^n_k$ of time-oriented null directions in $\mathbb{R}^{n+2}_{k+1}$ and the space $\hat{Q}^n_k$ is embedded in $\mathbb{R}^{n+2}_{k+1}$ by

$$\begin{array}{rccc} i : & \hat{Q}^n_k & \hookrightarrow & \mathbb{R}^{n+2}_{k+1} \quad , \\ & \mathbb{R}_+ \cdot x & \mapsto & \sqrt{\dfrac{2}{\langle x, x\rangle^{n+2}_0}} \cdot x \end{array}$$

where $\langle\ ,\ \rangle^{n+2}_0$ is the Euclidean scalar product. One can see that the space $\hat{Q}^n_k$ is homeomorphic to $S^k \times S^{n-k}$, since the image $i(\hat{Q}^n_k)$ is the Carthesian product of the unit spheres in the semi-Riemannian product $\mathbb{R}^{n+2}_{k+1} \cong \mathbb{R}^{k+1}_{k+1} \times \mathbb{R}^{n-k+1}_0$. Notice that $\hat{Q}^n_0$ isn't connected and $\hat{Q}^n_1$ isn't simply connected.

We denote by $\hat{c}$ the conformal structure on $\hat{Q}^n_k$ that arises from the induced metric $i^*(\ ,\ )^{n+2}_{k+1}$. This conformal structure $\hat{c}$ is flat, since the induced metric is conformally flat. The conformally flat structure $\hat{c}$ projects to a conformally flat structure c on the quadric Q^n_k. For $k \neq 0, n$ we denote the arising conformel spaces by

$$C^n_k := (Q^n_k, c) \text{ and } \hat{C}^n_k := (\hat{Q}^n_k, \hat{c}).$$

In case that $k = 0, n$ we use the exceptional notation

$$\hat{C}^n_0 = C^n_0 := (Q^n_0, c),$$

since $\hat{Q}^n_0$ isn't connected. The conformally flat space C^n_k is the conformal compactification of $\mathbb{R}^n_k$ and is called the (pseudo)-Möbius sphere of dimension n and index k (comp. [Sch97]). The space $\hat{C}^n_k$ is a conformal spin manifold, since the normal bundle of $i(Q^n_k)$ in $\mathbb{R}^{n+2}_{k+1}$ admits a globally defined orthonormal frame field (comp. section 4). We call the model $\hat{C}^n_k$ the conformal spin space of dimension n and index k. The space C^n_k isn't in general conformally spin (comp. section 4).

We want to describe the models $\hat{C}^n_k$ as homogenous spaces. Let us denote

$$GSO(n,k) := \begin{cases} SO(n+2,k+1) : k \neq 0, n \\ SO_o(n+2,k+1) : k = 0, n \end{cases}.$$

The group $GSO(n,k)$ acts naturally on $\hat{C}^n_k$ through its embedding in $\mathbb{R}^{n+2}_{k+1}$. This action is conformal, orientation-preserving and transitive. With respect to some point $e \in \hat{C}^n_k$ the isotropy group of the $GSO(n,k)$-action on $\hat{C}^n_k$ is the subgroup

$$BSO(n,k) := \left\{ \begin{pmatrix} a^{-1} & q & b \\ 0 & A & r \\ 0 & 0 & a \end{pmatrix} \mid A \in SO(n,k), a \in \mathbb{R}_+, q \in \mathbb{R}^{n*} \right\}$$

of $GSO(n,k)$, where $I^n_k := \begin{pmatrix} -I_k & 0 \\ 0 & I_{n-k} \end{pmatrix}$, $r = -aAI^n_k q^T$ and $b = -\frac{a}{2}\langle q^T, q^T \rangle^n_k$, and it holds $\hat{C}^n_k \cong GSO(n,k)/BSO(n,k)$. The kernel of the linear isotropy representation of $BSO(n,k)$ on the tangent space $T_e\hat{C}^n_k$ is the vector group

$$B_1(n,k) := \left\{ \begin{pmatrix} 1 & q & b \\ 0 & I & -I^n_k q^T \\ 0 & 0 & 1 \end{pmatrix} \in GSO(n,k) \mid q \in \mathbb{R}^{n*}, b = -\frac{1}{2}\langle q^T, q^T \rangle^n_k \right\},$$

which is a normal subgroup of $BSO(n,k)$. The group $BSO(n,k)/B_1(n,k)$ is isomorphic to the conformal group $CSO(n,k) := \mathbb{R}_+ \times SO(n,k)$, which is embedded in $GSO(n,k)$ by

$$\begin{array}{rccc} i_c : & CSO(n,k) & \hookrightarrow & GSO(n,k) \\ & aA & \mapsto & \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix} \end{array}.$$

The Lie algebra $\mathfrak{so}(n+2,k+1)$ is semisimple and $|1|$-graded:

$$\mathfrak{so}(n+2,k+1) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where

$$\mathfrak{g}_{-1} := \left\{ \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ 0 & -p^T I^n_k & 0 \end{pmatrix} \mid p \in \mathbb{R}^n \right\},$$

$$\mathfrak{g}_0 := \left\{ \begin{pmatrix} -a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix} \mid A \in \mathfrak{so}(n,k), a \in \mathbb{R} \right\},$$

$$\mathfrak{g}_1 := \left\{ \begin{pmatrix} 0 & q & 0 \\ 0 & 0 & -I^n_k q^T \\ 0 & 0 & 0 \end{pmatrix} \mid q \in \mathbb{R}^{n^*} \right\}.$$

The subalgebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the Lie algebra of $BSO(n,k)$, the subalgebra $\mathfrak{g}_1$ is the Lie algebra of $B_1(n,k)$ and $\mathfrak{g}_0 \cong \mathfrak{cso}(n,k)$. With the definitions in the previous section we see that $(GSO(n,k), BSO(n,k))$ is a Hermitian symmetric geometry and $\hat{C}^n_k \cong GSO(n,k)/BSO(n,k)$ is a Hermitian symmetric space for every dimension n and index k. In particular, it holds that the $\mathfrak{g}_0$-modules $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are dual to each other with respect to the Killing form on $\mathfrak{so}(n+2,k+1)$ and that

$$BSO(n,k) = CSO(n,k) \rtimes B_1(n,k)$$

is a semi-direct product.

