Weyl function and spectral properties of self-adjoint extensions

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We characterize the spectra of self-adjoint extensions of a symmetric operator with equal deficiency indices in terms of boundary values of their Weyl functions. A complete description is obtained for the point and absolutely continuous spectrum while for the singular continuous spectrum additional assumptions are needed. The results are illustrated by examples.

Key Words: symmetric operators, self-adjoint extensions, abstract boundary conditions, Weyl function

1. INTRODUCTION

Let A be a closed symmetric operator on a separable Hilbert space $\mathfrak{H}$. If the operator A has equal deficiency indices $n_{\pm}(A) = \dim(\mathfrak{H} \ominus \operatorname{ran}(A \pm iI))$, then the symmetric operator A has a lot of self-adjoint extensions. These self-adjoint extensions can be labeled by the so-called Weyl function $M(\cdot)$. This function was introduced by Weyl [30, 31, 32] in order to investigate Sturm-Liouville problems. However, later it became clear that the Weyl function makes sense not only for the Sturm-Liouville problem but also for an arbitrary closed symmetric operator with equal deficiency indices. For this purpose it was necessary to generalized the method of boundary conditions used in the theory of Sturm-Liouville problems to arbitrary closed symmetric operators with equal deficiency indices.

This generalization is based on the concept of a boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* being an abstract generalization of the Green's identity (see [22] and references therein). Here $\mathcal{H}$ is a separable Hilbert space with $\dim(\mathcal{H}) = n_{\pm}(A)$ and Γ_0 and Γ_1 are linear mappings from $\operatorname{dom}(A^*)$ to $\mathcal{H}$ so that Green's identity is satisfied, see below (10). One naturally associates with this triple two self-adjoint extensions A_i defined by $\operatorname{dom}(A_i) = \ker(\Gamma_i)$ $(i = 0, 1)$.

On this basis in [14] the notion of Weyl function was generalized to closed symmetric operators with equal deficiency indices, see also [15, 16]. In the next section we give a brief introduction to the abstract theory of boundary triples and Weyl functions.

If A is a simple closed symmetric operator in $\mathfrak{H}$, then the Weyl function $M(\cdot)$ corresponding to a boundary triple Π determines the pair $\{A, A_0\}$ (as well as the pair $\{A, A_1\}$) up to unitary equivalence. Therefore the problem arises to describe the spectrum and certain spectral subsets as point, singular, singular continuous and absolutely continuous spectrum of self-adjoint extensions of A in terms of the Weyl function. Of course, since in the abstract sense self-adjoint extensions of A and Weyl functions are identical objects the spectral information has to be encoded in the Weyl function. Indeed, Lemma 3.2 shows that it is so. However, under practical aspects this correspondence is useless since the reconstruction of the spectral properties is very implicit.

Our aim is to describe spectral subsets by the boundary values of Weyl function at the real axis. So the problem is the following: Let $M(\cdot)$ be the Weyl function of a certain self-adjoint extension A_0 of A. Introducing the associated scalar Weyl function $M_h(\cdot) := (M(\cdot)h, h)$, $h \in \mathcal{H}$, is it possible to localize the different spectral subsets of A_0 knowing the boundary values $M_h(x + i0)$, $x \in \mathbb{R}$, of the associated scalar Weyl function for a suitable set $\mathcal{T}$ of elements of $\mathcal{H}$. It turns out that the point and absolutely continuous spectrum can be exactly localized in this way, see Theorem 3.3 and Theorem 4.1. We also mention Theorem 4.2 and Corollary 4.2 which indicate sufficient conditions for A_0 to be purely absolutely continuous within an arbitrary interval (a, b).

With respect to the singular continuous spectrum we get in general an incomplete information. That means, we are only able to indicate a subset in which the singular continuous spectrum is contained, see Theorem 3.3. However, under certain circumstances we are able to localized the singular continuous spectrum too, see Theorem 4.1 and Corollary 4.1. We note in this connection that in our paper [8] we applied Theorem 4.1 in order to construct self-adjoint extension A_0 with arbitrary pure singular spectrum within the gap (α, β) of a symmetric operator A.