Since the model $\hat{C}^n_k$ is conformally spin, it is usefull to give a spin description of the Hermitian symmetric geometry $(GSO(n,k), BSO(n,k))$. We denote

$$GSpin(n,k) := \begin{cases} Spin(n+2,k+1) : k \neq 0,n \\ Spin_o(n+2,k+1) : k = 0,n \end{cases}.$$

The group $GSpin(n,k)$ acts on $\hat{C}^n_k$ via the group-covering $\lambda : GSpin(n,k) \to GSO(n,k)$. There exists a uniquely defined closed subgroup $BSpin(n,k)$ of $GSpin(n,k)$, which covers $BSO(n,k)$ in $GSO(n,k)$ by λ, such that

$$\hat{C}^n_k \cong GSpin(n,k)/BSpin(n,k).$$

The pair $(GSpin(n,k), BSpin(n,k))$ is a Hermitian symmetric geometry. There is a uniquely defined embedding

$$i_{cs} : CSpin(n,k) \hookrightarrow GSpin(n,k)$$

of the conformal spin group $CSpin(n,k) := \mathbb{R}_+ \times Spin(n,k)$ such that $\lambda \circ i_{cs} = i_c \circ \lambda$. Then we have

$$BSpin(n,k) = CSpin(n,k) \rtimes B_1(n,k),$$

where in this case $B_1(n,k)$ should be understood as the simply connected normal vector subgroup in $BSpin(n,k)$ corresponding to the subalgebra $\lambda_*^{-1}\mathfrak{g}_1 \subset \mathfrak{spin}(n+2,k+1)$. The differential of λ is given by

$$\begin{array}{lll}
\lambda_* : & \mathfrak{spin}(n+2,k+1) & \to \mathfrak{so}(n+2,k+1)\ , \\
& \frac{1}{2}f_{n+1}f_i & \mapsto \begin{pmatrix} 0 & 0 & 0 \\ e_i & 0 & 0 \\ 0 & -e_i^T I_k^n & 0 \end{pmatrix} \\
& \frac{1}{2}f_0 f_i & \mapsto \begin{pmatrix} 0 & 0 & 0 \\ e_i & 0 & 0 \\ 0 & -e_i^T I_k^n & 0 \end{pmatrix} \\
& \frac{1}{2}f_0 f_i & \mapsto \begin{pmatrix} 0 & -e_i^T I_k^n & 0 \\ 0 & 0 & e_i \\ 0 & 0 & 0 \end{pmatrix} \\
& \frac{1}{2}f_0 f_{n+1} & \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
& \frac{1}{2}f_i f_j & \mapsto \quad E_{ij}
\end{array}$$

where $\{f_i : i = 0, \dots, n+1\}$ is the standard basis of $\mathbb{R}^{n+2}$, $\{e_i : i = 1, \dots, n\}$ is the standard basis of $\mathbb{R}^n$ and E_{ij} is the standard basis of $\mathfrak{so}(n,k)$. It holds

$$\begin{aligned}
\lambda_*^{-1}(\mathfrak{g}_{-1}) &= Span\{f_{n+1}f_i : i = 1, \dots, n\}, \\
\lambda_*^{-1}(\mathfrak{g}_0) &= Span\{f_i f_j\} \oplus \mathbb{R} f_0 f_{n+1} \cong \mathfrak{cspin}(n,k), \\
\lambda_*^{-1}(\mathfrak{g}_1) &= Span\{f_0 f_i : i = 1, \dots, n\}.
\end{aligned}$$

3 Twistor Spinors and Normal Conformal Cartan Connections

It is well-known that a twistor spinor $\psi \in \Gamma(S)$ on a semi-Riemannian spin manifold (M,g) may be interpreted as a parallel section in the two-fold spinor bundle $S \oplus S$ with respect to a certain covariant derivative $\nabla^{S\oplus S}$ (see [BFGK91]). It was already proved in [Fri76] that on a space-time the 'twistor connection' is induced by the canonical normal Cartan connection of conformal geometry. We give here a prove of this fact on semi-Riemannian spin manifolds with arbitrary signature using our approach to normal Cartan connections due to [CSS].

Let (M^n, c) be an oriented C^∞-manifold of dimension n with conformal structure c of index k, that is a class of conformally equivalent (pseudo)-Riemannian metrics with index k on M^n. The conformal structure c is equivalently described by the conformal frame bundle $CSO(M)$, which is a $CSO(n,k)$-principal fiber subbundle of the first order linear frame bundle $Gl^{(1)}(M)$. The first order conformal structure $CSO(M)$ gives rise via the canonical first prolongation to the almost Hermitian symmetric structure $(BSO(M), \theta)$ of type $(GSO(n,k), BSO(n,k))$, where the soldering form θ has no torsion, i.e.

$$T := d\theta_{-1} - [\theta_{-1}, \theta_0] = 0.$$

This implies that $BSO(M)$ is a subbundle of the second order frame bundle $Gl^{(2)}(M)$ over M and θ is the restriction of the canonical form on $Gl^{(2)}(M)$ (comp. [CSS], I.6.2., II.1.). On the other hand, let $BSO(M)$ be a $BSO(n,k)$-subbundle of $Gl^{(2)}(M)$ over a C^∞-manifold M and θ the restriction of the canonical soldering form to $BSO(M)$. Then the induced $CSO(n,k)$-principal fiber bundle $BSO(M)/B_1(n,k)$ can be interpreted in a natural way as subbundle of

$Gl^{(1)}(M)$ (comp. [CSS] I.3.). This leads to

Definition 3.1 *(1) A conformal structure of index k on an oriented C^∞-manifold M^n of dimension n is an almost Hermitian symmetric structure $(BSO(M),\theta)$ of type $(GSO(n,k), BSO(n,k))$, whose soldering form θ has vanishing torsion.*

(2) A conformal spin structure on M^n of index k is a $BSpin(n,k)$-reduction $(BSpin(M), f)$ of a conformal structure $(BSO(M),\theta)$, i.e. it holds

$$f(b)\cdot\lambda(q) = f(b\cdot q) \quad \forall b \in BSpin(M),\ q \in BSpin(n,k),$$

where $f : BSpin(M) \to BSO(M)$ is a two-fold covering map. The 1-form $f^*\theta$ is the soldering form on $BSpin(M)$.

A conformal spin structure $(BSpin(M), f)$ on M^n in the sense of Definition 3.1 induces a $CSpin(n,k)$-reduction of the first order conformal frame bundle $CSO(M)$:

$$CSpin(M) := BSpin(M)/B_1(n,k) \xrightarrow{f'} CSO(M),$$

where f' is induced by F.

For the rest of this section we assume (M^n, c) to be a conformal spin manifold of index k and by $(BSpin(M), f)$ we denote a conformal spin structure on M. We want to define conformal spinor bundles on (M,c). So let $\Delta_k^n \cong \mathbb{C}^{[\frac{n}{2}]}$ denote the standard (complex) spinor modul with $Spin(n,k)-$action

$$\kappa : Spin(n,k) \times \Delta_k^n \to \Delta_k^n$$

and Clifford multiplication

$$\mu : \mathbb{R}_k^n \otimes \Delta_k^n \to \Delta_k^n$$

(comp. [Baum81]). With κ_ℓ we denote the $CSpin(n,k)$-representation

$$\begin{array}{llll} \kappa_\ell : & CSpin(n,k) \times \Delta_k^n & \to & \Delta_k^n \\ & ((t,s),v) & \mapsto & t^\ell \cdot sv \end{array}$$

of weight $\ell \in \mathbb{R}$. Consider now the spinor modul Δ_{k+1}^{n+2} and the null vectors $f_0, f_{n+1} \in \mathbb{R}_{k+1}^{n+2}$. Let

$$\begin{array}{lll} Ann(f_0) & := \{v \in \Delta_{k+1}^{n+2} \mid f_0 \cdot v = 0\} & \text{and} \\ Ann(f_{n+1}) & := \{v \in \Delta_{k+1}^{n+2} \mid f_{n+1} \cdot v = 0\} & \end{array}$$

denote the annulation spaces with respect to Clifford multiplication by f_0 resp. f_{n+1}. Since every $v \in \Delta_{k+1}^{n+2}$ decomposes to $f_0 w + f_1 w$ for some $w \in \Delta_{k+1}^{n+2}$ and

$$x f_i = -f_i x \quad \forall x \in \mathbb{R}_k^n \cong Span\{f_1, \ldots, f_n\} \subset \mathbb{R}_{k+1}^{n+2}, \quad i = 0, n+1,$$

we can conclude that the annulation spaces $Ann(f_0)$ and $Ann(f_{n+1})$ are isomorphic to Δ_k^n as $Spin(n,k)$-representations:

$$\Delta_{k+1}^{n+2}|_{Spin(n,k)} = Ann(f_0) \oplus Ann(f_{n+1}) \cong \Delta_k^n \oplus \Delta_k^n.$$

We fix an isomorphism

$$\alpha : Ann(f_0) \xrightarrow{\sim} \Delta_k^n$$

of $Spin(n,k)$-representations and define the corresponding isomorphism

$$\begin{array}{cccc} \beta : & Ann(f_{n+1}) & \xrightarrow{\sim} & \Delta_k^n \quad . \\ & v & \mapsto & -\frac{1}{n}\alpha(f_0 v) \end{array}$$

Then it holds

$$\begin{aligned} \beta(\tfrac{1}{2} f_{n+1} \cdot x \cdot \alpha^{-1}(v)) &= \tfrac{1}{n} x \cdot v \quad \forall x \in \mathbb{R}^n_k,\ v \in \Delta^n_k, \\ \alpha(\tfrac{1}{2} f_0 \cdot x \cdot \beta^{-1}(w)) &= \tfrac{n}{2} x \cdot w \quad \forall x \in \mathbb{R}^n_k,\ w \in \Delta^n_k. \end{aligned}$$

Moreover, since

$$v f_0 = t^{-1} f_0 v \text{ and } v f_{n+1} = t f_{n+1} v \quad \forall v = i_{cs}((t,s)) \in GSpin(n,k),$$

we have an isomorphism of $CSpin(n,k)$-representations:

$$(\Delta^{n+2}_{k+1}, \kappa|_{CSpin(n,k)}) \cong (\Delta^n_k, \kappa_{-1}) \oplus (\Delta^n_k, \kappa_1).$$

Let

$$GSpin(M) := BSpin(M) \times_{BSpin(n,k)} GSpin(n,k)$$

denote the enlarged $GSpin(n,k)$-principal fiber bundle. We define then

$$\begin{aligned} E &:= GSpin(M) \times_\kappa \Delta^{n+2}_{k+1}, \\ T &:= GSpin(M) \times_{\lambda \circ \kappa} \mathbb{R}^{n+2}_{k+1}, \\ S^c_\ell &:= CSpin(M) \times_{\kappa_\ell} \Delta^n_k, \qquad \ell \in \mathbb{R}. \end{aligned}$$

On $BSO(M)$ exists a canonical normal connection, which we denote by ω_{NC}. The canonical normal connection on $BSpin(M)$ is then given by

$$\omega_{NCS} := f^* \omega_{NC}.$$

Furthermore, we have the connection $\bar{\omega}_{NCS}$ on the enlarged bundle $GSpin(M)$ and this connection induces a derivative ∇^E on the conformal spinor bundle E.

Let $\sigma \in \Gamma(CSpin(M); BSpin(M))$ be a $CSpin(n,k)$-equivariant section. The section σ induces an isomorphism

$$\begin{array}{ccc} S^c_{-1} \oplus S^c_1 \cong CSpin(M) \times_{\kappa \circ i_{cs}} \Delta^{n+2}_{k+1} & \to & E \quad . \\ [l,v] & \mapsto & [[\sigma(l), e], v] \end{array}$$

It is known that in conformal geometry there exists a bijective correspondence between $CSO(n,k)$-equivariant sections and torsion-free connections (Weyl structures) on the conformal frame bundle $CSO(M)$. So let us fix a metric g in the conformal class c on M. With $(Spin(M,g), f')$ we denote the corresponding semi-Riemannian spin structure on (M, g). The usual spinor bundle on (M, g) is denoted by

$$S^g := Spin(M,g) \times_\kappa \Delta_k^n$$

and ∇^{S^g} denotes the spinor derivative on S^g. The Levi-Civita connection to g on the bundle of orthonormal frames $SO(M, g)$ induces a torsion free connection ω_g on the $CSpin(n, k)$-principal bundle $CSpin(M)$. But then exists a uniquely determined $CSpin(n, k)$-equivariant section $\sigma^g \in \Gamma(CSpin(M); BSpin(M))$ such that

$$\sigma^{g^*} \circ f^*\theta_0 = \omega_g,$$

where $f^*\theta_0$ is the $\mathfrak{g}_0$-valued component of the soldering form $f^*\theta$ on $BSpin(M)$ (comp. [CSS] II.1.7.). It holds

$$\omega_{NCS} = \omega^{\sigma^g} - \Gamma^{\sigma^g} \circ f^*\theta_{-1},$$

where $\omega^{\sigma^g} = f^*\theta + \omega_1^{\sigma^g}$ is the admissible connection to the section σ^g and Γ^{σ^g} is the pullback of the Schouten tensor

$$L := \frac{1}{n-2}\left(\frac{s}{2(n-1)} - \mathrm{Ric}\right)$$

to $BSpin(M)$ (comp. [CSS] I.6.4.).