The paper is organized as follows. In Section 2 we define and recall the notions of R-function, boundary triple, Weyl function and γ-field which are important ingredients for the next sections. In Section 3 we investigate the relation between the boundary values of the Weyl function and the spectral properties of the self-adjoint extensions. Finally, in Section 4 we prove a refinement of the results for the absolutely continuous spectrum and indicate sufficient conditions which allow to localize exactly the singular continuous spectrum. In Section 5 we give examples which illustrate the abstract observations of the sections above. In particular we consider here an infinite-dimensional Dirac operator $A_0 = A_0^* = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \otimes \frac{d}{dx} + \begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix}$ in $L^2(\mathbb{R}, \mathcal{H})$, $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$, with an arbitrary nonnegative bounded operator T in $\mathcal{K}$. Applying Theorem 4.2 we establish that A_0 is purely absolutely continuous. We finish the paper with some general remarks in Section 6.

In a forthcoming paper we apply our general results from Sections 3 and 4 to inverse spectral theory (in the sense of [2], [6]-[12]) of a symmetric operator with several gaps.

In conclusion the reader is refered to recent publications [19, 20] which are closed to our paper. In [19, 20] a spectral analysis of matrix-valued Herglotz (Nevanlinna) functions as well as their application to the well-known Aronszajn-Donoghue theory is considered.

2. PRELIMINARIES

2.1. R-functions

Let $\mathcal{H}$ be a separable Hilbert space. Recall that an operator function $F(\cdot)$ with values in $[\mathcal{H}]$ is said to be a Herglotz or Nevanlinna function [1, 5, 19, 20, 22] as well as a R-function [25] if it is holomorphic in $\mathbb{C}_+$ and for every $z \in \mathbb{C}_+$ the operator $F(z)$ in $\mathcal{H}$ is dissipative, i.e.

$$\Im\mathrm{m}(F(z)) := \frac{F(z) - F(z)^*}{2i} \geq 0. \tag{1}$$

In the following we prefer the notion R-function. The class of R-functions with values in $[\mathcal{H}]$ will be denoted by $(R_{\mathcal{H}})$. If $F(\cdot) \in (R_{\mathcal{H}})$, then there exist bounded self-adjoint operators C_0, $C_1 \geq 0$, a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$, a self-adjoint operator L in $\mathcal{K}$, a bounded non-negative operator $R \geq 0$ with $R|\mathcal{K} \ominus \mathcal{H} = 0$ such that

$$F(z) = C_0 + C_1 z + R^{1/2}(I_{\mathcal{K}} + zL)(L - z)^{-1} R^{1/2}|\mathcal{H}, \quad z \in \mathbb{C}_+, \qquad (2)$$

see [26]. Denoting by $E_L(\cdot)$ the spectral measure of the self-adjoint operator L one immediately obtains from (2) the representation

$$F(z) = C_0 + C_1 z + \int_{-\infty}^{\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^2}\right) d\Sigma_F(t), \quad z \in \mathbb{C}_+, \qquad (3)$$

where $\Sigma_F(\cdot)$ is an operator-valued Borel measure on $\mathbb{R}$ given by

$$d\Sigma_F(t) = (1 + t^2) R^{1/2} dE_L(\cdot) R^{1/2}, \qquad t \in \mathbb{R}. \qquad (4)$$

Obviously, the measure $\Sigma_F(\cdot)$ is self-adjoint and obeys

$$\int_{-\infty}^{+\infty} \frac{1}{1 + t^2} d\Sigma_F(t) \in [\mathcal{H}]. \qquad (5)$$

In contrast to spectral measures of self-adjoint operators it is not necessarily true that $\mathrm{ran}(\Sigma(\delta_1))$ is orthogonal to $\mathrm{ran}(\Sigma(\delta_2))$ for disjoint Borel sets δ_1 and δ_2. However, the measure $\Sigma_F(\cdot)$ is uniquely determined by the R-function $F(\cdot)$. The integral in (3) is understood in the strong sense. In the following the operator-valued measure $\Sigma_F(\cdot)$ is called the spectral measure of $F(\cdot)$.

Usually, with the operator-valued measure $\Sigma_F(\cdot)$ one associates an distribution operator-valued function $\Sigma_F(\cdot)$ which is defined by

$$\Sigma_F(t) = \begin{cases} \Sigma_F([0, t)) & : \ t > 0 \\ 0 & : \ t = 0 \\ -\Sigma_F([t, 0)) & : \ t < 0. \end{cases} \qquad (6)$$

It turns out that the distribution function $\Sigma_F(\cdot)$ is strongly left continuous and satisfies the conditions

$$\Sigma_F(t) = \Sigma_F(t)^*, \quad \Sigma_F(s) \leq \Sigma_F(t), \quad -\infty < s \leq t < \infty. \qquad (7)$$

In the following the distribution function $\Sigma_F(\cdot)$ is called the spectral function of $F(\cdot)$.