With the help of the section σ^g to $g \in c$ we are able to reduce the structure group of the bundles E and T to $Spin(n, k)$:

$$\begin{array}{rcc} Spin(M,g) \times_{\kappa\circ i_{cs}(Spin(n,k))} \Delta_{k+1}^{n+2} & \cong & E, \\ [l,v] & \mapsto & [[\sigma^g(l),e],v] \\ Spin(M,g) \times_{\lambda\circ\kappa\circ i_{cs}} \mathbb{R}_{k+1}^{n+2} & \cong & T. \\ [l,x] & \mapsto & [[\sigma^g(l),e],v] \end{array}$$

There are canonical sections in the bundle $T \to M$ with respect to g:

$$\begin{array}{rcl} \zeta_0^g & := & [[\sigma^g(l),e],f_0] \in \Gamma(T), \\ \zeta_{n+1}^g & := & [[\sigma^g(l),e],f_{n+1}] \in \Gamma(T). \end{array}$$

We have a decomposition of the bundle E into the annulation spaces:

$$\begin{array}{rcl} E & = & Ann(\zeta_0^g) \oplus Ann(\zeta_{n+1}^g), \\ [[\sigma^g(l),e],v] & = & [[\sigma^g(l),e],f_0 w] \oplus [[\sigma^g(l),e],f_{n+1}w] \end{array}$$

where $v = f_0 w + f_{n+1} w \in \Delta_{k+1}^{n+2}$. Moreover, there is the isomporhism

$$\begin{array}{rccc} \Phi^g : & E & \cong & S^g \oplus S^g \quad . \\ & [[\sigma^g(l),e],v] & \mapsto & [l,\alpha(f_0 w)] \oplus [l,\beta(f_{n+1}w)] \end{array}$$

Let $\tilde{g} = \rho^2 \cdot g \in c$, $\rho : M \to \mathbb{R}_+$, be a conformally equivalent metric to g. The corresponding $CSpin(n, k)$-equivariant section $\sigma^{\tilde{g}} \in \Gamma(CSpin(M); BSpin(M))$ is related to σ^g by

$$\sigma^{\tilde{g}}(l) = \sigma^g(l) \cdot \exp(\mathrm{grad}^g(\rho)(l)), \quad l \in CSpin(M),$$

where $f(l) = (f(l)_1, \dots, f(l)_n) \in CSO(M)$ and

$$\mathrm{grad}^g(\rho)(l) := \frac{1}{2} f_0 \cdot \sum_{i=1}^{n} \varepsilon_i d\rho(f(l)_i) f_i \in \lambda_*^{-1}(\mathfrak{g}_1) \in \mathfrak{spin}(n+2, k+1).$$

We obtain an isomorphism via the bundle E of the two-fold spin bundles:

$$\begin{array}{cccl} \Xi_g^{\tilde{g}}: & S^g \oplus S^g & \xrightarrow{\sim} & S^{\tilde{g}} \oplus S^{\tilde{g}} \quad . \\ & [l, \alpha(f_0 w)] \oplus [l, \beta(f_{n+1} w)] & \mapsto & \sigma^{-\frac{1}{2}} \cdot [\frac{1}{\sigma} l, \alpha(f_0 w)] + \frac{n}{2} \sigma^{-\frac{1}{2}} \mathrm{grad}^{\tilde{g}}(\sigma) \cdot [\frac{1}{\sigma} l, \beta(f_{n+1} w)] \\ & & & \oplus \ \sigma^{\frac{1}{2}} \cdot [\frac{1}{\sigma} l, \beta(f_{n+1} w)] \end{array}$$

After we have defined the appropriate bundles, we come now to the definition of twistor spinors. The twistor operator $\mathcal{T}^g$, which depends on the choosen metric $g \in c$ and acts on $\Gamma(S^g)$, is the first order differential operator defined by

$$\mathcal{T}^g := proj_{ker\mu} \circ \nabla^S : \Gamma(S^g) \to \Gamma(ker\mu),$$

where $\ker \mu$ is the kernel of the Clifford multiplication $\mu : TM \otimes S^g \to S^g$. A spinor field $\psi \in \Gamma(S^g)$, which lies in the kernel of $\mathcal{T}^g$, i.e. $\mathcal{T}^g \psi = 0$, is called a twistor spinor on (M, g). The Dirac operator on S^g is defined by the composition

$$D^g : \Gamma(S) \xrightarrow{\nabla^{S^g}} \Gamma(TM^* \otimes S) \xrightarrow{g} \Gamma(TM \otimes S) \xrightarrow{\mu} \Gamma(S).$$

Proposition 3.1 ([BFGK91]) *(1) For a spinor field $\psi \in \Gamma(S^g)$ the following conditions are equivalent:*

i) ψ is a twistor spinor.

ii) ψ satisfies the so-called twistor-equation

$$\nabla_X^{S^g} \psi + \frac{1}{n} X \cdot D^g \psi = 0 \quad \forall X \in TM.$$

iii) There exists a spinor field $\varphi \in \Gamma(S^g)$ such that

$$\varphi = g(X, X) X \cdot \nabla_X^S \psi$$

for all vectors $X \in TM$ with $\mid g(X, X) \mid = 1$.

(2) Let $\psi \in \Gamma(S^g)$ be a twistor spinor. Then

$$\nabla_X^{S^g} D^g \psi = \frac{n}{2} L(X) \cdot \psi$$

for all vector fields $X \in TM$.

Let us consider a conformally equivalent metric $\tilde{g} = \rho^2 \cdot g \in c$ and let

$$\begin{array}{cccl} \sim: & S^g & \to & S^{\tilde{g}} \\ & \psi = [s, v] & \mapsto & \tilde{\psi} = [\frac{1}{\rho} s, v] \end{array}$$

be a bundle isomorphism. The conformally changed twistor and Dirac operators are related by

$$\begin{aligned} D^{\tilde{g}}\tilde{\varphi} &= \rho^{-\frac{n+1}{2}} \cdot (D^g(\rho^{\frac{n-1}{2}}\varphi))^{\sim}, \\ \mathcal{T}^{\tilde{g}}\varphi &= \rho^{-\frac{1}{2}} \cdot (\mathcal{T}^g(\rho^{-\frac{1}{2}}\varphi))^{\sim}. \end{aligned}$$

In particular, the kernel of $\mathcal{T}^g$ is conformally invariant and we have the following conformally invariant definition of twistor spinors on conformal spin manifolds. A spinor field $\psi \in \Gamma(S^c_{\frac{1}{2}}) \cong \Gamma(S^g)$ is called twistor spinor on (M,c) if $\mathcal{T}^g\psi = 0$ for some $g \in c$.

Consider the bundle $S^g \oplus S^g$ on (M,g). On $S^g \oplus S^g$ there exists a covariant derivative $\nabla^{S^g\oplus S^g}$ defined by

$$\nabla^{S^g\oplus S^g}_X := \begin{pmatrix} \nabla^{S^g}_X & \frac{1}{n}X \\ -\frac{n}{2}L(X) & \nabla^{S^g}_X \end{pmatrix}.$$

It is proved in [BFGK91] that $\psi \in \Gamma(S^g)$ is a twistor spinor iff $(\psi, D^g\psi) \in \Gamma(S^g \oplus S^g)$ is parallel with respect to $\nabla^{S^g\oplus S^g}$. Let us denote by

$$proj^g_i : E \to Ann(\zeta^g_i), \quad i = 0, n+1,$$

the projections to the annulation spaces.

Theorem 3.2 *Let $\varphi^E \in \Gamma(E)$ be a spinor field on (M,c) and g a metric in the conformal class c on M. The spinor field φ^E is parallel with respect to the normal Cartan connection, $\nabla^E\varphi^E = 0$, if and only if*

$$\psi := \Phi^g \circ proj^g_{n+1}(\varphi^E)$$

is a twistor spinor on (M,g) and

$$D^g\psi = \Phi^g \circ proj^g_0(\varphi^E).$$

Proof. It is

$$\varphi^E = [[\sigma^g(l), e], v] + [[\sigma^g(l), e], w]$$

for some functions $v : M \to Ann(f_0)$ and $w : M \to Ann(f_{n+1})$. Let

$$\varrho^g : U \subset M \to Spin(M,g)$$

be a local section and let denote

$$\varrho^c := \sigma^g \circ \varrho^g : U \to GSpin(M), \qquad f' \circ \varrho^g(x) = (e_1(x), \dots, e_n(x)) \in SO(M,g),\ x \in U.$$

Remember that ${\sigma^g}^*\theta_0$ is the Levi-Civita connection to g on $CSpin(M)$. We have locally

$$\begin{aligned}
\Phi^g(\nabla^E_X\varphi^E) &= \Phi^g[\varrho^c, d(v+w)+\kappa_*\circ\omega_{NCS}\circ d\varrho^c(X)(v+w)] \\
&= \Phi^g[\varrho^c, dv+\kappa_*\circ\theta_0\circ d\varrho^c(X)v]+\Phi^g[\varrho^c, dw+\kappa_*\circ\theta_0\circ d\varrho^c(X)w] \\
&+ \Phi^g[\varrho^c, \kappa_*\circ\theta_{-1}\circ d\varrho^c(X)v]-\Phi^g[\varrho^c,\kappa_*\circ\Gamma^{\sigma^g}\circ\theta_{-1}\circ d\varrho^c(X)w] \\
&= \nabla^{S^g}_X(\Phi^g\circ proj^g_{n+1}(\varphi^E))+\Phi^g[\varrho^c,\frac{1}{2}f_{n+1}\cdot\big(\sum_{i=1}^n\varepsilon_i g(X,e_i)f_i\big)\cdot v] \\
&\oplus \nabla^{S^g}_X(\Phi^g\circ proj^g_0(\varphi^E))+\Phi^g[\varrho^c,-\frac{1}{2}f_0\cdot\big(\sum_{i=1}^n\varepsilon_i g(L(X),e_i)f_i\big)\cdot w] \\
&= \nabla^{S^g}_X(\Phi^g\circ proj^g_{n+1}(\varphi^E))+\frac{1}{n}X\cdot\Phi^g\circ proj^g_0(\varphi^E) \\
&\oplus \nabla^{S^g}_X(\Phi^g\circ proj^g_0(\varphi^E))-\frac{n}{2}L(X)\cdot\Phi^g\circ proj^g_{n+1}(\varphi^E).
\end{aligned}$$

It follows that $\nabla^E_X\varphi^E=0$ iff

$$\begin{aligned}
&\nabla^{S^g}_X(\Phi^g\circ proj^g_{n+1}(\varphi^E))+\tfrac{1}{n}X\cdot\Phi^g\circ proj^g_0(\varphi^E)=0 \quad \text{and} \\
&\nabla^{S^g}_X(\Phi^g\circ proj^g_0(\varphi^E))-\tfrac{n}{2}L(X)\cdot\Phi^g\circ proj^g_{n+1}(\varphi^E)=0,
\end{aligned}$$

i.e. $\nabla^E_X\varphi^E=0$ if and only if

$$\Phi^g\circ proj^g_{n+1}(\varphi^E)\in\Gamma(S^g)$$

is a twistor spinor and

$$D^g(\Phi^g\circ proj^g_{n+1}(\varphi^E))=\Phi^g\circ proj^g_0(\varphi^E)$$

(comp. Proposition 3.1). □

Let us consider a conformally equivalent metric $\tilde{g}=\rho^2\cdot g\in c$. If $(\psi, D^g\psi)$ is parallel in $S^g\oplus S^g$ with respect to $\nabla^{S^g\oplus S^g}$ then the pair

$$Q:=(\rho^{\frac{1}{2}}\tilde{\psi},\rho^{-\frac{1}{2}}\widetilde{D^g\psi}+\frac{n}{2}\rho^{\frac{1}{2}}(\mathrm{grad}^g(\rho)\cdot\psi)\tilde{}\ )$$