We note that the spectral function $\Sigma_F(\cdot)$ of $F(\cdot)$ can be obtained by the Stieltjes transformation:

$$\frac{1}{2}(\Sigma_F(t+0)+\Sigma(t))-\frac{1}{2}(\Sigma(s+0)+\Sigma(s))= \qquad (8)$$
$$w-\lim_{y\downarrow 0}\frac{1}{\pi}\int_s^t \Im\mathrm{m}(F(x+iy))dx, \quad t,s\in\mathbb{R}$$

where it is used that the spectral function is strongly left continuous.

2.2. Boundary triples and self-adjoint relations

In the following A will always denote a closed symmetric operator with equal deficiency indices $n_+(A)=n_-(A)$. Without loss of generality we can assume that A is simple. This means that A has no self-adjoint parts.

Our approach to the inverse spectral theory of self-adjoint extensions is based on the concept of boundary triples (see [22] and references therein) which may be considered as an abstract version of the second Green's formula.

Definition 2.1. A triple $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ consisting of an auxiliary Hilbert space $\mathcal{H}$ and linear mappings

$$\Gamma_i : \mathrm{dom}(A^*) \longrightarrow \mathcal{H}, \quad i=0,1, \qquad (9)$$

is called a boundary triple for the adjoint operator A^* of A if the following two conditions are satisfied:

(i) The second Green's formula

$$(A^*f,g)-(f,A^*g)=(\Gamma_1 f,\Gamma_0 g)-(\Gamma_0 f,\Gamma_1 g), \quad f,g\in\mathrm{dom}(A^*), \qquad (10)$$

takes place and

(ii) the mapping

$$\Gamma:\mathrm{dom}(A^*)\longrightarrow\mathcal{H}\oplus\mathcal{H}, \quad \Gamma f:=\{\Gamma_0 f,\Gamma_1 f\}, \qquad (11)$$

is surjective.

Let us make the following remarks.

Remark 2. 1. (i) The boundary triple for the adjoint A^* of the symmetric operator A is not unique.

(ii) It is easily seen that for each self-adjoint extension $\tilde{A}$ of A there exists a boundary triple $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ such that

$$\mathrm{dom}(\tilde{A})=\ker(\Gamma_0) \qquad (12)$$

is valid.

(iii) Moreover (see [16]) for each transversal pair $A_0, A_1 \in Ext_A$ (that means $\mathrm{dom}(A_0) \cap \mathrm{dom}(A_1) = \mathrm{dom}(A)$ and $\mathrm{dom}(A_0) + \mathrm{dom}(A_1) = \mathrm{dom}(A^*)$) there exists a boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ such that the condition

$$\mathrm{dom}(A_i) = \ker(\Gamma_i), \quad i = 0, 1. \tag{13}$$

is satisfied.

(iv) For each boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of A^* and each bounded self-adjoint operator B in $\mathcal{H}$ also $\Pi_B = \{\mathcal{H}, \Gamma_0^B, \Gamma_1^B\}$ is a boundary triple for A^* where

$$\Gamma_1^B := \Gamma_0, \quad \Gamma_0^B := B\Gamma_0 - \Gamma_1. \tag{14}$$

We set $A_0^B := \ker(\Gamma_0^B)$.

Once a boundary triple for A^* is fixed one can give explicitly a one-to-one correspondence between the set Ext_A of self-adjoint extensions of A and the set of self-adjoint linear relations Θ in $\mathcal{H}$. To explain this we recall the definition of self-adjoint relations:

Definition 2.2. (i) A closed linear relation Θ in $\mathcal{H}$ is a closed subspace Θ of $\mathcal{H} \oplus \mathcal{H}$.

(ii) The closed linear relation Θ is symmetric if the condition

$$(g_1, f_2) - (f_1, g_2) = 0, \quad \{f_1, g_1\}, \{f_2, g_2\} \in \Theta. \tag{15}$$

is satisfied.

(iii) The closed linear relation Θ is self-adjoint if it is maximal symmetric, i.e., Θ is symmetric and there does not exist a closed symmetric relation $\tilde{\Theta}$ such that Θ is properly contained in $\tilde{\Theta}$.

Let us illustrate closed linear relations by simple examples.

Example 2.1. (i) Let B be a