is parallel in $S^{\tilde{g}}\oplus S^{\tilde{g}}$ with respect to $\nabla^{S^{\tilde{g}}\oplus S^{\tilde{g}}}$ and in fact

$$\Xi^{\tilde{g}}_g(\psi, D^g\psi)=Q.$$

4 Twistor Spinors on Conformally Flat Manifolds

In section 2 we introduced the conformal spin spaces $\hat{C}^n_k$. It is

$$\hat{C}^n_k=GSO(n,k)/BSO(n,k)=GSpin(n,k)/BSpin(n,k).$$

Let $\omega_{GSpin(n,k)}$ be the Maurer-Cartan form of $GSpin(n,k)$. The canonical form on $GSpin(n,k)\to\hat{C}^n_k$ is

$$\theta_{GSpin(n,k)}:=\omega_{GSpin(n,k)-1}+\omega_{GSpin(n,k)0}$$

and $\omega_{GSpin(n,k)}$ is the flat normal Cartan connection on $(GSpin(n,k), \theta_{GSpin(n,k)}) \to \hat{C}^n_k$. The enlarged $GSpin(n,k)$-principal fiber bundle on $\hat{C}^n_k$ is defined by

$$\overline{GSpin}(n,k) := GSpin(n,k) \times_{BSpin(n,k)} GSpin(n,k).$$

The bundle $\overline{GSpin}(n,k)$ is equipped with the induced flat connection $\bar{\omega}_{GSpin(n,k)}$. It is easy to give a canonical flat section γ in $\overline{GSpin}(n,k) \to \hat{C}^n_k$:

$$\gamma(a \cdot BSpin(n,k)) = [a, a^{-1}] \in \Gamma(\overline{GSpin}(n,k)).$$

The parallel sections in the spinor bundle $E = \overline{GSpin}(n,k) \times_{GSpin(n,k)} \Delta^{n+2}_{k+1}$ on $\hat{C}^n_k$ are given by

$$\psi_v(a \cdot BSpin(n,k)) := [[a, a^{-1}], v] \in \Gamma(E)$$

for arbitrary $v \in \Delta^{n+2}_{k+1}$. Let $g \in c$ be a conformally flat metric on $\hat{Q}^n_k$ and $\sigma^g : CSpin(\hat{C}^n_k) \to GSpin(n,k)$ the corresponding $CSpin(n,k)$-equivariant section. It is

$$\psi_v(\pi(l)) = [[\sigma^g(l), e], \sigma^g(l) \cdot v]$$

and the parallel section ψ_v induces a twistor spinor on $\hat{C}^n_k$ given by

$$\hat{\psi}_v(\pi(l)) := [l, \beta \circ proj^g_{n+1}(\sigma^g(l) \cdot v)] \in \Gamma(S^c_{\frac{1}{2}}),$$

where $l \in Spin(\hat{Q}^n_k, g)$. Since the bundle $E \to \hat{C}^n_k$ is trivial, we can conclude that the space of twistor spinors on $\hat{C}^n_k$ has maximal dimension $2^{[\frac{n}{2}]+1}$.

There is another way of constructing twistor spinors on $\hat{C}^n_k$. In general, let $M^{n+2}_{k+1} := (M, h_1)$ be a semi-Riemannian spin manifold of dimension $n+2$ and index $k+1$ with spin structure $(Spin(M^{n+2}_{k+1}), f)$. Further, let $F^n_k := (F, h_2)$ be a semi-Riemannian manifold of dimension n and index k and let $i : F^n_k \hookrightarrow M^{n+2}_{k+1}$ be an embedding such that the normal bundle NF^n_k of F^n_k in M^{n+2}_{k+1} can be trivialized by a global orthonormal frame field. Under these assumptions a spin structure $(Spin(M^{n+2}_{k+1}), f)$ on M^{n+2}_{k+1} induces in a natural way a spin structure on F^n_k denoted by

$$(Spin(F^n_k), f').$$

We denote the spinor bundles on M^{n+2}_{k+1} and F^n_k by S^M resp. S^F. Let (ζ_0, ζ_{n+1}) be any global orthonormal frame field in NF^n_k. The restricted spinor bundle $S^M|_F$ on F^n_k decomposes to the annulation spaces

$$Ann(\zeta_0 + \zeta_{n+1}) \oplus Ann(\zeta_0 - \zeta_{n+1})$$

(comp. section 3). Each of these annulation spaces is isomorphic to the spinor bundle S^F on F^n_k, i.e. we have an isomorphism:

$$S^M|_F \cong S^F \oplus S^F.$$

Let $\varphi \in \Gamma(S^M)$ be a spinor field and

$$\varphi|_F = \varphi_1 \oplus \varphi_2 \in \Gamma(S^F \oplus S^F).$$

The spinor derivatives ∇^M and ∇^F are related by

$$\begin{aligned}\nabla^M_X\varphi|_F &= \nabla^F_X\varphi_1 \oplus \nabla^F_X\varphi_2 + [\frac{1}{2}(\nabla^M_X\zeta_1)\cdot\zeta_1\cdot\varphi - \frac{1}{2}(\nabla^M_X\zeta_2)\cdot\zeta_2\cdot\varphi \\ &\quad +\frac{1}{2}g(\nabla^M_X\zeta_2,\zeta_1)\cdot\zeta_2\cdot\zeta_1\cdot\varphi]|_F \quad \forall X\in TF^n_k\end{aligned}$$

(comp. [BFGK91] for a similar formula in codimension 1).

We can apply this formula to the embedding

$$\begin{aligned} i: \quad \hat{Q}^n_k &\to (\mathbb{R}^{n+2}, \langle\ ,\ \rangle^{n+2}_{k+1}) \ . \\ \mathbb{R}_+\cdot x &\mapsto \sqrt{\frac{2}{\langle x,x\rangle^{n+2}_0}}\cdot x \end{aligned}$$

Notice that this embedding isn't the same as given in section 2. There is a natural orthonormal frame in the normal bundle of $\hat{Q}^n_k$ in $\mathbb{R}^{n+2}$ given by

$$\begin{aligned}\zeta_0(x) &:= (x_0,\dots,x_k,0,\dots,0),\\ \zeta_{n+1}(x) &:= (0,\dots,0,x_{k+1},\dots,x_{n+1}), \quad x\in\mathbb{R}^{n+2}.\end{aligned}$$

Let $\varphi_v(x) := x\cdot v$, $v\in\Delta^{n+2}_{k+1}$, be a twistor spinor in the spinor bundle $\mathbb{R}^{n+2}\times\Delta^{n+2}_{k+1}$ on $(\mathbb{R}^{n+2},\langle\ ,\ \rangle^{n+2}_{k+1})$. We denote

$$\varphi_v|_{\hat{Q}^n_k} = \varphi_{v_1}\oplus\varphi_{v_2}\in\Gamma(Ann(\zeta_0+\zeta_1)\oplus Ann(\zeta_0-\zeta_1)).$$

It holds

$$\begin{aligned}& g(\nabla^{\mathbb{R}^{n+2}}_X\zeta_0,\zeta_{n+1}) = 0,\\ & \nabla^{\mathbb{R}^{n+2}}_X\zeta_0 = (x_0,\dots,x_k,0,\dots,0) =: proj_T X,\\ & \nabla^{\mathbb{R}^{n+2}}_X\zeta_{n+1} = (0,\dots,0,x_{k+1},\dots,x_{n+1}) =: proj_S X\end{aligned}$$

for all $X=(x_0,\dots,x_{n+1})\in Ti(\hat{Q}^n_k)$ and

$$(\zeta_0+\zeta_{n+1})(x)\cdot\varphi_v|\hat{Q}^n_k = x\cdot x\cdot v|_{\hat{Q}^n_k} = 0 \quad \forall x\in i(\hat{Q}^n_k).$$

We obtain $\varphi_{v_2}\equiv 0$ and

$$\nabla^{\mathbb{R}^{n+2}}_X\varphi_v|_{\hat{Q}^n_k} = \nabla^{\hat{Q}^n_k}_X\varphi_{v_1} + [\frac{1}{2}proj_T X\cdot\zeta_0\cdot\varphi - \frac{1}{2}proj_S X\cdot\zeta_{n+1}\cdot\varphi_v]|_{\hat{Q}^n_k},$$

i.e. the term $g(X,X)X\cdot\nabla^{\hat{Q}^n_k}_X\varphi_{v_1}$ is independent of $X\in T\hat{Q}^n_k$ and with Proposition 3.1 we can conclude that

$$\varphi_{v_1}\in\Gamma(Ann(\zeta_0+\zeta_{n+1}))\cong\Gamma(S^{\hat{Q}^n_k})\cong\Gamma(S^{\hat{C}^n_k}_{\frac{1}{2}})$$

is a twistor spinor on $\hat{C}^n_k$ for any $v\in\Delta^{n+2}_{k+1}$.

In section 1 we introduced in general the notations of development and holonomy representation for flat Cartan geometries. Let (M,c) be an oriented and conformally flat manifold with canonical normal Cartan geometry $(BSO(M),\omega)$. Let $(\tilde{M},\tilde{c})$ denote the universal covering space of (M,c) with induced conformally flat structure $\tilde{c}$. The manifold $(\tilde{M},\tilde{c})$ is conformally spin. Then we have a development

$$\begin{array}{cccc} \bar{\delta}: & GSpin(\tilde{M}) & \rightarrow & \overline{GSpin}(n,k) \\ & \downarrow & & \downarrow \\ \bar{\delta}: & GSO(\tilde{M}) & \rightarrow & \overline{GSO}(n,k) \\ & \downarrow & & \downarrow \\ \delta^{\tilde{M}}: & \tilde{M} & \rightarrow & \hat{C}^n_k \end{array}$$

and a corresponding holonomy representation

$$\rho : \pi_1(M) \rightarrow GSO(n,k)$$

of the funfamental group of M into the automorphism group $GSO(n,k)$ of $(GSO(n,k),\omega_{GSO(n,k)})$. The manifold (M,c) is conformally spin if and only if ρ admits a lift

$$\tilde{\rho} : \pi_1(M) \rightarrow GSpin(n,k)$$

with respect to $\lambda : GSpin(n,k) \rightarrow GSO(n,k)$. The set of conformal spin structures on (M,c) correspond bijectively to the set of lifts of the holonomy representation ρ (comp. [Baum81]).

Theorem 5.1 *Let (M,c) be a conformally flat spin manifold with development δ and holonomy representation ρ and let*

$$\tilde{\rho} : \pi_1(M) \rightarrow GSpin(n,k)$$

be a lift of ρ, which defines a spin structure on (M,c). Let $V \subset \Delta^{n+2}_{k+1}$ be the maximal subspace, on which the representation $\tilde{\rho}$ acts trivially. Then every $v \in V$ gives rise to a twistor spinor φ_v on $(M,c,\tilde{\rho})$. Moreover, the space $J(M,c,\tilde{\rho})$ of twistor spinors on $(M,c,\tilde{\rho})$ and the space V have the same dimension.

Proof. Let $\delta^{\tilde{M}} : (\tilde{M},\tilde{c}) \rightarrow \hat{C}^n_k$ be the development, which induces the holonomy representation ρ. The development $\delta^{\tilde{M}}$ lifts to a bundle morphism

$$\delta^{\tilde{M}}_* : E(\tilde{M},\tilde{c}) \rightarrow E(\hat{C}^n_k)$$

of the conformal spinor bundles. Let $v \in V$ be arbitrary, ψ_v the corresponding parallel section in $E(\hat{C}^n_k)$ and let $\tilde{\varphi}_v$ be the parallel section in $E(\tilde{M},\tilde{c})$ such that $\delta^{\tilde{M}}_*(\tilde{\varphi}_v) = \psi_v$. Since ρ acts trivially on V, the parallel section $\tilde{\varphi}_v$ projects to a parallel section $\varphi_v := \pi_*\tilde{\varphi}_v$ in $E(M,c,\tilde{\rho})$. Conversely, let φ be a parallel section in $E(M,c,\tilde{\rho})$ and $\tilde{\varphi}$ its lift to $E(\tilde{M},\tilde{c})$. Then, it exists a unique spinor $v \in V$ such that $\psi_v = \delta^{\tilde{M}}_*(\tilde{\varphi})$. The fact that parallel sections in $E(M,c,\tilde{\rho})$ correspond bijectively to twistor spinors on $(M,c,\tilde{\rho})$ proves the theorem $\square$

A twistor spinor on the Riemannian sphere $\hat{C}^n_0 = S^n$ has at most one zero. This observation leads to the following Corollary of Theorem 5.1, which is a result of Kühnel and Rademacher proved in [KR96]:

Corollary 5.2 *Let (M, g) be an oriented conformally flat Riemannian manifold with holonomy repesentation*

$$\rho : \pi_1(M) \to SO_o(n+2, 1).$$

Then (M, g) admits a twistor spinor with zero if and only if the following conditions are satisfied:

i) The holonomy representation ρ fixes a point $p \in \hat{C}^n_0 = S^n$.

ii) The linear holonomy representation ρ_{*p} in $p \in \hat{C}^n_0$ is orthogonal:

$$\rho_{*p} : \pi_1(M) \to SO(n).$$

iii) The linear holonomy representation ρ_{*p} has a lift

$$\tilde{\rho}_{*p} : \pi_1(M) \to Spin(n)$$

with respect to λ.

iv) The representation $\tilde{\rho}_{*p}$ acts trivially on a non-trivial subspace V of the spinor modul Δ^n_0.

Example: Twistor spinors on a Möbius sphere C^n_k for $1 < k < n-1$
The two-fold covering space $\hat{C}^n_k$ of C^n_k is simply connected and consequently it holds $\pi_1(M) = \mathbb{Z}_2$. The corresponding holonomy representation is given by

$$\begin{array}{rcl} \rho : & \mathbb{Z}_2 & \to SO(n+2, k+1) \ . \\ & \{\pm 1\} & \to \{\pm I\} \end{array}$$

Let

$$\gamma := \frac{1}{2}(f_0 - f_{n+1}) \cdot f_1 \cdot \ldots \cdot f_n \cdot (f_0 + f_{n+1}) \in Spin(n+2, k+1).$$

It holds $\lambda(\gamma) = -I$. The holonomy representation ρ admits a lift to $Spin(n+2, k+1)$ if and only if $\gamma^2 = 1$, which is equivalent to the condition

$$\begin{array}{lll} n = 2 \bmod 4, & k = 1 \bmod 2 & \text{or} \\ n = 2 \bmod 4, & k = 0 \bmod 2. & \end{array}$$

In this case there are two conformal spin structures on C^n_k, which are characterized by the following two lifts of the holonomy representations:

$$\begin{array}{rcl} \tilde{\rho}_+ : & \mathbb{Z}_2 & \to Spin(n+2, k+1), \\ & \{1, -1\} & \mapsto \{1, \gamma\} \\ \tilde{\rho}_- : & \mathbb{Z}_2 & \to Spin(n+2, k+1). \\ & \{1, -1\} & \mapsto \{1, -\gamma\} \end{array}$$

Moreover, the spinor modul Δ^{n+2}_{k+1} splits into a positive and a negative part:

$$\begin{aligned}\Delta^{n+2}_{k+1} &= \Delta^{n+2}_{k+1+} \oplus \Delta^{n+2}_{k+1-},\\ \gamma \cdot \Delta^{n+2}_{n+1+} &\subset \Delta^{n+2}_{k+1-},\\ \gamma \cdot \Delta^{n+2}_{n+1-} &\subset \Delta^{n+2}_{k+1+}.\end{aligned}$$

Let us define

$$\begin{aligned}V_+ &:= \{v + \gamma v : v \in \Delta^{n+2}_{k+1+}\} \subset \Delta^{n+2}_{k+1},\\ V_- &:= \{v - \gamma v : v \in \Delta^{n+2}_{k+1+}\} \subset \Delta^{n+2}_{k+1}.\end{aligned}$$

The spaces V_+ and V_- are the eigenspaces of γ to the eigenvalues 1 resp. -1, i.e. $\tilde{\rho}_+$ acts trivially on V_+, whereas $\tilde{\rho}_-$ acts trivially on V_-. Since $\dim V_+ = \dim V_- = \frac{1}{2} \dim \Delta^{n+2}_{n+1} = 2^{[\frac{n}{2}]}$, it follows that on $(C^n_k, \tilde{\rho}_+)$ and $(C^n_k, \tilde{\rho}_-)$ there exist $2^{[\frac{n}{2}]}$ linearly independent twistor spinors. This is one half of the maximal possible dimension.

Example: In the Lorentzian case, where $k = 1$, it holds

$$\pi_1(\hat{C}^n_1) = \mathbb{Z}, \quad H^1(\hat{C}^n_1, \mathbb{Z}) = \mathbb{Z}_2.$$

This implies that two conformal spin structures exist on $\hat{C}^n_1$. The natural one is induced by the embedding of $\hat{C}^n_1$ in $\mathbb{R}^{n+2}_{k+1}$ and admits a set of twistor spinors of maximal dimension, whereas on $\hat{C}^n_1$ furnished with the 'non-natural' conformal spin structure no twistor spinors exist.

References

[CSS] A. Čap, J. Slovak, V. Souček. *Invariant Operators on Manifolds with Almost Hermitian Symmetric Structures I & II.* Acta. Math. Univ. Comen., New Ser. 66, No. 1, p. 33-69 & No. 2, p. 203-220(1997).

[Sha96] R.W. Sharpe. *Differential Geometry.* Graduate Texts in Mathematics 166. Springer-Verlag New York, 1972.

[KR96] W. Kühnel and H.-B. Rademacher. *Twistor Spinors on Conformally Flat Manifolds.* Illinois. J. Math. 41(1997), 495-503.

[Baum81] Helga Baum. *Spin-Strukturen und Dirac-Operatoren über pseudo-Riemannschen Mannigfaltigkeiten.* Nummer 41 in Teubner-Texte zur Mathemtik. Teubner, 1981.

[Fri76] Helmut Friedrich. *Twistor Connection and Normal Conformal Cartan Connection.* General Relativity and Gravitation, Vol. 8. No. 5(1977), 303-312.

[BFGK91] H. Baum, T. Friedrich, R. Grunewald and I. Kath. *Twistor and Killing spinors on Riemannian manifolds.* Volume 124 of Teubner-Texte zur Mathematik. Teubner-Verlag Stuttgart/Leipzig, 1991.

[Sch97] M. Schottenloher. *A Mathematical Introduction to Conformal Field Theory.* Springer Verlag Berlin Heidelberg, 1997.

[Kob72] S. Kobayashi. *Transformation Groups in Differential Geometry.* Springer-Verlag Berlin Heidelberg, 1972.

[Och70] T. Ochiai. *Geometry associated with semisimple flat homogenous spaces.* Trans. Amer. Math. Soc. 152(1970), 159-193.

[PM72] R. Penrose, M. A. MacCallum. Phys. Rep., 6C, 241.